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ALGEBRAIC APPROACH TO THE MORSE OSCILLATORS

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Abstract. In this paper we obtain the ladder operators for the $1D$ and $3D$ Morse potential. Then we show that these operators satisfy $\mathrm{SU}(2)$ commutation relation. Finally we obtain the Hamiltonian in terms of the $\mathfrak{su}(2)$ algebra.

1. Introduction

In the recent years, Lie algebraic methods have been the subject of interest in many of fields of physics. For example the algebraic methods provide a way to obtain wave functions of polyatomic molecules [15, 16, 18, 20–22]. These methods provide a description to Dunham-type expansions and to force-field variational methods [17]. It is clear that systems displaying a dynamical symmetry can be treated by algebraic methods [1, 2, 19, 23]. For details concerning the ladder operators of quantum systems with some important potentials such as Morse potential the Pöschel-Teller one, the pseudo harmonic one, the infinitely square-well one and other quantum systems we refer to [3–13].

The Morse potential is a solvable potential, hence the interest to deal with it using different approaches, in particular factorization approach [1, 4, 19]. According to these methods as $\mathfrak{su}(1,1)$ algebra has been found in [4, 9, 19]. The Morse potential has been studied in terms of $\mathrm{SO}(2,1)$ and $\mathrm{SU}(2)$ groups [8, 13]. In fact $\mathrm{SU}(2)$ is the symmetry group associated with the bounded region of the spectrum [12].

In this paper we study the dynamical symmetry for the one and three-dimensional Morse oscillator by another algebraic approach. We establish the creation and annihilation operators directly from the eigenfunctions for this system, and that the ladders operators construct the dynamical algebra $\mathfrak{su}(2)$.

2. Algebraic Method in One-Dimensional Potential

We consider the Schrödinger equation with the Morse potential

$$\left(-\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_D(x)\right)\psi_n(x) = E_n\psi_n(x) \tag{1}$$

i.e.,

$$V_D = V_0(1 - \exp(-ax))^2 \tag{2}$$

and where V_0 and a are constants. By considering the following change of coordinates

$$r = 2\exp(-ax) \tag{3}$$

we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - (\frac{1}{4} - \frac{1}{r} - \frac{\varepsilon_n}{r^2})\right)\phi_n(r) = 0 \tag{4}$$

where $\varepsilon_n = \frac{E_n}{V_0}$ and $a^2 = \frac{2M V_0}{\hbar^2}$. From the behavior of the wave functions at the origin and at infinity, we can consider the following ansatz for $\phi_n(r)$

$$\phi_n(r) = N r^s \exp(-\frac{r}{2})\, {}_1F_1(-n + \frac{1}{2}, 2n, r) \tag{5}$$

in which

$$s = \sqrt{-\varepsilon_n} := n, \qquad \varepsilon_n = -n^2 \tag{6}$$

${}_1F_1(-n + \frac{1}{2}, 2n, r)$ is the hypergeometric function, and N is the normalization factor. From consideration of the finiteness of the wave function (5), it is shown that equation (6) that the general quantum condition is

$$-s + \frac{1}{2} = -m \tag{7}$$

and therefore we can write the wave function as

$$\phi_m(r) = N_m r^s \exp(-\frac{r}{2}) L_m^{2s}(r) \tag{8}$$

where the $L_m^{2s}(r)$ are the associated Laguerre polynomials. Here we have used the following relation between hypergeometric functions and Laguerre polynomials

$$L_m^{2s} = \frac{\Gamma(2s + m + 1)}{m!\,\Gamma(2s + 1)}\, {}_1F_1(-m, 2s + 1, r) \tag{9}$$

and in this way we can obtain the normalization factor N_m that is given by the formula

$$N_m = \sqrt{\frac{m!}{\Gamma(m + 2s + 1)}}. \tag{10}$$

Now we introduce the ladder operators in the form

$$\hat{L}_\pm \phi_m(r) = l_\pm \phi_{m\pm1}(r). \tag{11}$$

By considering the following ansatz for ladder operator

$$\hat{L}_{\pm} = A_{\pm}(r)\frac{\mathrm{d}}{\mathrm{d}r} + B_{\pm} \tag{12}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\,\phi_m(r) = \frac{s}{r}\,\phi_m(r) - \frac{1}{2}\,\phi_m(r) + \frac{m}{r}\,\phi_m(r) - \frac{m+2s}{r}\,\frac{N_m}{N_{m-1}}\,\phi_{m-1}(r). \tag{13}$$

Further on, we can rewrite the above equation as

$$\left(-r\frac{\mathrm{d}}{\mathrm{d}r} + (s+m) - \frac{r}{2}\right)\phi_m(r) = (m+2s)\frac{N_m}{N_{m-1}}\,\phi_{m-1}(r) \tag{14}$$

in order to obtain the operator

$$\hat{L}_{-} = -r\frac{\mathrm{d}}{\mathrm{d}r} + (s+m) - \frac{r}{2} \tag{15}$$

with eigenvalues

$$l_{-} = \sqrt{m(m+2s)}. \tag{16}$$

Similarly one can obtain

$$\hat{L}_{+} = r\frac{\mathrm{d}}{\mathrm{d}r} + (s+m+1) - \frac{r}{2}, \qquad l_{+} = \sqrt{(m+1)(m+2s+1)}. \tag{17}$$

Now, using relation (7) we have for the ladder operators

$$\hat{L}_{-} = -r\frac{\mathrm{d}}{\mathrm{d}r} + (2m+\frac{1}{2}) - \frac{r}{2}, \qquad \hat{L}_{+} = r\frac{\mathrm{d}}{\mathrm{d}r} + (2m+\frac{3}{2}) - \frac{r}{2} \tag{18}$$

with eigenvalues

$$l_{-} = \sqrt{m(3m+1)}, \qquad l_{+} = \sqrt{(m+1)(3m+2)}. \tag{19}$$

Now, we obtain the algebra associated with the operators $\hat{L}_{-}$, $\hat{L}_{+}$. Using equation (11, 16, 17) we can calculate their commutator

$$[\hat{L}_{-}, \hat{L}_{+}]\phi_m = 2(m+s+\frac{1}{2}). \tag{20}$$

Now, we define the operator $\hat{L}_0$ as

$$\hat{L}_0 = 2(\hat{m}+s+\frac{1}{2}) \tag{21}$$

where the operator $\hat{m}$ is defined by the following relations

$$\hat{m}\phi_m(r) = m\phi_m(r) \tag{22}$$

and therefore one can rewrite the eigenvalues of $\hat{L}_0$ as

$$l_0 = 2(m+s+\frac{1}{2}) \tag{23}$$

The operators $\hat{L}_\pm$, $\hat{L}_0$ satisfy the commutation relations of the Lie algebra of $\mathfrak{su}(2)$

$$[\hat{L}_-, \hat{L}_+] = 2\hat{L}_0, \qquad [\hat{L}_0, \hat{L}_-] = -\hat{L}_-, \qquad [\hat{L}_0, \hat{L}_+] = \hat{L}_+. \tag{24}$$

Then we notice that in terms of the $\mathfrak{su}(2)$ algebra the hamiltonian has following form

$$\hat{\mathcal{H}} = -\frac{V_0}{16}\hat{L}_0^2 \tag{25}$$

with eigenvalues

$$E_n = -\frac{V_0}{4}(2m+1)^2. \tag{26}$$

Using equations (6), (7) we can rewrite the eigenvalues as

$$E_n = -\frac{V_0}{4}(2s)^2 = -V_0 n^2 \tag{27}$$

which is consistent with the definition of E_n in equation (1), where $\varepsilon_n = \frac{E_n}{V_0}$, and ε_n given by equation (6).

3. Three-Dimensional Morse Potential

In this section we extend the algebraic approach of previous section to the three-dimensional Morse potential. The Morse potential in three-dimension is

$$V(r) = V_0(\exp(-2ar) - 2\exp(-ar)) \tag{28}$$

and therefore the radial part of the Hamiltonian given by

$$H = -\frac{\hbar^2}{2M}(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}) + V_0(\exp(-2ar) - 2\exp(-ar)). \tag{29}$$

Now by using $x = a\,r$ the above equation can be rewriten in the dimensionless form

$$H = -\frac{\hbar^2 a^2}{2M}(\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x}) + V_0(\exp(-2x) - 2\exp(-x)). \tag{30}$$

If in the Schrödinger equation

$$\hat{H}\Psi_m(x) = E_m\Psi_m(x) \tag{31}$$

we take $\frac{\hbar^2 a^2}{2M} = V_0$, $\frac{E_m}{V_0} = \varepsilon_m$, then we have

$$\left(-\left(\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x}\right) + \Big(\exp(-2x) - 2\exp(-x)\Big)\right)\Psi_m(x) = \varepsilon_m\Psi_m(x). \tag{32}$$

Introducing

$$\Psi_m(x) := \frac{\Phi_m(x)}{x} \tag{33}$$

we can rewrite equation (32) as

$$\left(-\frac{\partial^2}{\partial x^2} + (\exp(-2x) - 2\exp(-x)\right)\Phi_m(x) = \varepsilon_m\Phi_m(x) \tag{34}$$

which is the one-dimensional Morse potential problem from the previous section. The solutions of the above equation are

$$\Phi_m(\rho) = N_m \rho^s \exp(-\frac{\rho}{2})\, L_m^{2s}(\rho) \tag{35}$$

in which $\rho := 2\exp(-x)$ and N_m is again the normalization factor. Then we obtain the following relation for the functions $\Psi_m(\rho)$

$$\Psi_m(\rho) = N_m \frac{\rho^s \mathrm{e}^{-\frac{\rho}{2}}}{\ln\frac{2}{\rho}}\, L_m^{2s}(\rho). \tag{36}$$

Since $0 \leq x \leq +\infty$, then, $0 \leq \rho \leq 2$, we obtain the normalization factor from the formula

$$N_m^2 \int_0^2 \left(\frac{\rho^s \mathrm{e}^{-\frac{\rho}{2}}}{\ln\frac{2}{\rho}}\, L_m^{2s}(\rho)\right)^2 \mathrm{d}\rho = 1. \tag{37}$$

Now we want to obtain the ladder operators and for this purpose we use the relation given in [14]

$$\rho\,\frac{\mathrm{d}}{\mathrm{d}\rho} L_n^\alpha(\rho) = n\, L_n^\alpha(\rho) - (n+\alpha) L_{n-1}^\alpha(\rho) \tag{38}$$

$$\rho\,\frac{\mathrm{d}}{\mathrm{d}\rho} L_n^\alpha(\rho) = (n+1)\, L_{n+1}^\alpha(\rho) - (n+\alpha+1-\rho) L_n^\alpha(\rho) \tag{39}$$

where $L_n^\alpha(\rho)$ are the associated Laguerre function. By the action of the differential operator $\frac{\mathrm{d}}{\mathrm{d}\rho}$ on the wave functions (36) and using equation (38)

$$\left(\rho\,\frac{\mathrm{d}}{\mathrm{d}\rho} - (s+m) + \frac{\rho}{2} - \frac{1}{\ln\frac{2}{\rho}}\right) \Psi_m(\rho) = -(m+2s)\frac{N_m}{N_{m-1}} \Psi_{m-1}(\rho) \tag{40}$$

we can define the lowering operator

$$\hat{L_-} = -\rho\,\frac{\mathrm{d}}{\mathrm{d}\rho} + (s+m) - \frac{\rho}{2} + \frac{1}{\ln\frac{2}{\rho}} \tag{41}$$

with eigenvalues

$$l_- = (m+2s)\frac{N_m}{N_{m-1}}\cdot \tag{42}$$

Now we proceed to find the corresponding creation operators. Here we should make use of equation (39)

$$\left(\rho\,\frac{\mathrm{d}}{\mathrm{d}\rho} + (s+m+1) - \frac{\rho}{2} - \frac{1}{\ln\frac{2}{\rho}}\right) \Psi_m(\rho) = (m+1)\frac{N_m}{N_{m+1}} \Psi_{m+1}(\rho) \tag{43}$$

Then, we can define the operator

$$\hat{L_+} = \rho\,\frac{\mathrm{d}}{\mathrm{d}\rho} + (s+m+1) - \frac{\rho}{2} - \frac{1}{\ln\frac{2}{\rho}} \tag{44}$$

which has as eigenvalues

$$l_+ = (m+1)\frac{N_m}{N_{m+1}}\cdot \tag{45}$$

Now we investigate the algebra associated with the operators $\hat{L}_+$, $\hat{L}_-$. Based on the equations (40, 42) and(43, 45) we can calculate their commutator $[\hat{L}_-, \hat{L}_+]$

$$[\hat{L}_-, \hat{L}_+]\Psi_m(\rho) = 2(m+s+\frac{1}{2})\Psi_m(\rho) \tag{46}$$

suggesting to introduce the eigenvalues

$$l_0 = 2(m+s+\frac{1}{2}). \tag{47}$$

In this way, we can define the operator

$$\hat{L}_0 = (\hat{m}+s+\frac{1}{2}). \tag{48}$$

The operators $\hat{L}_+$, $\hat{L}_-$, $\hat{L}_0$ satisfy the commutation relations

$$[\hat{L}_-, \hat{L}_+] = 2\hat{L}_0, \qquad [\hat{L}_0, \hat{L}_-] = -\hat{L}_-, \qquad [\hat{L}_0, \hat{L}_+] = \hat{L}_+. \tag{49}$$

4. Conclusion

In this paper we have obtained the raising and lowering operators for the 1D and 3D Morse potentials. We have shown that $\mathrm{SU}(2)$ is the dynamical group associated with the bounded region of the spectrum. Also we have obtained the Hamiltonian and eigenvalues of the Hamiltonian in terms of the $\mathfrak{su}(2)$ algebra.

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