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# SEMI-DISCRETE CONSTANT MEAN CURVATURE SURFACES OF REVOLUTION IN MINKOWSKI SPACE

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**Abstract.** In this paper we describe semi-discrete isothermic constant mean curvature surfaces of revolution with smooth profile curves in Minkowski three-space. Unlike the case of semi-discrete constant mean curvature surfaces in Euclidean three-space, they might have certain types of singularities in a sense defined by the second author in a previous work. We analyze the singularities of such surfaces.

*MSC*: 53A10, 52C99
*Keywords*: Discrete differential geometry, isothermic surfaces, singularity, surface of revolution

## 1. Introduction

Constant mean curvature (CMC) surfaces of revolution have been well-studied ever since Delaunay found that any profile curve of a CMC surface of revolution (except for sphere) in Euclidean three-space $\mathbb{R}^3$ can be obtained as the trace of one focal point of a quadric (see [4] for example), and these surfaces are now called *Delaunay surface*. As a generalization, Kenmotsu [11] described surfaces of revolution with prescribed mean curvature in $\mathbb{R}^3$. Similarly, Hano and Nomizu [7] showed that the profile curves of spacelike CMC surfaces of revolution with timelike or spacelike axes in Minkowski three-space $\mathbb{R}^{2,1}$ can be also obtained as traces of one focal point of quadrics, and Ishihara and Hara [9] derived explicit parametrizations of spacelike (or timelike) surfaces of revolution with prescribed mean curvature (see also [8], [12] and [19]). On the other hand, unlike the case of CMC surfaces of revolution in $\mathbb{R}^3$, spacelike CMC surfaces of revolution may have singularities (see [8] for singularities of spacelike Delaunay surfaces and their associate families, and see also [13], [5] in the case of spacelike maximal surfaces).

In [15], Wallner and the first author described semi-discrete isothermic surfaces in $\mathbb{R}^3$ using integrable systems techniques. Starting from [15], there have been various works in this direction (see [2], [18], [21] for example). In particular, the first author [14] investigated semi-discrete CMC surfaces in $\mathbb{R}^3$.

Our ultimate goal is to establish a more complete theory for semi-discrete surface theory with singularities. As a first step, the second author [21] investigated semi-discrete surfaces with vanishing mean curvature in $\mathbb{R}^{2,1}$, which are called semi-discrete maximal surfaces, and analyzed their singularities. From a different viewpoint, singularities of several semi-discrete surfaces were analyzed in [22] (see also [17]). Except for these examples, there were no known semi-discrete surfaces with singularities. In order to achieve our goal, making new examples with singularities is an important step.

In this paper we briefly introduce semi-discrete surface theory and construct semi-discrete CMC surafces of revolution with smooth spacelike profile curve in $\mathbb{R}^{2,1}$ (semi-discrete spacelike Delaunay surfaces, for short). When the profile curve is discrete, it is difficult to derive explicit parametrizations, because of freedom of choices for the discrete profile curve. On the other hand, when the profile curve is smooth, we can derive explicit parametrizations. When the mean curvature of a semi-discrete surface in $\mathbb{R}^{2,1}$ is identically zero, as shown in [21], such a semi-discrete surface generally has certain singularities, and the types of their singularities are completely classified. On the other hand, like in the smooth case, semi-discrete spacelike Delaunay surfaces may have singularities. We classify all possible semi-discrete spacelike Delaunay surfaces in $\mathbb{R}^{2,1}$. When the surface has singularities, we analyze these singularities.

This paper is organized as follows: In Section 2, we briefly introduce semi-discrete surface theory. In Section 3, we introduce several results on semi-discrete maximal surfaces in $\mathbb{R}^{2,1}$. In Section 4, we derive explicit parametrizations of all possible semi-discrete spacelike Delaunay surfaces. Finally in Section 5, we analyze their singularities and suggest several related problems.

## 2. Preliminaries

Let $\mathbb{R}^{2,1} := (\mathbb{R}^3, \langle\cdot,\cdot\rangle)$ be 3-dimensional Minkowski space with the Lorentz metric

$$\langle (x_1, x_2, x_0)^t, (y_1, y_2, y_0)^t \rangle = x_1 y_1 + x_2 y_2 - x_0 y_0$$

for $(x_1, x_2, x_0)^t, (y_1, y_2, y_0)^t \in \mathbb{R}^{2,1}$. For fixed $c_1 \in \mathbb{R}$ and vector $\nu \in \mathbb{R}^{2,1} \setminus \{0\}$, a plane $\mathcal{P} = \{X \in \mathbb{R}^{2,1};\ \langle X, \nu \rangle = c_1\}$ is *spacelike* (respectively *timelike*, *lightlike*) when $n$ is timelike (respectively spacelike, lightlike).

In Definition 1, in order to define semi-discrete isothermic surfaces in $\mathbb{R}^{2,1}$, any element $(x_1, x_2, x_0)^t \in \mathbb{R}^{2,1}$ is identified with the matrix form $\begin{pmatrix} \mathrm{i}x_0 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & -\mathrm{i}x_0 \end{pmatrix}$ (see also [2], [3]).

**Definition 1.** *Let $x : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^{2,1}$ be a semi-discrete surface parametrized by $(k,t) \in \mathbb{Z} \times \mathbb{R}$. Then $x$ is a* semi-discrete isothermic *surface if each plane containing $\partial x, \partial x_1, \partial \Delta x$ is spacelike and $cr(x, x_1) = -\frac{\tau}{\sigma} I$, where*

$$x = x(k,t), \qquad x_1 := x(k+1,t), \qquad \partial x := \frac{\mathrm{d}x}{\mathrm{d}t}, \qquad \Delta x := x_1 - x$$
$$cr(x, x_1) := (\partial x) \cdot (\Delta x)^{-1} \cdot (\partial x_1) \cdot (\Delta x)^{-1}$$

*$I$ is the $2 \times 2$ unit matrix, and $\tau$ (respectively $\sigma$) is a positive scalar function depending on only $t$ (respectively $k$). We call $cr(x, x_1)$ the* tangent cross ratio *of $x$.*

**Remark 2.** *Henceforth, if the tangent cross ratio is $cI$ for $c \in \mathbb{R}$, $cI$ is identified with the scalar value $c$. So, the tangent cross ratio of a semi-discrete isothermic surface is simply written as $-\frac{\tau}{\sigma}\cdot$*

In order to describe semi-discrete spacelike Delaunay surfaces in $\mathbb{R}^{2,1}$, here we define Gaussian and mean curvatures of semi-discrete surfaces in $\mathbb{R}^{2,1}$, which were originally introduced in [10] (see also [21]). In the following definition, $\mathbb{H}^2$ denotes the hyperbolic two-plane, that is,

$$\mathbb{H}^2 := \{X \in \mathbb{R}^{2,1};\ \langle X, X \rangle = -1\}.$$

**Definition 3.** *Let $x : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^{2,1}$ be a semi-discrete circular surface (for the definition of semi-discrete circular surfaces, see* [15], [21]*), and let $n : \mathbb{Z} \times \mathbb{R} \to \mathbb{H}^2$ be a semi-discrete circular surface satisfying $\partial x \parallel \partial n$ and $\Delta x \parallel \Delta n$ (we call $n$ the* Gauss map *of $x$). Then*

- *$K$ and $H$ defined by*

$$\begin{aligned} K &= \frac{\det(\partial n + \partial n_1, \Delta n, \tilde{\nu})}{\det(\partial x + \partial x_1, \Delta x, \tilde{\nu})} \\ H &= -\frac{\det(\partial x + \partial x_1, \Delta n, \tilde{\nu}) + \det(\partial n + \partial n_1, \Delta x, \tilde{\nu})}{2\det(\partial x + \partial x_1, \Delta x, \tilde{\nu})} \end{aligned} \tag{1}$$

  *are called the* Gaussian *and* mean curvatures *of $x$ (with respect to $n$), where $\tilde{\nu}$ is a unit vector perpendicular to the plane containing $\partial x, \Delta x$ and $\partial \Delta x$.*
- *Let $x$ be a semi-discrete circular surface and let $n$ be its Gauss map. Then the scalar functions $\kappa_k(t)$, $\kappa_{k,k+1}(t)$ given by*

$$\partial n(k,t) = -\kappa_k(t)\partial x, \qquad \Delta n(k,t) = -\kappa_{k,k+1}(t)\Delta x(k,t)$$

*are called the* principal curvatures *of* $x$. *Here we abbreviate* $\kappa = \kappa_k(t),\ \kappa_1 = \kappa_{k+1}(t),\ \kappa_{01} = \kappa_{k,k+1}(t)$.

As already shown in [21], there are non-trivial relations between Gaussian or mean curvatures and principal curvatures as follows:

**Proposition 4.** *Let* $x$ *be a semi-discrete surface in* $\mathbb{R}^{2,1}$, *let* $\kappa, \kappa_{01}$ *be the principal curvatures of* $x$, *and let* $K$ *and* $H$ *be the Gaussian and mean curvatures of* $x$. *Then* $K$ *and* $H$ *are expressed as*

$$K = \frac{\kappa_{01}}{\kappa_1 + \kappa - 2\kappa_{01}}(2\kappa\kappa_1 - \kappa\kappa_{01} - \kappa_1\kappa_{01}), \qquad H = \frac{\kappa\kappa_1 - \kappa_{01}^2}{\kappa_1 + \kappa - 2\kappa_{01}}. \tag{2}$$

**Proof:** In [22], it is shown that the Gaussian curvature of a semi-discrete circular surface is of the form in Proposition 4, and the proof is almost the same. Here we derive the form of the mean cuvature. By definition, we have

$$\begin{aligned} H &= -\frac{\det(\partial x + \partial x_1, \Delta n, \tilde{\nu}) + \det(\partial n + \partial n_1, \Delta x, \tilde{\nu})}{2\det(\partial x + \partial x_1, \Delta x, \tilde{\nu})} \\ &= \frac{\det(\kappa_{01}(\partial x + \partial x_1) + \kappa\partial x + \kappa_1\partial x_1, \Delta x, \tilde{\nu})}{2\det(\partial x + \partial x_1, \Delta x, \tilde{\nu})}. \end{aligned} \tag{3}$$

Now, differentiation gives

$$\Delta n = -\kappa_{01}\Delta x \Rightarrow (\partial\kappa_{01})\Delta x = (\kappa_1 - \kappa_{01})\partial x_1 - (\kappa - \kappa_{01})\partial x.$$

Here we assume that $\partial\kappa_{01} \neq 0$. Substituting $\Delta x$ in the above form into equation (3), $H$ is written as

$$\begin{aligned} H &= \frac{\det\left(\kappa_{01}(\partial x + \partial x_1) + \kappa\partial x + \kappa_1\partial x_1, \frac{\kappa_1 - \kappa_{01}}{\partial\kappa_{01}}\partial x_1 - \frac{\kappa - \kappa_{01}}{\partial\kappa_{01}}\partial x, \tilde{\nu}\right)}{2\det\left(\partial x + \partial x_1, \frac{\kappa_1 - \kappa_{01}}{\partial\kappa_{01}}\partial x_1 - \frac{\kappa - \kappa_{01}}{\partial\kappa_{01}}\partial x, \tilde{\nu}\right)} \\ &= \frac{\kappa_{01}\left(\frac{\kappa_1 - \kappa_{01}}{\partial\kappa_{01}} + \frac{\kappa - \kappa_{01}}{\partial\kappa_{01}}\right) + \frac{\kappa(\kappa_1 - \kappa_{01})}{\partial\kappa_{01}} + \frac{\kappa_1(\kappa - \kappa_{01})}{\partial\kappa_{01}}}{2\left(\frac{\kappa_1 - \kappa_{01}}{\partial\kappa_{01}} + \frac{\kappa - \kappa_{01}}{\partial\kappa_{01}}\right)} = \frac{\kappa\kappa_1 - \kappa_{01}^2}{\kappa_1 + \kappa - 2\kappa_{01}}. \end{aligned}$$

Thus $H$ is of the form in Proposition 4. Similarly, when $\partial\kappa_{01} = 0$, we have the same $H$ of the form in Proposition 4. For $K$, see [22]. ■

At first glance, relations between the princpal curvatures and the Gaussian or mean curvature are not elegant. However, we can interpret these relations as follows: Assume that $\kappa, \kappa_1$ converge to $k_1$ and $\kappa_{01}$ converges to $k_2$ by taking a "formal" limit (note that we do not actually use such a limit in this work). Then, the Gaussian and mean curvatures $K, H$ of a semi-discrete circular surface converge to

$$K \to k_1 k_2, \qquad H \to \frac{k_1 + k_2}{2}.$$

In this sense, the forms of $K$ and $H$ are natural.

## 3. Semi-Discrete Maximal Surfaces with Singularities in $\mathbb{R}^{2,1}$

In this section we briefly introduce three results on semi-discrete maximal surfaces in $\mathbb{R}^{2,1}$. A semi-discrete isothermic surface $x$ is called a *semi-discrete maximal surface* if the mean curvature $H$ of $x$ identically vanishes. The following fact is shown in [21].

**Theorem 5.** *Any semi-discrete maximal surface $x$ can be locally constructed using a semi-discrete holomorphic function $g$ by solving*

$$\partial x = -\operatorname{Re}\left(\frac{\tau}{2\partial g}\begin{pmatrix}1+g^2\\ \mathrm{i}(1-g^2)\\ -2g\end{pmatrix}\right), \qquad \Delta x = \operatorname{Re}\left(\frac{\sigma}{2\Delta g}\begin{pmatrix}1+gg_1\\ \mathrm{i}(1-gg_1)\\ -(g+g_1)\end{pmatrix}\right) \quad (4)$$

*with $\tau$ and $\sigma$ determined from $g$, where a semi-discrete isothermic surface $g : \mathbb{Z}\times\mathbb{R}\to\mathbb{R}^2\cong\mathbb{C}$ is called a* semi-discrete holomorphic function.

The interesting point is that the Weierstrass-type representation (4) for semi-discrete maximal surfaces in $\mathbb{R}^{2,1}$ is almost the same as the one for semi-discrete minimal surfaces in $\mathbb{R}^3$ (see [18]), but the global behavior is quite different. In fact, unlike the case of semi-discrete minimal surfaces in $\mathbb{R}^3$, semi-discrete surfaces described by equation (4) are locally semi-discrete maximal but not globally, leading us to the necessity to consider "singularities" of semi-discrete maximal surfaces. Here we define singularities of semi-discrete maximal surfaces as follows.

**Definition 6.** *Let $x$ be a semi-discrete surface in $\mathbb{R}^{2,1}$. Then an edge $[x, x_1]$ with endpoints $x, x_1$ is a* singular edge *if the plane containing $\partial x, \partial x_1, \partial\Delta x$ is not spacelike. In particular, when at least one of the directions of $\partial x, \partial x_1, \Delta x$ is lightlike, a singular edge is called* non-generic*, and otherwise called* generic.

We introduce criteria for singular edges of semi-discrete maximal surfaces in $\mathbb{R}^{2,1}$. Details can be found in [21] (and see Fig. 1).

**Theorem 7.** *Let $g : \mathbb{Z}\times\mathbb{R}\to\mathbb{C}$ be a semi-discrete holomorphic function and let $x$ be a semi-discrete surface described by equation* (4) *using a semi-discrete holomorphic function $g$. Then the edge $[x, x_1]$ is a singular edge if and only if a tangent circle $\mathfrak{C}$ of $[g, g_1]$ corresponding to $[x, x_1]$ intersects the unit circle $\mathbb{S}^1\subset\mathbb{C}$.*

Admitting singular edges, by Theorem 5, we can construct semi-discrete maximal surfaces of revolution with timelike, spacelike, or lightlike axis. Explicit choices of the corresponding semi-discrete holomorphic functions can be found in [21], so here we only state the following result:

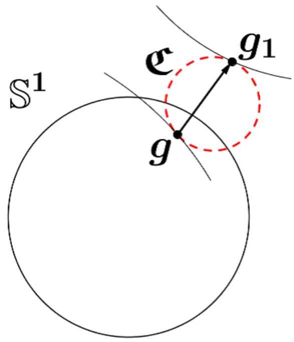


**Figure 1.** The tangent circle $\mathfrak{C}$ of $[g, g_1]$ intersects $\mathbb{S}^1$ (the case where a singular edge appears).

**Proposition 8.** *Semi-discrete maximal surfaces of revolution with smooth profile curves have the same profile curves as smooth maximal surfaces.*

**Remark 9.** *In addition to* Proposition 8*, we have shown that semi-discrete maximal surfaces of revolution with discrete profile curves have the same profile curves as discrete maximal surfaces of revolution in the sense of* [20]*. In this paper, because of the simplicity of deriving the explicit parametrizations, we only focus on semi-discrete surfaces of revolution with smooth profile curves.*

## 4. Semi-Discrete Spacelike Delaunay Surfaces in $\mathbb{R}^{2,1}$

In this section we derive explicit parametrizations of the semi-discrete spacelike Delaunay surfaces with smooth profile curve and timelike (respectively spacelike, lightlike) axes, which we call the semi-discrete *t-Delaunay* (respectively *s-Delaunay*, *l-Delaunay*) surfaces. First we state that surfaces of revolution in $\mathbb{R}^{2,1}$ might be semi-discrete isothermic.

**Lemma 10.** *Let $x$ be a surface of revolution with smooth profile curve and timelike axis (respectively spacelike, lightlike) in $\mathbb{R}^{2,1}$ parametrized as*

$$x(k,t) = (f(t)\cos\alpha_k, f(t)\sin\alpha_k, g(t))^t$$

*respectively*

$$x(k,t) = (f(t)\sinh\alpha_k, g(t), f(t)\sin\alpha_k, f(t)\cosh\alpha_k)^t$$
$$x(k,t) = (f(t) - (1-\alpha_k^2)g(t), 2\alpha_k g(t), f(t) + (1+\alpha_k^2)g(t))^t$$

*where $f(t), g(t)$ are scalar functions depending only on $t$, and $\alpha_k$ is a scalar function depending only on $k$. Then*

$$cr(x, x_1) = -\frac{(f')^2 - (g')^2}{4f^2 \sin^2\left(\frac{\alpha_1 - \alpha}{2}\right)}$$

*respectively*

$$cr(x, x_1) = -\frac{-(f')^2 + (g')^2}{4f^2 \sin^2\left(\frac{\alpha_1 - \alpha}{2}\right)}, \quad cr(x, x_1) = \frac{f'g'}{g^2(\alpha_1 - \alpha)^2}$$

*where $f' := \frac{\mathrm{d}f}{\mathrm{d}t}$, $g' := \frac{\mathrm{d}g}{\mathrm{d}t}\cdot$ If the corresponding profile curve is spacelike, $cr(x, x_1)$ are negative, implying that $x$ could be semi-discrete isothermic.*

**Remark 11.** *Even if the tangent cross ratio of a semi-discrete surface of revolution in $\mathbb{R}^{2,1}$ is negative in* Lemma 10*, $x$ is not necessarily semi-discrete isothermic.*

Take a parametrization of a semi-discrete t-Delaunay surface as follows

$$x(k, t) = (f(t)\cos\alpha_k, f(t)\sin\alpha_k, g(t))^t$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are functions satisfying $(f')^2 - (g')^2 = 1$, that is, the profile curve is arc-length parametrized. Without loss of generality, we can choose the Gauss map $n$ of $x$ as

$$n(k, t) = (-g'(t)\cos\alpha_k, -g'(t)\sin\alpha_k, -f'(t))^t.$$

Then the arc-length parametrization condition gives that $f'' = \frac{g'g''}{f'}$, implying $\partial n = -\frac{g''}{f'}\partial x$. Thus we have $\kappa = \frac{g''}{f'}$. And we have $\Delta n = -\frac{g'}{f}\Delta x$, implying $\kappa_{01} = \frac{g'}{f}\cdot$ Substituting $\kappa_0, \kappa_1, \kappa_{01}$ into equation (1), we have $H = \frac{1}{2}\Big(\frac{g'}{f} + \frac{g''}{f'}\Big)$. Again by the arc-length parametrization condition, we have

$$\begin{aligned} H &= \frac{1}{2}\Big(\frac{g'}{f} + \frac{g'' + (g')^2 g'' - g' \cdot g' g''}{f'}\Big) \\ &= \frac{1}{2}\Big(\frac{g'}{f} + \frac{(f')^2 g'' - f' f'' g'}{f'}\Big) = \frac{1}{2}\Big(\frac{g'}{f} + f'g'' - f''g'\Big). \end{aligned}$$

Therefore, we have

$$2H(t)f(t) - g'(t) - f(t)(f'(t)g''(t) - f''(t)g'(t)) = 0.$$

This equation is the same as equation (3.1) in [9]. This implies that the profile curve of a semi-discrete t-Delaunay surface is the same as the one of a smooth t-Delaunay surface with the same constant mean curvature (more precisely, the same prescribed mean curvature).

In the same vein, we derive explicit parametrizations of semi-discrete s-Delaunay surfaces. Take the parametrization of a semi-discrete s-Delaunay surface as follows

$$x(k,t) = (f(t)\sinh\alpha_k, g(t), f(t)\cosh\alpha_k)^t$$

with $-(f')^2 + (g')^2 = 1$. We can choose the Gauss map $n$ of $x$ as

$$n(k,t) = (g'(t)\sinh\alpha_k, f'(t), g'(t)\cosh\alpha_k)^t.$$

By a similar computation as in the case of semi-discrete t-Delaunay surfaces, $\kappa = -\frac{g''}{f'}$ and $\kappa_{01} = -\frac{g'}{f}\cdot$ Substituting $\kappa_0, \kappa_1, \kappa_{01}$ into equation (1), we have $H = -\frac{1}{2}\Big(\frac{g'}{f} + \frac{g''}{f'}\Big)$, and by the arc-length parametrization condition, we have

$$H = -\frac{1}{2}\Big(\frac{g'}{f} + \frac{g'' - (g')^2 g'' + g'\cdot g' g''}{f'}\Big) = -\frac{1}{2}\Big(\frac{g'}{f} - f'g'' + f''g'\Big).$$

In summary, we have

$$2H(t)f(t) + g'(t) + f(t)(f''(t)g'(t) - f'(t)g''(t)) = 0.$$

This equation is the same as equation (3.2) in [9]. This implies that the profile curve of a semi-discrete s-Delaunay surface is the same as that of a smooth s-Delaunay surface with the same constant mean curvature.

Finally, we derive parametrizations of semi-discrete l-Delaunay surfaces. Take the parametrization of a semi-discrete l-Delaunay surface as follows

$$x(k,t) = (f(t) - (1-\alpha_k^2)g(t), 2\alpha_k g(t), f(t) + (1+\alpha_k^2)g(t))^t$$

with $-4f'g' = 1$. And we can choose the Gauss map $n$ of $x$ as

$$n(k,t) = (-f'(t) + (1-\alpha_k^2)g'(t), -2\alpha_k g'(t), -f'(t) + (1+\alpha_k^2)g'(t))^t.$$

By a similar computation as in the cases of semi-discrete t- and s-Delaunay surfaces, $\kappa = -\frac{g''}{g'}$ and $\kappa_{01} = -\frac{g'}{g}\cdot$ Substituting $\kappa_0, \kappa_1, \kappa_{01}$ into equation (1) and deforming the equation, we have

$$2H(t)g(t) + g'(t) - 2g(t)(f'(t)g''(t) - f''(t)g'(t)) = 0.$$

This equation is the same as equation (3.3) in [9], implying that the profile curve of a semi-discrete l-Delaunay surface is the same as that of a smooth l-Delaunay surface with the same constant mean curvature.

Thus we have the following theorem.

**Theorem 12.** *Semi-discrete surfaces of revolution with prescribed mean curvature having smooth profile curves have the same prescribed mean curvature as their* *smooth counterpart surfaces with the same profile curves. In particular, semi-discrete* CMC *surfaces of revolution with smooth profile curves have the same collection of profile curves as the smooth* CMC *surfaces of revolution.*

Profile curves of all possible semi-discrete Delaunay surfaces are listed in Table 1, and an example of a semi-discrete spacelike Delaunay surface is shown in Fig. 2.

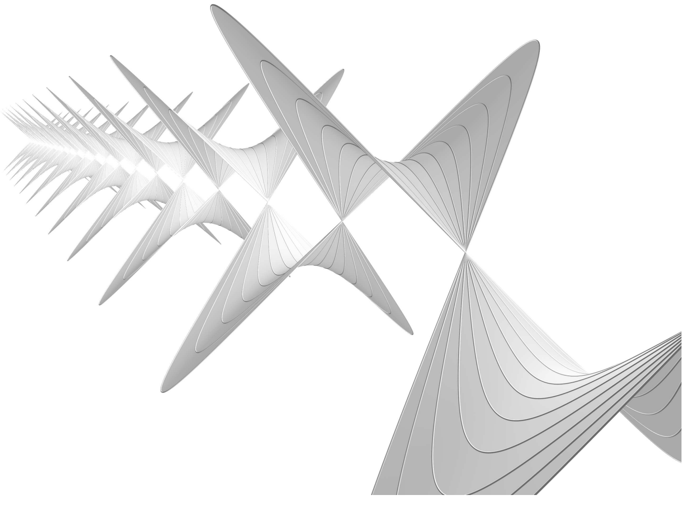

**Figure 2.** A semi-discrete s-Delaunay surface. This has singular edges around cone points.

## 5. Singularities of Semi-Discrete Spacelike Delaunay Surfaces in $\mathbb{R}^{2,1}$

Here we analyze singular edges of semi-discrete spacelike Delaunay surfaces with smooth profile curve in $\mathbb{R}^{2,1}$. Like the case of semi-discrete maximal surfaces of revolution, semi-discrete spacelike Delaunay surfaces may have singular edges. By a similar argument in [21], we have the following criteria for singular edges of semi-discrete CMC surfaces in $\mathbb{R}^{2,1}$ to appear.

**Theorem 13.** *Let $x$ be a semi-discrete isothermic* CMC *surfaces in $\mathbb{R}^{2,1}$ and let $n$ be its Gauss map. Then an edge $[x, x_1]$ is a singular edge if and only if the tangent circle of $[\phi \circ n, \phi \circ n_1]$ corresponding to $[x, x_1]$ intersects $\mathbb{S}^1$, where $\phi : \mathbb{H}^2 \to \mathbb{R}^2 \cong \mathbb{C}$ is a stereographic projection with south pole $(0, 0, -1)^t \in \mathbb{R}^{2,1}$.*

Applying Theorem 13, we can analyze singular edges of semi-discrete spacelike Delaunay surfaces. In fact, edges around cone points in Fig. 2 satisfy Theorem 13. For other cases, if a semi-discrete spacelike Delaunay surface has a cone point, one can confirm that edges around this point are singular.

**Table 1.** The list of abbreviation of symbols and profile curves of possible semi-discrete spacelike Delaunay surfaces in $\mathbb{R}^{2,1}$. All possible smooth spacelike Delaunay surfaces were classified in [9] (see also [8]), and the explicit parametrizations here are cited from [8].

| Abbreviation of symbols | profile curve |
|---|---|
| $U_{\mathfrak{t}}(t)=\sqrt{\mathrm{e}^{2Ht}+k\mathrm{e}^{-2Ht}-k-1}$ | $\left(\frac{U_{\mathfrak{t}}(t)}{2H},0,\pm\int\frac{\mathrm{e}^{2Ht}+k\mathrm{e}^{-2Ht}-2}{2U_{\mathfrak{t}}(t)}\mathrm{d}t\right)$ |
| $U_{\mathfrak{sc}}(t)=\sqrt{\mathrm{e}^{2Ht}+k\mathrm{e}^{-2Ht}+k+1}$ | $\left(0,\int\frac{\mathrm{e}^{2Ht}+k\mathrm{e}^{-2Ht}-2}{2U_{\mathfrak{sc}}(t)}\mathrm{d}t,\pm\frac{U_{\mathfrak{sc}}}{2H}\right)$ |
| $U_{\mathfrak{si}}(t)=\sqrt{\mathrm{e}^{2Ht}+k\mathrm{e}^{-2Ht}-k-1}$ | $\left(0,\int\frac{\mathrm{e}^{2Ht}+k\mathrm{e}^{-2Ht}-2}{2U_{\mathfrak{si}}(t)}\mathrm{d}t,\pm\frac{U_{\mathfrak{si}}}{2H}\right)$ |
| – | $\left(0,\frac{t}{2H},\pm\frac{1}{2H}\right)$ |
| $\xi_{\mathfrak{lt}}(t)=\sqrt{\mathrm{e}^{2Ht}-1}$<br>$\eta_{\mathfrak{lt}}(t)=\frac{\pm1}{8H^2}\left(\frac{\pm\xi_{\mathfrak{lt}}(t)}{\mathrm{e}^{2Ht}}-\arctan\xi(t)\right)$ | $(\eta_{\mathfrak{lt}}(t)+\xi_{\mathfrak{lt}}(t),0,\eta_{\mathfrak{lt}}(t)-\xi_{\mathfrak{lt}}(t))$ |
| $\xi_{\mathfrak{lsc}}(t)=\sqrt{\mathrm{e}^{2Ht}+1}$<br>$\eta_{\mathfrak{lsc}}(t)=\frac{\pm1}{8H^2}\left(\frac{\pm\xi_{\mathfrak{lsc}}(t)}{\mathrm{e}^{2Ht}}-\operatorname{arccoth}\xi(t)\right)$ | $(\eta_{\mathfrak{lsc}}(t)+\xi_{\mathfrak{lsc}}(t),0,\eta_{\mathfrak{lsc}}(t)-\xi_{\mathfrak{lsc}}(t))$ |
| $\xi_{\mathfrak{lsi}}(t)=\sqrt{-\mathrm{e}^{2Ht}+1}$<br>$\eta_{\mathfrak{lsi}}(t)=\frac{\pm1}{8H^2}\left(\frac{\pm\xi_{\mathfrak{lsi}}(t)}{\mathrm{e}^{2Ht}}+\operatorname{arctanh}\xi(t)\right)$ | $(\eta_{\mathfrak{lsi}}(t)+\xi_{\mathfrak{lsi}}(t),0,\eta_{\mathfrak{lsi}}(t)-\xi_{\mathfrak{lsi}}(t))$ |
| – | $\left(\frac{\pm\mathrm{e}^{-Ht}}{4H^2}\pm\mathrm{e}^{Ht},0,\frac{\pm\mathrm{e}^{-Ht}}{4H^2}\mp\mathrm{e}^{Ht}\right)$ |

On the other hand, unlike the case of semi-discrete maximal surfaces of revolution, the profile curves of semi-discrete spacelike spacelike Delaunay surfaces may come close to lightlike lines asymptotically. So singular edges might appear around infinity. However, there is no twisted singular edge, where a singular edge of $x$ is called *twisted* if $\partial x$ and $\partial x_1$ lie to the opposite sides in a plane (spanned by $\partial x, \partial x_1, \Delta\partial x$) of the straight line containing $\Delta x$.

This observation leads us to the following conjecture:

**Conjecture.** Semi-discrete CMC surface in $\mathbb{R}^{2,1}$ might come asymptotically close to a lighlike plane around infinity. On the other hand, there is no twisted singular edge around infinity.

Finally we conclude this paper with two interesting related problems.

**Problems.**

- Like in the smooth case, can we consider associated families of semi-discrete maximal surfaces? This problem is highly related to the work in [3]. In addition, is there any Lax representation for semi-discrete CMC surfaces? If yes, we can also consider their associated families.
- If we can set up the associated families, how do edges of the associated families corresponding to singular edges of the original semi-discrete maximal or CMC surface behave? In particular, in the smooth case, there are dualities between singularities of a maximal surface and its conjugate surface (see [6] for example). So it is interesting to see whether there are also dualities between singularities of a semi-discrete maximal surface and its conjugate surface.

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