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GEODESIC MAPPINGS OF MANIFOLDS WITH AFFINE CONNECTION ONTO SYMMETRIC MANIFOLDS

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Abstract. In this paper we study fundamental equations of geodesic mappings of manifolds with affine connection onto symmetric manifolds. We obtain fundamental equations of this problem. At the end of our paper we demonstrate example of studied mappings.

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1. Introduction

This paper is devoted to the theory of geodesic mappings. This theory was initiated by Levi-Civita in [17]. He studied and solved problem about finding metrics of Riemannian spaces with common geodesics. It is very interesting because these problems are closely linked to dynamical equations of a mechanical systems.

After that the theory of geodesic mappings, it was further developed by Thomas, Weyl, Shirokov, Solodovnikov, Sinyukov and others [1, 4–9, 13–15, 19–24, 26, 27, 29, 30, 32, 35, 37–41].

Some authors, see [12, 29, 30, 37], decided to study geodesic mappings on symmetric spaces, which was introduced by Cartan [3].

We note that geodesic and generalized mappings of symmetric, recurrent and general recurrent spaces studied many authors, e.g. [2, 13, 16, 18, 20–24, 29, 30, 33–37].

In our paper the fundamental equations of geodesic mappings of manifolds with affine connection onto symmetric manifolds were obtained as closed differential equations system of Cauchy type in covariant derivative. The essential number of parameters on which depend the general solutions was determined too.

We suppose that the studied spaces are simply connected and for their dimension it pays that $n > 2$. Next, we will suppose that geometric objects are continuous and sufficiently differentiable.

2. Geodesic Mappings of Manifolds With Affine Connection

A diffeomorphism f between manifolds A_n and $\bar{A}_n$ is **geodesic mapping** if f maps any geodesic in A_n onto a geodesic in $\bar{A}_n$.

Let us suppose that A_n admits geodesic mapping onto $\bar{A}_n$ and coordinate system (U, x) at a point p is "common" of f, i.e., image $f(p)$ has the same coordinate $(x^1, x^2, \ldots, x^n)$ such point p.

Let we have the deformation tensor of diffeomorphism f

$$P^h_{ij}(x) = \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x) \tag{1}$$

where $\Gamma^h_{ij}(x)$ and $\bar{\Gamma}^h_{ij}(x)$ are components of affine connection A_n and $\bar{A}_n$, respectively.

It is well known [6, 29, 30, 32, 37] that diffeomorphism $f\colon A_n \to \bar{A}_n$ is geodesic if and only if in common coordinate system $(x^1, x^2, \ldots, x^n)$ the deformation tensor of connections has the following form

$$P^h_{ij}(x) = \psi_i(x)\delta^h_j + \psi_j(x)\delta^h_i \tag{2}$$

where δ^h_i is the Kronecker symbol and $\psi_i(x)$ are components of a covector field.

The geodesic mapping is *non-trivial* if $\psi_i(x) \not\equiv 0$.

Evidently any manifold with affine connection A_n admit non-trivial geodesic mapping onto a certain other manifold with affine connection $\bar{A}_n$. This mapping is determined by the vector field ψ. This proposition does not valid generally if manifold $\bar{A}_n$ is Riemannian.

We obtained [23], see [30, p. 283], fundamental linear differential equations system of Cauchy type for geodesic mappings of spaces with affine connection onto (pseudo-) Riemannian spaces.

3. Geodesic Mappings of Manifolds With Affine Connection Onto Symmetric Manifolds

Manifold with affine connection is called (locally) **symmetric space**, if its curvature tensor is absolutely parallel (P. Shirokov [35], É. Cartan [3], Helgason [10]), see [29, 30, 32, 37].

These manifolds are playing important role in the general theory of relativity and the theory of Killing and general Killing vector fields [32].
We suppose that manifold $\bar{A}_n$ is symmetric, i.e., we have the following condition

$$\bar{R}^h_{ijk|m}(x) = 0 \tag{3}$$

where $\bar{R}^h_{ijk}(x)$ is the Riemannian (curvature) tensor $\bar{A}_n$, the symbol " | " denote a covariant derivative in $\bar{A}_n$.
Because

$$\bar{R}^h_{ijk|m} = \frac{\partial \bar{R}^h_{ijk}}{\partial x^m} + \bar{\Gamma}^h_{i\alpha}\bar{R}^\alpha_{ijk} - \bar{\Gamma}^\alpha_{mi}\bar{R}^h_{\alpha jk} - \bar{\Gamma}^\alpha_{mj}\bar{R}^h_{i\alpha k} - \bar{\Gamma}^\alpha_{mk}\bar{R}^h_{ij\alpha}$$

then from formula (1) it follows

$$\bar{R}^h_{ijk|m} = \bar{R}^h_{ijk,m} + P^h_{m\alpha}\bar{R}^\alpha_{ijk} - P^\alpha_{mi}\bar{R}^h_{\alpha jk} - P^\alpha_{mj}\bar{R}^h_{i\alpha k} - P^\alpha_{mk}\bar{R}^h_{ij\alpha} \tag{4}$$

where the symbol " , " denote a covariant derivative in A_n.
Insomuch as manifold $\bar{A}_n$ is symmetric than from formulas (3) and (4), we obtain

$$\bar{R}^h_{ijk,m} = -P^h_{m\alpha}\bar{R}^\alpha_{ijk} + P^\alpha_{mi}\bar{R}^h_{\alpha jk} + P^\alpha_{mj}\bar{R}^h_{i\alpha k} + P^\alpha_{mk}\bar{R}^h_{ij\alpha}. \tag{5}$$

When A_n admit geodesic mapping onto $\bar{A}_n$ formula (2) holds. Then from (5) follows

$$\bar{R}^h_{ijk,m} = 2\psi_m\bar{R}^h_{ijk} + \psi_j\bar{R}^h_{imk} + \psi_k\bar{R}^h_{ijm} - \delta^h_m\psi_\alpha\bar{R}^\alpha_{ijk}. \tag{6}$$

Between the curvature tensors of A_n and $\bar{A}_n$ the following formula

$$\bar{R}^h_{ijk} = R^h_{ijk} + P^h_{ik,j} - P^h_{ij,k} + P^\alpha_{ik}P^h_{j\alpha} - P^\alpha_{ij}P^h_{k\alpha} \tag{7}$$

holds [29, 30, 37].
Using $P^h_{ij,k} = \psi_{i,k}(x)\delta^h_j + \psi_{j,k}(x)\delta^h_i$ and from formula (7) after some calculation we obtain

$$\bar{R}^h_{ijk} = R^h_{ijk} - \delta^h_j\psi_{i,k} + \delta^h_k\psi_{i,j} - \delta^h_i\psi_{j,k} + \delta^h_i\psi_{k,j} + \delta^h_j\psi_i\psi_k - \delta^h_k\psi_i\psi_j. \tag{8}$$

Contracting (8) with respect to indices h and k, we get

$$\bar{R}_{ij} = R_{ij} + n\,\psi_{i,j} - \psi_{j,i} + (1-n)\,\psi_i\psi_j \tag{9}$$

where R_{ij} and $\bar{R}_{ij}$ are the Ricci tensors on A_n and $\bar{A}_n$, respectively.
Alternating above formula we have

$$\bar{R}_{[ij]} = R_{[ij]} + (n+1)\,(\psi_{i,j} - \psi_{j,i}) \tag{10}$$

where square bracket denote alternating of indices without dividing.
From formula (10) it follows

$$\psi_{i,j} - \psi_{j,i} = \frac{1}{n+1}\,(\bar{R}_{[ij]} - R_{[ij]}). \tag{11}$$

Using formulas (9) and (11) we obtain

$$\psi_{i,j} = \psi_i\psi_j + \frac{1}{n-1}\,(\bar{R}_{ij} - R_{ij}) - \frac{1}{n^2-1}\,(\bar{R}_{[ij]} - R_{[ij]}). \tag{12}$$

It is clear to see that equations (6) and (12) on the manifold A_n form a closed differential equations system of Cauchy type respective unknown functions $\bar{R}^h_{ijk}$ and $\psi_i(x)$, which naturally satisfy the following algebraic conditions

$$\bar{R}^h_{i(jk)} = 0 \quad \text{and} \quad \bar{R}^h_{(ijk)} = 0 \tag{13}$$

where brackets denote symmetrization with respect to indices without dividing.
In addition on conditions (13), it must valid conditions for curvature tensor $\bar{R}$ as well. These conditions are derived from condition (8) if we exclude $\psi_{i,j}$ using equations (12). Thus the conditions have following form

$$\begin{aligned} \bar{R}^h_{ijk} = R^h_{ijk} - \delta^h_j \left[\tfrac{1}{n-1}\,(\bar{R}_{ik} - R_{ik}) - \tfrac{1}{n^2-1}\,(\bar{R}_{[ik]} - R_{[ik]})\right] \\ +\delta^h_k \left[\tfrac{1}{n-1}\,(\bar{R}_{ij} - R_{ij}) - \tfrac{1}{n^2-1}\,(\bar{R}_{[ij]} - R_{[ij]})\right] \\ +\delta^h_i \tfrac{1}{n+1}\,(\bar{R}_{[jk]} - R_{[jk]}). \end{aligned} \tag{14}$$

On the base of above mentioned we obtain the following theorem.

Theorem 1. *A manifold A_n admits geodesic mappings onto a symmetric manifold $\bar{A}_n$ if and only if on A_n exists solution respective unknown functions $\bar{R}^h_{ijk}$ and $\psi_i(x)$ of a closed differential equations system of Cauchy type*

$$\begin{aligned} \bar{R}^h_{ijk,m} &= 2\psi_m \bar{R}^h_{ijk} + \psi_j \bar{R}^h_{imk} + \psi_k \bar{R}^h_{ijm} - \delta^h_m \psi_\alpha \bar{R}^\alpha_{ijk} \\ \psi_{i,j} &= \psi_i\psi_j + \tfrac{1}{n-1}\,(\bar{R}_{ij} - R_{ij}) - \tfrac{1}{n^2-1}\,(\bar{R}_{[ij]} - R_{[ij]}). \end{aligned}$$

with algebraic conditions (14).

We remark that equation (6) is a completely integrable.
By direct substitution, we convince, that from conditions (14) follow conditions (13).
On the base of algebraic properties (14) of unknown functions $\bar{R}^h_{ijk}$ the general solution of Theorem 1 depends no more than

$$n(n+1)$$

essential parameters. For these initial conditions at point x_0 it pays, that

$$\psi_i(x_0) = \psi^0_i \quad \text{and} \quad \bar{R}_{ij}(x_0) = \bar{R}^0_{ij}.$$

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