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ON THE JACOBIAN GROUP FOR MÖBIUS LADDER AND PRISM GRAPHS

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Abstract. The notion of the Jacobian group of graph (also known as Picard group, sandpile group, critical group) was independently given by many authors. This is a very important algebraic invariant of a finite graph. In particular, the order of the Jacobian group coincides with the number of spanning trees for a graph. The latter number is known for the simplest families of graphs such as Wheel, Fan, Prism, Ladder and Möbius ladder graphs. At the same time the structure of the Jacobian group is known only in several cases. The aim of this paper is to determine the structure of the Jacobian group of the Möbius ladder and Prism graphs.

1. Introduction

We define a Möbius ladder M_n of order n as the cubic circulant graph $C_{2n}(1, n)$ with $2n$ vertices. In this case, M_n can be considered as a regular $2n$-gon whose n pairs of opposite vertices are joint by an edge. One can also realize M_n as a ladder with n steps on the Möbius band. A **Prism graph** Pr_n, sometimes also called a **circular ladder graph**, is a cubic graph with $2n$ vertices, which are connected as shown in Fig. 1. Notice, that Prism graph Pr_{2n} is a double cover of Möbius ladder M_n. It is a discrete version of the statement that the cylinder is a double cover of the Möbius band.

The aim of the present paper is to find the structure of the Jacobian group of the Möbius ladder M_n and Prism graph Pr_n.

The notion of the Jacobian group of a graph (also known as Picard group, sandpile group, critical group) was independently given by many authors [1–3, 5]. This is a very important algebraic invariant of a finite graph. In particular, the order of the Jacobian group coinsides with the number of spanning trees for a graph. The latter number is known for the simplest families of graphs such as Wheel, Fan, Prism, Ladder and Möbius ladder graphs [4]. In the same time the structure of the Jacobian group is known only in a several cases (see [9] for references).

Following Baker-Norine [2] we define the Jacobian (or the Picard) group of a graph as follows.

Let G be a graph. Throughout this paper we suppose that G is finite, connected multigraph without loops. Let $V(G)$ and $E(G)$ be the sets of vertices and edges of G, respectively. Denote by $\mathrm{Div}(G)$ a free Abelian group on $V(G)$. We refer to elements of $\mathrm{Div}(G)$ as divisors on G. Each element $D \in \mathrm{Div}(G)$ can be uniquely presented as $D = \sum_{x \in V(G)} D(x)(x)$, $D(x) \in \mathbb{Z}$. We define the degree of D by the formula $\deg(D) = \sum_{x \in V(G)} D(x)$. Denote by $\mathrm{Div}^0(G)$ the subgroup of $\mathrm{Div}(G)$ consisting of divisors of degree zero.

Let f be a $\mathbb{Z}$-valued function on $V(G)$. We define the divisor of f by the formula

$$\mathrm{div}(f) = \sum_{x \in V(G)} \sum_{xy \in E(G)} (f(x) - f(y))(x).$$

The divisor $\mathrm{div}(f)$ can be naturally identified with the graph-theoretic Laplacian of f. Divisors of the form $\mathrm{div}(f)$, where f is as above are called principal divisors. Denote by $\mathrm{Prin}(G)$ the group of principal divisors of G. It is easy to see that every principal divisor has a degree zero, so that $\mathrm{Prin}(G)$ is a subgroup of $\mathrm{Div}^0(G)$.

The **Jacobian group** of G is defined to be quotient group

$$\mathrm{Jac}(G) = \mathrm{Div}^0(G)/\mathrm{Prin}(G).$$

By making use of the Kirchhoff Matrix-Tree Theorem [8] one can show that $\mathrm{Jac}(G)$ is a finite Abelian group of order $\tau(G)$, where $\tau(G)$ is number of spanning trees of G. Moreover, any finite Abelian group is the Jacobian group of some graph.

For a fixed base point $x_0 \in V(G)$ we define the Abel-Jacobi map $S_{x_0} : G \to \mathrm{Jac}(G)$ by the formula $S_{x_0}(x) = [(x) - (x_0)]$, where $[d]$ is an equivalence class of divisor d in $\mathrm{Jac}(G)$.

We endow each edge of G by two possible orientations. Since G has no loops it is well-defined procedure. Let $\vec{E} = \vec{E}(G)$ be the set of oriented edges of G. For $e \in \vec{E}$ we denote initial vertex $o(e)$ and terminus vertex $t(e)$, respectively. We define the flow of e by the formula $\omega(e) = [t(e) - o(e)]$. We note that

$$\omega(e) = [(t(e) - x_0) - (o(e) - x_0)] = [[t(e) - x_0] - [o(e) - x_0]] = S_{x_0}(t(e)) - S_{x_0}(o(e))$$

does not depend of the choice of initial point x_0. By virtue of Lemma 1.6 in [2] (see also [1]) the Jacobian group $\mathrm{Jac}(G)$ is an Abelian group generated by flows

$\omega(e), e \in \vec{E}$, whose defining relations are given by the following two Kirchhoff laws.

I) The flow through each vertex of G is equal to zero. It means that

$$\sum_{e\in\vec{E},t(e)=x} \omega(e) = 0, \qquad \text{for all} \qquad x \in V(G).$$

II) The flow along each closed orientable walk W in G is equal to zero. That is

$$\sum_{e\in W} \omega(e) = 0.$$

Recall that the closed orientable walk in G is a sequence of orientable edges $e_i \in \vec{E}(G)$, $i = 1, \ldots, n$ such that $t(e_i) = o(e_{i+1})$ for $i = 1, \ldots, n-1$ and $t(e_n) = o(e_1)$.

2. Preliminary

Let $a_1, a_2, \ldots, a_m \in \mathbb{Z}$. Denote by GCD $(a_1, a_2, \ldots, a_m) = (a_1, a_2, \ldots, a_m)$ the greatest common divisor of $a_1, a_2, \ldots, a_m$ in the ring of integers $\mathbb{Z}$. We will use the following evident properties of GCD.

i) $(a, a+b) = (a, b) = (a, a-b)$

ii) $(a, b, c) = (a, (b, c))$

iii) $(k\,a, k\,b) = k(a, b)$.

The **Chebyshev polynomials** of the first and of the second kind are defined by the formulas

$$T_n(x) = \cos(n \arccos(x)), \qquad U_{n-1}(x) = \sin(n \arccos(x))/\sin(\arccos(x)).$$

respectively. Recall the following basic properties of these polynomials

$1^\circ \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \qquad T_0(x) = 1, \qquad T_1(x) = x$

$2^\circ \quad U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \qquad U_0(x) = 1, \qquad U_1(x) = 2x.$

In this paper we will be mainly interesting in particular values of Chebyshev polynomials at the point $x = 2$. In this case $T_n(2) = ((2+\sqrt{3})^n + (2-\sqrt{3})^n)/2$ and $U_{n-1}(2) = ((2+\sqrt{3})^n - (2-\sqrt{3})^n)/(2\sqrt{3})$.

We will use the following version of the fundamental theorem of finite Abelian group (see for instance [7], p. 344).

Theorem 1. *Let $\mathcal{A}$ be a finite Abelian group generated by $x_1, x_2, \ldots, x_n$ and satisfying the system of relations*

$$\sum_{j=1}^{n} a_{ij}x_j = 0, \qquad i = 1, \ldots, m$$

where $A = \{a_{ij}\}$ is an integer $m \times n$ matrix. Set $B_j,\ j = 1,\ \dots,\ r = \min(n,\ m)$, for the greatest common divisor of all $j \times j$ minors of A. Then,

$$\mathcal{A} \cong \mathbb{Z}_{B_1} \oplus \mathbb{Z}_{B_2/B_1} \oplus \mathbb{Z}_{B_3/B_2} \oplus \cdots \oplus \mathbb{Z}_{B_r/B_{r-1}}.$$

The latter decomposition is known as the Smith Normal Form (see [10], Ch. 3.22 for details of calculations).

3. Main Result

3.1. Calculation of the Structure of Jacobian Group for Prism Graph

Consider the Prism graph Pr_n as graph shown on the Fig.1 with vertices labeled by $1, 2, \dots, n, n+1, , \dots, 2n$. Denote by $d_i, i = 1, \dots, n$ the flow along orientable edge $(i, i+n)$ with initial vertex i and terminal vertex $i+n$. We also denote by X_i and x_i the flows along orientable edges $(i, i+1)$ and $(i+n, i+n+1)$, respectively.

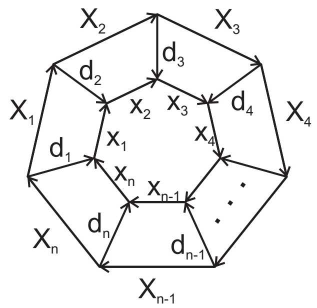

Figure 1. Prism graph Pr_n.

By the first Kirchhoff law we have the following equations

$$\begin{aligned} d_1 &= x_1 - x_n, & d_i &= x_i - x_{i-1}, & i &= 2, \dots, n \\ d_1 &= X_n - X_1, & d_i &= X_{i-1} - X_i, & i &= 2, \dots, n. \end{aligned} \tag{1}$$

Applying the second Kirchhoff's law for the closed walks $W_i = (i, n+i, n+i+1, i+1)$, we get the following system of equations

$$\begin{aligned} d_i + x_i - d_{i+1} - X_i &= 0, & i &= 1, \dots, n-1 \\ d_n + x_n - d_1 - X_n &= 0. \end{aligned} \tag{2}$$

Excluding d_i, $i = 1, \dots, n$, from first line of system (1) and putting them in (2) we get the following relations between X_i and x_i

$$\begin{pmatrix} X_1 \\ X_2 \\ \dots \\ X_{n-1} \\ X_n \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & 0 & \dots & 0 & -1 \\ -1 & 3 & -1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & -1 & 3 & -1 \\ -1 & 0 & \dots & 0 & 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_{n-1} \\ x_n \end{pmatrix}. \tag{3}$$

Substituting these identities into expressions $x_1 - x_n = X_n - X_1,\ x_i - x_{i-1} = X_{i-1} - X_i,\ i = 2, \dots, n-1$ of system (1)we obtain $n-1$ equations of x_i, $i = 1, \dots, n$. Notice that by the second Kirchhoff law for the closed walk $(n+1, \dots, 2n)$ we have the equation

$$\sum_{i=1}^{n} x_i = 0.$$

Thus the Jacobian $\mathrm{Jac}(\mathrm{Pr}_n)$ is an Abelian group generated by $x_1, x_2, \dots, x_n$ satisfying the relations

$$\begin{pmatrix} 1 & -5 & 5 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & -5 & 5 & -1 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & -5 & 5 & -1 \\ 5 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & -5 \\ -5 & 5 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix} = 0. \tag{4}$$

Now we reduce the number of generators of the group $\mathrm{Jac}(\mathrm{Pr}_n)$ from n to 3. Namely, we will show that the group $\mathrm{Jac}(\mathrm{Pr}_n)$ is generated by x_1, x_2, x_3 satisfying three linear equations. To do this we note that the generators $x_1, x_2, \dots, x_n$ satisfy the following recursive relation

$$x_j - 5x_{j+1} + 5x_{j+2} - x_{j+3} = 0, \qquad j = 1, 2, \dots, n-3.$$

The characteristic polynomial for the above equation is

$$1 - 5q + 5q^2 - q^3 = 0.$$

The roots of this polynomial are $q_1 = 1$, $q_{2,3} = 2 \pm \sqrt{3}$. Hence, the general solution of recursion is given by $x_j = C_1 + C_2 q^j + C_3 q^{-j}$, where $q = 2 + \sqrt{3}$ and C_1, C_2, C_3 are constants dependant only of initial values x_1, x_2, x_3. As a result, we obtained $x_4, x_5, \dots, x_n$ as linear combinations of x_1, x_2 and x_3 whose coefficients can be found explicitly. Substituting the obtained equations into the last three lines of system (4) we get

$$\begin{aligned}\widetilde{a}_{11}x_1+\widetilde{a}_{12}x_2+\widetilde{a}_{13}x_3&=0\\ \widetilde{a}_{21}x_1+\widetilde{a}_{22}x_2+\widetilde{a}_{23}x_3&=0\\ \widetilde{a}_{31}x_1+\widetilde{a}_{32}x_2+\widetilde{a}_{33}x_3&=0.\end{aligned}\tag{5}$$

We note that $T_n(2)=(q^n+q^{-n})/2$ and $U_{n-1}(2)=(q^n-q^{-n})/(2\sqrt{3})$. By straightforward calculations we obtain the following explicit formulae for $\widetilde{a}_{ij},\ i,j=1,2,3$.

$$\begin{aligned}
&\widetilde{a}_{11}=-7L+12U, && \widetilde{a}_{12}=9L-15U, && \widetilde{a}_{13}=-2L+3U\\
&\widetilde{a}_{21}=\frac{11}{2}L-\frac{19}{2}U, && \widetilde{a}_{22}=-7L+12U, && \widetilde{a}_{23}=\frac{3}{2}L-\frac{5}{2}U\\
&\widetilde{a}_{31}=n-2L+\frac{7}{2}U-\frac{n}{2}, && \widetilde{a}_{32}=\frac{5}{2}L-\frac{9}{2}U+2n, && \widetilde{a}_{33}=-\frac{1}{2}L+U-\frac{n}{2}
\end{aligned}$$

where $L=T_n(2)-1$ and $U=U_{n-1}(2)$.
Now we are able to proof the following lemma.

Lemma 2. *Let B_1 be the greatest common divisor of $\widetilde{a}_{ij},\ i,j=1,2,3$. Then*

$$B_1=\mathrm{GCD}(n,L,U)/\mathrm{GCD}(2,n).$$

Proof: We have

$$\begin{aligned}
B_1&=\mathrm{GCD}(\widetilde{a}_{ij})=\mathrm{GCD}(\widetilde{a}_{11},\widetilde{a}_{12}-5\widetilde{a}_{13},\widetilde{a}_{13},\widetilde{a}_{21}-4\widetilde{a}_{23},\widetilde{a}_{22},\widetilde{a}_{23},\widetilde{a}_{31},\widetilde{a}_{32},\widetilde{a}_{33})\\
&=\mathrm{GCD}(\widetilde{a}_{11},-L,\widetilde{a}_{13},-\frac{1}{2}L-\frac{1}{2}U,\widetilde{a}_{23},\widetilde{a}_{31},\widetilde{a}_{32},\widetilde{a}_{33})\\
&=\mathrm{GCD}(-L,3U,-\frac{L+U}{2},-4U,\ \frac{7U-n}{2},\ \frac{L-9U}{2}+2n,\ -\frac{L+n}{2}+U)\\
&=\mathrm{GCD}(L,U,-\frac{L+U}{2},\ \frac{U-n}{2},\frac{L-U}{2}+2n,-\frac{L+n}{2})\\
&=\mathrm{GCD}(L,U,-\frac{L+U}{2},\ \frac{U-n}{2}-\frac{L+n}{2},\ \frac{L-U}{2}+2n,-\frac{L+n}{2})\\
&=\mathrm{GCD}(L,U,-\frac{L+U}{2},\ -\frac{L-U}{2}-n,\ \frac{L-U}{2}+2n,-\frac{L+n}{2})\\
&=\mathrm{GCD}(L,U,n,\frac{L+U}{2},\ \frac{L-U}{2},\ \frac{L+n}{2})\\
&=\mathrm{GCD}(L,U,n,\frac{U+n}{2},\ \frac{L+n}{2}).
\end{aligned}$$

From the main recursive relations for the Chebyshev polynomials 1° and 2° we have the following properties. The numbers $L=T_n(2)-1$ and $U=U_{n-1}(2)$ are of the same parity as n.

Let us consider two cases n is odd and n is even. In the first case, we have

$$\begin{aligned} B_1 &= \mathrm{GCD}(L, U, n, \frac{U+n}{2}, \frac{L+n}{2}) = \mathrm{GCD}(L, U, n, U+n, L+n) \\ &= \mathrm{GCD}(L, U, n) = \mathrm{GCD}(n, L, U)/\mathrm{GCD}(2, n). \end{aligned}$$

In the second case, n is even. Using properties of the Chebyshev polynomials 1° and 2° we obtain

$$\begin{aligned} B_1 &= \mathrm{GCD}(L, U, n, \frac{U+n}{2}, \frac{L+n}{2}) \\ &= \mathrm{GCD}(n, L, U)/2 = \mathrm{GCD}(n, L, U)/\mathrm{GCD}(2, n). \end{aligned}$$

■

Now our aim is to find the greatest common divisor of two-by-two minors of matrix $\widetilde{A} = \{\widetilde{a}_{ij}\}_{i,j=1,2,3}$. Denote by m_{ij} the two-by-two minor of $\widetilde{A}$ obtained by removing i-th row and j-th column of $\widetilde{A}$. By direct calculations we have

$$\begin{aligned} m_{11} &= \frac{1}{2}(n+1)L - nU, & m_{12} &= -(\frac{1}{2}+2n)L + \frac{7n}{2}U \\ m_{13} &= \frac{1}{2}(1+15n)L - 13nU & & \\ m_{21} &= -(\frac{n}{2}+1)L + \frac{3n}{2}U, & m_{22} &= (\frac{5n}{2}+1)L - \frac{9n}{2}U \\ m_{23} &= -(\frac{19n}{2}+1)L + \frac{33n}{2}U, & m_{31} &= -m_{32} = m_{33} = L. \end{aligned}$$

We assert that the following lemma is true.

Lemma 3. *Let B_2 be the greatest common divisor of m_{ij}, $i, j = 1, 2, 3$. Then*

$$B_2 = \mathrm{GCD}(L, nU)/\mathrm{GCD}(2, n).$$

Proof: By virtue of explicit formulae for m_{ij} we have

$$\begin{aligned} B_2 &= \mathrm{GCD}(m_{11},\, m_{12},\, m_{13},\, m_{21},\, m_{22},\, m_{23},\, m_{31}) \\ &= \mathrm{GCD}(m_{11},\, m_{12} + 2nm_{31},\, m_{13} - 7nm_{31},\, m_{21} + m_{31}, \\ &\quad m_{22} - (2n+1)m_{31},\, m_{23} + (9n+1)m_{31},\, m_{31}) \\ &= \mathrm{GCD}(\frac{n+1}{2}L - nU,\, \frac{7nU - L}{2}, \frac{n+1}{2}L - 13nU,\, -\frac{n}{2}L + \frac{3n}{2}U, \\ &\quad \frac{n}{2}L - \frac{9n}{2}U,\, -\frac{n}{2}L + \frac{33n}{2}U,\, L) \end{aligned}$$

$$\begin{aligned}
&= \mathrm{GCD}(\frac{n+1}{2}L - nU,\ \frac{7nU-L}{2},\ -12nU,\ \frac{3nU-nL}{2},\ -3nU,\ 15nU,\ L)\\
&= \mathrm{GCD}(4nU,\ \frac{7nU-L}{2},\ \frac{3nU-nL}{2},\ -3nU,\ L)\\
&= \mathrm{GCD}(nU,\ \frac{7nU-L}{2},\ \frac{3nU-nL}{2},\ L)\\
&= \mathrm{GCD}(nU,\ \frac{nU-L}{2},\ -\frac{n}{2}L + \frac{n}{2}U,\ L).
\end{aligned}$$

Now, let us consider the case $n = 2m$ is even. Then

$$\begin{aligned}
B_2 &= \mathrm{GCD}(nU,\ \frac{nU-L}{2},\ -\frac{n}{2}L + \frac{n}{2}U,\ L)\\
&= \mathrm{GCD}(2mU,\ \frac{2mU-L}{2},\ -mL + mU,\ L)\\
&= \mathrm{GCD}(2mU,\ \frac{2mU-L}{2},\ mU,\ L)\\
&= \mathrm{GCD}(\frac{1}{2}L,\ mU) = \mathrm{GCD}(\frac{1}{2}L,\ \frac{2m}{2}U)\\
&= \mathrm{GCD}(L,\ nU)/2 = \mathrm{GCD}(L,\ nU)/\mathrm{GCD}(2,n).
\end{aligned}$$

On the other hand, if $n = 2m+1$ is odd, then both L and U are odd. We have

$$\begin{aligned}
B_2 &= \mathrm{GCD}(nU,\ -\frac{1}{2}L + \frac{n}{2}U,\ -\frac{n}{2}L + \frac{n}{2}U,\ L)\\
&= \mathrm{GCD}((2m+1)U,\ -\frac{1}{2}L + \frac{2m+1}{2}U,\ -\frac{2m+1}{2}L + \frac{2m+1}{2}U,\ L)\\
&= \mathrm{GCD}((2m+1)U,\ -L + (2m+1)U,\ -(2m+1)L + (2m+1)U,\ L)\\
&= \mathrm{GCD}((2m+1)U,\ L) = \mathrm{GCD}(nU,\ L).
\end{aligned}$$

■

Let B_3 be the determinant of matrix $\{\widetilde{a}_{ij}\}_{i,j=1,2,3}$. By the Kirchhoff Matrix-Tree Theorem B_3 coincides with the number of spanning trees of Prism graph Pr_n. This number is well known and was calculated independently by many authors (J. Sedláček, J. Moon, N. Biggs and others) [4]. We represent the result as follows.

Lemma 4. *Let B_3 be the determinant of matrix $\{\widetilde{a}_{ij}\}_{i,j=1,2,3}$. Then B_3 is given by the formula*

$$B_3 = nL$$

where $L = T_n(2) - 1$ and $T_n(2) = ((2+\sqrt{3})^n + (2-\sqrt{3})^n)/2$ is the Chebyshev polynomial of the first kind.

By the fundamental theorem of finite Abelian groups (Theorem 1) we have the following decomposition for the Jacobian group of Pr_n

$$\mathrm{Jac}(\mathrm{Pr}_n) = \mathbb{Z}_{B_1} \oplus \mathbb{Z}_{B_2/B_1} \oplus \mathbb{Z}_{B_3/B_2}.$$

Taking into account Lemma 2, Lemma 3 and Lemma 4, we have the following theorem

Theorem 5. *The Jacobian group of Prism graph* Pr_n *has the following presentation*

$$\mathrm{Jac}(\mathrm{Pr}_n) = \mathbb{Z}_{\frac{(n,L,U)}{(2,n)}} \oplus \mathbb{Z}_{\frac{(L,nU)}{(n,L,U)}} \oplus \mathbb{Z}_{\frac{(2,n)nL}{(L,nU)}} \tag{6}$$

where $(l,m,n) = \mathrm{GCD}(l,m,n)$, $L = T_n(2) - 1$, $U = U_{n-1}(2)$ *and* $T_n(2) = ((2+\sqrt{3})^n + (2-\sqrt{3})^n)/2$, $U_{n-1}(2) = ((2+\sqrt{3})^n + (2-\sqrt{3})^n)/(2\sqrt{3})$ *are Chebyshev polynomials of the first and second kind, respectively.*

4. The Structure of Jacobian Group for Möbius Ladder

For details of calculations of the structure of Jacobian group for Möbius ladder one can see our paper [11].

Theorem 6. *The Jacobian group of Möbius ladder* M_n *has the following presentation*

$$\mathrm{Jac}(M_n) = \mathbb{Z}_{\frac{(n,T,U)}{(2,n)}} \oplus \mathbb{Z}_{\frac{(T,nU)}{(n,T,U)}} \oplus \mathbb{Z}_{\frac{(2,n)nT}{(T,nU)}} \tag{7}$$

where $(l,m,n) = \mathrm{GCD}(l,m,n)$, $T = T_n(2) + 1$, $U = U_{n-1}(2)$ *and*
$T_n(2) = ((2+\sqrt{3})^n + (2-\sqrt{3})^n)/2$, $U_{n-1}(2) = ((2+\sqrt{3})^n + (2-\sqrt{3})^n)/(2\sqrt{3})$ *are Chebyshev polynomials of the first and second kind, respectively.*

Let us notice that $L = T_n(2) - 1$ and $T = T_n(2) + 1$. Formula (7) is obtained from formula (6) by replacing L with T. This is related to topological fact that Prism graph Pr_{2n} is a double cover of Möbius ladder M_n.

We note that the structure of the Jacobian groups $\mathrm{Jac}(\mathrm{Pr}_n)$ and $\mathrm{Jac}(M_n)$ were independently calculated in [6] and [12], respectively. It was done by completely different methods.

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