Sixteenth International Conference on
Geometry, Integrability and Quantization
June 6–11, 2014, Varna, Bulgaria
Ivaïlo M. Mladenov, Andrei Ludu
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2015, pp 36–72
doi: 10.7546/giq-16-2015-36-72

THE TWO APPARENTLY DIFFERENT BUT HIDDENLY RELATED EULER ACHIEVEMENTS: RIGID BODY AND IDEAL FLUID. OUR UNIFYING GOING BETWEEN: AFFINELY-RIGID BODY AND AFFINE INVARIANCE IN PHYSICS

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Abstract. Reviewed are ideas underlying our concept of affinely-rigid body. We do this from the perspective of the Leonhard Euler two main achievements, known under his name: the mechanics of rigid body and the dynamics of incompressible ideal fluid. But we formulate the theory which is somehow placed between those two models. Our scheme is a finite-dimensional dynamical system, but admitting deformative degrees of freedom. We also stress the connection with the general idea of affine invariance in physics.

MSC: 37J05, 70H33, 70E05, 76B99
Keywords: rigid body, incompressible ideal fluid, affine invariance, finite-dimensional dynamical systems, deformative degrees of freedom

CONTENTS

1. Introduction

It is an apparently accidental fact that it was the same person, namely Leonhard Euler, who derived equations known under his name and concerning two completely different theories in mathematical physics, describing two seemingly different objects, namely the rigid body and the incompressible ideal fluid. However, quite recently it turned out that there is no accidence here. Namely, the free rigid body is a left-invariant Hamiltonian system on the Lie group $\mathrm{SO}(3,\mathbb{R})$, whereas the incompressible ideal fluid is a right-invariant system under the group of all volume-preserving diffeomorphisms of $\mathbb{R}^3$. The difference is just the mentioned left- and right. A more serious difference is that $\mathrm{SDiff}(\mathbb{R}^3)$ being infinite-dimensional is not a Lie group in the classical sense, nevertheless the invariance properties are quite analogous. Our idea below is "to go between" two models. So we admit deformations, but they are finite-dimensional, i.e., ruled by $\mathrm{GL}(3,\mathbb{R})$, or more generally under $\mathrm{GL}(n,\mathbb{R})$. Discussed are some problems concerning the left- and right-invariance. This enables one to understand better the deep and correct ideas of Euler. Discussed are some problems connected with the left and right affine invariance of the kinetic energy. In connection with this we mention the idea of the basic affine invariance in fundamental physics. This is connected with certain modern attempts of formulating affinely invariant physical theories, in the spirit of Thales of Miletus [4, 5, 15, 26–28, 34, 41, 42, 55–57, 59, 61, 70, 71].

Let us begin with the three-dimensional rigid body without translational motion, e.g., fastened at the center of mass. Its configuration space Q may be identified with the orthogonal group in three dimensions

$$Q = \mathrm{SO}(3,\mathbb{R}) = \left\{\varphi \in \mathrm{GL}(3,\mathbb{R})\,;\, \varphi^T\varphi = I,\ \det\varphi = +1\right\}.$$

Therefore, only proper rotations are admitted, translations neglected, and reflections are forbidden. Motions are described by curves in Q

$$\mathbb{R} \ni t \mapsto \varphi(t) \in Q.$$

Angular velocities are non-holonomic, namely in the spatial representations they are given by [2, 3, 7, 8, 14, 15]

$$\Omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\varphi^{-1} = \varphi\widehat{\Omega}\varphi^{-1} = -\Omega^T.$$

The co-moving representation is given by

$$\widehat{\Omega} = \varphi^{-1}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \varphi^{-1}\Omega\varphi = -\widehat{\Omega}^T. \tag{1}$$

They are elements of the Lie algebra $\mathfrak{so}(3)$, so as written above they are skew-symmetric. And only in the exceptional case of the two-dimensional body, $n = 2$, they are holonomic. And for any $n > 2$ they fail to be so.

In three dimensions there is an isomorphism between skew-symmetric second-order tensors and axial vectors [2, 3]

$$\Omega = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}, \qquad \widehat{\Omega} = \begin{bmatrix} 0 & -\widehat{\Omega}_3 & \widehat{\Omega}_2 \\ \widehat{\Omega}_3 & 0 & -\widehat{\Omega}_1 \\ -\widehat{\Omega}_2 & \widehat{\Omega}_1 & 0 \end{bmatrix}$$

where obviously

$$\Omega^i = \varphi^i{}_A \widehat{\Omega}^A.$$

Kinetic energy of rotations is given by

$$T = \frac{1}{2} I_{AB} \widehat{\Omega}^A \widehat{\Omega}^B = \sum_{A=1}^{3} \frac{I_A}{2} \left(\widehat{\Omega}^A \right)^2$$

where I_{AB} are constant co-moving components of the inertial tensor [8, 9]. The extreme right hand side of this equation is valid in material coordinates diagonalizing the tensor of inertia. The constancy of I_{AB}, I_A is very important. Namely, it implies that for any orthogonal transformation $U \in \mathrm{SO}(3, \mathbb{R})$ the corresponding left translation

$$\varphi \mapsto U\varphi \tag{2}$$

preserves $\widehat{\Omega}$ and therefore, the kinetic energy T is also conserved. The transformations acting on the left (in the commonly used definition of the superposition of mappings) describe spatial rotations, therefore in the geodetic case (no external forces) one obtains the spin conservation from the left translations. For the right-invariant geodetic models which fail to be simultaneously left-invariant this conservation law does not hold. Obviously, geodetic models which are simultaneously left- and right- invariant correspond to the spherical top, i.e., such one that

$$I_1 = I_2 = I_3 = I$$

this is the spherical top.

Let the vector N_i denote the spatial torque (moment of forces) and $\widehat{N}_A$ - its co-moving representation. Roughly speaking, $\widehat{N}_A$ are projections of N onto the main co-moving axes

$$\widehat{N} = \varphi^{-1} N.$$

We speak here about vectors in the Euclidean space, so there are no reasons to distinguish between contravariant and covariant vectors. It is convenient to write down equations of motion in the co-moving terms

$$
\begin{aligned}
I_1 \frac{\mathrm{d}\widehat{\Omega}_1}{\mathrm{d}t} &= (I_2 - I_3)\,\widehat{\Omega}_2\widehat{\Omega}_3 + \widehat{N}_1 \\
I_2 \frac{\mathrm{d}\widehat{\Omega}_2}{\mathrm{d}t} &= (I_3 - I_1)\,\widehat{\Omega}_3\widehat{\Omega}_1 + \widehat{N}_2 \\
I_3 \frac{\mathrm{d}\widehat{\Omega}_3}{\mathrm{d}t} &= (I_1 - I_2)\,\widehat{\Omega}_1\widehat{\Omega}_2 + \widehat{N}_3.
\end{aligned}
\tag{3}
$$

In geodetic case, when $N_i = 0$ (i.e., also $\widehat{N}_A = 0$) those equations become $\widehat{\Omega}$-autonomous, i.e., independent of φ. The same holds when $\widehat{N}_A$-s are constant or dependent only on $\widehat{\Omega}$, not on φ. For those reasons this form of equations is convenient: we begin (in principle) from solving them with respect to $\widehat{\Omega}$, and later on one obtains $\varphi(t)$ by solving (1) with respect to φ. In any case this is the programme. But in this form they are useful in control problems. Let us stress that geodetic equations are invariant under (2).

Let us reformulate these results in terms of the co-moving and spatial components of spin (internal angular momentum) with respect to the centre of mass

$$
\widehat{\Sigma}_A = \frac{\partial T}{\partial \widehat{\Omega}^A} = I_{AB}\Omega^B = I_{\underline{A}}\Omega^{\underline{A}}, \qquad \Sigma_a = \widehat{\Sigma}_B \left(\varphi^{-1}\right)^B{}_a .
$$

The corresponding expression for kinetic energy has the form

$$
\mathcal{T} = \frac{1}{2}\widetilde{I}^{AB}\widehat{\Sigma}_A\widehat{\Sigma}_B = \sum_{A=1}^{3} \frac{1}{2I_A}\widehat{\Sigma}_A^2 .
$$

Let us remind the following expressions for the Poisson brackets of “sigmas”

$$
\{\Sigma_a, \Sigma_b\} = \varepsilon_{ab}{}^c\Sigma_c, \qquad \{\widehat{\Sigma}_A, \widehat{\Sigma}_B\} = -\varepsilon_{AB}{}^C\widehat{\Sigma}_C, \qquad \{\Sigma_a, \widehat{\Sigma}_B\} = 0.
$$

Let us be aware of the difference in sign on the right-hand sides of the first and second equations. The fact that spatial and co-moving components of spin have different in sign Poisson brackets has deep geometric reasons, namely one has to do respectively with the representation and anti-representation action of $\mathrm{SO}(3,\mathbb{R})$ on the configuration space.

When expressed in spin terms, the Euler equations (3) acquire the following form [2, 3]

$$\begin{aligned}
\frac{\mathrm{d}\widehat{\Sigma}_1}{\mathrm{d}t} &= \left(\frac{1}{I_3} - \frac{1}{I_2}\right)\widehat{\Sigma}_2\widehat{\Sigma}_3 + \widehat{N}_1\\
\frac{\mathrm{d}\widehat{\Sigma}_2}{\mathrm{d}t} &= \left(\frac{1}{I_1} - \frac{1}{I_3}\right)\widehat{\Sigma}_3\widehat{\Sigma}_1 + \widehat{N}_2\\
\frac{\mathrm{d}\widehat{\Sigma}_3}{\mathrm{d}t} &= \left(\frac{1}{I_2} - \frac{1}{I_1}\right)\widehat{\Sigma}_1\widehat{\Sigma}_2 + \widehat{N}_3.
\end{aligned} \tag{4}$$

When the rigid body is spherical, $I_1 = I_2 = I_3 = I$, i.e., its kinetic energy is invariant under both left and right translations, then obviously (4) reduces to

$$\frac{\mathrm{d}\widehat{\Sigma}_a}{\mathrm{d}t} = \widehat{N}_a.$$

Let us observe that in the spatial representation the balance law

$$\frac{\mathrm{d}\Sigma_a}{\mathrm{d}t} = N_a$$

is universally valid, independently on the particular values of I_a. Nevertheless, those values are hidden in the structure of equations.

Geodetic motions of the spherical rigid body are given by the exponential formula

$$\varphi(t) = \exp(t\omega)\varphi(0) = \varphi(0)\exp(t\widehat{\omega}), \quad \widehat{\omega} = \varphi(0)^{-1}\omega\varphi(0)$$

where ω, $\widehat{\omega}$ are arbitrary. Quite a different situation occurs when the inertial co-moving tensor is anisotropic. Then there exist exponential solutions of the geodetic problem, nevertheless they are exceptional and correspond to rotations about the axes of inertia. But their knowledge is very desirable and contains much of information about the stability problem.

Let us now review the main ideas of the group approach to the hydrodynamics of incompressible fluids [2, 3, 36, 37]. The basic equation is given by

$$\varrho\frac{\mathrm{d}\overline{v}}{\mathrm{d}t} = \varrho\left(\frac{\partial\overline{v}}{\partial t} + \left(\overline{v}\cdot\overline{\nabla}\right)\overline{v}\right) = -\overline{\nabla}p + \varrho\overline{g}$$

where $\overline{v}$ is the Euler velocity field, ϱ is the Euler density of fluid, p is the field of pressure, and $\overline{g}$ denotes the Earth acceleration. The assumed iso-entropic condition may be formulated as

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\partial s}{\partial t} + \overline{v}\ \mathrm{grad}\ s = 0$$

i.e.,

$$\frac{\partial\left(\varrho s\right)}{\partial t} + \mathrm{div}\left(\varrho s\overline{v}\right) = 0.$$

Equations of motion may be written as

$$\frac{\partial}{\partial t}\varrho v^i = -\frac{\partial \Pi^{ik}}{\partial x^k}, \qquad \Pi^{ik} = pg^{ik} + \varrho v^i v^j.$$

The scalar product of the volume preserving vector fields is given by

$$\langle \overline{v}_1, \overline{v}_2 \rangle = \int\limits_D \overline{v}_1 \cdot \overline{v}_2 \,\mathrm{d}x$$

where D is the region occupied by fluid, $\overline{v}$ is tangent to the boundary of D, and div $\overline{v} = 0$ inside D. Therefore, in the incompressible case the kinetic energy is given by

$$T = \frac{\varrho}{2}\langle \overline{v}, \overline{v} \rangle = \frac{\varrho}{2}\int\limits_D g_{ij} v^i v^j \mathrm{d}_n(x; g)$$

where $\mathrm{d}_n(x; g)$ denotes the g-Riemannian element of volume in the physical space. At the time instant t the fluid configuration is given by $g_t \in \mathrm{SDiff}(D)$. At the time instant $t + \tau$ the configuration is $\exp(\overline{v}\tau)\, g_t$, where τ is small. The velocity field $\overline{v}$ obtained from the vector $\dot{g}$ tangent at g to the group $\mathrm{SDiff}(D)$ is invariant under the right action in this group. But be careful: the left/right actions are not the only difference between both cases. Namely, $\mathrm{SDiff}(M)$ is infinite-dimensional, so it is not a Lie group in the literal sense. Nevertheless, some a general Lie group techniques may be used and they lead to certain correctly looking expressions for the fluid motion. Once guessed in this way, they may be independently proved to be correct. Without the mentioned "heuristic" level it would be simply impossible to find them. And at the same time the similarity between two approaches: rigid body and incompressible ideal fluid is striking and convincing. Nevertheless, it is interesting to discuss finite-dimensional models admitting deformations. In a sense, it turns out that such models were suggested in a quite different aspect by C. Eringen in his micromorphic theory [10, 11, 16–18], and many years earlier by some Russian mathematician [53].

2. Our Idea: to Admit Finite-Dimensional Deformations. Affine Philosophy of Thales of Miletus. GAff-Invariance

Let us begin with remarks concerning dynamical systems on Lie groups or rather on their cotangent bundles. We assume that our Lie group G is linear, e.g., $G \subset \mathrm{GL}(N, \mathbb{R})$ or $G \subset \mathrm{GL}(N, \mathbb{C})$ (but being real, e.g., $\mathrm{U}(n)$).

Lie algebra of G will be denoted by $\mathfrak{g}$, it is respectively the linear subspace of $\mathrm{L}(n, \mathbb{R})$, $\mathrm{L}(n, \mathbb{C})$

$$\mathfrak{g} \subset \mathrm{L}(n, \mathbb{R}) = T_e\mathrm{GL}(n, \mathbb{R}), \qquad \mathfrak{g} \subset \mathrm{L}(n, \mathbb{C}) = T_e\mathrm{GL}(n, \mathbb{C}).$$

In exponential representation the relationship between Lie algebras and Lie groups is exponential

$$q\left(t^1,\ldots,t^k\right)=\exp\left(t^kE_k\right)\in \mathrm{GL}(n,\mathbb{R}),\mathrm{GL}(n,\mathbb{C})$$

where $t^1,\ldots,t^k$ are canonical coordinates on our k-dimensional Lie groups. Obviously, the structure constants are given by

$$[E_k,E_j]=C^m{}_{kj}E_m.$$

Let us remind that in matrix manifolds

$$\exp(A)=\sum_{n=0}^{\infty}\frac{1}{n!}A^n.$$

The Lie co-algebra consists of linear functions on $\mathfrak{g}$. Although very often, in best-known special examples. $\mathfrak{g}^*\simeq\mathfrak{g}$, namely in the sense of

$$\langle f,x\rangle=\mathrm{Tr}(fx).$$

Motions are described by curves $\mathbb{R}\ni t\mapsto q(t)\in G$. Usually one operates with Lie-algebraic velocities in two representations: "spatial" and "co-moving"

$$\Omega(t)=\dot{q}(t)q(t)^{-1},\qquad \widehat{\Omega}=q(t)^{-1}\dot{q}(t).$$

Obviously, the following holds

$$\Omega(t)=\mathrm{Ad}_{q(t)}\widehat{\Omega}(t),\qquad \mathrm{Ad}_q(x)=qxq^{-1}.$$

In terms of dual bases $\{\ldots,E_a,\ldots\}$, $\{\ldots,E^a,\ldots\}$ in $\mathfrak{g}$ and $\mathfrak{g}^*$ we have

$$\Omega=\Omega^a(t)E_a,\qquad \widehat{\Omega}=\widehat{\Omega}^a(t)E_a,\qquad \Omega^a(t)=\left(\mathrm{Ad}_{q(t)}\right)^a{}_b\widehat{\Omega}^b(t).$$

If G is non-Abelian, quasi-velocities Ω^a, $\widehat{\Omega}^a$ are non-holonomic i.e., they fail to be time derivatives of any generalized coordinates. Nevertheless, they are very convenient in applications. Let us remind the concept of angular velocity.

Left-invariant kinetic energy forms are given by

$$T=\frac{1}{2}\gamma_{ab}\widehat{\Omega}^a\widehat{\Omega}^b=\frac{1}{2}\gamma\left(\widehat{\Omega},\widehat{\Omega}\right)$$

where γ_{ab} are constants, so we are dealing with the algebraic quadratic forms of $\widehat{\Omega}$. Tangent and cotangent bundles are trivial

$$TG\simeq G\times\mathfrak{g},\qquad T^*G\simeq G\times\mathfrak{g}^*.$$

Using the cotangent language we can write

$$\Sigma=\Sigma_aE^a,\qquad \widehat{\Sigma}=\widehat{\Sigma}_aE^a$$

with the following trivialization mappings

$$\Sigma_a=\Sigma_a{}^i(q)p_i,\qquad \widehat{\Sigma}_a=\widehat{\Sigma}_a{}^i(q)p_i$$

where

$$\Sigma_a \Omega^a = \widehat{\Sigma}_a \widehat{\Omega}^a = p_i \dot{q}^i.$$

These quantities have the following transformation properties under the left-acting and right-acting group translations in G.

In the left translations

$$L_g : x \mapsto gx$$

we have

$$\begin{aligned} \Omega \mapsto g\Omega g^{-1} &= \mathrm{Ad}_g \Omega, \qquad \widehat{\Omega} \mapsto \widehat{\Omega} \\ \Sigma \mapsto g\Sigma g^{-1} &= \mathrm{Ad}_g^{*-1} \Sigma, \quad \widehat{\Sigma} \mapsto \widehat{\Sigma}. \end{aligned}$$

And conversely, under the right translations

$$R_g : x \mapsto xg$$

we have

$$\begin{aligned} \Omega \mapsto \Omega, &\quad \widehat{\Omega} \mapsto g^{-1}\widehat{\Omega} g = \mathrm{Ad}_g^{-1} \widehat{\Omega} \\ \Sigma \mapsto \Sigma, &\quad \widehat{\Sigma} \mapsto g^{-1}\widehat{\Sigma} g = \mathrm{Ad}_g^{*} \widehat{\Sigma}. \end{aligned}$$

This explains the difference in the structure of left-invariant and right-invariant kinetic energies, i.e., metric tensor on G.

Similarly, Poisson brackets have the following forms

$$\{\Sigma_i, \Sigma_j\} = C^k{}_{ij} \Sigma_k, \qquad \{\widehat{\Sigma}_i, \widehat{\Sigma}_j\} = -C^k{}_{ij} \widehat{\Sigma}_k, \qquad \{\Sigma_i, \widehat{\Sigma}_j\} = 0.$$

Let us stress the difference in sign on the right-hand sides of Σ_i and $\widehat{\Sigma}_i$-brackets. It is analogous to the corresponding difference in the rigid body case and has exactly the same group-theoretical origin. It is also interesting to quote the Poisson brackets between Σ-s and configuration-dependent quantities

$$\{\Sigma_a, f(q)\} = -\Sigma_a{}^i(q) \frac{\partial f}{\partial q^i}, \qquad \{\widehat{\Sigma}_a, f(q)\} = -\widehat{\Sigma}_a{}^i(q) \frac{\partial f}{\partial q^i}.$$

Together with

$$\{F(q), G(q)\} = 0$$

they are sufficient to calculate any other Poisson bracket just on the basis of its formal properties.

Let us stress that Σ_i are Hamiltonian generators of the group of left translations L_G and $\widehat{\Sigma}_i$ are generators of the right group R_G. In other words, they are momentum mappings of those groups. This is the reason of the above commutation relations.

Non-holonomic representation of Legendre transformations is given by the following equivalent formulas

$$\Sigma_a = \frac{\partial T}{\partial \Omega^a}, \qquad \widehat{\Sigma}_a = \frac{\partial T}{\partial \widehat{\Omega}^a}.$$

The left-invariant and right-invariant kinetic energies are given in the canonical language as follows

$$\mathcal{T} = \frac{1}{2}\widetilde{\gamma}^{ab}\widehat{\Sigma}_a\widehat{\Sigma}_b, \qquad \mathcal{T} = \frac{1}{2}\widetilde{\gamma}^{ab}\Sigma_a\Sigma_b$$

where the contravariant metric $\widetilde{\gamma}$ is reciprocal to the covariant γ

$$\widetilde{\gamma}^{ac}\gamma_{cb} = \delta^a{}_b.$$

Equations of motion written in terms of the Poisson brackets have the form

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\}, \qquad H = \mathcal{T} + \mathcal{V}(q).$$

Euler equations for the left-invariant model of T have the following form

$$\frac{\mathrm{d}\widehat{\Sigma}_a}{\mathrm{d}t} = -\widetilde{\gamma}^{cd}C^b{}_{ac}\widehat{\Sigma}_d\widehat{\Sigma}_b + \widehat{N}_a$$

where in the potential case

$$\widehat{N}_a = \widehat{\Sigma}_a{}^i(q)\frac{\partial\mathcal{V}}{\partial q^i}$$

but one can also include a dissipative, $\widehat{\Sigma}$-dependent term. In terms of $\widehat{\Omega}$ these equations become

$$\gamma_{ab}\frac{\mathrm{d}\widehat{\Omega}^b}{\mathrm{d}t} = -\gamma_{bd}C^b{}_{ac}\widehat{\Omega}^c\widehat{\Omega}^d + \widehat{N}_a.$$

In mixed terms

$$\frac{\mathrm{d}\widehat{\Sigma}_a}{\mathrm{d}t} = -C^b{}_{ac}\widehat{\Omega}^c\widehat{\Sigma}_b + \widehat{N}_a. \tag{5}$$

In the geodetic case, when $\widehat{N}_a = 0$, those equations are automatically solved with respect to $\widehat{\Sigma}$ or $\widehat{\Omega}$. Then the configuration evolution may be found by solving the equation

$$\frac{\mathrm{d}q}{\mathrm{d}t} = q(t)\widehat{\Omega}$$

with respect to $q(t)$ with $\widehat{\Omega}$ substituted from the solution of (5).

For the right-invariant models, when

$$T = \frac{1}{2}\gamma_{ab}\Omega^a\Omega^b$$

equations of motion have the form

$$\frac{\mathrm{d}\Sigma_a}{\mathrm{d}t} = \widetilde{\gamma}^{cd}C^b{}_{ac}\Sigma_d\Sigma_b + N_a$$

where in the potential case

$$N_a = \Sigma_a{}^i(q)\frac{\partial\mathcal{V}}{\partial q^i}$$

but of course non-geodetic ones are also admissible.

Let us observe that for the left-invariant geodetic models, but only for them, we have

$$\frac{\mathrm{d}\Sigma_a}{\mathrm{d}t} = 0$$

but for the general left-invariant case the following equations of motion are satisfied

$$\frac{\mathrm{d}\Sigma_a}{\mathrm{d}t} = N_a.$$

Let us mention, there are also doubly-invariant models. Obviously in the Abelian groups and semisimple Lie groups and also in their certain Cartesian products. Particularly interesting are semi-simple Lie groups. Their Killing metric tensors are given by the following non-degenerate expressions [31]

$$\gamma_{ab} = C^k{}_{la} C^l{}_{kb}.$$

The quantity C is then totally antisymmetric with respect to γ

$$C^{ijk} = C^i{}_{ab} \widetilde{\gamma}^{aj} \widetilde{\gamma}^{bk} = -C^{jik} = -C^{kji} = -C^{ikj}.$$

In the geodetic case the general solution is then exponential

$$q(t) = \exp(\Omega t) q(0) = q(0) \exp(\widehat{\Omega} t)$$

where obviously

$$\widehat{\Omega} = q(0)^{-1} \Omega q(0) = \mathrm{Ad}^{-1}_{q(0)} \Omega \tag{6}$$

and Ω, $\widehat{\Omega}$ are quite arbitrary. Let us repeat that in the case of non-Killing one-side symmetry such solutions, so-called stationary ones do exist only for some special values of Ω, $\widehat{\Omega}$.

Let us also stress that (6) is a general solution for arbitrary initial conditions Ω, $\widehat{\Omega}$ also in the situation when

$$G = X^N_{k=1} G_k$$

where the subgroups G_k are simple and the metric γ is given by

$$\gamma = \oplus^N_{K=1} C_K \pi^\star_K \gamma_K.$$

Obviously, $\pi_K : G \to G_K$ is the natural projection, γ_K's are Killing metrics on the groups G_K, and C_K are arbitrary constants.

3. Affinely-Rigid Body. Homogeneously Deformable Gyroscope

It is convenient to begin with the general case $Q = \mathrm{GL}(n, \mathbb{R})$ and to specify n to $3, 2, 1$ only at some final stage. Or, to be quite honest, it is definitely better to replace the group structure of Q by some homogeneous space with trivial, nevertheless, not canonically defined isotropy groups [23, 35–37, 71, 74, 77, 78, 84, 85]. So, we are dealing with two affine spaces: the material space $(N, U, \to \eta)$ and the physical one $(M, V, \to g)$. Here N denotes the manifold of material points, U –

the linear space of translations acting in it, $\rightarrow: N \times N \rightarrow U$ – the linear space of radius-vectors in N, and $\eta \in U^{\star} \otimes U^{\star}$-the material metric tensor in N. And similarly, M is the affine manifold of instantaneous positions of material points in the physical space, V – the linear space of translations acting in M, $\rightarrow: M \times M \rightarrow V$ – the linear space of radius-vectors in the affine space M, and $g \in V^{\star} \otimes V^{\star}$ – the metric tensor of M.

It is easy to see that the configuration space of affine motion may be canonically identified with the manifold

$$Q = M \times \mathrm{LI}(U, V) \tag{7}$$

where $\mathrm{LI}(U, V)$ denotes the set of linear isomorphisms of U onto V. Obviously, the first factor in (7) refers to translational motion, i.e., to the position of the center of mass in M, and the second one corresponds to internal degrees of freedom (or, using more careful terms, to degrees of freedom of the relative motion). When for simplicity we put: $M = N = U = V$, then Q becomes identified with the semi-direct product

$$Q = \mathrm{GL}(n, \mathbb{R}) \times_s \mathbb{R}^n \tag{8}$$

or, when we deal with the continuous medium and insist on the orientation-preserving motions, we replace (8) by

$$Q = \mathrm{GL}^{+}(n, \mathbb{R}) \times_s \mathbb{R}^n.$$

Let us notice however that this restriction is too strong when we deal with discrete systems of material points. Analytically, for any $\Phi \in Q$ we can write

$$\Phi(t, a)^i = \varphi^i{}_K(t) a^K + x^i(t). \tag{9}$$

Let us repeat that $x^i(t)$ are coordinates of the centre of mass position in M and $\varphi^i{}_K(t)$ are internal coordinates.

Inertial properties of the body are described by the total mass m and by the second-order co-moving inertial moment with respect to the centre of mass [59, 60]

$$m = \int_N \mathrm{d}\mu(a), \qquad J^{KL} = \int_N a^K a^L \mathrm{d}\mu(a).$$

J^{KL} is analytically equivalent to the co-moving tensor of inertia, although literally different from it. Let us stress that the measure μ is constant and defined on N. Similarly, J^{KL} is constant and expressed through the integration of Lagrange variables with respect to μ, thus also in the material space. Our assumption that the material centre of mass is placed at $a^K = 0$ implies that the dipole momentum of μ vanishes

$$J^K = \int_N a^K \mathrm{d}\mu(a) = 0. \tag{10}$$

Let us observe that (10) is based exactly on the assumption that the material center of mass is given by $a^K = 0$.

One can ask about higher-order material inertial tensor like

$$J^{K_1 \ldots K_m} = \int_N a^{K_1} \ldots a^{K_m} \mathrm{d}\mu(a).$$

However, in the usual dynamics of affine bodies we do not need them because neither the kinetic energy nor anything else essential for affine motion depends on them, One can also show that when releasing affine constraints and still wishing to use the multipole expansion for the dependence of Φ^i on a^K, we must admit Φ^i to be general analytic functions.

The kinetic energy obtained from the substituting constraints (9) to the usual expression for the kinetic energy of a general multi-particle system has the form

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{m}{2} g_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^j}{\mathrm{d}t} + \frac{1}{2} g_{ij} \frac{\mathrm{d}\varphi^i{}_A}{\mathrm{d}t} \frac{\mathrm{d}\varphi^j{}_B}{\mathrm{d}t} J^{AB}. \tag{11}$$

The Legendre transformation corresponding to $L = T - V(x, \varphi)$ has the following obvious form

$$p_i = m g_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}t}, \qquad p^A{}_i = g_{ij} \frac{\mathrm{d}\varphi^j{}_B}{\mathrm{d}t} J^{AB}.$$

The corresponding kinetic part of the Hamiltonian is given by

$$\mathcal{T} = \frac{1}{2m} g^{ij} p_i p_j + \frac{1}{2} \widetilde{J}_{AB} p^A{}_i p^B{}_j g^{ij} \tag{12}$$

where

$$\widetilde{J}_{AC} J^{CB} = \delta_A{}^B, \qquad g_{ik} g^{kj} = \delta^j_j.$$

We shall also need some formulas for the covariant and its reciprocal Cauchy and Green deformation tensors. The Cauchy tensors are given by

$$C_{ij} = \eta_{AB} \left(\varphi^{-1}\right)^A{}_i \left(\varphi^{-1}\right)^B{}_j, \qquad C^{ij} = \varphi^i{}_A \varphi^j{}_B \eta^{AB}.$$

And similarly, for the Green tensors we have:

$$G_{AB} = g_{ij} \varphi^i{}_A \varphi^j{}_B, \qquad G^{AB} = \left(\varphi^{-1}\right)^A{}_i \left(\varphi^{-1}\right)^B{}_j g^{ij}.$$

It is clear that the pairs of metric and metric-like tensors (g_{ij}, C_{ij}), (η_{AB}, G_{AB}) give rise to the mixed tensors

$$\widehat{C}^i{}_j = g^{ik} C_{kj}, \qquad \widehat{G}^A{}_B = \eta^{AC} G_{CB}. \tag{13}$$

And those quantities generate scalars known as deformation invariants:

$$\mathrm{Tr}\ \widehat{C}^p, \qquad \mathrm{Tr}\ \widehat{G}^p, \qquad p = 1, ..., n. \tag{14}$$

There are n independent among them and in a consequence of the Caylay-Hamilton theorem the invariants (14) form a functional basis of the set of invariants. Let us remind that invariants are defined as functions on $Q_{\rm int} = LI(UV)$ non-sensitive with respect to the spatial and material rotations. From the algebraic point of view their set is infinite-dimensional, nevertheless there are always only n functionally independent among them. The choices (14) belong to the infinity of all possibilities, sometimes most convenient ones, but not always.

Let us quote some non-doubtful range of physical applications of our model

$$L = T - V(\varphi), \qquad H = \mathcal{T} + V(\varphi).$$

Those are, e.g.,

- macroscopic elasticity when the length of excited waves is comparable with the linear size of the body, e.g., soap bubbles [6, 19–22, 24, 25, 32, 33, 38–40, 43–52, 54, 58, 60, 62–66, 69, 72, 73, 75, 76, 81, 86],
- micromorphic continua with internal degrees of freedom [10, 11, 17, 36, 37, 68, 70, 71, 79, 86],
- molecular vibrations, dynamics of molecular crystals[16–18, 29, 60, 69–71],
- nuclear dynamics (collective droplet model of the atomic nuclei) [46, 67, 68, 87],
- astrophysical objects, vibrating stars, shape of Earth [6, 60, 68, 70, 71],
- integrable one-dimensional lattices and n-dimensional affinely rigid body [68, 70, 71].

There are, however some obvious drawbacks of the model based on kinetic energy (11), (12). Let us quote them

1. First of all, geodetic models, i.e., ones without potentials, only with $L = T$ (11) are completely non-physical, predict both the unlimited expansion and concentration to the situations with the vanishing volume. Of course, both the situations are completely non-physical. To prevent them, one must introduce appropriate models of the potential energy. But in the usual (metrically) rigid body such a procedure, although admitted and often looking necessary is not so-to-speak qualitatively necessary to obtain sensibility-looking solutions. Is it possible to help this?
2. There is no dynamical affine invariance of equations of motion. Again only kinematical one. Quite unlike the theory of metrically rigid body. but this is very important. Namely, all advantages of the group structure are lost. In the theory of systems on Lie groups, in general the invariance of the dynamics under the group which rules the geometry of degrees of freedom is possible, although not necessary. In the mechanics of affinely rigid body based on (11), (12) the dynamics of affine motion simply must be not affinely-invariant. Therefore the following system of questions appears:

> What would be affinely-invariant dynamical models? Do they exist formally? And if so, are they realistic? What is relationship between models based on the d'Alembert constraints and the affinely-invariant models of the kinetic energy?

Let us begin again from the geometric analysis of the material and purely affine aspects of the mechanics of affine bodies. But now let us use again carefully the language of homogeneous spaces, not one of Lie groups.

Our canonical objects are coordinates $\left(x^i, \varphi^i{}_A\right)$ and their canonical object, namely

$$p_i,\ p^A{}_i \qquad \text{conjugate to} \qquad x^i,\ \varphi^i{}_A.$$

For Lagrangians $L = T - V(x, \varphi)$ the Legendre transformation is given by:

$$\begin{aligned} p_i &= \frac{\partial T}{\partial v^i} = m g_{ij} v^j = m g_{ij} \dot{x}^j \\ p^A{}_i &= \frac{\partial T}{\partial \dot{\varphi}^i{}_A} = g_{ij} \dot{\varphi}^j{}_B J^{BA} = g_{ij} \mathcal{V}^j{}_B J^{BA} \end{aligned} \tag{15}$$

Internal Lie-algebraic objects, i.e., affine velocities are given by following expressions:

$$\Omega = \dot{\varphi}\varphi^{-1} \in L(V), \qquad \widehat{\Omega} = \varphi^{-1}\dot{\varphi} \in L(U).$$

Eringen used for them the term "gyration". Obviously, Ω is a spatial and $\widehat{\Omega}$- a material representation

$$\Omega = \varphi\widehat{\Omega}\varphi^{-1}, \qquad \widehat{\Omega} = \varphi^{-1}\Omega\varphi.$$

Analytically they are given by

$$\Omega^i{}_j = \dot{\varphi}^i{}_A \varphi^{-1A}{}_j, \qquad \widehat{\Omega}^A{}_B = \varphi^{-1A}{}_i \dot{\varphi}^i{}_B.$$

They are evidently non-holonomic velocities, i.e., they fail to be time derivatives of any generalized coordinates. Nevertheless, they are not only geometrically distinguished, but also very convenient practically. For example, the Euler field of velocities in M is given by:

$$V^i(\xi) = v^i + \Omega^i_j(\xi^j - x^j).$$

The g- and η-skew-symmetric part of Ω, $\widehat{\Omega}$ are angular velocities, respectively in the spatial and co-moving representations

$$\begin{aligned} \omega^i{}_j &= \Omega^i{}_j - \Omega_j{}^i = \Omega^i_j - g_{jk} g^{il} \Omega^k_l \\ \widehat{\omega}^A{}_B &= \widehat{\Omega}^A{}_B - \widehat{\Omega}_B{}^A = \widehat{\Omega}^A{}_B - \eta_{BC}\eta^{AD}\widehat{\Omega}^C{}_D. \end{aligned}$$

To be completely honest, $\widehat{\omega}$ is not literally the system of co-moving components of ω (although $\widehat{\Omega}$ is so for Ω). The next important concept is affine spin, also in the spatial and co-moving representation

$$\Sigma = \varphi\pi, \qquad \widehat{\Sigma} = \pi\varphi$$

i.e., analytically

$$\Sigma^i{}_j = \varphi^i{}_A \pi^A{}_j, \qquad \widehat{\Sigma}^A{}_B = \pi^A{}_i \varphi^i{}_B.$$

Obviously, π-s are canonical momenta conjugate to φ.

Let us observe that Σ, $\widehat{\Sigma}$ are defined all over the configuration space, unlike affine velocities which are geometrically built of φ^{-1} . Obviously, Σ and $\widehat{\Sigma}$ are Hamiltonian generators, i.e., momentum mappings of the groups $GL(V)$, $GL(U)$ acting on Q_{int}

$$\begin{aligned} \varphi &\mapsto A\varphi, & \varphi &\mapsto \varphi B \\ A &\in \mathrm{GL}(V), & B &\in \mathrm{GL}(U). \end{aligned}$$

Their g-and η-skew-symmetric parts are respectively generators of spatial and material isomorphisms

$$\begin{aligned} S^i{}_j &= \Sigma^i{}_j - \Sigma_j{}^i = \Sigma^i{}_j - g^{ik} g_{jl} \Sigma^l_k \\ V^A{}_B &= \widehat{\Sigma}^A{}_B - \widehat{\Sigma}_B{}^A = \widehat{\Sigma}^A{}_B - \eta_{BC} \eta^{AD} \widehat{\Sigma}^C{}_D. \end{aligned}$$

Warning: $V^A{}_B$, Σ^i_j are not related to each other via φ, i.e., their relationship is much more complicated than in the case of metrically-rigid motion.

The quantities Σ, $\widehat{\Sigma}$ satisfy the following transformation rules under linear mappings: $\varphi \to A\varphi$, $\varphi \to B\varphi$

$$\begin{aligned} A : \Sigma &\mapsto A\Sigma A^{-1}, & \widehat{\Sigma} &\mapsto \widehat{\Sigma} \\ B : \Sigma &\mapsto \Sigma, & \widehat{\Sigma} &\mapsto B^{-1}\widehat{\Sigma} B \\ A : \Omega &\mapsto A\Omega A^{-1}, & \widehat{\Omega} &\mapsto \widehat{\Omega} \\ B : \Omega &\mapsto \Omega, & \widehat{\Omega} &\mapsto B^{-1}\widehat{\Omega} B. \end{aligned}$$

It is clear that the $\Omega-$ and Σ-objects transform in an analogous way.

Let us also introduce the co-moving translational objects

$$\widehat{v}^A = \varphi^{-1A}{}_i v^i, \qquad \widehat{p}_A = p_i \varphi^i{}_A.$$

The resulting Poisson brackets have the following form

$$\begin{aligned} \{\Sigma^i{}_j, \Sigma^k{}_l\} &= \delta^i{}_l \Sigma^k{}_j - \delta^k{}_j \Sigma^i{}_l \\ \{\widehat{\Sigma}^A{}_B, \widehat{\Sigma}^C{}_D\} &= \delta^C{}_B \widehat{\Sigma}^A{}_D - \delta^A{}_D \widehat{\Sigma}^C{}_B \\ \{\Sigma^i{}_j, \widehat{\Sigma}^A{}_B\} &= 0, \qquad \{\widehat{\Sigma}^A{}_B, \widehat{p}_C\} = \delta^A{}_C \widehat{p}_B \\ \{I(\mathcal{O})^i{}_j, p_k\} &= \{\Lambda(\mathcal{O})^i{}_j, p_k\} = \delta^i{}_k p_j \end{aligned}$$

where, obviously

$$I(\mathcal{O})^i{}_j := \Lambda(\mathcal{O})^i{}_j + \Sigma^i{}_j, \qquad \Lambda(\mathcal{O})^i{}_j := x^i p_j$$

and x^i are Cartesian coordinates of the $\mathcal{O}$-radius vector of the current position of the centre of mass in M.

For any function F depending only on the configuration variables the following Poisson relations hold

$$\{F, \Sigma^i{}_j\} = \varphi^i{}_A \frac{\partial F}{\partial \varphi^j{}_A}, \qquad \{F, \Lambda^i{}_j\} = x^i \frac{\partial F}{\partial x^j}, \qquad \{F, \widehat{\Sigma}^A{}_B\} = \varphi^i{}_B \frac{\partial F}{\partial \varphi^i{}_A}\cdot$$

The dipole distribution of linear momentum, i.e., canonical affine spin, is given by

$$K^{ij} = \int \left(y^i - x^i\right)\left(\dot{y}^j - \dot{x}^j\right) \mathrm{d}\mu_\varphi(y) = \int \left(y^i - x^i\right) \dot{\varphi}^j{}_K a^K \mathrm{d}\mu(a)$$

$$= \varphi^i{}_A \frac{\mathrm{d}\varphi^j{}_B}{\mathrm{d}t} J^{AB}$$

Obviously, this is an affine spin with respect to the instantaneous position of the center of mass. Similarly, the affine torque, i.e., affine moment of forces with respect to the center of mass is defined as

$$N^{ij} = \int \left(y^i - x^i\right) \mathcal{F}^j\left(y\right) \mathrm{d}\mu(y)$$

where $\mathcal{F}^j$ is the distribution of forces per unit mass and the total force acting on the center of mass is given by

$$F^i = \int \mathcal{F}^j(y) \mathrm{d}\mu(y).$$

Equations of motion following from the d'Alembert principle applied to affine motion imply that

$$m \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = F^i, \qquad \varphi^i{}_A \frac{\mathrm{d}^2 \varphi^j{}_B}{\mathrm{d}t^2} J^{AB} = N^{ij}. \tag{16}$$

In the potential case in Euclidean space we have

$$F^i = -g^{ij} \frac{\mathrm{d}V}{\mathrm{d}x^j}, \qquad N^{ij} = -\varphi^i{}_A \frac{\partial V}{\partial \varphi^k{}_A} g^{kj}. \tag{17}$$

Let us mention that in the potential case we must base on the formulas (13), (14). Or to be more honest-deformation invariants are very often the arguments of the potential function.

Incidentally, let us quote a few remarks concerning the concept of deformation invariants. As mentioned above, there are only n functionally independent among them, e.g.,

$$\mathcal{K}_a[\varphi] = \mathrm{Tr}\left(\widehat{G}[\varphi]^a\right) = \mathrm{Tr}\left(\widehat{C}[\varphi]^{-a}\right), \qquad a = 1 \ldots, n. \tag{18}$$

As mentioned, the range of a in (18) may be any set of subsequent n integers, not necessarily $1, \ldots, n$; this a consequence of the Caylay-Hamilton theorem. In general, one can say that deformation invariants are functions invariant under

$$\varphi \to L\varphi R, \qquad L, R \in O(n, \mathbb{R})$$

So they are defined on the manifold of double cosets

$$O(n, \mathbb{R}) \backslash GL(n, \mathbb{R}) \,/ O(n, \mathbb{R}).$$

For example, they are often defined as coefficients in the following eigenequations for λ

$$\det\left(\widehat{G}[\varphi] - \lambda[\varphi] I_n\right) = \sum_{k=0}^{n} (-1)^k \, I_{n-k}\,(\varphi)\, \lambda^k.$$

Obviously, $I_0 = 1$ is standard. I_{n-k} is a sum of products of $(n-k)$ different eingenvalues

$$I_0 = 1, \quad I_1 = \sum_i \lambda_i, \ldots, I_n = \prod_i \lambda_i = \lambda_1, \ldots \lambda_n.$$

In the physical three-dimensional case we have

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3, \qquad I_2 = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2, \qquad I_3 = \lambda_1\lambda_2\lambda_3.$$

Let us observe that in mechanics of systems of affinely deformable bodies we can also use something which may be called mutual deformation tensors. Namely, if φ, Ψ are linear isomorphisms of U onto V, then the corresponding Green and Cauchy mutual deformations $G \in U^* \otimes U^*$, $C \in V^* \otimes V^*$ are given by the following analytical formulas

$$G[\Psi, \varphi]_{AB} = g_{ij} \Psi^i{}_A \varphi^j{}_B, \qquad C[\Psi, \varphi]_{ij} = \eta_{AB}\; \varphi^{-1\,A}{}_i\; \Psi^{-1\,B}{}_j$$

and similarly for their inverses

$$G[\Psi, \varphi]^{-1\,AB} = \Psi^{-1\,A}{}_i\; \varphi^{-1\,B}{}_j g^{ij}, \qquad C[\Psi, \varphi]^{-1\,ij} = \varphi^i{}_A\; \Psi^j{}_B \eta^{AB}.$$

Obviously

$$G[\Psi, \Psi] = g[\Psi], \qquad C[\Psi, \Psi] = C[\Psi]. \tag{19}$$

Clearly, for any $A \in \mathrm{O}(V, g)$, $B \in \mathrm{O}(U, \eta)$ the following holds

$$G[A\Psi, A\varphi] = G[\Psi, \varphi], \qquad C[\Psi B, \varphi B] = C[\Psi, \varphi].$$

But there is no lucid expression if A, B are non-orthogonal. For any linear mappings $A \in GL(U)$, $B \in GL(V)$, the following holds

$$G[\Psi A, \varphi A] = A_* G[\Psi, \varphi], \qquad C[B\Psi, B\varphi] = B_*^{-1} C[\Psi, \varphi].$$

In terms of coordinates

$$\begin{aligned} G[\Psi A, \varphi A]_{KL} &= G[\Psi, \varphi]_{MN} A^M{}_K A^N{}_L \\ C[B\Psi, B\varphi]_{ij} &= C[\Psi, \varphi]_{rs}\, B^{-1\,r}{}_i\, B^{-1\,s}{}_j . \end{aligned} \tag{20}$$

However, also other kind of mutual displacement-kind quantities may be defined, namely

$$\Gamma[\Psi, \varphi] = \Psi^{-1}\varphi, \qquad \Sigma[\Psi, \varphi] = \varphi\Psi^{-1}.$$

If Ψ and φ are orthogonal mappings, then this definition reduces to the previous one. Namely, when $\Psi, \varphi \in O(U, \eta : V, g)$

$$\Gamma[\Psi, \varphi] = G[\Psi, \varphi], \qquad \Sigma[\Psi, \varphi] = C[\Psi, \varphi].$$

Those quantities may be exactly interpreted as group-theoretical analogues of the displacement vector in the Abelian group of translational degrees of freedom. And at the same time they are not only material, but also affine invariants. Namely, for any $A \in GL(U)$, $B \in GL(V)$ they satisfy

$$\begin{aligned} \Gamma[A\Psi, A\varphi] &= \Gamma[\Psi, \varphi], \qquad \Sigma[A\Psi, A\varphi] = A_* \Sigma[\Psi, \varphi] \\ \Gamma[\Psi B, \varphi B] &= B_*^{-1} \Gamma[\Psi, \varphi], \qquad \Sigma[\Psi B, \varphi B] = \Sigma[\Psi, \varphi] \end{aligned} \tag{21}$$

The parts of (21) with the transformation rules A_*, B_*^{-1} may be written down analytically as follows

$$\begin{aligned} \Sigma[A\Psi, A\varphi]^i{}_j &= A^i{}_k \Sigma[\Psi, \varphi]^k{}_m\, A^{-1\,m}{}_j \\ \Gamma[\Psi B, \varphi B]^K{}_L &= B^{-1\,K}{}_D \Gamma[\Psi, \varphi]^D{}_E B^E{}_L . \end{aligned}$$

This means that Γ is invariant under spatial affine mappings and satisfies the inverse adjoint rule under material affine mappings. Conversely, Σ transforms under spatial affine mappings and is invariant under material affine transformations. It is important that the quantities $G[\Psi, \varphi], C[\Psi, \varphi], \Gamma[\Psi, \varphi], \Sigma[\Psi, \varphi]$ generate scalar variables which may be used as independent variables (arguments) of the potential energy of mutual interactions between different affine bodies.Typical metrical scalars are

$$\mathcal{K}_a[\Psi, \varphi] = \mathrm{Tr}\left(G[\Psi, \varphi]^a\right) = \mathrm{Tr}\left(C[\Psi, \varphi]^{-a}\right), \qquad a = 1, \ldots, n.$$

They are invariant under spatial and material rotations, so that for any $A \in O(V, g)$, $B \in O(U, \eta)$ the following equation is satisfied

$$\mathcal{K}_a[A\Psi B, A\varphi B] = \mathcal{K}_a[\Psi, \varphi].$$

The quantities Γ, Σ give rise to more general invariants which are non-sensitive with respect to all affine transformations, not only with respect to rigid rotations. They are given by

$$\mathcal{M}_a[\Psi, \varphi] = \mathrm{Tr}\left(\Gamma[\Psi, \varphi]^a\right) = \mathrm{Tr}\left(\Sigma[\Psi, \varphi]^a\right).$$

It follows from their construction that they are affinely-invariant, i.e., for any $A \in GL(U)$, $B \in GL(V)$

$$\mathcal{M}_a[A\Psi\varphi, A\varphi B] = \mathcal{M}_a[\Psi, \varphi]. \tag{22}$$

Therefore, it is possible to construct the hierarchy of invariants non-sensitive with respect to affine groups like in (22), or to their isometric subgroups. This extends the aprioric hierarchy of our models and in particular, enables one to discuss the influence and physical meaning of affinely-invariant part of mutual potentials.

Equations of motion (16), (17) follow from the d'Alembert principle and from the following obvious expression for the power of forces in affine motion

$$\mathcal{P} = \mathcal{P}_{\rm tr} + \mathcal{P}_{\rm int} = F_i v^i + N^{ij}\Omega_{ij} = F_i v^i + N^i{}_j \Omega^j{}_i. \tag{23}$$

Hence F^i, N^{ij} denote respectively the total force and affine moment of forces acting on the body. In they are built of given forces and reactions of affine constraints. But according to the d'Alembert principle the expression (20) with reactions $F^{(R)}{}_i$, $N^{(R)\,i}{}_j$ substituted to the expression produces a zero result. Therefore, equations (16), (17) contain only given forces and it is just the reason of their form. Namely, reactions, although non-vanishing, are automatically removed from them.

Let us observe that equations of affine motion may be written as balance laws for the affine spin $\mathcal{K}^{ij}$ and the usual linear momentum $k^i = g^{ij}p_j$ and the affine spin $\mathcal{K}^{ij}$

$$\begin{aligned} \frac{\mathrm{d}k^i}{\mathrm{d}t} &= F^i \\ \frac{\mathrm{d}K^{ij}}{\mathrm{d}t} &= \frac{\mathrm{d}\varphi^i{}_A}{\mathrm{d}t}\frac{\mathrm{d}\varphi^j{}_B}{\mathrm{d}t}J^{AB} + N^{ij}. \end{aligned} \tag{24}$$

Why this balance form is so essential? Why without solving with respect to $\mathrm{d}^2\varphi^i{}_A/\mathrm{d}t^2$? It is again the formula (23) for the power forces. The form (**??**), (24) enables one to introduce constraints of group-theoretical origin in an almost automatic way.

For example, when the motion is metrically rigid, then $\widehat{\Omega}^i{}_j$ is the g-antisymmetric part of equations of motion in the form (24)

$$\frac{\mathrm{d}S^{ij}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(K^{ij} - K^{ji}\right) = N^{ij} - N^{ji} = \mathcal{N}^{ij}$$

i.e.,

$$\varphi^i{}_A\frac{\mathrm{d}^2\varphi^j{}_B}{\mathrm{d}t^2} - \varphi^j{}_A\frac{\mathrm{d}^2\varphi^i{}_B}{\mathrm{d}t^2} = N^{ij} - N^{ji} = \mathcal{N}^{ij}.$$

Similarly,when affine motion is incompressible, equations of motion are given by the trace-less part of original equations of motion

$$\varphi^i{}_A \frac{\mathrm{d}^2\varphi^j{}_B}{\mathrm{d}t^2}\, J^{AB} - \frac{1}{n}\, g_{ab}\, \varphi^a{}_A \frac{\mathrm{d}^2\varphi^b{}_B}{\mathrm{d}t^2} J^{AB} g^{ij} = N^{ij} - \frac{1}{n}\, g_{ab}\, N^{ab}\, g^{ij}.$$

And now something really new example of constraints, namely constraints of spatially rotation-less motion. Let us remind that constraints of the purely rotational motion, may be written both in the usual holonomic way

$$\varphi \in O(U, \eta; V, g)$$

i.e., than φ is an isometry, but also in apparently non-holonomic way

$$\Omega^i{}_j + \Omega_j{}^i = \Omega^i{}_j + g_{jk}\, g^{il}\, \Omega^k{}_l = 0. \tag{25}$$

Strictly speaking, (25) are semi-holonomic constrains, i.e., they describe the foliation of the configuration space by the family holonomic constraints of gyroscopic type . Therefore the constraints of rotation-free motion may be defined as opposite, complementary to (25), i.e., such ones that Ω is $g-$symmetric

$$\Omega^i{}_j - \Omega_j{}^i = \Omega^i{}_j - g_{jk}\, g^{il}\, \Omega^k{}_l = 0. \tag{26}$$

The corresponding equations of motion are then given by

$$\varphi^i{}_A \frac{\mathrm{d}^2\varphi^j{}_B}{\mathrm{d}t^2}\, J^{AB} + \varphi^j{}_A \frac{\mathrm{d}^2\varphi^i{}_B}{\mathrm{d}t^2}\, J^{AB} = N^{ij} + N^{ji} \tag{27}$$

or, more precisely, they are given by the system (26), (27). It is very surprising that these constraints are non-holonomic. Indeed, $g-$symmetric matrices Ω^i_j do not from a Lie algebra and the Pfaff system (26) is non-integrable. Therefore, the rotation-less motions do not generate any manifold of rotation-less configurations. This way was naturally expected because $g-$symmetric matrices do not from a Lie subalgebra or Lie subgroups. And certainly it would be incorrect to define the set of mutually non-rotated configurations by anything like the symmetry demand for the matrix $\left[\varphi^i{}_A\right]$ because the indices i, A refer to different linear spaces and the symmetry would be a completely artificial, in any case non-transitive feature. Constraints of rotation-less motion may appear in the study of affine motion of inclusion injected into very viscous fluid.

Let us also mention that there is also another kind of rotation-less motion, namely, materially non-rotational one. The corresponding constraints are obviously also non-holonomic but the equation (26) is then replaced by another, non-equivalent material equation

$$\widehat{\Omega}^A{}_B - \widehat{\Omega}_B{}^A = \widehat{\Omega}^A{}_B - \eta_{BC}\, \eta^{AD}\, \widehat{\Omega}^C{}_D.$$

And further one should again proceed according to the d'Alembert principle of ideal, i.e., non-working reaction forces.

Let us go back to our general equations of affine motion. In gyroscopic case they were balance equations of the Euler type, reducing to the angular momentum conservation laws in the geodetic case. But for the affine motion it is no longer the case-affine symmetry of degrees of freedom becomes reduced to orthogonal one

$$\frac{\mathrm{d}K^{ij}}{\mathrm{d}t} = 2\frac{\partial T_{\mathrm{int}}}{\partial g_{ij}} + N^{ij}, \qquad 2\frac{\partial T_{\mathrm{int}}}{\partial g_{ij}} = \frac{\mathrm{d}\varphi^{i}{}_{A}}{\mathrm{d}t}\frac{\mathrm{d}\varphi^{j}{}_{B}}{\mathrm{d}t}J^{AB}.$$

This is not a conservation law even in the geodetic case. Let us write explicitly the system of equations of motion in terms of balance laws

$$\begin{aligned}\frac{\mathrm{d}\widehat{k}^{A}}{\mathrm{d}t} &= -\widehat{k}^{B}\widetilde{J}_{BC}\widehat{K}^{CA} + \widehat{F}^{A}\\ \frac{\mathrm{d}\widehat{K}^{AB}}{\mathrm{d}t} &= -\widehat{K}^{AC}\widetilde{J}_{CD}\widehat{K}^{DB} + \widehat{N}^{AB}\end{aligned} \tag{28}$$

In many problems it is convenient to use the velocity representation of those equations

$$\begin{aligned} m\frac{\mathrm{d}\widehat{v}^{A}}{\mathrm{d}t} &= -m\widehat{\Omega}^{A}{}_{B}\widehat{v}^{B} + \widehat{F}^{A}\\ J^{AC}\frac{\mathrm{d}\widehat{\Omega}^{B}{}_{C}}{\mathrm{d}t} &= -\widehat{\Omega}^{B}{}_{D}\widehat{\Omega}^{D}{}_{C}J^{CA} + \widehat{N}^{AB}.\end{aligned} \tag{29}$$

Of course, (28), (29) are written in terms of material (co-moving representation), nevertheless they have never the rigorous Euler form. The main circumstance is that they are never affinely invariant. Only geometry of internal degrees of freedom is ruled by affine group, but no longer their dynamics.

4. What Would Be Affine Models?

So, we are faced with the following challenge: we must answer the question if affinely-invariant dynamics does exist formally. And if yes, one should formulate some hypotheses concerning the physical applications of the corresponding models.

It is very easy to answer the existence questions when one does assume again the kinetic energy in the form of the sum of internal and translational parts

$$T = T_{\mathrm{int}}\left(\varphi;\dot{\varphi}\right) + T_{\mathrm{tr}}\left(x,\varphi;\dot{x}\right). \tag{30}$$

The affine groups $\mathrm{GAff}(M)$, $\mathrm{GAff}(N)$ act on $Q = \mathrm{AffI}(N, M)$ trough the left and right regular translations

$$\mathcal{A} \in \mathrm{GAff}(M), \qquad \mathcal{B} \in \mathrm{GAff}(N) : Q \ni \Phi \mapsto \mathcal{A}\circ\Phi\circ\mathcal{B} \in Q.$$

To be more precise, because the material center of mass in N is distinguished, we are interested rather in the action of $\mathrm{GAff}(N)$ and $\mathrm{GL}(U)$ according to

$$\mathcal{A} \in \mathrm{GAff}(M), \quad \mathcal{B} \in \mathrm{GL}(U) : (x, \varphi) \mapsto (\mathcal{A}(x), L[\mathcal{A}]\varphi), \quad (x, \varphi) \mapsto (x, \varphi\mathcal{B})$$

where $L[\mathcal{A}] \in \mathrm{GL}(V)$ is the linear part of $\mathcal{A}$ in the sense

$$\overrightarrow{\mathcal{A}(x)\mathcal{A}(y)} = L[\mathcal{A}]\,\overrightarrow{xy}.$$

Obviously, $\overrightarrow{xy}$ as usual denotes the vector from x to y.
Then the left-invariant kinetic energy is given by

$$T_{\mathrm{int}} = \frac{1}{2}\mathcal{L}^{B}{}_{A}{}^{D}{}_{C}\widehat{\Omega}^{A}{}_{B}\widehat{\Omega}^{C}{}_{D}, \qquad T_{\mathrm{tr}} = \frac{m}{2}C_{ij}\frac{\mathrm{d}x^{i}}{\mathrm{d}t}\frac{\mathrm{d}x^{j}}{\mathrm{d}t} = \frac{m}{2}\eta_{AB}\widehat{v}^{A}\widehat{v}^{B} \tag{31}$$

where $\mathcal{L}^{B}{}_{A}{}^{D}{}_{C}$ are constants and $\mathcal{L}^{B}{}_{A}{}^{D}{}_{C} = \mathcal{L}^{D}{}_{C}{}^{B}{}_{A}$.
Leta us observe that in T_{tr} the role of the spatial metric tensor is played by the Cauchy deformation tensor. This is responsible for the affine invariance of T_{tr} in M. The corresponding equations of motion have the form

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = Q_i, \qquad \frac{\mathrm{d}\Sigma^{i}{}_{j}}{\mathrm{d}t} = -\frac{1}{m}\,C^{-1ik}p_k p_j + Q^{i}{}_{j}$$

where

$$p_i = C_{ij}\frac{\mathrm{d}x^{j}}{\mathrm{d}t}$$

and Q_i, $Q^{i}{}_{j}$ are generalized forces. In the potential case they are given by

$$Q_i = -\frac{\partial V}{\partial x^{i}}, \qquad Q^{i}{}_{j} = -\varphi^{i}{}_{A}\frac{\partial V}{\partial \varphi^{j}{}_{A}}.$$

One can write them in a more concise form

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = Q_i, \qquad \frac{\mathrm{d}I(\mathcal{O})^{i}{}_{j}}{\mathrm{d}t} = Q_{\mathrm{tot}}(\mathcal{O})^{i}{}_{j}$$

with the following natural definitions

$$\begin{aligned} I(\mathcal{O})^{i}{}_{j} &= \Lambda(\mathcal{O})^{i}{}_{j} + \Sigma^{i}{}_{j} = x^{i}p_j + \Sigma^{i}{}_{j} \\ Q_{\mathrm{tot}}(\mathcal{O})^{i}{}_{j} &= Q_{\mathrm{tr}}(\mathcal{O})^{i}{}_{j} + Q^{i}{}_{j} = x^{i}Q_j + Q^{i}{}_{j}. \end{aligned}$$

For the right-invariant models equations of motion have the following form

$$\frac{\mathrm{d}\hat{p}_A}{\mathrm{d}t} = \widehat{Q}_A, \quad \text{i.e.,} \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = Q_i, \qquad \frac{\mathrm{d}\widehat{\Sigma}^{A}{}_{B}}{\mathrm{d}t} = \widehat{Q}^{A}{}_{B}$$

where

$$\widehat{Q}_A = Q_i\varphi^{i}{}_{A}, \qquad \widehat{Q}^{A}{}_{B} = \varphi^{-1A}{}_{i}Q^{i}{}_{j}\varphi^{j}{}_{B}.$$

Unlike in the case of the usual, i.e., metrically-rigid body, there are no kinetic energy, i.e., non-degenerate metric models which would be simultaneously left- and right-invariant under the total affine group. The reason is that the affine group is non-semisimple in a rather malicious way. There are only doubly-invariant models on $Q_{\rm int}$, the submanifold of internal degrees of freedom

$$T_{\rm int} = \frac{A}{2}\,\Omega^{i}{}_{j}\Omega^{j}{}_{i} + \frac{B}{2}\,\Omega^{i}{}_{i}\Omega^{j}{}_{j} = \frac{A}{2}\,\widehat{\Omega}^{K}{}_{L}\widehat{\Omega}^{L}{}_{K} + \frac{B}{2}\,\widehat{\Omega}^{K}{}_{K}\widehat{\Omega}^{L}{}_{L}. \tag{32}$$

However, independently on the choice of constants A and B this model is never positively – nor negatively – definite. Unlike the first, bad impression this need not disqualify them, moreover, they may be physically interesting. The main term of (32) is the A-term, the B-term itself is degenerate and plays only the role of some correction. However, the translational term $T_{\rm tr}$ always reduces the symmetry group.

Let us observe that the right affinely-invariant models of kinetic energy have the form (30) where

$$T_{\rm int} = \frac{1}{2}\,\mathcal{R}^{j}{}_{i}{}^{l}{}_{k}\,\Omega^{i}{}_{j}\Omega^{k}{}_{l}, \qquad T_{\rm tr} = \frac{m}{2}\,g_{ij}\frac{{\rm d}x^{i}}{{\rm d}t}\frac{{\rm d}x^{j}}{{\rm d}t} \tag{33}$$

where $\mathcal{R}^{j}{}_{i}{}^{l}{}_{k}$, g_{ij} are constants symmetric respectively in the pairs of indices and in indices. Obviously, the expression (32) is a very special case of the first formulas of (31), (33). In particular, it gives rise to two special, highly-symmetric cases of the total formulas for $T = T_{\rm tr} + T_{\rm int}$. We denote them by $T^{\rm aff-met}$ and $T^{\rm met-aff}$. The first of them is invariant under the spatial affine group ${\rm GAff}(M)$ and the material Euclidean group acting in N (or in U). Writing this explicitly

$$\begin{aligned} T^{\rm aff-met} &= \frac{m}{2}\,\eta_{KL}\hat{v}^{K}\hat{v}^{L} + \frac{I}{2}\,\eta_{KL}\widehat{\Omega}^{K}{}_{M}\widehat{\Omega}^{L}{}_{N}\eta^{MN} \\ &\quad + \frac{A}{2}\,\widehat{\Omega}^{K}{}_{L}\widehat{\Omega}^{L}{}_{K} + \frac{B}{2}\,\widehat{\Omega}^{K}{}_{K}\widehat{\Omega}^{L}{}_{L} \end{aligned} \tag{34}$$

$$T^{\rm met-aff} = \frac{m}{2}\,g_{ij}v^{i}v^{j} + \frac{I}{2}\,g_{ij}\Omega^{i}{}_{k}\Omega^{j}{}_{l}g^{kl} + \frac{A}{2}\,\Omega^{i}{}_{j}\Omega^{j}{}_{i} + \frac{B}{2}\,\Omega^{i}{}_{i}\Omega^{j}{}_{j}. \tag{35}$$

Let us stress again that the last two expressions in both formulas are equal but written in apparently different forms for aestetical reasons. Unlike this, the second terms are different. And similarly the first terms. Let us respect that

$$\frac{m}{2}\,\eta_{KL}\widehat{v}^{K}\widehat{v}^{L} = \frac{m}{2}\,C_{ij}(\varphi)v^{i}v^{j} \tag{36}$$

so that the metric tensor is replaced by the Cauchy deformation tensor depending on φ. This resembles the procedure of introducing the tensor of effective mass in solid state physics [29]. Let us observe that in the potential forces case the formulas (34), (36) tell us that according to Legendre transformation

$$p_{i} = C_{ij}(\varphi)\frac{{\rm d}x^{j}}{{\rm d}t} \neq g_{ij}\frac{{\rm d}x^{j}}{{\rm d}t}\cdot \tag{37}$$

But if there are no translational forces, p_i is a constant of motion. Unlike this, as seen from (37), due to the dependence on deformation variables φ, the translational velocity v^i fails to be constant, both in the sense of direction and magnitude. This strange phenomenon may be called the "drunk missile effect". Of course, it does not occur when (35), i.e., $T^{\mathrm{met-aff}}$ is used as a kinetic energy term. The corresponding model, i.e., (35) may be used as a discretized, final-dimensional, version of the Arnold description of the ideal fluid. More precisely: it may be so when Q_{int} is replaced by the hypersurface of isochoric, i.e., volume-preserving configurations.

It is interesting at least from the purely academic point of view to ask for the geodetic motions corresponding to (34), (35) or to their internal φ-parts. If we consider the doubly-invariant, i.e., left-invariant and right-invariant affine models, then it is clear that the general solution is given by the exponential expression:

$$\varphi(t) = \exp\left(Et\right)\varphi_0 = \varphi_0 \exp\left(\widehat{E}t\right), \qquad \widehat{E} = {\varphi_0}^{-1} E \varphi_0$$

where φ_0 and E or $\widehat{E}$ are arbitrary initial data.

But the situation becomes more complicated when we admit only one-side affine invariance and one-side metrical invariance. Then there are some stationary conditions.

In the case of (31), i.e., left affinely-invariant and right metrically-invariant kinetic energy we also start from looking on exponential solutions

$$\varphi(t) = \varphi_0 \exp(Ft).$$

It may be shown that such solutions exist for any initial configuration φ_0 but only for the η-normal F-s;

$$\left[F, F^{\eta T}\right] = FF^{\eta T} - F^{\eta T}F = 0$$

where, by definition

$$\left(F^{\eta T}\right)^{A}{}_{B} = \eta_{BD}\, F^{D}{}_{C}\, \eta^{CA}.$$

This holds in particular when

$$F^{\eta T} = -F, \qquad F^{\eta T} = F.$$

Let us now assume (33), i.e., left metrically-invariant and right affinely-invariant model of T. We are looking for the stationary solutions given by

$$\varphi(t) = \exp(Et)\varphi_0.$$

It turns out that such solutions exist for an arbitrary initial configuration φ_0, but E is to be g-normal, i.e.,

$$\left[E, E^{gT}\right] = EE^{gT} - E^{gT}E = 0$$

where

$$\left(E^{gT}\right)^{i}{}_{j} = g_{jl}\, E^{l}{}_{k}\, g^{ki}.$$

This holds, e.g., in special cases when

$$F^{gT} = -F, \qquad F^{gT} = F.$$

Obviously, the metric tensors on Q underlying (34), (35) are very special, just distinguished by the assumed invariance requirements. But because of this one can say something about the solutions of equations of motion following from the corresponding Lagrangians. The question appears however concerning the physical motivation and applicability of those models invariant under the left or right action of the affine group on the configuration space Q. In what are the corresponding metric tensors better than (11) following from the d'Alembert principle? Let us notice that (34) (35) may be also interpreted in d'Alembert terms, however with a new metric tensor on Q, not necessarily one derived from the spatial or material metric g, η. Again the concept of effective mass in solid state physics suggests something like that [29]. The more so, in such applications like the nuclear fluid or neutrons star the usual mechanism of constraints need not be justified. Affine motion may be some aspect of the averaged behavior of system, different than constraints imposed on the configuration space. And then it is only the general invariance assumption that may offer some guiding hints concerning the structure of the kinetic energy expression.

Let us observe that after performing the Legendre transformation on the internal part of the kinetic energy (34), (35) it becomes respectively

$$\mathcal{T}^{\text{aff met}}_{\text{int}} = \frac{1}{2\alpha}\,\mathrm{Tr}\left(\widehat{\Sigma}^{2}\right) + \frac{1}{2\beta}\left(\mathrm{Tr}\,\widehat{\Sigma}\right)^{2} - \frac{1}{2\mu}\,\mathrm{Tr}\left(V^{2}\right) \tag{38}$$

$$\mathcal{T}^{\text{met aff}}_{\text{int}} = \frac{1}{2\alpha}\,\mathrm{Tr}\left(\Sigma^{2}\right) + \frac{1}{2\beta}\left(\mathrm{Tr}\,\Sigma\right)^{2} - \frac{1}{2\mu}\,\mathrm{Tr}\left(S^{2}\right) \tag{39}$$

where, as usual, the skew-symmetric spatial and material tensors S, V denote the canonical spin and vorticity. The parameters α, β, μ are expressed by the primary constants A, B, C as follows

$$\alpha = I + A, \qquad \beta = -\frac{1}{B}(I+A)(I+A+nB), \qquad \mu = \frac{1}{I}(I^{2} - A^{2}).$$

Denoting by $C(k)$ the k-th degree Casimir quantity we can rewrite (38) and (39) as follows

$$\mathcal{T}_{\text{int}} = \frac{1}{2\alpha}\,C(2) + \frac{1}{2\beta}\,C(1)^{2} + \frac{1}{2\mu}\,\|V\|^{2} \tag{40}$$

$$\mathcal{T}_{\text{int}} = \frac{1}{2\alpha}\,C(2) + \frac{1}{2\beta}\,C(1)^{2} - \frac{1}{2\mu}\,\|S\|^{2} \tag{41}$$

where $\|V\|$, $\|S\|$ denote the scalar magnitudes of V and S

$$\|V\|^2 = \frac{1}{2}\,\mathrm{Tr}\left(V^2\right), \qquad \|S\|^2 = \frac{1}{2}\,\mathrm{Tr}\left(S^2\right).$$

Obviously, the Casimir invariants $C(k)$ are given by

$$C(k) = \mathrm{Tr}\left(\widehat{\Sigma}^k\right) = \mathrm{Tr}\left(\Sigma^k\right).$$

It is seen that the first two doubly-affine expressions in (40), (41) are respectively identical. The difference between (40), (41) consists in the third term proportional to the Casimir invariants of the material and spatial group of (metrically) rigid rotations. From the quantum point of view $\|S\|^2$, $\|V\|^2$ may be interpreted respectively as the squared magnitude of spin and of the isospin. This suggests us to introduce a systematic description of the left- and right-rotationally invariant models with the hierarchic ordering of rotationally-invariant and purely affinely-invariant terms. Then instead of (40), (41) we obtain a linear combination of the family of terms involving both $\|S\|^2$ and $\|V\|^2$. It seems that the most general form of the left- and right-isotropic constituents is given by

$$\begin{aligned} T = {} & \frac{1}{2}\left(m_1 G_{AB} + m_2 \eta_{AB}\right) \hat{v}^A \hat{v}^B + \frac{1}{2}\left(I_1 G_{KL} G^{MN} + I_2 \eta_{KL} \eta^{MN}\right. \\ & \left. + I_3 G_{KL} \eta^{MN} + I_4 \eta_{KL} G^{MN}\right) \widehat{\Omega}^K{}_M \widehat{\Omega}^L{}_N + \frac{A}{2}\,\widehat{\Omega}^I{}_J \widehat{\Omega}^J{}_I + \frac{B}{2}\,\widehat{\Omega}^I{}_I \widehat{\Omega}^J{}_J. \end{aligned} \tag{42}$$

This expression may be rewritten in the following equivalent form

$$\begin{aligned} T = {} & \frac{1}{2}\left(m_1 g_{ij} + m_2 C_{ij}\right) v^i v^j \\ & + \frac{1}{2}\left(I_1 g_{kl} g^{mn} + I_2 C_{kl} C^{mn} + I_3 g_{kl} C^{mn} + I_4 C_{kl} g^{mn}\right) \Omega^k{}_m \Omega^l{}_n \\ & + \frac{A}{2}\,\Omega^i{}_j \Omega^j{}_i + \frac{B}{2}\,\Omega^i{}_i \Omega^j{}_j. \end{aligned} \tag{43}$$

The simultaneous left- and right-invariance of T is impossible when translational degrees of freedom are active. It becomes possible when we neglect them, i.e., put $m_1 = 0$, $m_2 = 0$ and in addition $I_1 = 0$, $I_2 = 0$, $I_3 = 0$, $I_4 = 0$. The metric tensor on Q_{int} becomes then affinely-invariant both on the right (i.e., materially, in the body) and on the left (in the physical space). The full affine invariance in the right-hand side material sense appears when $m_1 = 0$, $I_2 = 0$, $I_3 = 0$, $I_4 = 0$. The kinetic energy becomes spatially affine-invariant when we put in (42), (43) that $m_1 = 0$, $I_1 = 0$, $I_3 = 0$, $I_4 = 0$. And of course, for any fixed constants the expression (42), (43) is simultaneously space and material isotropic. All such metrics are Riemannian, i.e., curved, for any choice of constants excepting $m_2 = 0$, $I_1 = 0$, $I_2 = 0$, $I_4 = 0$, $A = 0$, $B = 0$. In this particular situation the metric becomes (11) with the special substitution $J^{AB} = I\eta^{AB}$ corresponding to the spherical symmetry of inertia.

Let us summarize (42), (43) in the following way in the formula for the general doubly-isotropic metric tensor Γ on the configuration space Q

$$\begin{aligned}\Gamma = {} & \frac{1}{2}\,(m_1 g_{ij} + m_2 C_{ij})\,\mathrm{d}x^i \otimes \mathrm{d}x^j \\ & + \big(I_1 g_{ij} G^{-1\,AB} + I_2 C_{ij}\eta^{AB} + I_3 g_{ij}\eta^{AB} + I_4 C_{ij} G^{-1\,AB} \\ & + A\varphi^{-1A}{}_j \varphi^{-1B}{}_i + B\varphi^{-1A}{}_j \varphi^{-1B}{}_i\big)\,\mathrm{d}\varphi^i{}_A \otimes \mathrm{d}\varphi^j{}_B.\end{aligned}$$

Let us now discuss some analytic questions. There are two metric-like tensors in any of linear spaces U, V, and equivalently in any of affine spaces N, M. They are $G[\varphi], \eta \in U^* \otimes U^*$ and $C[\varphi], g \in V^* \otimes V^*$. Raising their first indices respectively with the help of η, g, one obtains the following tensors, once contravariant, once covariant

$$\widehat{G}[\varphi]^A{}_B = \eta^{AC}\, G[\varphi]_{CB}, \qquad \widehat{C}[\varphi]^i{}_j = g^{ik}\, C[\varphi]_{kj}.$$

They are respectively members of $U \otimes U^*$, $V \otimes V^*$. We can consider the following eigenequations

$$\widehat{G} R_a = \lambda_a R_a = \exp(2q^a)\, R_a, \qquad \widehat{C} L_a = \lambda_a^{-1} L_a = \exp(-2q^a)\, L_a.$$

Obviously, λ_a are real numbers, whereas R_a, L_a are basis vectors in U, V. The quantities λ_a are deformation invariants, and q^a are exponential deformation invariants. The bases of V, U given by $L = (\ldots, L_a, \ldots)$, $R = (\ldots, R_a, \ldots)$ may be identified as usual with linear isomorphisms

$$L : \mathbb{R}^n \to V, \qquad R : \mathbb{R}^n \to U.$$

And conversely, the dual bases $(\ldots, L^a, \ldots)$, $(\ldots, R^a, \ldots)$ are identical with the inverse isomorphisms

$$L^{-1} : V \to \mathbb{R}^n, \qquad R^{-1} : U \to \mathbb{R}^n.$$

The diagonal matrix $\mathrm{Diag}\,(\ldots, \exp(q^a), \ldots)$ may be obviously identified with the linear mapping

$$D : \mathbb{R}^n \to \mathbb{R}^n.$$

So, we can write the so-called two-polar decomposition as follows

$$\varphi = LDR^{-1}. \tag{44}$$

Here L, R are orthogonal, whereas D is real-diagonal. This means that $\varphi \in \mathrm{LI}(U, V)$ is represented as a pair of rigid bodies (in the material sense $\mathbb{R}^n$) and with the n-tuple of one-dimensional oscillatory coordinates q^a which are interpreted as logarithmic deformation invariants.

To separate the dilatational part of motion one introduces its logarithmic part

$$q = \frac{1}{n}\left(q^1 + \ldots + q^n\right).$$

Roughly speaking, q is the "centre of mass" of logarithmic deformation invariants $q^1, \ldots, q^n$. The canonical momentum conjugate to q will be denoted by p, and of course

$$p = p_1 + \ldots + p_n$$

where p_i are canonical momenta conjugate to q^i. The co-moving angular velocities of our fictitious rigid bodies are denoted respectively by

$$\widehat{\chi}^a{}_b = L^a{}_i \frac{\mathrm{d}L^i{}_b}{\mathrm{d}t}, \qquad \widehat{\vartheta}^a{}_b = R^a{}_K \frac{\mathrm{d}R^K{}_b}{\mathrm{d}t}$$

where the small first Latin letter refer to the space $\mathbb{R}^n$. And $[L^a{}_i]$, $[R^a{}_K]$ are respectively components of the inverse matrices of $\left[L^i{}_a\right]$, $\left[R^K{}_b\right]$. Their conjugate affine spins have respectively the matrices $[\widehat{\rho}^a{}_b]$, $[\widehat{\tau}^a{}_b]$, where again the indices refer to the arithmetic space $\mathbb{R}^n$. It is convenient to introduce the partially diagonalizing quantities

$$M^a{}_b = -\widehat{\rho}^a{}_b - \widehat{\tau}^a{}_b, \qquad N^a{}_b = \widehat{\rho}^a{}_b - \widehat{\tau}^a{}_b.$$

The term "partially diagonalizing" is justified by the fact that the second-order Casimir invariant

$$C(2) = \mathrm{Tr}\left(\Sigma^2\right) = \mathrm{Tr}\left(\widehat{\Sigma}^2\right)$$

has the following nice form

$$C(2) = \sum_a p_a^2 + \frac{1}{16}\sum_{a,b} \frac{(M^a{}_b)^2}{\mathrm{sh}^2 \frac{q^a - q^b}{2}} - \frac{1}{16}\sum_{a,b} \frac{(N^a{}_b)^2}{\mathrm{ch}^2 \frac{q^a - q^b}{2}}.$$

This means that for the special affine-affine model, when $I = 0$ and $B = 0$, the kinetic energy has the following nice-looking lattice structure

$$\mathcal{T}_{\mathrm{latt}} = \frac{1}{2\alpha}\sum_a p_a^2 + \frac{1}{32\alpha}\sum_{a,b} \frac{(M^a{}_b)^2}{\mathrm{sh}^2 \frac{q^a - q^b}{2}} - \frac{1}{32\alpha}\sum_{a,b} \frac{(N^a{}_b)^2}{\mathrm{ch}^2 \frac{q^a - q^b}{2}}. \tag{45}$$

The one-dimensional lattice structure is easily seen here. It is similar to the hyperbolic Sutherland lattice. The difference is that it is not positively-definite. Namely, in a consequence of the non-compactness of $\mathrm{GL}(n, \mathbb{R})$, the Killing form contains the negative contribution, describing something like the apparently strange "centrifugal attraction". There is a competition with the positive centrifugal repulsion. In a consequence of this the purely geodetic affine-affine models may predict not necessarily centrifugal repulsion but also the attraction which may prevail and lead to the purely geodetic elastic vibrations, ever without any attractive potential. We mean of course the elastic vibrations of distances between deformation invariants

q^i. When we admit in addition the I- and B-terms in the kinetic energy, then this hyperbolic Sutherland structure may be written down as follows

$$\begin{aligned}
\mathcal{T}^{\mathrm{aff-aff}}_{\mathrm{int}} &= \frac{1}{4An}\sum_{a,b}(p_a - p_b)^2 + \frac{1}{32A}\sum_{a,b}\frac{(M^a{}_b)^2}{\operatorname{sh}^2\frac{q^a-q^b}{2}} \\
&\quad -\frac{1}{32A}\sum_{a,b}\frac{(N^a{}_b)^2}{\operatorname{ch}^2\frac{q^a-q^b}{2}} + \frac{1}{2n(A+nB)}p^2 \\
\mathcal{T}^{\mathrm{aff-metr}}_{\mathrm{int}} &= \mathcal{T}^{\mathrm{aff}}_{\mathrm{int}}\left[A \to I + A\right] + \frac{I}{2(I^2 - A^2)}\|V\|^2 \\
\mathcal{T}^{\mathrm{metr-aff}}_{\mathrm{int}} &= \mathcal{T}^{\mathrm{aff}}_{\mathrm{int}}\left[A \to I + A\right] + \frac{I}{2(I^2 - A^2)}\|S\|^2
\end{aligned} \tag{46}$$

where $\mathcal{T}^{\mathrm{aff-aff}}_{\mathrm{int}}[A \mapsto I + A]$ denotes the expression $\mathcal{T}^{\mathrm{aff-aff}}_{\mathrm{int}}$ in which the constant A is replaced by $I + A$. Obviously, these formulas are very special, concerning the model one-side affinely invariant and the other side–metrically invariant. However, the natural question appears as to the hypothetic models combining the affine-affine term with both terms proportional to $\|V\|^2$, $\|S\|^2$. Of course, one can introduce them without any problems on the Hamiltonian level. It is interesting what would be corresponding Lagrangians. We do not go into this problem here. It seems that the corresponding expressions should be searched in the form (42), (43).

For the comparison let us remind the usual trigonometric Sutherland lattice

$$\begin{aligned}
\mathcal{T}_{\mathrm{int}} &= \frac{1}{2A}\sum_{a} p_a^2 - \frac{B}{2A(A+nB)}p^2 \\
&\quad + \frac{1}{32A}\sum_{a,b}\frac{(M^a{}_b)^2}{\sin^2\frac{q^a-q^b}{2}} + \frac{1}{32A}\sum_{a,b}\frac{(N^a{}_b)^2}{\cos^2\frac{q^a-q^b}{2}}.
\end{aligned}$$

When we put $B = 0$, this formula becomes quite analogous to (45) with the important difference however that the hyperbolic functions are replaced by the trigonometric once and the sign at the $\cos^{-2}$ part is positive. But, let us notice, in the case (47) it has nothing to do with the repulsive/attractive action of the coincidence q^a and q^b. Simply, in the usual trigonometric Sutherland lattice case, we still use the two-binary decomposition (44), however with the substitution

$$D_{aa} = \exp\left(\mathrm{i}q^a\right).$$

But then, in the resulting circular topology of deformation invariants, there is no essential difference between their attraction and repulsion.

It is important to stress that in the models of kinetic energy (46)–(47) one can expect the stationary-oscillating motion in the purely geodetic framework, without any use of potential energy or other non–potential forces. Of course, they are admissible, with the special stress on rotationally doubly (i.e., left- and right-) invariant models, nevertheless they are able to describe elastic vibrations only due to inertial forces. It is not the case with the Calogaro-Moser lattices when the kinetic energy in the doubly isotropic-invariant models is given by

$$\mathcal{T}_{\mathrm{int}} = \frac{1}{2I}\sum_a P_a^2 + \frac{1}{8I}\sum_{a,b}\frac{(M^a{}_b)^2}{(Q^a - Q^b)^2} + \frac{1}{8I}\sum_{a,b}\frac{(N^a{}_b)^2}{(Q^a + Q^b)^2}. \tag{47}$$

This is the special case of (11), or rather of its internal part, with $J^{AB} = I\eta^{AB}$. P_a's are canonical momenta conjugate to Q^a so that

$$P_a = \mathrm{e}^{-q^a} p_a, \qquad p_a = \mathrm{e}^{q^a} P_a$$

(so summation over a). It is clear from (11) that without any additionally introduced potential $\mathcal{V}$, the geodetic Lagrangian

$$L = \mathcal{T}_{\mathrm{int}}$$

is unable to describe elastic vibrations. The general solution of equations of motion would be

$$\varphi^i{}_A(t) = \varphi^i{}_A(0) + \xi^i{}_A(0)t$$

so either escaping to infinity or coming through the zero–volume configuration. Therefore, the d'Alembert's expression (47) for the kinetic energy must be always completed by some potential energy. The affinely–invariant models of $\mathcal{T}$ may be, but need not be combined with any potential energy to describe elastic vibrations.

let us illustrate this fact on the simplest plane model when $n = 2$. This is the special, but very instructive case of the "Flatland" [1]. The corresponding reduced Hamiltonian is given by

$$H^{\mathrm{eff}}_{M,N} = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + U^{\mathrm{centr}}_{M,N} + \mathcal{V}\left(q^1, q^2\right)$$

where $U^{\mathrm{centr}}_{M,N}$ is given by

$$U^{\mathrm{centr}}_{M,N} = \frac{M^2}{16m\,\mathrm{sh}^2\frac{q^1-q^2}{2}} - \frac{N^2}{16m\,\mathrm{ch}^2\frac{q^1-q^2}{2}} \tag{48}$$

and $\mathcal{V}\left(q^1, q^2\right)$ is an additional potential energy depending only on deformation invariants. Therefore

$$\mathcal{V}\left(q^1, q^2\right) = \mathcal{V}\left(q^2, q^1\right)$$

because in general any doubly-isotropic potential energy, i.e., one depending only on deformation invariants. $V\left(q^1, \ldots, q^n\right)$ must be invariant under any permutation of its arguments q^i.

Let us now again introduce the variables

$$
\begin{aligned}
x &= q^2 - q^1, & q &= \frac{1}{2}\left(q^1 + q^2\right) \\
p_x &= \frac{1}{2}\left(p_2 - p_1\right), & p &= p_1 + p_2 \\
M &= M^1{}_2, & N &= N^1{}_2.
\end{aligned}
$$

Then we obtain

$$
\mathcal{T}^{\mathrm{aff-aff}}_{\mathrm{int}} = \mathcal{T}^{\mathrm{aff-aff}}_{\mathrm{int}}[x] + U^{\mathrm{centr}}_{M,N}(x) = \frac{p_x^2}{A} + \frac{M^2}{16A\mathrm{sh}^2\frac{x}{2}} - \frac{N^2}{16A\mathrm{ch}^2\frac{x}{2}} + \frac{p^2}{4(A+2B)}
$$

where, as usual, $U^{\mathrm{centr}}_{M,N}$ is given by (48).

In the case of potential energy separated into the x- and q-parts

$$
\mathcal{V}\left(q^1, q^2\right) = V(x) + W(q)
$$

we obtain finally

$$
\mathcal{H} = \frac{p_x^2}{A} + U^{\mathrm{centr}}_{M,N}(x) + V(x) + \frac{p^2}{4(A+2B)} + W(q)
$$

where $U^{\mathrm{centr}}_{M,N}$ is given by (48), (49).

W is chosen so as to stabilize the dilatational motion, e.g., the harmonic oscillator or the potential well about $q = 0$. The internal potential $V(x)$ may, but as said above, need not be used. Depending on the mutual relationship between M, N the corresponding x-term of H may predict both the bounded (in general anharmonic) vibrations or the decaying behavior.

In the general dimension $n > 2$ neither $\widehat{\rho}^a{}_b$ or $\widehat{\tau}^a{}_b$, and therefore, neither $M^a{}_b$, $N^a{}_b$ are constants of motion. Therefore, there is no reduced autonomous dynamics for the q^a-degrees of freedom, i.e., no autonomous, although $M^a{}_b$, $N^a{}_b$-dependent reduced dynamics. Instead, in any doubly- (left- and right-) invariant dynamical models the dynamical quantities $(q^a, p_b, M^a{}_b, N^a{}_b)$ span what Caratheodory called [12] the “Funktionengruppe”, i.e., their Poisson brackets are their own functions. So, if for brevity we denote them or any of their functions by ξ^μ, then for the doubly invariant models the Poisson brackets of ξ^μ have the form

$$
\{\xi^\mu, \xi^\nu\} = C^{\mu\nu}(\xi).
$$

Therefore, in principle the system of equations

$$
\frac{\mathrm{d}\xi^\mu}{\mathrm{d}t} = \{\xi^\mu, H(\xi)\}
$$

is autonomously solvable. Then, performing the inverse Legendre transformation we can obtain (in principle) the time dependence of the rigid bodies angular velocities, $\hat{\chi}^a{}_b(t)$, $\hat{\vartheta}^a{}_b(t)$. Further, the time evolution of L, R may be found from the solution of the obvious time-dependent equations equivalent to the definition of quantities $\hat{\chi}$, $\hat{\vartheta}$

$$\frac{\mathrm{d}L}{\mathrm{d}t} = L\hat{\chi}, \qquad \frac{\mathrm{d}R}{\mathrm{d}t} = R\hat{\vartheta}.$$

So, in principle, the total system of equations of motion may be solved for the internal variables $\varphi(t)$.

In dimension $n = 2$, when $|N| > |M|$, then for $|M| > 2$ the singularity $q^1 - q^2 = 0$ is strongly repulsive, but at large distances $\left|q^1 - q^2\right|$ the second, attractive term of (48) prevails. Therefore, (48) has the characteristic shape of typical intermolecular forces. If $|N| < |M|$ the geometric potential (48) is purely repulsive. Therefore, the characteristic threshold typical for intermolecular forces occurs when $|N| = |M|$, the stable equilibrium when $|N| > |M|$, and the purely repulsive behavior for $|N| < |M|$. As mentioned, when $n > 2$, then $\widehat{\rho}$, $\widehat{\tau}$, and also M, N fail to be constants of motion in geodetic models. They depend on time, however the above general properties are still satisfied.

Acknowledgements

This paper partially contains results obtained within the framework of the research project N N501 049 540 financed from the Scientific Research Support Fund in the years 2011-2014. I appreciate to the Polish Ministry of Science and Higher Education for this financial support.

I am thankful to Professor Ivaïlo M. Mladenov and to all Organizers of the Sixteenth International Conference on Geometry, Integrability and Quantization Dedicated to my 70-th Anniversary, June 6-11, 2014, Sts. Constantine and Elena, Varna, Bulgaria, for their cordial hospitality.

I am also very deeply indebted to my younger colleagues, first of all to my former PhD students, for their co-working with me, their enthusiasm, and their permanent pressure on me to work, both myself and with them. First of all I have in mind Barbara Gołubowska, Vasyl Kovalchuk, Agnieszka Martens, and Ewa Eliza Rożko.

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