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DEFORMATION QUANTIZATION OF KÄHLER MANIFOLDS AND THEIR TWISTED FOCK REPRESENTATION

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Abstract. We introduce the notion of twisted Fock representations of noncommutative Kähler manifolds and give their explicit expressions. The so-called twisted Fock representation is a representation of the Heisenberg like algebra whose states are constructed by acting creation operators on a vacuum state. “Twisted" means that creation operators are not Hermitian conjugate of annihilation operators. In deformation quantization of Kähler manifolds with separation of variables formulated by Karabegov, local complex coordinates and partial derivatives of the Kähler potential with respect to coordinates satisfy the commutation relations between the creation and annihilation operators. Based on these relations, the twisted Fock representation of noncommutative Kähler manifolds is constructed.

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1. Introduction

Deformation quantization can be considered as a way to construct noncommutative manifolds. In this article, the deformation quantization with separation of variables is used to construct noncommutative Kähler manifolds, which was proposed by Karabegov [1–3]. The deformation quantization is an associative algebra on a set of formal power series of C^∞ functions with a star product between formal power series. In deformation quantization, the analytical techniques are available on noncommutative manifolds because star products are usually represented by using bidifferential operators. On the other hand, in analyses of field theories on noncommutative manifolds constructed by deformation quantization, physical quantities are given as formal power series of a noncommutative parameter, and thus it is difficult to give them physical interpretations. In order to solve the difficulties, we here make representations of the noncommutative algebra.

In this article, we construct the Fock representation of noncommutative Kähler manifolds which are given by deformation quantization with separation of variables. It is immediately realized that the noncommutative algebras constructed by the method contain Heisenberg like algebras. Namely, local complex coordinates and partial derivatives of a Kähler potential satisfy the commutation relations between creation and annihilation operators. A Fock space is spanned by a vacuum and states obtained by applying creation operators on this vacuum. We represent the algebras on noncommutative Kähler manifolds as linear operators acting on the Fock space. We call this representation the Fock representation. In representations investigated in this article, creation operators and annihilation operators are not Hermitian conjugate with each other, in general. Thus, the bases of the Fock space are not given as the Hermitian conjugates of those of the dual vector space. In this case, we call the representation the twisted Fock representation. We define the twisted Fock representation on a local coordinate chart, and then glue between the representations on charts with nonempty intersections. Therefore, we construct transition functions between the twisted Fock algebras on two charts having an overlapping region. We observe several examples, $\mathbb{C}^N$, $\mathbb{CP}^N$ and $\mathbb{CH}^N$.

2. Deformation Quantization of Kähler Manifolds

A general definition of deformation quantization is the following.

Definition 1 (Deformation quantization (weak sense)). Let M be a Poisson manifold. $\mathcal{F}$ is defined as a set of formal power series

$$\mathcal{F} := \left\{ f \ ; \ f = \sum_k f_k \hbar^k, \ f_k \in C^\infty(M) \right\}. \tag{1}$$

Deformation quantization is defined as a structure of associative algebra of $\mathcal{F}$ whose product is defined by a star product. The star product is defined as

$$f * g = \sum_k C_k(f, g) \hbar^k \tag{2}$$

such that the product satisfies the following conditions

1. $*$ is associative product.
2. C_k is a bidifferential operator.

3. C_0 and C_1 are defined as

$$C_0(f,g) = fg \tag{3}$$

$$C_1(f,g) - C_1(g,f) = \mathrm{i}\{f,g\} \tag{4}$$

where $\{f,g\}$ is the Poisson bracket.

4. $f * 1 = 1 * f = f$.

In particular, Karabegov [1–3] proposed the notion of deformation quantization with separation of variables for Kähler manifolds quantization.

Definition 2 (A star product with separation of variables). The operation $*$ is called a star product with separation of variables when

$$a * f = af \tag{5}$$

for a holomorphic function a and

$$f * b = fb \tag{6}$$

for an anti-holomorphic function b.

We here consider only this type of deformation quantization for Kähler manifolds. Let M be an N-dimensional complex Kähler manifold, Φ be its Kähler potential and ω be its Kähler two-form

$$\omega := \mathrm{i} g_{k\bar{l}} \mathrm{d}z^k \wedge \mathrm{d}\bar{z}^l, \qquad g_{k\bar{l}} := \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l}. \tag{7}$$

Here g is the Kähler metric and $z^i, \bar{z}^j$ $(i,j = 1, \cdots, N)$ are local coordinates on an open set $U \subset M$ which is diffeomorphic to a connected open subset of $\mathbb{C}^N$. In this article, the Einstein summation convention over repeated indices is used. The $g^{\bar{k}l}$ is the inverse of the metric $g_{k\bar{l}}$

$$g^{\bar{k}l} g_{l\bar{m}} = \delta_{\bar{k}\bar{m}}. \tag{8}$$

In the following, we use the following notations for notational simplicity

$$\partial_k = \frac{\partial}{\partial z^k}, \qquad \partial_{\bar{k}} = \frac{\partial}{\partial \bar{z}^k}. \tag{9}$$

We here briefly explain the method to construct a star product with separation of variables for Kähler manifolds [1, 2], For the left star multiplication by $f \in \mathcal{F}$, there exists a differential operator L_f such that

$$L_f g = f * g. \tag{10}$$

L_f is given as a formal power series in $\hbar$

$$L_f = \sum_{n=0}^{\infty} \hbar^n A^{(n)} \tag{11}$$

where $A^{(n)}$ is a differential operator which contains only partial derivatives by the holomorphic coordinates z^i $(i = 1, \cdots, N)$ and can be represented as the following form

$$A^{(n)} = \sum_{k \geq 0} a^{(n;k)}_{\bar{i}_1 \cdots \bar{i}_k} D^{\bar{i}_1} \cdots D^{\bar{i}_k} \tag{12}$$

where

$$D^{\bar{i}} = g^{\bar{i}j} \partial_j \tag{13}$$

and each $a^{(n;k)}_{\bar{i}_1 \cdots \bar{i}_k}$ is a C^∞ function on M. In particular, $a^{(n;0)}$ acts as a multiplication operator. The following relations which the differential operators $D^{\bar{i}}$ satisfy are useful

$$[D^{\bar{i}}, D^{\bar{j}}] = 0, \qquad [D^{\bar{i}}, \partial_{\bar{j}} \Phi] = \delta_{ij}. \tag{14}$$

The following theorem give the method to construct L_f concretely.

Theorem 2.1 (Karabegov[1, 2]). *L_f is uniquely determined by requiring the following conditions*

$$L_f 1 = f * 1 = f, \qquad [L_f, \partial_{\bar{i}} \Phi + \hbar \partial_{\bar{i}}] = 0. \tag{15}$$

The star product $*$ made by the method is associative

$$h * (g * f) = (h * g) * f. \tag{16}$$

Similarly, a differential operator R_f corresponding to the right star multiplication is also defined.

In particular, the left star product by $\partial_i \Phi$ and the right star product by $\partial_{\bar{i}} \Phi$ are written as

$$\begin{aligned} L_{\partial_i \Phi} &= \hbar \partial_i + \partial_i \Phi = \hbar \mathrm{e}^{-\Phi/\hbar} \partial_i \mathrm{e}^{\Phi/\hbar} \\ R_{\partial_{\bar{i}} \Phi} &= \hbar \partial_{\bar{i}} + \partial_{\bar{i}} \Phi = \hbar \mathrm{e}^{-\Phi/\hbar} \partial_{\bar{i}} \mathrm{e}^{\Phi/\hbar}. \end{aligned} \tag{17}$$

By a straightforward calculation, the following relations are derived

$$\begin{aligned} &[\frac{1}{\hbar} \partial_i \Phi,\ z^j]_* = \delta_{ij}, \qquad [z^i,\ z^j]_* = 0, \qquad [\partial_i \Phi,\ \partial_j \Phi]_* = 0 \\ &[\bar{z}^i,\ \frac{1}{\hbar} \partial_{\bar{j}} \Phi]_* = \delta_{ij}, \qquad [\bar{z}^i,\ \bar{z}^j]_* = 0, \qquad [\partial_{\bar{i}} \Phi,\ \partial_{\bar{j}} \Phi]_* = 0 \end{aligned} \tag{18}$$

where $[A,\ B]_* = A * B - B * A$. Hence, $\{z^i, \partial_j \Phi\ ;\ i, j = 1, 2, \cdots, N\}$ and $\{\bar{z}^i, \partial_{\bar{j}} \Phi\ ;\ i, j = 1, 2, \cdots, N\}$ constitute $2N$ sets of the creation and annihilation operators under the star product. However, it should be noted that operators in the set $\{z^i, \partial_j \Phi\}$ do not commute with ones in $\{\bar{z}^i, \partial_{\bar{j}} \Phi\}$, generally, e.g., $z^i * \bar{z}^j - \bar{z}^j * z^i \neq 0$.

3. Local Twisted Fock Representations

In this section we introduce the Fock space on an open set $U \subset M$ which is diffeomorphic to a connected open subset of $\mathbb{C}^N$ and an algebra as a set of linear operators acting on the Fock space.

As mentioned in Section 2, $\{z^i, \partial_j \Phi \ ; i,j = 1,2,\cdots,N\}$ and $\{\bar{z}^i, \partial_{\bar{j}} \Phi \ ; i,j = 1,2,\cdots,N\}$ satisfy the algebra between the creation and annihilation operators under the star product $*$, respectively. We introduce $a_i^\dagger, a_i, \underline{a}_i^\dagger$ and $\underline{a}_i$ by

$$a_i^\dagger = z^i, \qquad \underline{a}_i = \frac{1}{\hbar}\partial_i \Phi, \qquad a_i = \bar{z}^i, \qquad \underline{a}_i^\dagger = \frac{1}{\hbar}\partial_{\bar{i}} \Phi. \tag{19}$$

Then they satisfy the following commutation relations which are similar to the usual commutation relations for the creation and annihilation operators

$$\begin{array}{lll} [\underline{a}_i,\ a_j^\dagger]_* = \delta_{ij}, & [a_i^\dagger,\ a_j^\dagger]_* = 0, & [\underline{a}_i,\ \underline{a}_j]_* = 0 \\ [a_i,\ \underline{a}_j^\dagger]_* = \delta_{ij}, & [\underline{a}_i^\dagger,\ \underline{a}_j^\dagger]_* = 0, & [a_i,\ a_j]_* = 0. \end{array} \tag{20}$$

Here, it should be noted that $[a_i, a_i^\dagger]_*$ and $[\underline{a}_i,\ \underline{a}_j^\dagger]_*$ do not vanish in general, and $\underline{a}_i$ and $a_i^\dagger$ (a_i and $\underline{a}_i^\dagger$) are not Hermitian conjugate with each other.

We introduce the Fock space as a vector space spanned by the bases which are generated by applying $a_i^\dagger$ on the vacuum $|\vec{0}\rangle$

$$|\vec{n}\rangle = |n_1,\cdots,n_N\rangle = \frac{1}{\sqrt{\vec{n}!}}(a_1^\dagger)_*^{n_1} * \cdots * (a_N^\dagger)_*^{n_N} * |\vec{0}\rangle \tag{21}$$

where the vacuum $|\vec{0}\rangle = |0,\cdots,0\rangle$ is defined by

$$\underline{a}_i * |\vec{0}\rangle = 0 \ \ (i = 1,\cdots,N) \tag{22}$$

$(A)_*^n$ stands for $\overbrace{A * \cdots * A}^{n}$ and $\vec{n}! = n_1! n_2! \cdots n_N!$. Then, we define the basis of a dual vector space by applying $\underline{a}_i$ on $\langle\vec{0}|$

$$\underline{\langle\vec{m}|} = \underline{\langle m_1,\cdots,m_N|} = \langle\vec{0}| * (\underline{a}_1)_*^{m_1} * \cdots * (\underline{a}_N)_*^{m_N} \frac{1}{\sqrt{\vec{m}!}} \tag{23}$$

and

$$\langle\vec{0}| * a_i^\dagger = 0 \qquad i = 1,\cdots,N \tag{24}$$

The underline which is attached to the bra vectors means that $\underline{\langle\vec{m}|}$ is not Hermitian conjugate to $|\vec{m}\rangle$.

Definition 3. The local twisted Fock algebra (representation) F_U is defined as a algebra given by a set of linear operators acting on the Fock space defined on U

$$F_U := \{\sum_{\vec{n},\vec{m}} A_{\vec{n}\vec{m}} |\vec{n}\rangle \underline{\langle \vec{m}|} \ ; A_{\vec{n}\vec{m}} \in \mathbb{C}\} \tag{25}$$

and products between its elements are given by the star product $*$.

In the rest of this section, we give concrete expressions of functions which are elements of the local twisted Fock algebra.

Lemma 3.1 (Berezin)**.** *For arbitrary Kähler manifolds (M, ω), there exists a Kähler potential $\Phi(z^1, \ldots, z^N, \bar{z}^1, \ldots, \bar{z}^N)$ such that*

$$\Phi(0, \ldots, 0, \bar{z}^1, \ldots, \bar{z}^N) = 0, \qquad \Phi(z^1, \ldots, z^N, 0, \ldots, 0) = 0. \tag{26}$$

This is easily shown from the fact that Kähler potentials have ambiguities of adding holomorphic and anti-holomorphic functions.
It is shown that a vacuum projection operator $|\vec{0}\rangle\langle\vec{0}|$ for a Kähler manifold corresponds to the function $\mathrm{e}^{-\Phi/\hbar}$.

Proposition 3.2. *Let (M, ω) be a Kähler manifold, Φ be its Kähler potential with the property (26), and $*$ be a star product with separation of variables given in the previous section. Then the following function*

$$|\vec{0}\rangle\langle\vec{0}| := \mathrm{e}^{-\Phi/\hbar} \tag{27}$$

satisfies

$$\begin{gathered} \underline{a}_i * |\vec{0}\rangle\langle\vec{0}| = 0, \qquad |\vec{0}\rangle\langle\vec{0}| * a_i^\dagger = 0 \\ \left(|\vec{0}\rangle\langle\vec{0}|\right) * \left(|\vec{0}\rangle\langle\vec{0}|\right) = \mathrm{e}^{-\Phi/\hbar} * \mathrm{e}^{-\Phi/\hbar} = \mathrm{e}^{-\Phi/\hbar} = |\vec{0}\rangle\langle\vec{0}|. \end{gathered} \tag{28}$$

Outline of proof (A more detailed proof is given in [8].) It is easy to show that the following normal ordered quantity

$$: \mathrm{e}^{-\sum_i a_i^\dagger \underline{a}_i} : := \prod_{i=1_*}^{N} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (a_i^\dagger)_*^n * (\underline{a}_i)_*^n \tag{29}$$

is equal to the vacuum projection, $: \mathrm{e}^{-\sum_i a_i^\dagger \underline{a}_i} := |\vec{0}\rangle\langle\vec{0}|$, similarly to the case of the ordinary harmonic oscillator, Therefore, we next show that this quantity coincides with $\mathrm{e}^{-\Phi/\hbar}$

$$: \mathrm{e}^{-\sum_i a_i^\dagger \underline{a}_i} := \mathrm{e}^{-\Phi/\hbar}. \tag{30}$$

This can be done as follows

$$: \mathrm{e}^{-\sum_i a_i^\dagger \underline{a}_i} := \sum_{\vec{n}} \frac{(-1)^{|n|}}{\vec{n}!}(a^\dagger)_*^{\vec{n}} * (\underline{a})_*^{\vec{n}} = \sum_{\vec{n}} \frac{(-1)^{|n|}}{\vec{n}!\hbar^{|n|}}(z)_*^{\vec{n}} * (\partial\Phi)_*^{\vec{n}}. \tag{31}$$

In this article, we use the following notation: for an N-tuple $A_i, i = 1, 2, \cdots, N$ and an N-vector $\vec{n} = (n_1, n_2, \cdots, n_N)$

$$(A)_*^{\vec{n}} = (A_1)_*^{n_1} * (A_2)_*^{n_2} * \cdots * (A_N)_*^{n_N}, \quad \vec{n}! \ = n_1!n_2!\cdots n_N!, \quad |n| = \sum_{i=1}^N n_i.$$

By using $(z)_*^{\vec{n}} = (z)^{\vec{n}} = (z^1)^{n_1} \cdots (z^N)^{n_N}$, the right hand side of (31) is recast as

$$\begin{aligned} &\sum_{n_1,n_2,\ldots,n_N=0}^{\infty} \frac{1}{n_1!n_2!\cdots n_N!}(-z^1)^{n_1}\cdots(-z^N)^{n_N} \mathrm{e}^{-\frac{\Phi(z,\bar{z})}{\hbar}} \partial_1^{n_1}\cdots\partial_N^{n_N} \mathrm{e}^{\frac{\Phi(z,\bar{z})}{\hbar}} \\ &\qquad = \mathrm{e}^{-\frac{\Phi(z,\bar{z})}{\hbar}} \mathrm{e}^{\frac{\Phi(0,\bar{z})}{\hbar}} = \mathrm{e}^{-\frac{\Phi(z,\bar{z})}{\hbar}}. \end{aligned} \tag{32}$$

Here, the final equality follows from the condition (26). □

Similarly, the following relations hold with respect to a_i and $\underline{a}_i^\dagger$

$$\begin{aligned} &|\vec{0}\rangle\langle\vec{0}| = \mathrm{e}^{-\Phi/\hbar} =: \mathrm{e}^{-\sum_i \underline{a}_i^\dagger a_i} := \prod_{i=1_*}^N \sum_{n=0}^\infty \frac{(-1)^n}{n!}(\underline{a}_i^\dagger)_*^n * (a_i)_*^n \\ &a_i * |\vec{0}\rangle\langle\vec{0}| = 0, \qquad |\vec{0}\rangle\langle\vec{0}| * \underline{a}_i^\dagger = 0. \end{aligned} \tag{33}$$

We then expand a function $\exp \Phi(z,\bar{z})/\hbar$ as a power series of z^i and $\bar{z}^j$

$$\mathrm{e}^{\Phi(z,\bar{z})/\hbar} = \sum_{\vec{m},\vec{n}} H_{\vec{m},\vec{n}}(z)^{\vec{m}}(\bar{z})^{\vec{n}} \tag{34}$$

where $(z)^{\vec{n}} = (z^1)^{n_1}\cdots(z^N)^{n_N}$ and $(\bar{z})^{\vec{n}} = (\bar{z}^1)^{n_1}\cdots(\bar{z}^N)^{n_N}$. Since $\exp\Phi/\hbar$ is real and satisfies (26), the expansion coefficients $H_{\vec{m},\vec{n}}$ satisfy the followings

$$\bar{H}_{\vec{m},\vec{n}} = H_{\vec{n},\vec{m}}, \qquad H_{\vec{0},\vec{n}} = H_{\vec{n},\vec{0}} = \delta_{\vec{n},\vec{0}}.$$

Using this expansion, the following relations are obtained.

Proposition 3.3. *The right $*$-multiplication of $(\underline{a})_*^{\vec{n}} = (\partial\Phi/\hbar)_*^{\vec{n}}$ on $|\vec{0}\rangle\langle\vec{0}|$ is related to the right $*$-multiplication of $(a)_*^{\vec{n}} = (\bar{z})_*^{\vec{n}}$ on $|\vec{0}\rangle\langle\vec{0}|$ as follows*

$$\begin{aligned} |\vec{0}\rangle\langle\vec{0}| * (\underline{a})_*^{\vec{n}} &= |\vec{0}\rangle\langle\vec{0}| * \left(\frac{1}{\hbar}\partial\Phi\right)_*^{\vec{n}} \\ &= \vec{n}! \sum_{\vec{m}} H_{\vec{n},\vec{m}} |\vec{0}\rangle\langle\vec{0}| * (\bar{z})_*^{\vec{m}} = \vec{n}! \sum_{\vec{m}} H_{\vec{n},\vec{m}} |\vec{0}\rangle\langle\vec{0}| * (a)_*^{\vec{m}}. \end{aligned} \tag{35}$$

Similarly, the following relation holds

$$\begin{aligned}(\underline{a}^\dagger)^{\vec{n}}_* * |\vec{0}\rangle\langle\vec{0}| &= \left(\frac{1}{\hbar}\bar{\partial}\Phi\right)^{\vec{n}}_* * |\vec{0}\rangle\langle\vec{0}| \\ &= \vec{n}! \sum_{\vec{m}} H_{\vec{m},\vec{n}} (z)^{\vec{m}} * |\vec{0}\rangle\langle\vec{0}| = \vec{n}! \sum_{\vec{m}} H_{\vec{m},\vec{n}} (a^\dagger)^{\vec{m}}_* * |\vec{0}\rangle\langle\vec{0}|. \end{aligned} \tag{36}$$

If there exists the inverse matrix $H^{-1}_{\vec{m},\vec{n}}$, then the following relations also holds

Corollary 3.4.

$$\begin{aligned}|\vec{0}\rangle\langle\vec{0}| * (a)^{\vec{n}}_* &= \sum_{\vec{m}} \frac{1}{\vec{m}!} H^{-1}_{\vec{n},\vec{m}} |\vec{0}\rangle\langle\vec{0}| * (\underline{a})^{\vec{m}}_* \\ (a^\dagger)^{\vec{n}}_* * |\vec{0}\rangle\langle\vec{0}| &= \sum_{\vec{m}} \frac{1}{\vec{m}!} H^{-1}_{\vec{m},\vec{n}} (\underline{a}^\dagger)^{\vec{m}} * |\vec{0}\rangle\langle\vec{0}| \end{aligned} \tag{37}$$

where $H^{-1}_{\vec{n},\vec{m}}$ is the inverse matrix of the matrix $H_{\vec{n},\vec{m}}$, $\sum_{\vec{k}} H_{\vec{m},\vec{k}} H^{-1}_{\vec{k},\vec{n}} = \delta_{\vec{m},\vec{n}}$.

Now, we introduce bases of the Fock representation as follows

$$|\vec{m}\rangle\underline{\langle\vec{n}|} := \frac{1}{\sqrt{\vec{m}!\vec{n}!}} (a^\dagger)^{\vec{m}}_* * |\vec{0}\rangle\langle\vec{0}| * (\underline{a})^{\vec{n}}_* = \frac{1}{\sqrt{\vec{m}!\vec{n}!}} (z)^{\vec{m}}_* * \mathrm{e}^{-\Phi/\hbar} * \left(\frac{1}{\hbar}\partial\Phi\right)^{\vec{n}}_* . \tag{38}$$

By using (35), the bases are also written as

$$|\vec{m}\rangle\underline{\langle\vec{n}|} = \sqrt{\frac{\vec{n}!}{\vec{m}!}} \sum_{\vec{k}} H_{\vec{n},\vec{k}} (z)^{\vec{m}}_* * \mathrm{e}^{-\Phi/\hbar} * (\bar{z})^{\vec{k}}_* = \sqrt{\frac{\vec{n}!}{\vec{m}!}} \sum_{\vec{k}} H_{\vec{n},\vec{k}} (z)^{\vec{m}} (\bar{z})^{\vec{k}} \mathrm{e}^{-\Phi/\hbar}. \tag{39}$$

The completeness of the bases are formally shown as

$$\sum_{\vec{n}} |\vec{n}\rangle\underline{\langle\vec{n}|} = \sum_{\vec{m},\vec{n}} H_{\vec{n},\vec{m}} (z)^{\vec{n}} (\bar{z})^{\vec{m}} \mathrm{e}^{-\Phi/\hbar} = \mathrm{e}^{\Phi/\hbar}\mathrm{e}^{-\Phi/\hbar} = 1. \tag{40}$$

The bases satisfy the following orthogonality relation under the $*$-products

$$\begin{aligned}|\vec{m}\rangle\underline{\langle\vec{n}|} * |\vec{k}\rangle\underline{\langle\vec{l}|} &= \frac{1}{\sqrt{\vec{m}!\vec{n}!\vec{k}!\vec{l}!}} (a^\dagger)^{\vec{m}}_* * |\vec{0}\rangle\langle\vec{0}| * (\underline{a})^{\vec{n}}_* * (a^\dagger)^{\vec{k}}_* * |\vec{0}\rangle\langle\vec{0}| * (\underline{a})^{\vec{l}}_* \\ &= \delta_{\vec{n},\vec{k}} |\vec{m}\rangle\underline{\langle\vec{l}|}. \end{aligned} \tag{41}$$

It should be noted that in the twisted Fock representation, the behavior of the bases under the complex conjugation is different from usual

$$\overline{|\vec{m}\rangle\underline{\langle\vec{n}|}} = \sqrt{\frac{\vec{n}!}{\vec{m}!}} \sum_{\vec{k}} H_{\vec{k},\vec{n}} (z)^{\vec{k}} (\bar{z})^{\vec{m}} \mathrm{e}^{-\Phi/\hbar} = \sqrt{\frac{\vec{n}!}{\vec{m}!}} \sum_{\vec{k},\vec{l}} \sqrt{\frac{\vec{k}!}{\vec{l}!}} H_{\vec{k},\vec{n}} H^{-1}_{\vec{m},\vec{l}} |\vec{k}\rangle\underline{\langle\vec{l}|}.$$

The action of the creation and annihilation operators $a_i^\dagger, \underline{a}_i$ on the bases is calculated as

$$a_i^\dagger * |\vec{m}\rangle\underline{\langle\vec{n}|} = \sqrt{m_i+1}|\vec{m}+\vec{e}_i\rangle\underline{\langle\vec{n}|}, \qquad |\vec{m}\rangle\underline{\langle\vec{n}|} * a_i^\dagger = \sqrt{n_i}|\vec{m}\rangle\underline{\langle\vec{n}-\vec{e}_i|}$$
$$\underline{a}_i * |\vec{m}\rangle\underline{\langle\vec{n}|} = \sqrt{m_i}|\vec{m}-\vec{e}_i\rangle\underline{\langle\vec{n}|}, \qquad \vec{m}\rangle\underline{\langle\vec{n}|} * \underline{a}_i = \sqrt{n_i+1}|\vec{m}\rangle\underline{\langle\vec{n}+\vec{e}_i|}$$

where $\vec{e}_i$ is a unit vector, $(\vec{e}_i)_j = \delta_{ij}$. The action of a_i and $\underline{a}_i^\dagger$ is derived by the Hermitian conjugation of the above equations. From these relations, the representations of the creation and annihilation operators in the twisted Fock representation are derived

$$a_i^\dagger = \sum_{\vec{n}} \sqrt{n_i+1}|\vec{n}+\vec{e}_i\rangle\underline{\langle\vec{n}|}, \qquad a_i = \sum_{\vec{m},\vec{n},\vec{k}} \sqrt{\frac{\vec{m}!}{\vec{n}!}} H_{\vec{m},\vec{k}} H^{-1}_{\vec{k}+\vec{e}_i,\vec{n}} |\vec{m}\rangle\underline{\langle\vec{n}|}$$
$$\underline{a}_i = \sum_{\vec{n}} \sqrt{n_i+1}|\vec{n}\rangle\underline{\langle\vec{n}+\vec{e}_i|}, \qquad \underline{a}_i^\dagger = \sqrt{\frac{\vec{m}!}{\vec{n}!}}(k_i+1) H_{\vec{m},\vec{k}+\vec{e}_i} H^{-1}_{\vec{k},\vec{n}} |\vec{m}\rangle\underline{\langle\vec{n}|}. \tag{42}$$

Summarizing this section, we obtained the following dictionary Table 1 which contains the correspondence between functions on a noncommutative Kähler manifold and elements of the twisted Fock representations.

Table 1. Functions - Fock operators Dictionary.

Functions	Fock operators
$\mathrm{e}^{-\Phi/\hbar}$	$\lvert\vec{0}\rangle\underline{\langle\vec{0}\rvert}$
z_i	$a_i^\dagger$
$\frac{1}{\hbar}\partial_i\Phi$	$\underline{a}_i$
$\bar{z}^i$	$a_i = \sum \sqrt{\frac{\vec{m}!}{\vec{n}!}} H_{\vec{m},\vec{k}} H^{-1}_{\vec{k}+\vec{e}_i,\vec{n}} \lvert\vec{m}\rangle\underline{\langle\vec{n}\rvert}$
$\frac{1}{\hbar}\partial_{\bar{i}}\Phi$	$\underline{a}_i^\dagger = \sum \sqrt{\frac{\vec{m}!}{\vec{n}!}}(k_i+1) H_{\vec{m},\vec{k}+\vec{e}_i} H^{-1}_{\vec{k},\vec{n}} \lvert\vec{m}\rangle\underline{\langle\vec{n}\rvert}$

4. Transition Maps

Let $\{U_a\}$ with $M = \cup_a U_a$ be a locally finite open covering and $\{(U_a, \phi_a)\}$ be an atlas, where $\phi_a : U_a \to \mathbb{C}^N$. Consider the case $U_a \cap U_b \neq \emptyset$. Denote by $\phi_{a,b}$ the transition map from $\phi_a(U_a)$ to $\phi_b(U_b)$. The local coordinates $(z, \bar{z}) = (z^1, \cdots, z^N, \bar{z}^1, \cdots, \bar{z}^N)$ on U_a are transformed into the coordinates $(w, \bar{w}) = (w^1, \cdots, w^N, \bar{w}^1, \cdots, \bar{w}^N)$ on U_b by $(w, \bar{w}) = (w(z), \bar{w}(\bar{z}))$, where $w(z) = (w^1(z), \cdots, w^N(z))$ is a holomorphic function and $\bar{w}(\bar{z}) = (\bar{w}^1(\bar{z}), \cdots, \bar{w}^N(\bar{z}))$

is an anti-holomorphic function. Denote by $f *_a g$ and $f *_b g$ the star products defined in Section 2 on U_a and U_b, respectively. In general, there is a nontrivial transition maps T between two star products i.e. $f *_b g = T(f) *_a T(g)$. But it can be shown that the transition maps are trivial in our case.

Proposition 4.1. *For a non-empty overlap $U_a \cap U_b \neq \emptyset$*

$$f *_b g(w, \bar{w}) = \phi^*_{a,b}\, f *_a g(w, \bar{w}) = \phi^*_{a,b}\, f(w(z), \bar{w}(\bar{z})) *_a g(w(z), \bar{w}(\bar{z})). \tag{43}$$

Here $\phi^*_{a,b}$ is the pull back of $\phi_{a,b}$.

In other words, $a^b_{n,\alpha}(f)$ transforms as a tensor

$$a^b_{n,\vec{\bar{j}}}(f) \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\vec{\bar{j}}}_{\vec{\bar{i}}} = a^a_{n,\vec{\bar{i}}}(f). \tag{44}$$

We consider the transition function between twisted Fock representations. From Lemma 3.1, we can choose Kähler potentials $\Phi_a(z, \bar{z})$ on U_a and $\Phi_b(w, \bar{w})$ on U_b such that

$$\Phi_a(0, \bar{z}) = \Phi_a(z, 0) = 0, \qquad \Phi_b(0, \bar{w}) = \Phi_b(w, 0) = 0. \tag{45}$$

Using these Kähler potentials, the vacuum projection $|\vec{0}\rangle_{pp}\langle\vec{0}|$ is defined as

$$|\vec{0}\rangle_{pp}\langle\vec{0}| = \mathrm{e}^{-\Phi_p/\hbar}, \qquad (p = a, b)$$

and the bases of twisted Fock representations $|\vec{m}\rangle_{p\underline{p\langle\vec{n}|}}$ are defined by

$$|\vec{m}\rangle_{a\underline{a\langle\vec{n}|}} = \frac{1}{\sqrt{\vec{m}!\vec{n}!}} (z)^{\vec{m}}_* * \mathrm{e}^{-\Phi_a/\hbar} * \left(\frac{1}{\hbar}\partial\Phi_a\right)^{\vec{n}}_*$$
$$|\vec{m}\rangle_{b\underline{b\langle\vec{n}|}} = \frac{1}{\sqrt{\vec{m}!\vec{n}!}} (w)^{\vec{m}}_* * \mathrm{e}^{-\Phi_b/\hbar} * \left(\frac{1}{\hbar}\partial\Phi_b\right)^{\vec{n}}_*.$$

Let us consider the case when on the non-empty overlap $U_a \cap U_b$ the coordinate transition function $w(z)$, $\bar{w}(\bar{z})$, and the functions $\exp(\phi(w)/\hbar)$ and $\exp(\bar{\phi}(\bar{w})/\hbar)$ are given by analytic functions. Then the products $(w(z))^{\vec{\alpha}} \exp(-(\phi(w)/\hbar))$ and $(\bar{w}(\bar{z}))^{\vec{\alpha}} \exp(-(\bar{\phi}(\bar{w})/\hbar))$ are also analytic functions

$$(w(z))^{\vec{\alpha}} \mathrm{e}^{-\phi(w)/\hbar} = \sum_{\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}} z^{\vec{\beta}}, \qquad (\bar{w}(\bar{z}))^{\vec{\alpha}} \mathrm{e}^{-\bar{\phi}(\bar{w})/\hbar} = \sum_{\vec{\beta}} \bar{C}^{\vec{\alpha}}_{\vec{\beta}} \bar{z}^{\vec{\beta}}. \tag{46}$$

By using (35), the bases are rewritten as

$$
\begin{aligned}
|\vec{m}\rangle_{a}\underline{{}_a\langle\vec{n}|} &= \sqrt{\frac{\vec{n}!}{\vec{m}!}}\sum_{\vec{k}} H^{a}_{\vec{n},\vec{k}}(z)^{\vec{m}}(\bar{z})^{\vec{k}}\mathrm{e}^{-\Phi_a/\hbar} \\
|\vec{m}\rangle_{b}\underline{{}_b\langle\vec{n}|} &= \sqrt{\frac{\vec{n}!}{\vec{m}!}}\sum_{\vec{k}} H^{b}_{\vec{n},\vec{k}}(w)^{\vec{m}}(\bar{w})^{\vec{k}}\mathrm{e}^{-\Phi_b/\hbar}.
\end{aligned}
\tag{47}
$$

From the (46), $|\vec{m}\rangle_{b}\underline{{}_b\langle\vec{n}|}$ can be written as the following function of z^i and $\bar{z}^i$

$$
|\vec{m}\rangle_{b}\underline{{}_b\langle\vec{n}|} = \sqrt{\frac{\vec{n}!}{\vec{m}!}}\sum_{\vec{k}} H^{b}_{\vec{n},\vec{k}}\Big(\sum_{\vec{\alpha}} C^{\vec{m}}_{\vec{\alpha}} z^{\vec{\alpha}}\Big)\Big(\sum_{\vec{\beta}} \bar{C}^{\vec{k}}_{\vec{\beta}}\bar{z}^{\vec{\beta}}\Big)\mathrm{e}^{-\Phi_a/\hbar}. \tag{48}
$$

Representing the right hand side of the equation by using the bases $|\vec{m}\rangle_{a}\underline{{}_a\langle\vec{n}|}$ on U_a, we obtain the transformation between the bases

$$
T^{ab} : F_{U_a} \to F_{U_b} \tag{49}
$$

as

$$
|\vec{m}\rangle_{b}\underline{{}_b\langle\vec{n}|} = \sum_{\vec{i},\vec{j}} T^{ba,\vec{i}\vec{j}}_{\vec{m}\vec{n}}|\vec{i}\rangle_{a}\underline{{}_a\langle\vec{j}|} \tag{50}
$$

where

$$
T^{ba,\vec{i}\vec{j}}_{\vec{m}\vec{n}} = \sqrt{\frac{\vec{n}!}{\vec{m}!}}\sqrt{\frac{\vec{i}!}{\vec{j}!}}\sum_{\vec{k}} H^{b}_{\vec{n},\vec{k}}(C^{\vec{m}}_{\vec{i}})\Big(\sum_{\vec{\beta}} \bar{C}^{\vec{k}}_{\vec{\beta}} H^{a-1}_{\vec{\beta}\vec{j}}\Big). \tag{51}
$$

Using this transformation, the twisted Fock representation can be extended to M. We call it the twisted Fock representation of M.

5. Examples

In this section, some examples of the Fock representations are given.

Example 1. Fock representation of $\mathbb{C}^N$.

The first example is $\mathbb{C}^N$. The Kähler potential is given by

$$
\Phi_{\mathbb{C}^N} = \sum_{i=1}^{N} |z^i|^2. \tag{52}
$$

Following the procedure given in Section 2, the star product is easily calculated as

$$
f * g = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!}\delta^{k_1 l_1}\cdots\delta^{k_n l_n}(\partial_{\bar{k}_1}\cdots\partial_{\bar{k}_n} f)(\partial_{l_1}\cdots\partial_{l_n} g).
$$

This star product was given in [2]. We define

$$a_i^\dagger = z^i, \qquad \underline{a}_i = \frac{1}{\hbar}\bar{z}^i, \qquad a_i = \bar{z}^i, \qquad \underline{a}_i^\dagger = \frac{1}{\hbar}z^i. \tag{53}$$

Then they satisfy the following commutation relations

$$[\underline{a}_i,\ a_j^\dagger]_* = \delta_{ij}, \qquad [a_i,\ \underline{a}_j^\dagger]_* = \hbar\delta_{ij} \tag{54}$$

and the others vanish. Since in this case the operators with the underline are essentially equal to those without the underline, we omit the underline of the bra vectors. The basis of the twisted Fock algebra is given by

$$|\vec{m}\rangle\langle\vec{n}| = \frac{1}{\hbar^{|\vec{n}|}\sqrt{\vec{m}!\vec{n}!}}(z)^{\vec{m}}\mathrm{e}^{-\Phi/\hbar}(\bar{z})^{\vec{n}}. \tag{55}$$

As a result, the twisted Fock representation coincides with the ordinary Fock representation for noncommutative Euclidean spaces.

Example 2. Fock representation of noncommutative $\mathbb{CP}^N$.

We give an explicit expression of the twisted Fock representation of noncommutative of $\mathbb{CP}^N$. In this case, the twisted Fock representation on an open set is essentially the same as the representation given in [4–7].

Let denote ζ^a $(a = 0, 1, \dots, N)$ homogeneous coordinates and $\bigcup U_a$ $(U_a = \{[\zeta^0 : \zeta^1 : \cdots : \zeta^N]\}|\zeta^a \neq 0)$ an open covering of $\mathbb{C}P^N$. Inhomogeneous coordinates on U_a are defined as

$$z_a^0 = \frac{\zeta^0}{\zeta^a}, \ \cdots, \quad z_a^{a-1} = \frac{\zeta^{a-1}}{\zeta^a}, \quad z_a^{a+1} = \frac{\zeta^{a+1}}{\zeta^a}, \ \cdots, \quad z_a^N = \frac{\zeta^N}{\zeta^a}. \tag{56}$$

We choose a Kähler potential on U_a which satisfies the condition (26)

$$\Phi_a = \ln(1 + |z_a|^2) \tag{57}$$

where $|z_a|^2 = \sum_i |z_a^i|^2$. A star product on U_a is given in [5, 6]

$$f * g = \sum_{n=0}^{\infty} c_n(\hbar) g_{j_1\bar{k}_1}\cdots g_{j_n\bar{k}_n}\left(D^{j_1}\cdots D^{j_n} f\right) D^{\bar{k}_1}\cdots D^{\bar{k}_n} g \tag{58}$$

where

$$c_n(\hbar) = \frac{\Gamma(1 - n + 1/\hbar)}{n!\Gamma(1 + 1/\hbar)}, \qquad D^{\bar{i}} = g^{\bar{i}j}\partial_j, \qquad D^i = g^{i\bar{j}}\partial_{\bar{j}}. \tag{59}$$

On U_a, the creation and the annihilation operators are introduced as

$$a_{a,i}^\dagger = z_a^i, \qquad \underline{a_{a,i}} = \frac{1}{\hbar}\partial_i\Phi_a = \frac{\bar{z}_a^i}{\hbar(1 + |z_a|^2)}$$

$$a_{a,i} = \bar{z}_a^i, \qquad \underline{a_{a,i}}^\dagger = \frac{1}{\hbar}\partial_{\bar{i}}\Phi_a = \frac{z_a^i}{\hbar(1 + |z_a|^2)}$$

and then the vacuum becomes

$$|\vec{0}\rangle_{a}\underline{{}_a\langle\vec{0}|} = \mathrm{e}^{-\Phi_a/\hbar} = (1+|z_a|^2)^{-1/\hbar}. \tag{60}$$

Bases of the twisted Fock representation on U_a are constructed as

$$\begin{aligned} |\vec{m}\rangle_{a}\underline{{}_a\langle\vec{n}|} &= \frac{1}{\sqrt{\vec{m}!\vec{n}!}}(a_a^\dagger)_*^{\vec{m}} * |\vec{0}\rangle_{a}\underline{{}_a\langle\vec{0}|} * (\underline{a_a})_*^{\vec{n}} \\ &= \frac{1}{\sqrt{\vec{m}!\vec{n}!\hbar^{|n|}}}(z_a)_*^{\vec{m}} * \mathrm{e}^{-\Phi_a/\hbar} * (\partial\Phi_a)_*^{\vec{n}}. \end{aligned} \tag{61}$$

The following relation is shown in [5],

$$\begin{aligned} (\partial\Phi_a)_*^{\vec{n}} &= \frac{\hbar^{|n|}\Gamma(1/\hbar+1)}{\Gamma(1/\hbar-|n|+1)}(\partial\Phi_a)^{\vec{n}} \\ &= \frac{\hbar^{|n|}\Gamma(1/\hbar+1)}{\Gamma(1/\hbar-|n|+1)}\left(\frac{\bar{z}_a}{1+|z_a|^2}\right)^{\vec{n}}. \end{aligned} \tag{62}$$

Then, the bases can be explicitly written as

$$|\vec{m}\rangle_{a}\underline{{}_a\langle\vec{n}|} = \frac{\Gamma(1/\hbar+1)}{\sqrt{\vec{m}!\vec{n}!}\Gamma(1/\hbar-|n|+1)}(z_a)^{\vec{m}}(\bar{z}_a)^{\vec{n}}\mathrm{e}^{-\Phi/\hbar}. \tag{63}$$

By comparing this equation and (39), $H_{\vec{m},\vec{n}}$ is obtained as

$$H_{\vec{m},\vec{n}} = \delta_{\vec{m},\vec{n}}\frac{\Gamma(1/\hbar+1)}{\vec{m}!\Gamma(1/\hbar-|m|+1)} \tag{64}$$

and the relation $\mathrm{e}^{\Phi_a/\hbar} = \sum H_{\vec{m},\vec{n}}(z_a)^{\vec{m}}(\bar{z})^{\vec{n}}$ is shown formally.

Let us consider a transition map between the Fock representations on U_a and U_b ($a < b$). The transformations for the coordinates and the Kähler potentials on $U_a \bigcap U_b$ are

$$\begin{aligned} &z_a^i = \frac{z_b^i}{z_b^a}, \qquad z_a^b = \frac{1}{z_b^a}, \qquad i = 0,1,\ldots,a-1,a+1,\ldots,b-1,b+1,\ldots,N \\ &\Phi_a = \Phi_b - \ln z_b^a - \ln \bar{z}_b^a. \end{aligned} \tag{65}$$

Thus, $|\vec{m}\rangle_{a}\underline{{}_a\langle\vec{n}|}$ is written on $U_a \bigcap U_b$ as

$$\begin{aligned} |\vec{m}\rangle_{a}\underline{{}_a\langle\vec{n}|} &= \frac{\Gamma(1/\hbar+1)}{\sqrt{\vec{m}!\vec{n}!}\Gamma(1/\hbar-|n|+1)}\mathrm{e}^{-\Phi_b/\hbar} \\ &\times (z_b^0)^{m_0}\ldots(z_b^{a-1})^{m_{a-1}}(z_b^a)^{1/\hbar-|m|}(z_b^{a+1})^{m_{a+1}}\ldots(z_b^{b-1})^{m_{b-1}}(z_b^{b+1})^{m_{b+1}}\ldots(z_b^N)^{m_N} \\ &\times (\bar{z}_b^0)^{n_0}\ldots(\bar{z}_b^{a-1})^{n_{a-1}}(\bar{z}_b^a)^{1/\hbar-|n|}(\bar{z}_b^{a+1})^{n_{a+1}}\ldots(\bar{z}_b^{b-1})^{n_{b-1}}(\bar{z}_b^{b+1})^{n_{b+1}}\ldots(\bar{z}_b^N)^{n_N} \end{aligned} \tag{66}$$

where

$$\vec{m} = (m_0, \dots, m_{a-1}, m_{a+1}, \dots, m_N), \qquad \vec{n} = (n_0, \dots, n_{a-1}, n_{a+1}, \dots, n_N).$$

We should treat $(z_b^a)^{1/\hbar-|m|}$ and $(\bar{z}_b^a)^{1/\hbar-|n|}$ carefully, if they are not monomials. Here, to avoid such kind of complications concerning $(z_b^a)^{1/\hbar-|m|}$ and $(\bar{z}_b^a)^{1/\hbar-|n|}$, we introduce a slightly different representation from the above twisted Fock representation of $\mathbb{CP}^N$. Let us consider the case that the noncommutative parameter is the following value

$$1/\hbar = L \in \mathbb{Z}, \qquad L \geq 0. \tag{67}$$

Then, we define F_a^L on U_a as a subspace of a local twisted Fock algebra F_{U_a}

$$F_a^L = \{\sum_{\vec{m},\vec{n}} A_{\vec{m}\vec{n}} |\vec{m}\rangle_{a\underline{a}}\underline{\langle\vec{n}|} \;; A_{\vec{m}\vec{n}} \in \mathbb{C},\ |m| \leq L,\ |n| \leq L\}. \tag{68}$$

From (66), it is shown that the bases on U_a are related to those on U_b as

$$\sqrt{\frac{(L-|n|)!}{(L-|m|)!}} |\vec{m}\rangle_{a\underline{a}}\underline{\langle\vec{n}|} = \sqrt{\frac{(L-|n'|)!}{(L-|m'|)!}} |\vec{m'}\rangle_{b\underline{b}}\underline{\langle\vec{n'}|} \tag{69}$$

where

$$\begin{aligned}\vec{m'} &= (m_0, \dots, m_{a-1}, L-|m|, m_{a+1}, \dots, m_{b-1}, m_{b+1}, \dots, m_N) \\ \vec{n'} &= (n_0, \dots, n_{a-1}, L-|n|, n_{a+1}, \dots, n_{b-1}, n_{b+1}, \dots, n_N).\end{aligned}$$

Using the expression of (69), $|\vec{m}\rangle_{a\underline{a}}\underline{\langle\vec{n}|}$ can be defined on the whole of U_b. Therefore, the operators in F_a^L can be extended to the whole of $\mathbb{CP}^N$ by using the relations similar to (69).

Similarly to (42), we define a creation operator ${a_{a,i}^L}^\dagger$ and an annihilation operator $\underline{a}_{a,i}^L$ restricted on F_a^L by

$$\begin{aligned}{a_{a,i}^L}^\dagger &= \sum_{0\leq|n|\leq L-1} \sqrt{n_i+1} |\vec{n}+\vec{e}_i\rangle_{a\underline{a}}\underline{\langle\vec{n}|} = z_a^i \left[1 - \left(\frac{|z_a|^2}{1+|z_a|^2}\right)^L\right] \\ \underline{a}_{a,i}^L &= \sum_{0\leq|n|\leq L-1} \sqrt{n_i+1} |\vec{n}\rangle_{a\underline{a}}\underline{\langle\vec{n}+\vec{e}_i|} = L\frac{\bar{z}_a^i}{1+|z_a|^2}.\end{aligned} \tag{70}$$

By the restriction on F_a^L, ${a_{a,i}^L}^\dagger$ is shifted from z_a^i. These operators satisfy the following commutation relation which is modified from the ordinary ones

$$
\begin{aligned}
[\underline{a}_{a,i}^L,\ {a_{a,j}^L}^\dagger] &= \delta_{ij}\left(\sum_{0\le|n|\le L}|\vec{n}\rangle_{a\,a}\underline{\langle\vec{n}|} - \sum_{|n|=L}(n_i+1)|\vec{n}\rangle_{a\,a}\underline{\langle\vec{n}|}\right) \\
&= \delta_{ij} - \delta_{ij}\left(\frac{|z_a|^2}{1+|z_a|^2}\right)^L\left(1+L\frac{|z_a^i|^2}{|z_a|^2}\right).
\end{aligned}
\tag{71}
$$

Example 3. Fock representation of noncommutative $\mathbb{CH}^N$.

Here, we give an explicit expression of the Fock representation of noncommutative of $\mathbb{CH}^N$ [5,6].

We choose a Kähler potential which satisfies the condition (26)

$$\Phi = -\ln(1-|z|^2) \tag{72}$$

where $|z|^2 = \sum_i^N |z^i|^2$. A star product is given in [5,6]

$$f * g = \sum_{n=0}^{\infty} c_n(\hbar) g_{j_1\bar{k}_1}\dots g_{j_n\bar{k}_n}\left(D^{j_1}\dots D^{j_n} f\right) D^{\bar{k}_1}\dots D^{\bar{k}_n} g \tag{73}$$

where

$$c_n(\hbar) = \frac{\Gamma(1/\hbar)}{n!\Gamma(n+1/\hbar)}, \qquad D^{\bar{i}} = g^{\bar{i}j}\partial_j, \qquad D^i = g^{i\bar{j}}\partial_{\bar{j}}. \tag{74}$$

The creation and annihilation operators are introduced as

$$
\begin{aligned}
a_i^\dagger &= z^i, \qquad \underline{a_i} = \frac{1}{\hbar}\partial_i\Phi = \frac{\bar{z}^i}{\hbar(1-|z|^2)} \\
a_i &= \bar{z}^i, \qquad \underline{a_i}^\dagger = \frac{1}{\hbar}\partial_{\bar{i}}\Phi = \frac{z^i}{\hbar(1-|z|^2)}.
\end{aligned}
\tag{75}
$$

and the vacuum becomes

$$|\vec{0}\rangle\langle\vec{0}| = \mathrm{e}^{-\Phi/\hbar} = (1-|z|^2)^{1/\hbar}. \tag{76}$$

Bases of the Fock representation on $\mathbb{CH}^N$ are constructed as

$$|\vec{m}\rangle\underline{\langle\vec{n}|} = \frac{1}{\sqrt{\vec{m}!\vec{n}!}}(a^\dagger)_*^{\vec{m}} * |\vec{0}\rangle\underline{\langle\vec{0}|} * (\underline{a})_*^{\vec{n}} = \frac{1}{\sqrt{\vec{m}!\vec{n}!\hbar^{|n|}}}(z)_*^{\vec{m}} * \mathrm{e}^{-\Phi/\hbar} * (\partial\Phi)_*^{\vec{n}}. \tag{77}$$

By using the following relation which is shown in [5]

$$(\partial\Phi)_*^{\vec{n}} = \frac{(-\hbar)^{|n|}\Gamma(1/\hbar+|n|)}{\Gamma(1/\hbar)}\left(\frac{\bar{z}}{1-|z|^2}\right)^{\vec{n}} \tag{78}$$

the bases can be explicitly written as

$$|\vec{m}\rangle\underline{\langle\vec{n}|} = \frac{(-1)^{|n|}\Gamma(1/\hbar + |n|)}{\sqrt{\vec{m}!\vec{n}!}\Gamma(1/\hbar)}(z)^{\vec{m}}(\bar{z})^{\vec{n}}(1-|z|^2)^{1/\hbar}. \tag{79}$$

These are defined globally.

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