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LOCAL AND NON-LOCAL CONSERVATION LAWS FOR QUADRATIC CONSTRAINED LAGRANGIANS AND APPLICATIONS TO COSMOLOGY

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Abstract. We study the existence of conservation laws in constrained systems described by quadratic Lagrangians; the type of which is encountered in mini-superspace cosmology. As is well known, variational symmetries lead to conserved quantities that can be used in the classical and quantum integration of a system. Additionally - and due to the parametrization invariance of such Lagrangians - conditional symmetries defined on phase space can lead to non-local integrals of motion. The latter may be of importance in various cosmological configurations. As an example we present the case of scalar field cosmology with an arbitrary potential.

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1. Introduction

Symmetries at both the classical and the quantum level are of utmost importance in many physical theories. The same is also true for cosmology, especially in the context of a mini-superspace approximation. That is, when there can be constructed an equivalent mechanical system which exhibits the same dynamical evolution as the gravitational one. There is a series of works that deal with symmetries of these systems and how they are used to derive solutions or constrain the theory under consideration [2, 12, 13, 15]. When a mini-superspace approximation is adopted a constrained (or singular) system is obtained, meaning that not all of the equations of motion are independent. Usually, in the literature, a particular gauge fixing condition is being applied so as to treat these Lagrangians as regular. However, it is proven that such a procedure, when implemented prior to the derivation of symmetries, may result in losing some of them (see for example the Appendix of [4]).

The general theory behind the search of symmetries for both the action and the equations of motion is well known [11, 14] and with slight modifications it can also be used in the case of constrained systems [3]. Furthermore, it is proven that all of these symmetries are part of a larger class appearing only in the presence of constraints. Elements of this latter set lead to conserved quantities that can be non-local expressions, owed to involving integrals of functions of the configuration space variables. All aforementioned quantities can be exploited classically for the derivation of the solution. By following the main path of the canonical quantum theory, those that correspond to local integrals of motion may be applied at the quantum level as well. These results have been presented in a series of papers, not only for cosmological space-times [16] but also for black holes [4–6].

The layout of this paper is the following: In Section 2 we give the process of the reduction for the mini-superspace approximation. Then, in Section 3, we present the main result regarding the variational/Lie-point symmetries of the action/equations of motion. The Hamiltonian description and the derivation of a larger class of symmetries is given in Section 4. In Section 5 we give a brief description of how to apply some of the resulting symmetries at the quantum level. To exhibit how powerful is the use of the non-local conserved quantities we give an example in Section 6 involving an FLRW space-time in which a scalar field is minimally coupled to gravity. We manage to fully integrate the system for an arbitrary potential of the scalar field. In conclusion, we make some final remarks in the discussion.

2. The Mini-Superspace Description

Let us begin by considering Einstein's general theory of relativity with the line element of a four dimensional pseudo-Riemannian manifold $M = \mathbb{R} \times \Sigma$ given by

$$\mathrm{d}s^2 = g_{\alpha\beta}(x)\mathrm{d}x^\alpha \mathrm{d}x^\beta, \qquad \alpha, \beta = 0, ..., 3, \qquad x \in M$$

where the $g_{\alpha\beta}$ are the components of the metric tensor which extremizes the action functional

$$S = \int_M \mathrm{d}^4x\sqrt{-g}R + S_m \tag{1}$$

with $g = \det g_{\alpha\beta}$, R the Ricci scalar curvature and S_m the matter contribution to the action. Einstein's equations that are satisfied by $g_{\alpha\beta}$ are

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = T_{\alpha\beta}. \tag{2}$$

In its most generality, (2) is a system of ten partial differential equations. Four of them are constraints of the rest six, which means that only two are truly independent. This can be traced to the existence of the Bianchi identities stemming from the invariance of the theory under the group of the four dimensional diffeomorphisms.

In many cases, especially when treating cosmological configurations, a certain number of existing symmetries is adopted for the base manifold. As happens for example in the case of spatially homogeneous spacetimes, where a three dimensional group of motions acts simply transitively on the spatial slices Σ. The most general form of the line element in this case is

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + \gamma_{IJ}(t)\sigma^I_i(x)\sigma^J_j(x)\mathrm{d}x^i\mathrm{d}x^j \tag{3}$$

with the one-forms σ's satisfying

$$\sigma^I_{i,j} - \sigma^I_{j,i} = C^I_{JK}\sigma^K_i\sigma^J_j \tag{4}$$

and C^I_{JK} being the structure constants of the algebra corresponding to the above mentioned three dimensional group of isometries (all indices in (3) and (4) range from 1 to 3). The field equations (2), under the ansatz (3), reduce to ordinary differential equations. What is more, when the (3) is applied at the level of the action integral (1) and the non-dynamical spatial degrees of freedom are integrated out, one is left with an action of a mechanical system. The general form of the corresponding Lagrangian function is

$$L = \frac{1}{2N(t)}G_{\mu\nu}(q)\dot{q}^\mu(t)\dot{q}^\nu(t) - N(t)V(q), \qquad \mu,\nu = 1,...,d \tag{5}$$

where $\dot{} = \frac{\mathrm{d}}{\mathrm{d}t}$ while the q's denote the remaining degrees of freedom[1]. In the configuration space spanned by the q's, the tensor $G_{\mu\nu}$ is called the mini-superspace metric. Lagrangian (5) describes a singular system of $d+1$ degrees of freedom (counting in them the lapse function N) and is invariant under arbitrary time reparametrizations of the form $t \mapsto \tau = f(t)$ where the following transformation laws apply

$$N(t)\mathrm{d}t = \tilde{N}(\tau)\mathrm{d}\tau, \qquad q(t) = \tilde{q}(\tau).$$

It is a simple task to check that the action remains form invariant under such a transformation. This invariance is a remnant of the four dimensional diffeomorphism gauge group of the original gravitational system.

Whenever the corresponding Euler-Lagrange equations of (5)

$$E^0 := \frac{\partial L}{\partial N} = 0, \qquad E^\alpha := \frac{\partial L}{\partial q^\alpha} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) = 0$$

[1]The γ_{IJ}'s and possibly any additional matter field.

are equivalent to the Einstein equations (2) under the ansatz (3), we say that the system has a valid mini-superspace description, meaning that its dynamics may be investigated with the help of the mechanical system given by (5). Equation E^0 is quadratic in the velocities and is a constraint among the E^α's, of which only $d-1$ are truly independent.

3. Variational Symmetries of the Action and Lie-Point Symmetries of the Equations of Motion

Let us consider point transformations in the space spanned by (t, q, N) with a corresponding infinitesimal generator of the form

$$X = \chi(t,q,N)\frac{\partial}{\partial t} + \xi^\alpha(t,q,N)\frac{\partial}{\partial q^\alpha} + \omega(t,q,N)\frac{\partial}{\partial N}\cdot \tag{6}$$

A transformation of this type is a variational symmetry of the action if it alters the latter at most by a surface term, i.e., if $\delta(L\mathrm{d}t) = \mathrm{d}F$ [11]. In infinitesimal form this is expressed by the criterion

$$\mathrm{pr}^{(1)}X(L) + L\frac{\mathrm{d}\chi}{\mathrm{d}t} = \frac{\mathrm{d}F}{\mathrm{d}t} \tag{7}$$

where $\mathrm{pr}^{(1)}X$ is the first prolongation of generator (6) in the jet space containing the first derivatives of the dependent variables. In general for the k-th derivative extension we have

$$\mathrm{pr}^{(k)}X = X + \Phi_t^\alpha\frac{\partial}{\partial \dot{q}^\alpha} + \Omega_t\frac{\partial}{\partial \dot{N}} + \ldots + \Phi_{t^k}^\alpha\frac{\partial}{\partial(\partial_{t^k}q^\alpha)} + \Omega_{t^k}\frac{\partial}{\partial(\partial_{t^k}N)}$$

$$\begin{aligned}
\Phi_{t^k}^\alpha &= \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\xi^\alpha - \chi\dot{q}^\alpha\right) + \chi\frac{\mathrm{d}^{k+1}q^\alpha}{\mathrm{d}t^{k+1}}\\
\Omega_{t^k} &= \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\omega - \chi\dot{N}\right) + \chi\frac{\mathrm{d}^{k+1}N}{\mathrm{d}t^{k+1}}, \qquad k \in \mathbb{N}^*.
\end{aligned}$$

By applying criterion (7) on the Lagrangian (5) we get as a result a symmetry generator of the form $X = X_1 + X_2$, where

$$X_1 = \xi^\alpha(q)\frac{\partial}{\partial q^\alpha} + N\tau(q)\frac{\partial}{\partial N}, \qquad X_2 = \chi(t)\frac{\partial}{\partial t} - N\dot{\chi}(t)\frac{\partial}{\partial N}\cdot$$

The generator X_2 expresses the time re-parametrization invariance of the system, since $\chi(t)$ remains an arbitrary function. On the other hand, X_1 is a symmetry whenever

$$\mathcal{L}_\xi G_{\mu\nu} = -\frac{\mathcal{L}_\xi V}{V}G_{\mu\nu} \tag{8}$$

where $\mathcal{L}_\xi$ signifies the Lie derivative with respect to the configuration space vector $\xi = \xi^\alpha\frac{\partial}{\partial q^\alpha}\cdot$

Having obtained the symmetries of the action let us proceed to the Euler-Lagrange equations. The infinitesimal condition for Lie-point symmetries of the equations of motion - which states that the action of the generator of the symmetry over the equations must be zero modulo the equations themselves [11] - is expressed in our case as

$$pr^{(1)}X(E^0) = T(t,q,N)E^0$$
$$pr^{(2)}X(E^\alpha)\big|_{E^\alpha=0} = \Big(P^{\kappa}_{1\,\alpha}(t,q,N)\dot{q}^\alpha + P^{\kappa}_2(t,q,N)\dot{N} + P^{\kappa}_3(t,q,N)\Big)\,E^0$$

where on the right hand sides appear functions with such dependencies so that trivial terms do not emerge. By gathering the coefficients involving first derivatives of the dependent variables we are led to the following result: The vector field $X = \tilde{X}_1 + X_2$ is a symmetry generator, with

$$\tilde{X}_1 = X_1 + c\frac{\partial}{\partial N} = \xi^\alpha(q)\frac{\partial}{\partial q^\alpha} + N(\tau(q)+c)\frac{\partial}{\partial N}$$
$$X_2 = \chi(t)\frac{\partial}{\partial t} - N\dot{\chi}(t)\frac{\partial}{\partial N}$$

whenever

$$\mathcal{L}_\xi G_{\alpha\beta} = -\left(\frac{\mathcal{L}_\xi V}{V} + \tilde{c}\right) G_{\alpha\beta} \tag{9}$$

and c, $\tilde{c}$ are constants and $\chi(t)$ and arbitrary function. We see that the re-parametrisation invariance generator X_2 also appears in this case. As long as X_1 is concerned, by comparing results (8) with (9), we deduce that the second case is larger and contains the first (as is expected since variational symmetries of the action form a subgroup of the Lie-point symmetries).

To visualise the result more clearly, it is useful to turn to the constant potential parametrization. In a constrained system like (5) one can freely re-parameterize the lapse in an arbitrary way using any function of the configuration space variables. If we choose a new "lapse" function $n = N\,V(q)$ we are led to Lagrangian

$$\bar{L} = \frac{1}{2n(t)}\bar{G}_{\mu\nu}(q)\dot{q}^\mu(t)\dot{q}^\nu(t) - n(t) \tag{10}$$

with $\bar{G}_{\mu\nu} = V(q)G_{\mu\nu}$. It is a straightforward task to check that (5) and (10) are equivalent, i.e. the change $n \mapsto N$ maps the equations of motions of the latter to the ones of the former. In that particular parametrization results (8) and (9) become respectively

$$\mathcal{L}_\xi \bar{G}_{\mu\nu} = 0 \tag{11a}$$
$$\mathcal{L}_\xi \bar{G}_{\mu\nu} = \tilde{c}\,\bar{G}_{\mu\nu}. \tag{11b}$$

As we observe by these conditions, in order to have a variational symmetry, ξ must be a Killing vector of this scaled by the potential mini-supermetric $\bar{G}_{\mu\nu}$. But, to obtain a Lie-point symmetry of the Euler-Lagrange equations, apart from the case $\tilde{c} = 0$ one can additionally have a homothetic vector. Consequently, the maximum number of Noether symmetries is $d(d+1)/2$, while for Lie-point is the same raised by one.

4. Hamiltonian Description and Non-Local Symmetries

By following the Dirac-Bergmann [1], [8] algorithm we can write down the Hamiltonian for the constrained Lagrangians (5) and (10). For the sake of simplicity, we choose to work with Lagrangian (10) in the parametrization of the constant potential. Consequently, we are able to compare with results (11) which are "geometrized", in the sense that all the needed information is given with respect to the scaled by the potential metric $\bar{G}_{\mu\nu}$.

The total Hamiltonian corresponding to the system described by (10) is

$$H_T = n\mathcal{H} + u_n p_n \tag{12}$$

with $p_n \approx 0$ and

$$\mathcal{H} = \frac{1}{2}\bar{G}^{\mu\nu} p_\mu p_\nu + 1 \approx 0$$

being the primary and secondary constraints respectively. The first of them corresponding to the momentum for the degree of freedom n, which of course is zero $\frac{\partial \bar{L}}{\partial \dot{n}^\mu} = 0$. The other is the well known quadratic constraint, that is the equation of motion for n, $\frac{\partial \bar{L}}{\partial n} = 0$, written in phase space variables, where the $p_\mu = \frac{\partial \bar{L}}{\partial \dot{q}^\mu}$ are the conjugate momenta. The symbol "$\approx$" denotes a weak equality, i.e. the corresponding quantities are assigned to zero only after Poisson's brackets are calculated. For example p_n is zero only outside of a Poisson bracket, thus $\{n, p_n\} = 1$ but $\{n, p_n^2\} = 2\{n, p_n\}p_n = 2p_n \approx 0$.

Let as assume a quantity in phase space, which is at most linear in the momenta, $Q(t, q, p) = A(t, q)^\alpha(q)p_\alpha + B(t, q)$. We want to investigate under which conditions, Q is a conditional symmetry, i.e. an integral of motion due to the constraint $\mathcal{H} \approx 0$ [10]. In that case, the following condition must hold

$$\dot{Q} \approx 0 \Rightarrow \frac{\partial Q}{\partial t} + \{Q, H_T\} \approx 0.$$

It is straightforward to check that this leads to an integral of motion of the form

$$Q = \xi^\alpha(q)p_\alpha + \int n(t)\omega(q(t))\mathrm{d}t \tag{13}$$

whenever

$$\mathcal{L}_\xi \bar{G}_{\mu\nu} = \omega(q)\bar{G}_{\mu\nu}.$$

Henceforth, we can state that all conformal Killing vectors of $\bar{G}_{\mu\nu}$ generate integrals of motion. The Killing vector fields - variational symmetries (11a) - give rise to expressions of the form $Q = \xi^\alpha p_\alpha$, while the proper conformal Killing vectors, with $\omega \neq 0$, generate non-local conserved quantities given by (13). Among the latter is of course the Lie-point symmetry (11b) corresponding to the homothetic vector.

5. Canonical Quantization by Using Symmetries

In the process of canonical quantization one needs to assign operators to the classical phase space variables that preserve the canonical commutation relations, with the commutator being related to the classical Poisson brackets by the mapping $\{\cdot,\cdot\} \longrightarrow -\frac{\mathrm{i}}{\hbar}[\cdot,\cdot]$. We shall follow here the standard procedure of expressing the momenta as

$$p_\alpha \longmapsto \widehat{p}_\alpha = -\mathrm{i}\,\frac{\partial}{\partial q^\alpha}, \qquad p_n \longmapsto \widehat{p}_n = -\mathrm{i}\,\frac{\partial}{\partial n}$$

(for simplicity we choose to work in $\hbar = 1$ units), while the configuration space variables will act as multiplication operators . For the Hamiltonian (12) we follow Dirac's prescription when dealing with singular systems, which states that constrains must annihilate the wave function. For reasons that we will specify later we choose to perform the quantization in the constant potential parametrization. As a result one must require

$$\widehat{p}_n \Psi(q,n) = 0 \Rightarrow \Psi = \Psi(q) \tag{14a}$$

$$\widehat{\mathcal{H}}\Psi(q) = 0 \Rightarrow \left(-\frac{1}{2|\bar{G}|^{1/2}} \partial_\mu (|\bar{G}|^{1/2}\, \bar{G}^{\mu\nu} \partial_\nu) + 1 + \frac{d-2}{8\,(d-1)}\mathcal{R} \right) \Psi = 0 \tag{14b}$$

where $\mathcal{R}$ is the Ricci scalar of the mini-superspace and $\bar{G}$ the determinant of the metric $\bar{G}_{\mu\nu}$. The first equation (14a) is just the enforcement of the classical primary constraint $p_n \approx 0$ and states that the wave function cannot depend on n. Relation (14b) corresponds to the quadratic constraint $\mathcal{H} \approx 0$ and it defines the Wheeler-DeWitt equation in cosmology. For the latter we have chosen a specific factor ordering for the quadratic terms in the momenta, by adopting the conformal Laplacian (Yamabe operator). As a result $\widehat{\mathcal{H}}$, has now the property of commuting with the quantum analogues of the classical symmetries.

In order to promote to operators the classical linear in the momenta (local) expressions that are constants of motion, we choose the most general form of a first order Hermitian operator (with a vanishing on the boundary wave function) under the measure $|\bar{G}|^{1/2}$

$$\widehat{Q}_I = -\frac{\mathrm{i}}{2|\bar{G}|^{1/2}} \left(|\bar{G}|^{1/2}\, \xi_I^\alpha\, \partial_\alpha + \partial_\alpha\, |\bar{G}|^{1/2}\, \xi_I^\alpha \right) \tag{15}$$

with the index I counting the number of these quantities. The Killing vector fields of $\bar{G}_{\mu\nu}$, that lead to the classical conserved charges $Q_I = \xi_I^\alpha p_\alpha$, correspond to the simplified expression $\widehat{Q}_I = -\mathrm{i}\,\xi_I^\alpha\,\partial_\alpha$ and define the eigenequations

$$\widehat{Q}_I \Psi(q) = \kappa_I \Psi(q)$$

that can be used as supplementary conditions to the Wheeler-DeWitt equation (14b). This is owed to the fact that the Laplacian (and the Yamabe operator in general) have the property of exactly commuting with the Q_I's as defined by (15) whenever ξ is a Killing vector of $\bar{G}_{\mu\nu}$. At this point we have to note that this becomes possible in the constant potential parametrization, where the local symmetries correspond to Killing vector fields of the scaled by the potential mini-supermetric. It can also be easily verified that, with the previously made choice of operators, the Poisson algebra of the Q_I's becomes naturally a quantum algebra of the $\widehat{Q}_I$'s, i.e.,

$$\{Q_I, Q_J\} = C^M{}_{IJ} Q_M \Rightarrow [\widehat{Q}_I, \widehat{Q}_J] = \mathrm{i}\, C^M{}_{IJ} \widehat{Q}_M.$$

Of course not all of the Killing fields can be forced - through the $\widehat{Q}_I$'s - simultaneously over the wave function. The subalgebras that can be imposed are those whose structure constants satisfy the following integrability criterion

$$C^M{}_{IJ}\, \kappa_M = 0.$$

Note that this also allows for non-Abelian algebras [9].

6. A Classical Example of Integrability

We consider a FLRW space-time

$$\mathrm{d}s^2 = -N(t)^2 \mathrm{d}t^2 + a(t)^2 \left(\frac{1}{1 - k\, r^2} \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\varphi^2 \right) \tag{16}$$

in the context of Einstein's general relativity plus a minimally coupled scalar field

$$S = \int \mathrm{d}^4 x \sqrt{-g}\, (R + \epsilon\, \phi_{,\mu} \phi^{,\mu} + 2\, V(\phi))$$

with $\epsilon = \pm 1$ so that the model can describe a phantom scalar field as well. It can be seen that the Lagrangian

$$L = \frac{2a^2}{n} \left(a^2 V(\phi) - 3k \right) \left(-6\dot{a}^2 + \epsilon\, a^2 \dot{\phi}^2 \right) - n \tag{17}$$

where n is linked to the lapse function via

$$N = \frac{n}{2\, a\, (a^2 V(\phi) - 3k)}$$

correctly reproduces the set of Einstein's equations (2) under the ansatz (16), with the energy momentum tensor given by

$$T_{\alpha\beta} = \epsilon\, \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} \left(\epsilon\, \phi^{,\kappa} \phi_{,\kappa} - 2\, V(\phi) \right) g_{\alpha\beta}.$$

The scaled by the potential mini-supermetric that we can read from (17) is (note that we have expressed the Lagrangian in the constant potential parametrization):

$$G_{\mu\nu} = 4\,a^2\left(a^2 V(\phi) - 3k\right)\begin{pmatrix} -6 & 0 \\ 0 & \epsilon\,a^2 \end{pmatrix}.$$

This metric - for an arbitrary $V(\phi)$ - possess no Killing fields (unless certain conditions are enforced). We shall use a non-local symmetry to integrate the system of Euler-Lagrange equations, without having to assume a particular form for the potential. Notice that, due to the fact that the metric is two dimensional, there exist infinite conformal Killing fields, thus infinite non-local integrals of motion. Let us chose the vector field $\xi = \partial_\phi$ which has the corresponding conformal factor $\frac{a^2 V'(\phi)}{a^2 V(\phi)-3k}$. The following non-local integral of motion can be defined

$$\begin{aligned} Q &= p_\phi + \int \frac{a(t)^2 n(t) V'(\phi(t))}{a(t)^2 V(\phi(t)) - 3k}\mathrm{d}t = \frac{\partial L}{\partial \dot{\phi}} + \int \frac{a(t)^2 n(t) V'(\phi(t))}{a(t)^2 V(\phi(t)) - 3k}\mathrm{d}t \\ &= \frac{4\,\epsilon\,a^4 \dot{\phi}\left(a^2 V(\phi) - 3k\right)}{n} + \int \frac{a(t)^2 n(t) V'(\phi(t))}{a(t)^2 V(\phi(t)) - 3k}\mathrm{d}t. \end{aligned} \tag{18}$$

As a result, the relation $Q = \kappa$, where κ is constant holds. However, κ is not important for the analysis and without loss of generality can be put it equal to zero. Note that we need only solve this equation and the quadratic constraint $\frac{\partial L}{\partial n} = 0$ to completely integrate the system. The idea now is to fix the gauge by choosing $\phi = t$ (a possibility that is present in time parametrization invariant systems like the one under consideration). Then, we are able to parameterize the scalar field potential $V(\phi)$ as a function of time. At the same time, we want to perform a re-parametrization of the "lapse" function n in order to express (18) without the presence of the integral. In short we introduce a non-constant function h through

$$n(t) = \frac{2\dot{h}\left(a^2 V - 3k\right)}{a^2 \dot{V}}.$$

Without getting into too many details, it can be seen that with the appropriate parametrization of $h(t)$ and $V(t)$ the general solution can be derived by the two previously mentioned first order equations [7] and it reads

$$\begin{aligned} \mathrm{d}s^2 = & \frac{-\mathrm{e}^{\omega}\dot{\omega}^2}{36\left(2\,\mathrm{e}^{\omega - 6\int(\epsilon/\dot{\omega})\mathrm{d}t}\left(\tilde{c} + 3\,k\int \frac{\exp\left(6\int(\epsilon/\dot{\omega})\mathrm{d}t - \frac{\omega}{3}\right)}{\dot{\omega}}\mathrm{d}t\right) - k\mathrm{e}^{\frac{2\omega}{3}}\right)}\mathrm{d}t^2 \\ & + \mathrm{e}^{\omega/3}\left(\frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\right) \end{aligned} \tag{19}$$

with a corresponding the potential

$$V(t) = \frac{6}{e^{\omega}\dot{\omega}^2}\Big(\big(\dot{\omega}^2 - 6\epsilon\big)\, e^{\omega - 6\int(\epsilon/\dot{\omega})dt}\Big(3k\int\frac{\exp\big(6\int(\epsilon/\dot{\omega}) - \frac{\omega}{3}dt\big)}{\dot{\omega}}dt + c\Big) + 3k e^{\frac{2\omega}{3}}\Big) \tag{20}$$

where c, $\tilde{c}$ are integration constants and $\omega(t)$ is a non-constant function of time. The arbitrariness of the potential is expressed though this arbitrary function. The solution can be significantly simplified by performing the time transformation

$$t = \phi = \int\left[\frac{1}{6\,\epsilon}\left(\frac{S''(\omega)}{S'(\omega)} + \frac{1}{3}\right)\right]^{1/2} d\omega \tag{21}$$

where $S(\omega) = \exp\big(12\,k\int e^{F(\omega)-\omega/3}d\omega\big) - \frac{6\,c}{k}\cdot$ Under (21), the relations (19) and (20) become

$$ds^2 = -e^{F(\omega)}d\omega^2 + e^{\omega/3}\left(\frac{1}{1-kr^2}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\right) \tag{22}$$

and

$$V(\omega) = \frac{1}{12}e^{-F(\omega)}\big(1 - F'(\omega)\big) + 2\,k\,e^{-\omega/3} \tag{23}$$

respectively, with $F(\omega)$ expressing now the arbitrariness of the potential. For each function $F(\omega)$, line element (22) gives us the space-time that corresponds to the solution of a minimally coupled scalar field with gravity, having the corresponding potential given by (23). Thus, by using a non-local symmetry, we were able to integrate the system without making unnecessary restrictions over the potential.

7. Discussion

In this parer we investigated the conditional symmetries for constrained systems that usually appear in cosmology after a mini-superspace reduction. The variational and Lie-point symmetries of the action and of the equations of motion which are preserved in them are associated with Killing and homothetic vectors of the scaled by the potential mini-supermetric. However, conditional symmetries in their full generality are linked to conformal Killing fields of this metric, leading additionally to non-local expressions that are constants of motion and which involve integrals of functions of the configuration space variables.

The symmetries corresponding to the Killing vector fields of $\bar{G}_{\mu\nu}$ can also be used at the quantum level as supplementary conditions of the Wheeler-DeWitt equation. Given that the kinetic part of the quadratic constraint is expressed with the help of the Laplacian (or even by the conformal Laplacian), the linear Hermitian operators corresponding to these fields exactly commute with the Hamiltonian operator.

Thus, they may utilized to define eigenvalue equations which the solution of the Wheeler-DeWitt has to satisfy.

At the classical level the same symmetries, together with the larger class corresponding to conformal Killing vectors of $\bar{G}_{\mu\nu}$ can be used to integrate these configurations. It is important to note here that, only for constrained systems there exists the notion of conditional symmetries, i.e. expressions that are constants of motion due to constraints. The significance of the non-local integrals of motion at this level is paramount. We presented an example of a FLRW space-time with a scalar field minimally coupled to gravity (in reality a larger class of models can correspond to this e.g. $f(R)$, scalar-tensor gravity). There are many works devoted in the search of integrable models for certain expressions of $V(\phi)$. We proved that for any (smooth enough) potential the system is integrable and we derived the solution giving the corresponding space-time for each of these functions.

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