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TRAJECTORIES OF THE PLATE-BALL PROBLEM

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Abstract. The plate-ball problem concerns the shortest trajectories traced
by a rolling sphere on a horizontal plane between the prescribed initial and
final states meaning the positions and orientations of the sphere. Here we
present an explicit parametric representation of these trajectories in terms of
the Jacobian elliptic functions and elliptic integrals.

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1. Statement of the Problem

The problem of finding the shortest paths (optimal curves) traced by a spherical
ball while rolling without slipping about a horizontal axis on an infinite horizontal
plate was stated by Hammersley [8] in 1983. Since then this problem, as well as
its variants and generalizations, have become widely known as the *plate-ball prob-
lem*. In the version we are going to consider we have looked for a curved path with
a minimum length that delivers a sphere on a horizontal plane between two pre-
scribed initial and final states meaning the positions and orientations of the sphere.
Following Hammersley's original formulation in quaternions and calculus of vari-
ations settings, Arthurs and Walsh [2] showed, by making use of the Pontryagin
maximum principle, that the problem has a solution which is readily expressed via
elliptic functions. Their approach leads to an intrinsic equation for the curvature
of the shortest paths. Here we give an explicit solution of this equation in terms
of the Jacobian elliptic functions and elliptic integrals thereby parameterizing the
optimal curves of the considered plate-ball problem.

In what follows we are going to state the problem in an *optimal control* formulation of the article [2] adopting most of the notation used therein. We assume that the sphere has a unit radius and that it is rolling with an unit speed along a curved path Γ in a horizontal XOZ-plane about a horizontal axis in a way that there is no slipping between the sphere and the plane. We assume also that the path Γ starts at the origin $O(0,0)$ and ends at a prescribed final point $E(x_{\mathrm{e}}, 0)$ on the OX-axis. We write t for the arc length along Γ measured from $t = 0$ at the origin and T for the whole length of Γ between O and E. Following Hammersley [8] we introduce the quaternion function $q = q(t)$ such that the pair of quaternions $\pm q(t)$ represent the resultant orientation of the sphere at t. We take the initial and the final orientations of the sphere to be given by $q(0) = 1$ and $q(T) = q_{\mathrm{e}}$. We say that the *state* of the sphere at t is specified by its position $(x(t), z(t))$ in the plane and orientation $q(t)$ in the space.

Before going any further we recall some basic facts about the quaternions and the fundamental relationship between *quaternions and rotations* of $\mathbb{R}^3$ (for a comprehensive reading on the subject, see e.g., [4], [11] and [6]). Remember that the space of quaternions $\mathbb{H}$ is a four-dimensional real algebra that obeys, except for commutativity, the same algebraic properties as that of real and complex numbers. Every quaternion can be uniquely written in the form

$$q = \rho_0 + \rho_1 \mathbf{i} + \rho_2 \mathbf{j} + \rho_3 \mathbf{k}$$

where ρ_0, ρ_1, ρ_2 and ρ_3 are real numbers, and, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the quaternion imaginary units satisfying the fundamental equations of the quaternion multiplication

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

The *conjugate* of q is defined in analogous way to the conjugate of complex number

$$\bar{q} = \rho_0 - \rho_1 \mathbf{i} - \rho_2 \mathbf{j} - \rho_3 \mathbf{k}$$

and the *norm* of q is the same as the Euclidean norm on $\mathbb{H}$ considered as a vector space $\mathbb{R}^4$ i.e.,

$$|q| = \sqrt{q\bar{q}} = \sqrt{\rho_0^2 + \rho_1^2 + \rho_2^2 + \rho_3^2}.$$

A quaternion having norm one, i.e., $|q| = 1$, is said to be a *unit* quaternion. A quaternion that have a zero real part, $\rho_0 = 0$, is called a *pure* quaternion. When working with $\mathbb{H}$ it is usual to identify the set of pure quaternions with a three-dimensional Euclidean space $\mathbb{R}^3$ with elements of the form

$$\mathbf{v} = \rho_1 \mathbf{i} + \rho_2 \mathbf{j} + \rho_3 \mathbf{k}$$

where the imaginary units $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are regarded as unit vectors along the three mutually orthogonal axes of a given Cartesian coordinate system in $\mathbb{R}^3$. All vectors in $\mathbb{R}^3$ are interpreted and manipulated as pure quaternions, thus allowing every rotation of $\mathbb{R}^3$ to be obtained by two consecutive quaternion multiplications.

Rotations of $\mathbb{R}^3$ are orthogonal transformations with determinant one. All such transformations form the group $SO(3)$ under the composition of transformations, which is called the group of special orthogonal transformations or the *group of rotations* of $\mathbb{R}^3$. It can be shown that any element of $SO(3)$ can be obtained for some unit quaternion q by the *rotation map*

$$R_q : \mathbf{v} \mapsto q\mathbf{v}q^{-1}, \qquad \mathbf{v} \in \mathbb{R}^3, \qquad |q| = 1$$

and every such map is a rotation of $\mathbb{R}^3$. On the other hand the set of unit quaternions forms a (noncommutative) group under multiplication that coincides with the three dimensional unit sphere in $\mathbb{R}^4$

$$S^3 = \{q \in \mathbb{H} \, ; \, |q| = 1\}.$$

Based on the properties of the rotation map

$$R_{q_1 q_2} = R_{q_1} R_{q_2}, \qquad R_{-q} = R_q$$

it follows that the mapping $q \mapsto R_q$ is a surjective homomorphism from the group S^3 onto the group $SO(3)$, such that any two opposite unit quaternions $\pm q$ correspond to the same rotation. Just as for the complex numbers each unit quaternion $q \in \mathbb{H}$ can be written in the form

$$q = \mathrm{e}^{\varphi \mathbf{u}} = \cos \varphi + \sin \varphi \mathbf{u}, \qquad \mathbf{u}^2 = 1, \qquad \varphi \in [0, 2\pi]$$

where the condition $\mathbf{u}^2 = 1$ means that $\mathbf{u}$ belongs to the unit sphere S^2 in the space of pure quaternions $\mathbb{R}^3$. Any rotation from $SO(3)$ can be represented by specifying an axis $\mathbf{u}$ (a unit vector in $\mathbb{R}^3$) and the angle φ of rotation. A rotation through φ about $\mathbf{u}$ is obtained by applying the rotation map for one (no matter which) of the two unit quaternions

$$\pm q = \pm \mathrm{e}^{\frac{\varphi}{2} \mathbf{u}} = \pm (\cos \frac{\varphi}{2} + \sin \frac{\varphi}{2} \mathbf{u}).$$

This is a clockwise rotation when looking in the forward direction of $\mathbf{u}$.

On returning back to the plate-ball problem we fix a right-handed orthonormal coordinate basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ in $\mathbb{R}^3$ with the vectors $\mathbf{i}$ and $\mathbf{j}$ taken along the axes OX and OZ respectively, and the vector $\mathbf{k}$ directed vertically upwards (see Fig. 1). To handle rotations of the sphere we use quaternion multiplication treating the vectors in $\mathbb{R}^3$ as pure quaternions. Let $\mathbf{r}(t)$ is the position vector of the center of the sphere at t

$$\mathbf{r}(t) = x(t)\mathbf{i} + z(t)\mathbf{j} + \mathbf{k}$$

and let $\mathbf{h}(t)$ denotes the unit vector along the axis of rotation through that center

$$\mathbf{h}(t) = h_1(t)\mathbf{i} + h_2(t)\mathbf{j}.$$

The rolling of the sphere at each t will be regarded as a combination of two infinitesimal motions – a translation along the instantaneous velocity vector of the center of the sphere

$$\mathbf{v}(t) = \mathbf{h}(t)\mathbf{k} \tag{1}$$

and a rotation about the instantaneous axis of rotation $\mathbf{h}(t)$. The kinematics of these two motions are expressed separately by the equations [2, 8]

$$\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = \mathbf{h}(t)\mathbf{k}, \qquad \frac{\mathrm{d}q(t)}{\mathrm{d}t} = \frac{1}{2}\mathbf{h}(t)q(t)$$

where $q(t)$ is a unit quaternion that "records" the orientation of the sphere at t.
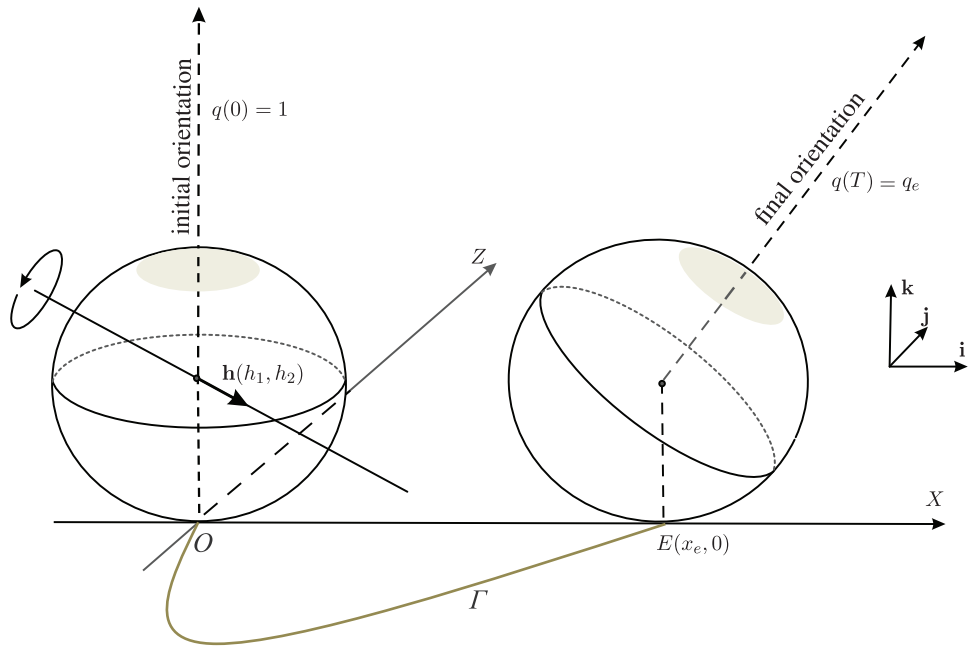

Figure 1. The initial and final positions and orientations of the sphere in the XOZ-plane.

It is clear from the above two equations that the function $\mathbf{h}(t)$ controls the motion of the sphere and will accordingly be called the *control function*. On passing to coordinate representation the kinematics equations take the form of six scalar

equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = h_2, \qquad\qquad \frac{\mathrm{d}z}{\mathrm{d}t} = -h_1$$

$$\frac{\mathrm{d}\rho_0}{\mathrm{d}t} = -\frac{1}{2}(h_1\rho_1 + h_2\rho_2), \qquad \frac{\mathrm{d}\rho_1}{\mathrm{d}t} = \frac{1}{2}(h_1\rho_0 + h_2\rho_3) \qquad (2)$$

$$\frac{\mathrm{d}\rho_2}{\mathrm{d}t} = \frac{1}{2}(-h_1\rho_3 + h_2\rho_0), \qquad \frac{\mathrm{d}\rho_3}{\mathrm{d}t} = \frac{1}{2}(h_1\rho_2 - h_2\rho_1)$$

which are the required *state equations* that under the controls

$$(h_1(t), h_2(t))$$

determine the *state* of the sphere at each moment t

$$(x(t), z(t), \rho_0(t), \rho_1(t), \rho_2(t), \rho_3(t)).$$

Now we can state the problem as a "minimum-time" optimal control problem: *Find in the XOZ-plane an optimal unit-speed curve*

$$\Gamma : (x(t), z(t))$$

defined by the state equations (2) *for some optimal control functions* $(h_1(t), h_2(t))$ *related by the constraint*

$$h_1^2(t) + h_2^2(t) = 1$$

that transfers a given initial state

$$x(0) = z(0) = 0, \qquad q(0) = 1 \qquad (3)$$

to a given final state

$$x(T) = x_\mathrm{e}, \qquad z(T) = 0, \qquad q(T) = q_\mathrm{e} \qquad (4)$$

and minimizes the integral

$$T = \int\limits_0^T 1 \, \mathrm{d}t.$$

2. Trajectories of the Plate-Ball Problem

Let Γ be an optimal solution (an optimal curve) of the optimal control problem stated in Section 1. Arthurs and Walsh [2] showed by applying the Pontryagin maximum principle (see e.g., [7]) that for any point $(x(t), z(t))$ lying on the curve

Γ, there exists a connection between the coordinate $z(t)$ and the curvature $\kappa(t)$ of Γ at this point, given by

$$\kappa(t) = -\lambda z(t) - \mu \tag{5}$$

where the constants $\lambda > 0$ and $\mu \in \mathbb{R}$ are to be determined in terms of x_e and q_e. If the points of Γ are considered to be defined by $(x, z(x))$, i.e., if the coordinate x is regarded as a parameter of the curve, then the above relation is equivalent to the equation

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = -(\lambda z + \mu)\left[1 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2\right]^{3/2}. \tag{6}$$

Now let us suppose that in the course of its motion in the plane the sphere reaches a maximum deflection from the OX-axis at the point $(\mathring{x}, z(\mathring{x}))$ at which the slope of the tangent line to Γ vanishes, namely

$$\frac{\mathrm{d}z}{\mathrm{d}x}\Big|_{x=\mathring{x}} = 0, \qquad z(\mathring{x}) = \eta, \qquad \mathring{x} \in (0, x_e) \tag{7}$$

where $\eta \in \mathbb{R}$ is the "maximum deflection parameter" of Γ. Under the condition (7) the equation (6) can be integrated once by multiplying both sides with $\mathrm{d}z/\mathrm{d}x$, giving the first order differential equation

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 = \frac{4 - \left[2 + \lambda(z^2 - \eta^2) + 2\mu(z - \eta)\right]^2}{\left[2 + \lambda(z^2 - \eta^2) + 2\mu(z - \eta)\right]^2}. \tag{8}$$

Recalling the arc length parameter t, it follows from (5) and (8) that Γ has an intrinsic equation of the form

$$\left(\frac{\mathrm{d}\kappa}{\mathrm{d}t}\right)^2 = \frac{1}{4}(\sigma^2 - \kappa^2)(\kappa^2 + 4\lambda - \sigma^2) \tag{9}$$

where

$$\sigma = \mu + \lambda\eta. \tag{10}$$

By substituting in (5) and (9) with

$$\zeta = z + \frac{\mu}{\lambda}, \qquad \nu = 1 - \frac{\sigma^2}{2\lambda} \tag{11}$$

where ζ is the translated coordinate z and $\nu < 1$, it can be readily shown that the functions $x(t)$ and $\zeta(t)$ satisfy a system of differential equations having the same form as the equations of the Eulerian elasticas [5, 12]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\lambda\zeta^2}{2} + \nu$$

$$\tag{12}$$

$$\left(\frac{\mathrm{d}\zeta}{\mathrm{d}t}\right)^2 = -\frac{\lambda^2\zeta^4}{4} - \lambda\nu\zeta^2 - \nu^2 + 1.$$

In obtaining the above equations it was taken into account the assumption of unit-speed parameterization

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 1.$$

Thus finally we are led to the conclusion: *The trajectories of the plate-ball problem, in the version considered in this paper, coincide with that of the Euler elasticas.*

Henceforth we can present the trajectories of the plate-ball problem, obtained as solutions of the system (12), and express them as follows (cf. [5])

for $\nu \in (-1, 1)$

$$x(t) = -\frac{2}{\sqrt{\lambda}} E(\mathrm{am}(\sqrt{\lambda}t + \alpha, k), k) + t + \beta$$

$$z(t) = a\,\mathrm{cn}(\sqrt{\lambda}t + \alpha, k) - \frac{\mu}{\lambda}$$

(13)

for $\nu = -1$

$$x(t) = -\frac{2\tanh\left(\sqrt{\lambda}t + \alpha\right)}{\sqrt{\lambda}} + t + \beta$$

$$z(t) = \frac{2\,\mathrm{sech}(\sqrt{\lambda}t + \alpha)}{\sqrt{\lambda}} - \frac{\mu}{\lambda}$$

(14)

and for $\nu < -1$

$$x(t) = -aE(\mathrm{am}(\frac{a\lambda}{2}t + \alpha, \hat{k}), \hat{k}) - \nu t + \beta$$

$$z(t) = a\,\mathrm{dn}(\frac{a\lambda}{2}t + \alpha, \hat{k}) - \frac{\mu}{\lambda}$$

(15)

where α and β are the constants of integrations. The rest of the three real constants, i.e., the elliptic moduli k and $\hat{k}$ of the Jacobian elliptic functions and a are specified by the formulas

$$k = \sqrt{\frac{1-\nu}{2}}, \qquad \hat{k} = \sqrt{\frac{2}{1-\nu}}, \qquad a = \sqrt{\frac{2(1-\nu)}{\lambda}}.$$

The functions used above are the hyperbolic functions $\text{sech}(\cdot)$ and $\tanh(\cdot)$, the incomplete elliptic integral of second order $E(\cdot\,,\,k)$ and the Jacobian elliptic functions – amplitude $\text{am}(\cdot\,,\,k)$, cosine $\text{cn}(\cdot\,,\,k)$ and delta $\text{dn}(\cdot\,,\,k)$. More about the Jacobian elliptic functions and the elliptic integrals, can be found e.g., in [1, 3].

By applying in succession the three boundary conditions $z(0) = 0$, $x(0) = 0$ and $z(T) = 0$, we obtain the constants of integration α and β, and the length T of Γ, expressed through the parameters λ, μ and η, namely

for $\nu \in (-1, 1)$

$$\alpha = -\text{cn}^{-1}(\frac{\mu}{\lambda a}, \, k), \qquad \beta = -\frac{2}{\sqrt{\lambda}} E(\arccos \frac{\mu}{\lambda a}, \, k)$$

(16)

$$T = \frac{2}{\sqrt{\lambda}} \text{cn}^{-1}(\frac{\mu}{\lambda a}, \, k)$$

for $\nu = -1$

$$\alpha = -\text{arccosh} \frac{2\sqrt{\lambda}}{\mu}, \qquad \beta = -\frac{\sqrt{4\lambda - \mu^2}}{\lambda}, \qquad T = \frac{2}{\sqrt{\lambda}} \text{arccosh} \frac{2\sqrt{\lambda}}{\mu} \quad (17)$$

and for $\nu < -1$

$$\alpha = -\text{dn}^{-1}(\frac{\mu}{\lambda a}, \, \hat{k}), \qquad \beta = -a\, E(\arcsin \frac{\sqrt{1 - (\frac{\mu}{\lambda a})^2}}{\hat{k}}, \, \hat{k})$$

(18)

$$T = \frac{4}{\lambda a} \text{dn}^{-1}(\frac{\mu}{\lambda a}, \, \hat{k}).$$

The optimal control functions are then found on the base of the relation (1) by differentiating the trajectory equations (13) – (15)

for $\nu \in (-1, 1)$

$$h_1(t) = a\sqrt{\lambda}\, \text{sn}(\sqrt{\lambda}t + \alpha, k)\, \text{dn}(\sqrt{\lambda}t + \alpha, k)$$

(19)

$$h_2(t) = 1 - 2\, \text{dn}^2(\sqrt{\lambda}t + \alpha, k)$$

for $\nu = -1$

$$h_1(t) = 2\, \text{sech}(\sqrt{\lambda}t + \alpha)\, \tanh(\sqrt{\lambda}t + \alpha)$$

(20)

$$h_2(t) = 1 - 2\, \text{sech}^2(\sqrt{\lambda}t + \alpha)$$

and for $\nu < -1$

$$h_1(t) = 2\,\mathrm{sn}(\frac{a\lambda}{2}t + \alpha, \hat{k})\,\mathrm{cn}(\frac{a\lambda}{2}t + \alpha, \hat{k})$$

(21)

$$h_2(t) = (\nu - 1)\,\mathrm{dn}^2(\frac{a\lambda}{2}t + \alpha, \hat{k}) - \nu.$$

Thus far we have classified the solutions of the plate-ball problem in three unified cases. The case to which a given optimal control and the relative optimal trajectory belong to, depends on the value of the parameter ν, which in turn is a function of the parameters λ, μ and η (see the defining equations (11) and (10)) being themselves determined by the boundary conditions (3) – (4). Consequently ν is a function of the initially prescribed quantities x_e and q_e (for more details concerning the boundary conditions see the next Section).

In obtaining the graphs of the plate-ball trajectories we follow a somewhat specific approach regarding the parameters involved. Let us recall, that as it follows from (11) and (10) the parameters λ, μ, η and ν are connected by the equation

$$\nu = 1 - \frac{(\mu + \lambda\eta)^2}{2\lambda}.$$

(22)

Another relation between these four parameters is provided by the boundary condition $x(T) = x_e$, which is represented for each one of the three cases by one of the following transcendental equations

for $\nu \in (-1, 1)$

$$\frac{2}{\sqrt{\lambda}}\,\mathrm{cn}^{-1}(\frac{\mu}{\lambda a}, k) - \frac{4}{\sqrt{\lambda}}\,E(\arccos\frac{\mu}{\lambda a}, k) - x_e = 0$$

(23)

for $\nu = -1$

$$\frac{2}{\sqrt{\lambda}}\,\mathrm{arccosh}\,\frac{2\sqrt{\lambda}}{\mu} - \frac{2\sqrt{4\lambda - \mu^2}}{\lambda} - x_e = 0$$

(24)

and for $\nu < -1$

$$2a\,E(\arcsin\frac{\sqrt{1 - (\frac{\mu}{\lambda a})^2}}{\hat{k}}, \hat{k}) + \frac{4\nu}{\lambda a}\,\mathrm{dn}^{-1}(\frac{\mu}{\lambda a}, \hat{k}) + x_e = 0.$$

(25)

Taking into account equation (22) in equations (23), (24) or (25), the number of the parameters for the respective trajectory is reduced effectively by two. By giving specific values to x_e, η and ν, and making use of the computer program *Mathematica*®, we succeeded in solving numerically for λ and μ some of the resulting transcendental systems of equations. Consequently we have obtained various

graphics of the optimal trajectories of the rolling sphere, produced by *Mathematica*®, having identical starting and ending points, but being with different maximum deflections from the OX-axis, the trajectories have different lengths (see Fig. 2 – Fig. 8 in which are given the graphics of three representative trajectories (left) and the type of the respective elastica curve (right)). Clearly such differences between the trajectories of the plate-ball problem are due to the different initial and final orientations of the sphere in the space.
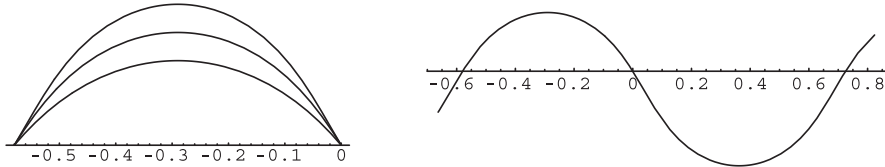
Figure 2. Parameters: $\nu = 0.5$, $x_{\text{e}} = -0.58$, and $\eta = 0.15, 0.2, 0.25$.
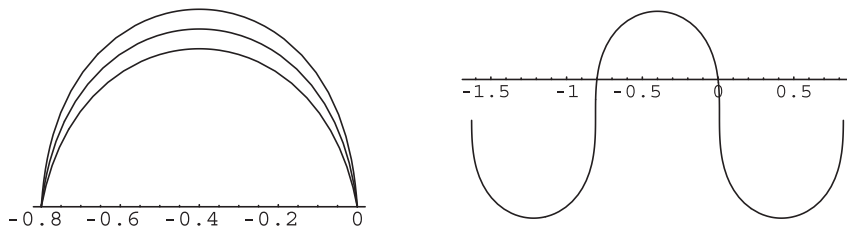
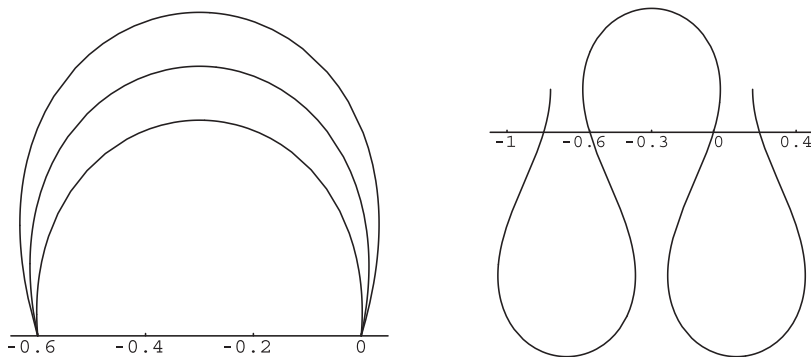**Figure 3.** Parameters: $\nu = 0$, $x_{\text{e}} = -0.8$ and $\eta = 0.4, 0.45, 0.5$.

Figure 4. Parameters: $\nu = -0.4$, $x_{\text{e}} = -0.6$ and $\eta = 0.4, 0.5, 0.6$.

Figure 5. Parameters: $\nu = -0.65223$, $x_{\mathrm{e}} = 0.12$ and $\eta = 0.3, 0.35, 0.4$.
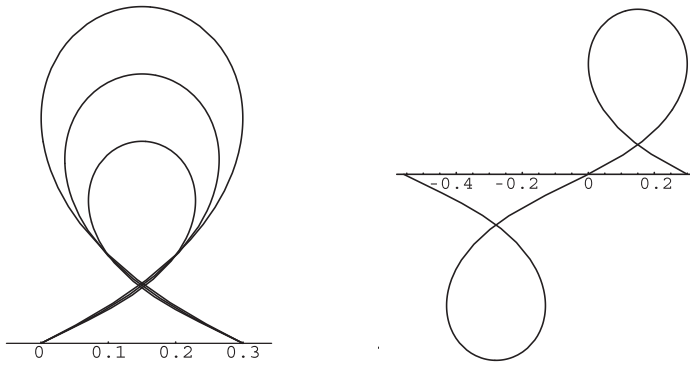
Figure 6. Parameters: $\nu = -0.9$, $x_{\mathrm{e}} = 0.3$ and $\eta = 0.3, 0.4, 0.5$.
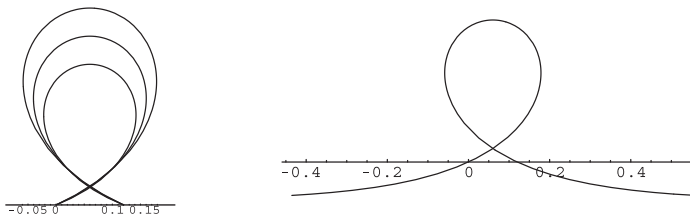

Figure 7. Parameters: $\nu = -1$, $x_{\mathrm{e}} = 0.12$ and $\eta = 0.25, 0.3, 0.35$.

By taking the final position and the maximum deflection of the sphere from the OX-axis to be $x_e = 0.12$ and $\eta = 0.3$, we solved numerically, with respect to λ and μ, the resulting system of equations for the two cases: the equations (22) and (23) for $\nu = -0.65223$, and the equations (22) and (24) for $\nu = -1$. We have used the so obtained values of λ and μ for producing via the parametrization formulas (13) and (14) (and *Mathematica*®) the graphs of the corresponding optimal trajectories of the sphere (the respective values of α and β have been calculated by (16) and (17)) (see Fig. 9).

Now, let us remark that in obtaining the formulas (16) – (18) and (23) – (25) we employed the boundary conditions related with the initial and final positions of the sphere. To this end, we have not taken into account the boundary conditions related with the orientation of the sphere. Despite that for drawing the graphs of the trajectories we freely choose the values of η and ν, these two parameters are by no means arbitrary – their values are connected via the boundary conditions with the prescribed initial and final orientation of the sphere in space. The conditions for the initial $q(0) = 1$ and the final $q(T) = q_e$ orientation of the sphere requires the quaternion function $q(t)$, which we have not found yet.
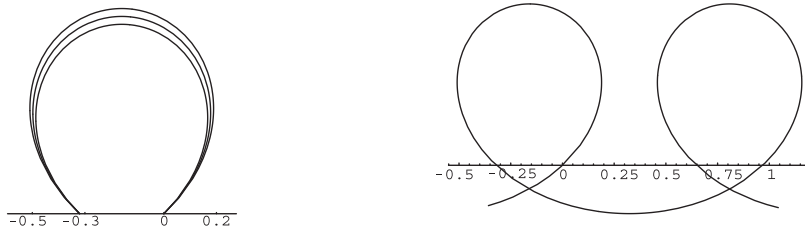

Figure 8. Parameters: $\nu = -1.2$, $x_e = -0.32$ and $\eta = 0.72, 0.75, 0.78$.
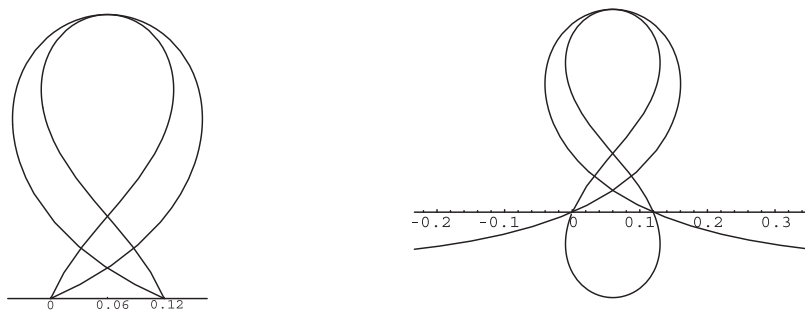

Figure 9. Parameters: $x_e = 0.12, \eta = 0.3, \nu = -1$ and $\nu = -0.65223$.

3. Concluding Remarks

We have considered the plate-ball problem, which is an optimal control problem of finding the shortest paths (optimal curves), traced by a sphere of unit radius while rolling on a horizontal plane about a horizontal axis without slipping or twisting between the two prescribed states – the initial and the final positions and orientations of the sphere in the space. Based on the result of Arthurs and Walsh [2], namely the equation for the curvature of the shortest paths (5), we derived an explicit parameterization for the trajectories of the plate-ball problem. By making use of the Jacobian elliptic functions and elliptic integrals, we obtained expressions for the optimal curves that coincide with the parameterization of the Euler elastics given in [5]. We presented the solution in three possible cases (13), (14) and (15) depending, via the parameter ν, on the prescribed values for the final position x_e and the final orientation $q_e = (\rho_{0e}, \rho_{1e}, \rho_{2e}, \rho_{3e})$ of the sphere. The plate-ball problem was solved by Jurdjevic in 1993 in a completely different context in connection with an optimal control problem suggested by R. Brockett and L. Dai [10].

Finally we will make some concluding remarks concerning the boundary conditions (3) – (4). Since $q(t)$ is a unit quaternion it obeys the constraint (for every $t \in [0, T]$)

$$\rho_0^2(t) + \rho_1^2(t) + \rho_2^2(t) + \rho_3^2(t) = 1 \tag{26}$$

from which it follows that only three of the quaternion coordinates ρ_{0e}, ρ_{1e}, ρ_{2e} and ρ_{3e} are independent – let us denote them by c_1, c_2 and c_3. Consequently the final state of the sphere is predetermined by four independent real quantities

$$(b, c_1, c_2, c_3)$$

where b stands for the final position x_e and (c_1, c_2, c_3) stand for the final orientation q_e of the sphere. It is worthwhile mentioning here that the parameters (c_1, c_2, c_3) are directly related with the coordinates of the so called *vector-parameter*, which in some other setting can also be effectively applied for manipulating with the three dimensional rotations.

As it can be inferred from (26), the total number of constants of integration in the solution of the system (2) is one less than the order of this system. For the determination of the rotational motion of the sphere one has to substitute with the obtained control functions $(h_1(t), h_2(t))$ in the equations for the quaternion coordinates in (2). For each one of the three cases of optimal controls (19), (20) and (21), the solution for the quaternion takes the form

$$\rho_i(t) = \rho_i(t, \gamma_1, \gamma_2, \gamma_3), \qquad i = 0, 1, 2, 3$$

where γ_1, γ_2 and γ_3 are the constants of integration. All integration constants involved in the solution of (2) are five: $(\alpha, \beta, \gamma_1, \gamma_2, \gamma_3)$, where α and β are the

constants of integration occurring in the solution for the translational motion of the sphere $(x(t, \alpha, \beta), z(t, \alpha, \beta))$ (see the solutions found in Section 2).

Now it is clear that we have nine unknown parameters

$$(\lambda, \ \mu, \ \eta, \ \alpha, \ \beta, \ \gamma_1, \ \gamma_2, \ \gamma_3, \ T) \tag{27}$$

that are to be determined as functions of the four prescribed quantities (b, c_1, c_2, c_3), which is to be fulfilled by the help of the boundary conditions (3) – (4). Since the relation (26), the boundary conditions (3) – (4) reduce from twelve to ten equations. Nine of these equations, no matter which of them, suffices for determining all the parameters (27) in terms of (b, c_1, c_2, c_3). Then the remainder tenth equation may be used to distinguish between the different possible cases of the solutions, in a way it has been done above via the parameter ν. The parameter ν, as it is clear now through (11) and (10), depends together with the parameters λ, μ and η on the predefined initial and final position and orientation of the sphere.

Acknowledgements

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