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SYMMETRIES OF HAMILTONIAN DYNAMICAL SYSTEMS, MOMENTUM MAPS AND REDUCTIONS

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Abstract. This text presents some basic notions in symplectic geometry, Poisson geometry, Hamiltonian systems, Lie algebras and Lie groups actions on symplectic or Poisson manifolds, momentum maps and their use for the reduction of Hamiltonian systems. It should be accessible to readers with a general knowledge of basic notions in differential geometry. Full proofs of many results are provided.

CONTENTS

1. Introduction

This text presents some basic notions in symplectic geometry, Poisson geometry, Hamiltonian systems, Lie algebras and Lie groups actions on symplectic or Poisson manifolds, momentum maps and their use for the reduction of Hamiltonian systems. It should be accessible to readers with a general knowledge of basic notions in differential geometry. Full proofs of many results are provided.

Of course this text is just an introduction. To extend his knowledge of the subject, the reader can consult the books by Abraham and Marsden [1], Arnold [3], Arnold and Khesin [4], Iglesias-Zemmour [17], Laurent-Gengoux, Pichereau and Vanhaecke [25], Kosmann-Schwarzbach (editor) [22] and Vaisman [42] on both the scientific and historical aspects of the development of modern Poisson geometry.

Our notations are those which today are generally used in differential geometry. For example, the tangent and the cotangent bundles to a smooth manifold M are denoted, respectively, by TM and by T^*M, and their canonical projections on M by $\tau_M : TM \to M$ and by $\pi_M : T^*M \to M$.

The spaces of smooth differential forms of degree p on a smooth manifold M is denoted by $\Omega^p(M)$, and the space of smooth multivectorr fields of degree p (that means the space of smooth sections of $\bigwedge^p TM$, the p-th external power of TM) by $A^p(M)$. The exterior algebras of smooth differential forms and of smooth multivector fields are $\Omega(M) = \oplus_{p\in\mathbb{Z}}\Omega^p(M)$ and $A(M) = \oplus_{p\in\mathbb{Z}}A^p(M)$, respectively, with the convention that $\Omega^0(M) = A^0(M) = C^\infty(M,\mathbb{R})$ (the space of smooth real functions on M), and that $\Omega^p(M) = 0$ and $A^p(M) = 0$ for $p < 0$ and for $p > \dim M$.

When $f : M \to N$ is a smooth map between two smooth manifolds M and N, the natural lift of f to the tangent bundles is denoted by $Tf : TM \to TN$. The same notation $Tf : \bigwedge^p TM \to \bigwedge^p TN$ is used to denote its natural prolongation to the p-th exterior power of TM. The pull-back by f of a smooth differential form $\alpha \in \Omega(N)$ is denoted by $f^*\alpha$.

When $f : M \to N$ is a smooth diffeomorphism, the push-forward f_*X of a a smooth vector field $X \in A^1(M)$ is the vector field $f_*X \in A^1(N)$ defined by

$$f_*X(y) = Tf\Bigl(X\bigl(f^{-1}(y)\bigr)\Bigr), \qquad y \in N$$

Similarly, the pull-back of a smooth vector field $Y \in A^1(N)$ is the vector field $f^*Y \in A^1(N)$ defined by

$$f^*Y(x) = Tf^{-1}\Big(Y\big(f(x)\big)\Big), \qquad x \in N.$$

The same notation is used for the push-forward of any smooth tensor field on M and the pull-back of any smooth tensor field on N.

2. Symplectic Manifolds

2.1. Definition and Elementary Properties

Definition 1. A **symplectic form** on a smooth manifold M is a bilinear skew-symmetric differential form ω on that manifold which satisfies the following two properties:

- the form ω is closed which it means that its exterior differential $\mathrm{d}\omega$ vanishes, i.e., $\mathrm{d}\omega = 0$
- the rank of ω is everywhere equal to the dimension of M which it means that for each point $x \in M$ and each vector $v \in T_xM$, $v \neq 0$, there exists another vector $w \in T_xM$ such that $\omega(x)(v, w) \neq 0$.

Equipped with the symplectic form ω, the manifold M is called a **symplectic manifold** and denoted (M, ω). One says also that ω determines a **symplectic structure** on the manifold M.

2.1.1. Elementary Properties of Symplectic Manifolds

Let (M, ω) be a symplectic manifold.

1. For each $x \in M$ and each $v \in T_xM$ we denote by $\iota(v)\omega : T_xM \to \mathbb{R}$ the map $w \mapsto \omega(x)(v, w)$, which is a linear form on the vector space T_xM, in other words an element of the cotangent space T_x^*M. Saying that the rank of ω is everywhere equal to the dimension of M amounts to say that the map $v \mapsto \iota(v)\omega$ is an isomorphism of the tangent bundle TM onto the cotangent bundle T^*M.

2. Let V be a finite-dimensional vector space, and $\eta : V \times V \to \mathbb{R}$ be a skew-symmetric bilinear form. As above, $v \mapsto \iota(v)\eta$ is a linear map defined on V, with values in its dual space V^*. The *rank* of η is the dimension of the image of that map. An easy result in linear algebra is that the rank of a skew-symmetric bilinear form is always an even integer. When (M, ω) is a symplectic manifold, for each $x \in M$ that result can be applied to the bilinear form $\omega(x) : T_xM \times T_xM \to \mathbb{R}$, and we see that the dimension of M must be an even integer $2n$.

3. The **Darboux theorem**, due to the French mathematician Gaston Darboux, states that any point in a $2n$-dimensional symplectic manifold (M, ω) has a neighbourhood on which there exists a system of local coordinates $(x^1, \ldots, x^{2n})$ in

which the $(2n) \times (2n)$-matrix $(\omega_{i\,j})$ $(1 \leq i,\ j \leq 2n)$ of components of ω is a constant, skew-symmetric invertible matrix. We recall that

$$\omega_{i\,j} = \omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

These local coordinates can even be chosen in such a way that

$$\omega_{ij} = \begin{cases} 1 & \text{if} \quad i-j \quad = n \\ -1 & \text{if} \quad i-j \quad = -n, \\ 0 & \text{if} \quad |i-j| \neq n \end{cases} \qquad 1 \leq i,\ j \leq 2n.$$

Local coordinates which satisfy this property are called *Darboux local coordinates*.

4. On the $2n$-dimensional symplectic manifold (M, ω), the $2n$-form ω^n (the n-th exterior power of ω) is a **volume form** (it means that it is everywhere $\neq 0$). Therefore a symplectic manifold always is *orientable*.

2.2. Examples of Symplectic Manifolds

2.2.1. Surfaces

A smooth orientable surface embedded in an Euclidean three-dimensional affine space, endowed with the area form determined by the Euclidean metric, is a symplectic manifold.

More generally, any two-dimensional orientable manifold, equipped with a nowhere vanishing area form, is a symplectic manifold.

2.2.2. Symplectic Vector Spaces

A **symplectic vector space** is a finite-dimensional real vector space E equipped with a skew-symmetric bilinear form $\omega : E \times E \to \mathbb{R}$ which is of rank equal to the dimension of E and therefore $\dim E$ is an even integer $2n$. Considered as a constant differential two-form on E, η is symplectic, which allows us to consider (E, η) as a symplectic manifold.

The canonical example of a symplectic vector space is the following. Let V be a real n-dimensional vector space and let V^* be its dual space. There exists on the direct sum $V \oplus V^*$ a natural skew-symmetric bilinear form

$$\eta\big((x_1, \zeta_1), (x_2, \zeta_2)\big) = \langle \zeta_1, x_2 \rangle - \langle \zeta_2, x_1 \rangle.$$

The rank of η being $2n$, $(V \oplus V^*, \eta)$ is a symplectic vector space.

Conversely, any $2n$-dimensional symplectic vector space (E, ω) can be identified with the direct sum of any of its n-dimensional vector subspaces V such that the symplectic form ω vanishes identically on $V \times V$, with its dual space V^*. In this identification, the symplectic form ω on E becomes identified with the above-defined symplectic form η on $V \oplus V^*$.

2.2.3. Cotangent Bundles

Let N be a smooth n-dimensional manifold. We denote by $\tau_N : TN \to N$ and by $\pi_N : T^*N \to N$ the canonical projections, respectively of the tangent bundle TN and of the cotangent bundle T^*N onto their common base N, by $\tau_{T^*N} : T(T^*N) \to T^*N$ the canonical projection of the tangent bundle $T(T^*N)$ onto its base T^*N, and by $T\pi_N : T(T^*N) \to TN$ the prolongation to vectors of the canonical projection $\pi_N : T^*N \to N$. We recall that the diagram

$$\begin{tikzcd} T(T^*N) \arrow[r, "T\pi_N"] \arrow[d, "\tau_{T^*N}"'] & TN \arrow[d, "\tau_N"] \\ T^*N \arrow[r, "\pi_N"] & N \end{tikzcd}$$

is commutative. For each $w \in T(T^*N)$, we can therefore write

$$\eta_N(w) = \bigl\langle \tau_{T^*N}(w), T\pi_N(w) \bigr\rangle .$$

This formula defines a differential one-form η_N on the manifold T^*N, called the **Liouville one-form**. Its exterior differential $\mathrm{d}\eta_N$ is a symplectic form, called the *canonical symplectic form* on the cotangent bundle T^*N.

Let $(x^1, \dots, x^n)$ be a system of local coordinates on the smooth manifold N, and $(x^1, \dots, x^n, p_1, \dots, p_n)$ be the corresponding system of local coordinates on T^*N. The local expressions of the Liouville form η_N and of its exterior differential $\mathrm{d}\eta_N$ are

$$\eta_N = \sum_{i=1}^{n} p_i \,\mathrm{d}x^i , \qquad \mathrm{d}\eta_N = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}x^i .$$

We see that $(x^1, \dots, x^n, p_1, \dots, p_n)$ is a system of Darboux local coordinates. Therefore any symplectic manifold is locally isomorphic to a cotangent bundle.

2.2.4. The Complex Plane

The complex plane $\mathbb{C}$ is naturally endowed with a Hermitian form

$$\eta(z_1, z_2) = z_1 \overline{z}_2 , \qquad z_1 \text{ and } z_2 \in \mathbb{C}$$

where $\overline{z}_2$ is the conjugate of the complex number z_2. Let us write $z_1 = x_1 + \mathrm{i}y_1$, $z_2 = x_2 + \mathrm{i}y_2$, where $x_1,\ y_1,\ x_2,\ y_2$ are real, and separate the real and imaginary parts of $\eta(z_1, z_2)$. We get

$$\eta(z_1, z_2) = (x_1x_2 + y_1y_2) + \mathrm{i}(y_1x_2 - y_2x_1) .$$

The complex plane $\mathbb{C}$ has an underlying structure of real, two-dimensional vector space, which can be identified with $\mathbb{R}^2$, each complex number $z = x + \mathrm{i}y \in \mathbb{C}$ being identified with $(x, y) \in \mathbb{R}^2$. The real and imaginary parts of the Hermitian form η on $\mathbb{C}$ are, respectively, the Euclidean scalar product g and the symplectic form ω on $\mathbb{R}^2$ such that

$$\begin{aligned}\eta(z_1, z_2) &= (x_1x_2 + y_1y_2) + \mathrm{i}(y_1x_2 - y_2x_1)\\ &= g\big((x_1, y_1), (x_2, y_2)\big) + \mathrm{i}\omega\big((x_1, y_1), (x_2, y_2)\big).\end{aligned}$$

2.2.5. Kähler Manifolds.

More generally, a n-dimensional **Kähler manifold** (i.e., a complex manifold of complex dimension n endowed with a Hermitian form whose imaginary part is a closed two-form), when considered as a real $2n$-dimensional manifold, is automatically endowed with a Riemannian metric and a symplectic form given, respectively, by the real and the imaginary parts of the Hermitian form.

Conversely, it is not always possible to endow a symplectic manifold with a complex structure and a Hermitian form of which the given symplectic form is the imaginary part. However, it is always possible to define, on a symplectic manifold, an *almost complex structure* and an *almost complex two-form* with which the properties of the symplectic manifold become similar to those of a Kähler manifold (but with change of chart functions which are not holomorphic functions).

This possibility was used by *Gromov* [12] in his theory of *pseudo-holomorphic curves*.

2.3. Remarkable Submanifolds of a Symplectic Manifold

Definitions 1. Let (V, ω) be a symplectic vector space, and W be a vector subspace of V. The **symplectic orthogonal** of W is the vector subspace

$$\operatorname{orth} W = \{\, v \in V \,;\, \omega(v, w) = 0 \text{ for all } w \in W \,\}.$$

The vector subspace W is said to be

- *isotropic* if $W \subset \operatorname{orth} W$
- *coisotropic* if $W \supset \operatorname{orth} w$
- *Lagrangian* if $W = \operatorname{orth} W$
- *symplectic* if $W \oplus \operatorname{orth} W = V$.

Exercises 1. 1. For any vector subspace W of the symplectic vector space (W, ω), we have $\operatorname{orth}(\operatorname{orth} W) = W$.

2. Let $\dim V = 2n$. For any vector subspace W of V, we have $\dim(\operatorname{orth} W) = \dim V - \dim W = 2n - \dim W$.

Therefore, if W is isotropic, $\dim W \leq n$; if W is coisotropic, $\dim W \geq n$; and if W is Lagrangian, $\dim W = n$.

3. Let W be an isotropic vector subspace of V. The restriction to $W \times W$ of the symplectic form ω vanishes identically. Conversely, if W is a vector subspace such that the restriction of ω to $W \times W$ vanishes identically, W is isotropic.

4. A Lagrangian vector subspace of V is an isotropic subspace whose dimension is the highest possible, equal to half the dimension of V.

5. Let W be a symplectic vector subspace of V. Since $W \cap \operatorname{orth} W = \{0\}$, the rank of the restriction to $W \times W$ of the form ω is equal to $\dim W$; therefore $\dim W$ is even, and, equipped with the restriction of ω, W is a symplectic vector space. Conversely if, when equipped with the restriction of ω, a vector subspace W of V is a symplectic vector space, we have $W \oplus \operatorname{orth} W = V$, and W is a symplectic vector subspace of V in the sense of the above definition.

6. A vector subspace W of V is symplectic if and only if $\operatorname{orth} W$ is symplectic.

Definitions 2. Let (M, ω) be a symplectic manifold. For each $x \in M$, $\bigl(T_xM, \omega(x)\bigr)$ is a symplectic vector space. A submanifold N of M is said to be

- *isotropic* if for each point $x \in N$, the space T_xN is an isotropic vector subspace of $\bigl(T_xM, \omega(x)\bigr)$
- *coisotropic* if for each $x \in N$, T_xN is a coisotropic vector subspace of $\bigl(T_xM, \omega(x)\bigr)$
- *Lagrangian* if for each $x \in N$, T_xN is a Lagrangian vector subspace of $\bigl(T_xM, \omega(x)\bigr)$
- *symplectic* if for each $x \in N$, T_xN is a symplectic vector subspace of $\bigl(T_xM, \omega(x)\bigr)$.

2.4. Hamiltonian Vector Fields on a Symplectic Manifold

Let (M, ω) be a symplectic manifold. We have seen that the map which associates to each vector $v \in TM$ the covector $\iota(v)\omega$ is an isomorphism from TM onto T^*M. So, for any given differential one-form α, there exists a unique vector field X such that $\iota(X)\omega = \alpha$. We are therefore allowed to state the following definitions.

Definitions 3. Let (M, ω) be a symplectic manifold and $f : M \to \mathbb{R}$ be a smooth function. The vector field X_f which satisfies

$$\iota(X_f)\omega = -\mathrm{d}f$$

is called the **Hamiltonian vector field** associated to f. The function f is called a **Hamiltonian** for the Hamiltonian vector field X_f.

A vector field X on M such that the one-form $\iota(X)\omega$ is closed

$$\mathrm{d}\iota(X)\omega = 0$$

is said to be **locally Hamiltonian.**

Remarks 1. The function f is not the unique Hamiltonian of the Hamiltonian vector field X_f as any function g such that $\iota(X_f)\omega = -\mathrm{d}g$ is another Hamiltonian for X_f. Given a Hamiltonian f of X_f, a function g is another Hamiltonian for X_f if and only if $\mathrm{d}(f-g) = 0$, or in other words if and only if $f - g$ keeps a constant value on each connected component of M.

Of course, a Hamiltonian vector field is locally Hamiltonian. The converse is not true when the cohomology space $H^1(M, \mathbb{R})$ is not trivial.

Proposition 1. *On a symplectic manifold (M, ω), a vector field X is locally Hamiltonian if and only if the Lie derivative $\mathcal{L}(X)\omega$ of the symplectic form ω with respect to X vanishes*

$$\mathcal{L}(X)\omega = 0.$$

The bracket $[X, Y]$ of two locally Hamiltonian vector fields X and Y is Hamiltonian, and has as a Hamiltonian the function $\omega(X, Y)$.

Proof: The well known formula which relates the the exterior differential d, the interior product $\iota(X)$ and the **Lie derivative** $\mathcal{L}(X)$ with respect to the vector field X

$$\mathcal{L}(X) = \iota(X)\mathrm{d} + \mathrm{d}\iota(X)$$

proves that when X is a vector field on a symplectic manifold (M, ω)

$$\mathcal{L}(X)\omega = \mathrm{d}\,\iota(X)\omega$$

since $\mathrm{d}\,\omega = 0$. Therefore $\iota(X)\omega$ is closed if and only if $\mathcal{L}(X)\omega = 0$.

Let X and Y be two locally Hamiltonian vector fields. We have

$$\begin{aligned}
\mathrm{i}\big([X, Y]\big)\omega &= \mathcal{L}(X)\iota(Y)\omega - \iota(Y)\mathcal{L}(X)\omega \\
&= \mathcal{L}(X)\iota(Y)\omega = \big(\iota(X)\mathrm{d} + \mathrm{d}\,\iota(X)\big)\iota(Y)\omega \\
&= \mathrm{d}\,\iota(X)\iota(Y)\omega = -\mathrm{d}\big(\omega(X, Y)\big)
\end{aligned}$$

which proves that $\omega(X, Y)$ is a Hamiltonian for $[X, Y]$. $\square$

2.4.1. Expression in a System of Darboux Local Coordinates

Let $(x^1, \ldots, x^{2n})$ be a system of Darboux local coordinates. The symplectic form ω can be locally writen as

$$\omega = \sum_{i=1}^{n} \mathrm{d}\,x^{n+i} \wedge \mathrm{d}\,x^i$$

so we see that the Hamiltonian vector field X_f associated to a smooth function f can be locally written as

$$X_f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^{n+i}} \frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^{n+i}}.$$

A smooth curve φ drawn in M parameterized by the real variable t is said to be a *solution* of the differential equation determined by X_f, or an *integral curve* of X_f, if it satisfies the equation, called the **Hamilton equation** for the Hamiltonian f

$$\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = X_f\bigl(\varphi(t)\bigr).$$

Its local expression in the considered system of Darboux local coordinates is

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\partial f}{\partial x^{n+i}}, \qquad \frac{\mathrm{d}x^{n+i}}{\mathrm{d}t} = -\frac{\partial f}{\partial x^i}, \qquad 1 \leq i \leq n. \tag{1}$$

Definition 2. Let $\Phi : N \to N$ be a diffeomorphism of a smooth manifold N onto itself. The *canonical lift* of Φ to the cotangent bundle is the transpose of the vector bundle isomorphism $T(\Phi^{-1}) = (T\Phi)^{-1} : TN \to TN$. In other words, denoting by $\widehat{\Phi}$ the canonical lift of Φ to the cotangent bundle, we have for all $x \in N$, $\xi \in T^*_x N$ and $v \in T_{\Phi(x)}N$

$$\bigl\langle \widehat{\Phi}(\xi), v \bigr\rangle = \bigl\langle \xi, (T\Phi)^{-1}(v) \bigr\rangle.$$

Remark 1. With the notations of Definition 2, we have

$$\pi_N \circ \widehat{\Phi} = \Phi \circ \pi_N,$$

where $\pi_N : T^*N \to N$ is the canonical projection.

2.4.2. The Flow of a Vector Field

Let X be a smooth vector field on a smooth manifold M. We recall that the *reduced flow* of X is the map Φ, defined on an open subset Ω of $\mathbb{R} \times M$ and taking its values in M, such that for each $x \in M$ the parameterized curve $t \mapsto \varphi(t) = \Phi(t, x)$ is the maximal integral curve of the differential equation

$$\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = X\bigl(\varphi(t)\bigr)$$

which satisfies

$$\varphi(0) = x.$$

For each $t \in \mathbb{R}$, the set $D_t = \{x \in M; (t, x) \in \Omega\}$ is an open subset of M and when D_t is not empty the map $x \mapsto \Phi_t(x) = \Phi(t, x)$ is a diffeomorphism of D_t onto D_{-t}.

Definitions 4. Let N be a smooth manifold, TN and T^*N be its tangent and cotangent bundles, $\tau_N : TN \to N$ and $\pi_N : T^*N \to N$ be their canonical projections. Let X be a smooth vector field on N and $\{\Phi^X_t\,; t \in \mathbb{R}\}$ be its reduced flow.

1. The *canonical lift* of X to the tangent bundle TN is the unique vector field $\bar X$ on TM whose reduced flow $\{\Phi_t^{\bar X}\,;t\in\mathbb{R}\}$ is the prolongation to vectors of the reduced flow of X. In other words, for each $t\in\mathbb{R}$

$$\Phi_t^{\bar X}=T\Phi_t^X$$

therefore, for each $v\in TN$

$$\bar X(v)=\frac{\mathrm{d}}{\mathrm{d}t}\bigl(T\Phi_t^X(v)\bigr)\,\bigr|_{t=0}\,.$$

2. The *canonical lift* of X to the cotangent bundle T^*N is the unique vector field $\widehat X$ on T^*M whose reduced flow $\{\Phi_t^{\widehat X}\,;t\in\mathbb{R}\}$ is the lift to the cotangent bundle of the reduced flow $\{\Phi_t^X\,;t\in\mathbb{R}\}$ of X. In other words, for each $t\in\mathbb{R}$

$$\Phi_t^{\widehat X}=\widehat{\Phi_t^X}$$

therefore, for each $\xi\in T^*N$

$$\widehat X(\xi)=\frac{\mathrm{d}}{\mathrm{d}t}\bigl(\widehat{\Phi_t^X}(\xi)\bigr)\,\bigr|_{t=0}\,.$$

Exercise 1. Let X be a smooth vector field defined on a smooth manifold N and $\bar X$ be its canonical lift to TN (cf Definition 4.1). Prove that

$$\bar X=\kappa_N\circ TX$$

where $\kappa_N:T(TN)\to T(TN)$ is the canonical involution of the tangent bundle to TN (see [41]).

Proposition 2. *Let $\Phi:N\to N$ be a diffeomorphism of a smooth manifold N onto itself and $\widehat\Phi:T^N\to T^*N$ the canonical lift of Φ to the cotangent bundle. Let η_N be the Liouville form on T^*N. We have*

$$\widehat\Phi^*\eta_N=\eta_N.$$

Let X be a smooth vector field on N, and $\widehat X$ be the canonical lift of X to the cotangent bundle. We have

$$\mathcal{L}(\widehat X)(\eta_N)=0.$$

Proof: Let $\xi\in T^*N$ and $v\in T_\xi(T^*N)$. We have

$$\widehat\Phi^*\eta_N(v)=\eta_N\bigl(T\widehat\Phi(v)\bigr)=\bigl\langle\tau_{T^*N}\circ T\widehat\Phi(v),T\pi_N\circ T\widehat\Phi(v)\bigr\rangle.$$

But $\tau_{T^*N}\circ T\widehat\Phi=\widehat\Phi\circ\tau_{T^*N}$ and $T\pi_N\circ T\widehat\Phi=T(\pi_N\circ\widehat\Phi)=T(\Phi\circ\pi_N)$. Therefore

$$\widehat\Phi^*\eta_N(v)=\bigl\langle\widehat\Phi\circ\tau_{T^*N}(v),T(\Phi\circ\pi_N)(v)\bigr\rangle=\bigl\langle\tau_{T^*N}(v),T\pi_N(v)\bigr\rangle=\eta_N(v)$$

since $\widehat\Phi=(T\Phi^{-1})^T$.

Now let X be a smooth vector field on N, $\{\Phi_t^X\,; t \in \mathbb{R}\}$ be its reduced flow, and $\widehat{X}$ be the canonical lift of X to the cotangent bundle. We know that the reduced flow of $\widehat{X}$ is $\{\widehat{\Phi_t^X}\,; t \in \mathbb{R}\}$, so we can write

$$\mathcal{L}(\widehat{X})\eta_N = \frac{\mathrm{d}}{\mathrm{d}t}\bigl(\widehat{\Phi_t^X}^*\eta_N\bigr)\,\Bigl|_{t=0}\,.$$

Since $\widehat{\Phi_t^X}^*\eta_N = \eta_N$ does not depend on t, $\mathcal{L}(\widehat{X})\eta_N = 0$ $\square$.

The following Proposition, which presents an important example of Hamiltonian vector field on a cotangent bundle, will be used when we will consider Hamiltonian actions of a Lie group on its cotangent bundle.

Proposition 3. *Let N be a smooth manifold, TN be its cotangent bundle, η_N be the Liouville form and $\mathrm{d}\eta_N$ be the canonical symplectic form on T^*N. Let X be a smooth vector field on N and $f_X : T^*N \to \mathbb{R}$ be the smooth function defined by*

$$f_X(\xi) = \Bigl\langle \xi, X\bigl(\pi_N(\xi)\bigr)\Bigr\rangle, \qquad \xi \in T^*N.$$

On the symplectic manifold $(T^*N, \mathrm{d}\eta_N)$, the vector field $\widehat{X}$, canonical lift to T^*N of the vector field X on N in the sense defined above (4)*, is a Hamiltonian field which has the function f_X as a Hamiltonian. In other words*

$$\iota(\widehat{X})\mathrm{d}\eta_N = -\mathrm{d}f_X.$$

Moreover

$$f_X = \iota(\widehat{X})\eta_N.$$

Proof: We have seen (Proposition 2) that $\mathcal{L}(\widehat{X})\eta_N = 0$. Therefore

$$\iota(\widehat{X})\mathrm{d}\eta_N = \mathcal{L}(\widehat{X})\eta_N - \mathrm{d}\iota(\widehat{X})\eta_N = -\mathrm{d}\iota(\widehat{X})\eta_N,$$

which proves that $\widehat{X}$ is Hamiltonian and admits $\iota(\widehat{X})\eta_N$ as Hamiltonian. For each $\xi \in T^*N$

$$\iota(\widehat{X})\eta_N(\xi) = \eta_N(\widehat{X})(\xi) = \Bigl\langle \xi, T\pi_N\bigl(\widehat{X}(\xi)\bigr)\Bigr\rangle = \Bigl\langle \xi, X\bigl(\pi_N(\xi)\bigr)\Bigr\rangle = f_X(\xi). \quad \square$$

2.5. The Poisson bracket

Definition 3. The **Poisson bracket** of an ordered pair (f, g) of smooth functions defined on the symplectic manifold (M, ω) is the smooth function $\{f, g\}$ defined by the equivalent formulae

$$\{f, g\} = \iota(X_f)\,\mathrm{d}g = -\iota(X_g)\,\mathrm{d}f = \omega(X_f, X_g).$$

Lemma 1. *Let (M,ω) be a symplectic manifold, let f and g be two smooth functions on M and let X_f and X_g be the associated Hamiltonian vector fields. The bracket $[X_f,X_g]$ is a Hamiltonian vector field which admits $\{f,g\}$ as Hamiltonian.*

Proof: This result is an immediate consequence of Proposition 1. $\square$

Proposition 4. *Let (M,ω) be a symplectic manifold. The Poisson bracket is a bilinear composition law on the space $C^\infty(M,\mathbb{R})$ of smooth functions on M, which satisfies the following properties*

1. *it is skew-symmetric: $\{g,f\}=-\{f,g\}$*
2. *it satisfies the Leibniz identity with respect to the ordinary product of functions:*
$$\{f,gh\}=\{f,g\}h+g\{f,h\}$$
3. *it satisfies the Jacobi identity, which is a kind of Leibniz identity with respect to the Poisson bracket itself*
$$\bigl\{f,\{g,h\}\bigr\}=\bigl\{\{f,g\},h\bigr\}+\bigl\{g,\{f,h\}\bigr\}$$
which can also be written, when the skew-symmetry of the Poisson bracket is taken into account
$$\bigl\{\{f,g\},h\bigr\}+\bigl\{\{g,h\},f\bigr\}+\bigl\{\{h,f\},g\bigr\}=0.$$

Proof: The proofs of Properties 1 and 2 are very easy and left to the reader. Let us prove Property 3. We have
$$\bigl\{\{f,g\},h\bigr\}=\omega(X_{\{f,g\}},X_h)=-\iota(X_{\{f,g\}})\mathrm{i}(X_h)\omega=\iota(X_{\{f,g\}})\mathrm{d}h.$$
By Lemma 1, $X_{\{f,g\}}=[X_f,X_g]$ so we have
$$\bigl\{\{f,g\},h\bigr\}=\iota\bigl([X_f,X_g]\bigr)\mathrm{d}h=\mathcal{L}\bigl([X_f,X_g]\bigr)h\,.$$
We also have
$$\bigl\{\{g,h\},f\bigr\}=-\mathcal{L}(X_f)\circ\mathcal{L}(X_g)h\,,\quad \bigl\{\{h,f\},g\bigr\}=\mathcal{L}(X_g)\circ\mathcal{L}(X_f)h\,.$$
Taking the sum of these three terms, and taking into account the identity
$$\mathcal{L}\bigl([X_f,X_g]\bigr)=\mathcal{L}(X_f)\circ\mathcal{L}(X_g)-\mathcal{L}(X_g)\circ\mathcal{L}(X_f)$$
we see that the Jacobi identity is satisfied. $\square$

Remarks 2.

1. In a system of Darboux local coordinates $(x^1,\dots,x^{2n})$, the Poisson bracket can be written in the form
$$\{f,g\}=\sum_{i=1}^n\left(\frac{\partial f}{\partial x^{n+i}}\,\frac{\partial g}{\partial x^i}-\frac{\partial f}{\partial x^i}\,\frac{\partial g}{\partial x^{n+i}}\right).$$

2. Let H be a smooth function on the symplectic manifold (M, ω), and X_H be the associaled Hamiltonian vector field. By using the Poisson bracket, one can write in a very concise way the Hamilton equation for X_H. Let $t \mapsto \varphi(t)$ be any integral curve of X_H. Then for any smooth function $f : M \to \mathbb{R}$

$$\frac{\mathrm{d}f\bigl(\varphi(t)\bigr)}{\mathrm{d}t} = \{H, f\}\bigl(\varphi(t)\bigr).$$

By succesively taking for f the coordinate functions $x^1, \ldots, x^{2n}$ of a system of Darboux local coordinates, we recover the equations (1).

3. Poisson Manifolds

3.1. The Inception of Poisson Manifolds

Around the middle of the XX-th century, several scientists felt the need of a frame in which Hamiltonian differential equations could be considered, more general than that of symplectic manifolds. Dirac for example had proposed such a frame in his famous 1950 paper *Generalized Hamiltonian dynamics* [10, 11].

In many applications in which, starting from a symplectic manifold, another manifold is built by a combination of processes (products, quotients, restriction to a submanifold, ...), there exists on that manifold a structure, more general than a symplectic structure, with which a vector field can be associated to each smooth function, and the bracket of two smooth functions can be defined. It was also known that on a (odd-dimensional) contact manifold one can define the bracket of two smooth functions.

Several generalizatons of symplectic manifolds were defined and investigated by Lichnerowicz during the years 1975–1980. He gave several names to these generalizations: *canonical, Poisson, Jacobi* and *locally conformally symplectic* manifolds [28, 29].

In 1976 Kirillov published a paper entitled *Local Lie Algebras* [19] in which he determined all the possible structures on a manifold allowing the definition of a bracket with which the space of smooth functions becomes a local Lie algebra. *Local* means that the value taken by the bracket of two smooth functions at each point only depends of the values taken by these functions on an arbitrarily small neighbourhood of that point. The only such structures are those called by Lichnerowicz *Poisson structures*, *Jacobi structures* and *locally conformally symplectic structures*.

In what follows we will mainly consider Poisson manifolds.

3.2. Definition and Structure of Poisson Manifolds

Definition 4. A **Poisson structure** on a smooth manifold M is the structure determined by a bilinear, skew-symmetric composition law on the space of smooth functions, called the *Poisson bracket* and denoted by $(f,g) \mapsto \{f,g\}$, satisfying the Leibniz identity

$$\{f,gh\} = \{f,g\}h + g\{f,h\}$$

and the Jacobi identity

$$\bigl\{\{f,g\},h\bigr\} + \bigl\{\{g,h\},f\bigr\} + \bigl\{\{h,f\},g\bigr\} = 0.$$

A manifold endowed with a Poisson structure is called a **Poisson manifold**.

Proposition 5. *On a Poisson manifold M, there exists a unique smooth bivector field Λ, called the* Poisson bivector field *of M, such that for any pair (f,g) of smooth functions defined on M, the Poisson bracket $\{f,g\}$ is given by the formula*

$$\{f,g\} = \Lambda(\mathrm{d}f, \mathrm{d}g).$$

Proof: The existence, uniqueness and skew-symmetry of Λ are easy consequences of the the Leibniz identity and of the skew-symmetry of the Poisson bracket. It does not depend on the Jacobi identity. □

Remark 2. The Poisson bivector field Λ determines the Poisson structure of M, since it allows the calculation of the Poisson bracket of any pair of smooth functions. For this reason a Poisson manifold M is often denoted by (M,Λ).

Definition 5. Let (M,Λ) be a Poisson manifold. We denote by $\Lambda^\sharp : T^*M \to TM$ the vector bundle homomorphism such that, for each $x \in M$ and each $\alpha \in T^*_xM$, $\Lambda^\sharp(\alpha)$ is the unique element in T_xM such that, for any $\beta \in T^*_xM$,

$$\bigl\langle \beta, \Lambda^\sharp(\alpha) \bigr\rangle = \Lambda(\alpha,\beta).$$

The subset $C = \Lambda^\sharp(T^*M)$ of the tangent bundle TM is called the **characteristic field** of the Poisson manifold (M,Λ)

The following theorem, due to Weinstein [44], proves that, loosely speaking, a Poisson manifold is the disjoint union of symplectic manifolds, arranged in such a way that the union is endowed with a differentiable structure.

Theorem 1. *Let (M,Λ) be a Poisson manifold. Its characteristic field C is a completely integrable generalized distribution on M. It means that M is the disjoint union of immersed connected submanifolds, called the* symplectic leaves *of (M,Λ), with the following properties: a leaf S is such that, for each $x \in S$, $T_xS = T_xM \cap C$. Moreover, S is maximal in the sense that any immersed connected submanifold S' containing S and such that for each $x \in S'$, $T_xS' = T_xM \cap C$, is equal to S.*

Moreover, the Poisson structure of M determines, on each leaf S, a symplectic form ω_S, such that the restriction to S of the Poisson bracket of two smooth functions defined on M only depends on the restrictions of these functions to S, and can be calculated as the Poisson bracket of these restrictions, using the symplectic form ω_S.

The reader may look at [44] or at [27] for a proof of this theorem.

3.2.1. The Schouten Bracket

Let M be a smooth n-dimensional manifold. For each integer p ($1 \leq p \leq n$) we denote by $\Omega^p(M)$ the space of differential forms of degree p, in other words the space of smooth sections of $\bigwedge^p(T^*M)$, the p-th exterior power of the cotangent bundle T^*M. By convention $\Omega^0(M) = C^\infty(M,\mathbb{R})$ is the space of smooth functions, and for $p < 0$ or $p > n$, we set $\Omega^p(M) = \{0\}$. The direct sum $\Omega(M) = \oplus_{p\in\mathbb{Z}}\Omega^p(M)$ is the exterior algebra of differential forms on M. It is endowed with a composition law, the *exterior product*, which associates to a pair (η,ζ), with $\eta \in \Omega^p(M)$ and $\zeta \in \Omega^q(M)$ the form $\eta\wedge\zeta \in \Omega^{p+q}(M)$. Moreover $\Omega(M)$ is endowed with a derivation of degree one, the **exterior differential** d, which is such that when $\eta \in \Omega^p(M)$, $\mathrm{d}\eta \in \Omega^{p+1}(M)$.

Similarly, for each integer p ($1 \leq p \leq n$) we denote by $A^p(M)$ the space of smooth multivector fields of degree p, in other words the space of smooth sections of $\bigwedge^p(TM)$, the p-th exterior power of the tangent bundle TM. By convention $A^0(M) = C^\infty(M,\mathbb{R})$ is the space of smooth functions, and for $p<0$ or $p>n$, we set $A^p(M) = \{0\}$. The direct sum $A(M) = \oplus_{p\in\mathbb{Z}}A^p(M)$ is the exterior algebra of smooth multivector fields on M. It is endowed with a composition law, the **exterior product**, which associates to a pair (P,Q), with $P \in A^p(M)$ and $Q \in A^q(M)$ the multivector field $P\wedge Q \in A^{p+q}(M)$.

There is a natural pairing of elements of same degree in $A(M)$ and in $\Omega(M)$. It is first defined for decomposable elements: let $\eta = \eta_1\wedge\cdots\wedge\eta_p \in \Omega^p(M)$ and $P = X_1\wedge\cdots\wedge X_p \in A^p(M)$. We set

$$\langle \eta, P\rangle = \det\bigl(\langle \eta_i, X_j\rangle\bigr).$$

Then this pairing can be uniquely extended to $\Omega^p(M)\times A^p(M)$ by bilinearity.

With any $P \in A^p(M)$ we can associate a graded endomorphism $\iota(P)$ of the exterior algebra of differential formls $\Omega(M)$, of degree $-p$, which means that when $\eta \in \Omega^q(M)$, $\iota(P)\eta \in \Omega^{q-p}(M)$. This endomorphism, which extends to multivector fields the interior product of forms with a vector field, is determined by the formula, in which $P \in A^p(M)$, $\eta \in \Omega^q(M)$ and $R \in A^{q-p}(M)$

$$\bigl\langle \iota(P)\eta, R\bigr\rangle = (-1)^{(p-1)p/2}\langle \eta, P\wedge Q\rangle.$$

Besides the exterior product, there exists on the graded vector space $A(M)$ of multivector fields another bilinear composition law, which naturally extends to multivector fields the Lie bracket of vector fields. It associates to $P \in A^p(M)$ and $Q \in A^q(M)$ an element denoted $[P,Q] \in A^{p+q-1}(M)$, called the **Schouten bracket** of P and Q. The Schouten bracket $[P,Q]$ is defined by the following formula, which gives the expression of the corresponding graded endomorphism of $\Omega(M)$

$$\mathrm{i}\bigl([P,Q]\bigr) = \Bigl[\bigl[\iota(P),\mathrm{d}\bigr],\iota(Q)\Bigr].$$

The brackets in the right hand side of this formula are the **graded commutators** of graded endomorphisms of $\Omega(M)$. Let us recall that if E_1 and E_2 are graded endomorphisms of $\Omega(M)$ of degrees e_1 and e_2 respectively, their graded commotator is

$$[E_1,E_2] = E_1\circ E_2 - (-1)^{e_1e_2}E_2\circ E_1.$$

The properties of the Schouten bracket can be deduced from the above formulae. For example, we easily see that if the degrees of P and Q are, respectively, p and q, the degree of $[P,Q]_S$ is $p+q-1$.

For more information about the Schouten bracket, the reader may look at [24] or [31].

Proposition 6. *Let Λ be a smooth bivector field on a smooth manifold M. Then Λ is a Poisson bivector field (and (M,Λ) is a Poisson manifold) if and only if $[\Lambda,\Lambda]=0$.*

Proof: We define the vector bundle homomorphism $\Lambda^\sharp : T^*M \to TM$ by setting, for all $x\in M$, α and $\beta \in T^*_xM$

$$\bigl\langle \beta, \Lambda^\sharp(\alpha)\bigr\rangle = \Lambda(\alpha,\beta).$$

For any pair (f,g) of smooth functions we set

$$X_f = \Lambda^\sharp(\mathrm{d}f), \qquad \{f,g\} = \iota(X_f)(\mathrm{d}g) = \Lambda(\mathrm{d}f,\mathrm{d}g).$$

This bracket is a bilinear skew-symmetric composition law on $C^\infty(M,\mathbb{R})$ which satisfies the Leibniz identity. Therefore Λ is a Poisson bivecctor field if and only if the above defined bracket of functions satisfies the Jacobi identity.

Let f, g and h be three smooth functions on M. We easily see that X_f and $\{f,g\}$ can be expressed in terms of the Schouten bracket. Indeed we have

$$X_f = -[\Lambda,f] = -[f,\Lambda], \qquad \{f,g\} = \bigl[[\Lambda,f],g\bigr].$$

Therefore

$$\bigl\{\{f,g\},h\bigr\} = \Bigl[\Bigl[\Lambda,\bigl[[\Lambda,f],g\bigr]\Bigr],h\Bigr].$$

By using the graded Jacobi identity satisfied by Schouten bracket, we see that

$$\Big[\Lambda, \big[[\Lambda, f], g\big]\Big] = -\big[[g, \Lambda], [f, \Lambda]\big] + 2\Big[\big[[\Lambda, \Lambda], f\big], g\Big].$$

Using the equalities $X_f = -[\Lambda, f] = -[f, \Lambda]$ and $X_g = -[\Lambda, g] = -[g, \Lambda]$ we obtain

$$\begin{aligned}\big\{\{f, g\}, h\big\} &= \big[[X_f, X_g], h\big] + 2\Big[\Big[\big[[\Lambda, \Lambda], f\big], g\Big], h\Big]\\ &= \mathcal{L}\big([X_f, X_g]\big)h + 2\Big[\Big[\big[[\Lambda, \Lambda], f\big], g\Big], h\Big].\end{aligned}$$

On the other hand, we have

$$\big\{\{g, h\}, f\big\} = -\mathcal{L}(X_f) \circ \mathcal{L}(X_g)h, \quad \big\{\{h, f\}, g\big\} = \mathcal{L}(X_g) \circ \mathcal{L}(X_f)h.$$

Taking into account the equality

$$\mathcal{L}\big([X_f, X_g]\big) = \mathcal{L}(X_f) \circ \mathcal{L}(X_g) - \mathcal{L}(X_g) \circ \mathcal{L}(X_f)$$

we obtain

$$\big\{\{f, g\}, h\big\} + \big\{\{g, h\}, f\big\} + \big\{\{h, f\}, g\big\} = 2\Big[\Big[\big[[\Lambda, \Lambda], f\big], g\Big], h\Big].$$

By using the formula which defines the Schouten bracket, we check that for any $P \in A^3(M)$

$$\big[\big[[P, f], g\big], h\big] = P(\mathrm{d}f, \mathrm{d}g, \mathrm{d}h).$$

Therefore

$$\big\{\{f, g\}, h\big\} + \big\{\{g, h\}, f\big\} + \big\{\{h, f\}, g\big\} = 2[\Lambda, \Lambda](\mathrm{d}f, \mathrm{d}g, \mathrm{d}h)$$

so Λ is a Poisson bivector field if and only if $[\Lambda, \Lambda] = 0$. □

3.3. Some Properties of Poisson Manifolds

Definitions 5. Let (M, Λ) be a Poisson manifold.

1. The **Hamiltonian vector field** associated to a smooth function $f \in C^\infty(M, \mathbb{R})$ is the vector field X_f on M defined by

$$X_f = \Lambda^\sharp(\mathrm{d}f).$$

The function f is called a *Hamiltonian* for the Hamiltonian vector field X_f.

2. A **Poisson vector field** is a vector field X which satisfies

$$\mathcal{L}(X)\Lambda = 0.$$

Example 1. On a symplectic manifold (M, ω) we have defined the Poisson bracket of smooth functions. That bracket endows M with a Poisson structure, said to be *associated* to its symplectic structure. The Poisson bivector field Λ is related to the symplectic form ω by

$$\Lambda(\mathrm{d}f, \mathrm{d}g) = \omega(X_f, X_g), \qquad f \text{ and } g \in C^\infty(M, \mathbb{R}).$$

The map $\Lambda^\sharp : T^*M \to TM$ such that, for any $x \in M$, α and $\beta \in T^*_xM$

$$\bigl\langle \beta, \Lambda^\sharp(\alpha) = \Lambda(\alpha, \beta)$$

is therefore the inverse of the map $\omega^\flat : TM \to T^*M$ such that, for any $x \in M$, v and $w \in T_xM$,

$$\bigl\langle \omega^\flat(v), w\bigr\rangle = -\bigl\langle \iota(v)\omega, w\bigr\rangle = \omega(w, v).$$

Hamiltonian vector fields for the symplectic structure of M coincide with Hamiltonian vector fields for its Poisson structure. The Poisson vector fields on the symplectic manifold (M, ω) are the locally Hamiltonian vector fields. However, on a general Poisson manifold, Poisson vector fields are more general than locally Hamiltonian vector fields: even restricted to an arbitrary small neighbourhood of a point, a Poisson vector field may not be Hamiltonian.

Remarks 3.

1. Another way in which the Hamiltonian vector field X_f associated to a smooth function f can be defined is by saying that, for any other smooth function g on the Poisson manifold (M, Λ),

$$\iota(X_f)(\mathrm{d}g) = \{f, g\}.$$

2. A smooth function g defined on the Poisson manifold (M, Λ) is said to be a **Casimir** if for any other smooth function h, we have $\{g, h\} = 0$. In other words, a Casimir is a smooth function g whose associated Hamiltonian vector field is $X_g = 0$. On a general Poisson manifold, there may exist Casimirs other than the locally constant functions.

3. A smooth vector field X on the Poisson manifold (M, Λ) is a Poisson vector field if and only if, for any pair (f, g) of smooth functions

$$\mathcal{L}(X)\bigl(\{f, g\}\bigr) = \bigl\{\mathcal{L}(X)f, g\bigr\} + \bigl\{f, \mathcal{L}(X)g\bigr\}.$$

Indeed we have

$$\begin{aligned}\mathcal{L}(X)\bigl(\{f, g\}\bigr) &= \mathcal{L}(X)\bigl(\Lambda(\mathrm{d}f, \mathrm{d}g)\bigr)\\ &= \bigl(\mathcal{L}(X(\Lambda)\bigr)(\mathrm{d}f, \mathrm{d}g) + \Lambda\bigl(\mathcal{L}(X)(\mathrm{d}f), \mathrm{d}g\bigr) + \Lambda\bigl(\mathrm{d}f, \mathcal{L}(X)(\mathrm{d}g)\bigr)\\ &= \bigl(\mathcal{L}(X)(\Lambda)\bigr)(\mathrm{d}f, \mathrm{d}g) + \bigl\{\mathcal{L}(X)f, g\bigr\} + \bigl\{f, \mathcal{L}(X)g\bigr\}.\end{aligned}$$

3. Any Hamiltonian vector field X_f is a Poisson vetor field. Indeed, if f is a Hamiltonian for X_f, g and h two other smooth functions,we have according to the Jacobi identity

$$\begin{aligned}\mathcal{L}(X_f)\bigl(\{g, h\}\bigr) &= \bigl\{f, \{g, h\}\bigr\} = \bigl\{\{f, g\}, h\bigr\} + \bigl\{g, \{f, h\}\bigr\}\\ &= \bigl\{\mathcal{L}(X_f)g, h\bigr\} + \bigl\{g, \mathcal{L}(X_f)h\bigr\}.\end{aligned}$$

4. Since the characteristic field of the Poisson manifold (M,Λ) is generated by the Hamiltonian vector fields, any Hamiltonian vector field is everywhere tangent to the symplectic foliation. A Poisson vector field may not be tangent to that foliation.

Proposition 7. *Let (M_1,Λ_1) and (M_2,Λ_2) be two Poisson manifolds and $\varphi : M_1 \to M_2$ a smooth map. The following properties are equivalent.*

1. *For any pair (f,g) of smooth functions defined on M_2*

$$\{\varphi^* f, \varphi^* g\}_{M_1} = \varphi^*\{f,g\}_{M_2}.$$

2. *For any smooth function $f \in C^\infty(M_2,\mathbb{R})$ the Hamiltonian vector fields $\Lambda_2^\sharp(\mathrm{d}f)$ on M_2 and $\Lambda_1^\sharp\big(\mathrm{d}(f\circ\varphi)\big)$ on M_1 are φ-compatible, which means that for each $x \in M_1$*

$$T_x\varphi\Big(\Lambda_1^\sharp\big(\mathrm{d}(f\circ\varphi)(x)\big)\Big) = \Lambda_2^\sharp\Big(\mathrm{d}f\big(\varphi(x)\big)\Big).$$

3. *The bivector fields Λ_1 on M_1 and Λ_2 on M_2 are φ-compatible, which means that for each $x \in M_1$*

$$T_x\varphi\big(\Lambda_1(x)\big) = \Lambda_2\big(\varphi(x)\big).$$

A map $\varphi : M_1 \to M_2$ which satisfies these equivalent properties is called a Poisson map.

Proof: Let f and g be two smooth functions defined on M_2. For each $x \in M_1$, we have

$$\begin{aligned}\{\varphi^* f, \varphi^* g\}_{M_1}(x) &= \{f\circ\varphi, g\circ\varphi\}(x) = \Lambda_1(x)\big(\mathrm{d}(f\circ\varphi)(x)\mathrm{d}(g\circ\varphi)(x)\big)\\ &= \Big\langle \mathrm{d}(g\circ\varphi)(x), \Lambda_1^\sharp\big(\mathrm{d}(f\circ\varphi(x))\big)\Big\rangle.\end{aligned}$$

We have also

$$\varphi^*\{f,g\}_{M_2}(x) = \{f,g\}_{M_2}\big(\varphi(x)\big) = \Big\langle \mathrm{d}g\big(\varphi(x)\big), \Lambda_2^*\big(\mathrm{d}f\big(\varphi(x)\big)\big)\Big\rangle.$$

These formulae show that Properties 1 and 2 are equivalent.
We recall that $T_x\varphi\big(\Lambda_1(x)\big)$ is, by its very definition, the bivector at $\varphi(x) \in M_2$ such that, for any pair (f,g) of smooth functions on M_2

$$T_x\varphi\big(\Lambda_1(x)\big)\Big(\mathrm{d}f\big(\varphi(x)\big), \mathrm{d}g\big(\varphi(x)\big)\Big) = \Lambda_1\big(\mathrm{d}(f\circ\varphi)(x), \mathrm{d}(g\circ\varphi)(x)\big).$$

The above equalities therefore prove that Properties 2 and 3 are equivalent. $\square$

Poisson manifolds often appear as quotients of symplectic manifolds, as indicated by the following Proposition, due to Paulette Libermann [26].

Proposition 8. *Let (M, ω) be a symplectic manifold and $\varphi : M \to P$ a surjective submersion of M onto a smooth manifold P whose fibres are conncted (it means that for each $y \in P$, $\varphi^{-1}(y)$ is connected). The following properties are equivalent.*

1. *On the manifold M, the distribution $\operatorname{orth}(\ker T\varphi)$ is integrable.*
2. *For any pair (f, g) of smooth functions defined on P, the Poisson bracket $\{f \circ \varphi, g \circ \varphi\}$ is constant on each fibre $\varphi^{-1}(y)$ of the submersion φ (with $y \in P$).*

When these two equivalent properties are satisfied, there exist on P a unique Poisson structure for which $\varphi : M \to P$ is a Poisson map (the manifold M being endowed with the Poisson structure associated to its symplectic structure).

Proof: On the manifold M, $\ker T\varphi$ is a an integrable distribution of rank $\dim M - \dim P$ whose integral submanifolds are the fibres of the submersion φ. Its symplectic orthogonal $\operatorname{orth}(\ker T\varphi)$ is therefore a distribution of rank $\dim P$. Let f and g be two smooth functions defined on M_2. On M_1, the Hamiltonian vector fields $X_{f\circ\varphi}$ and $X_{g\circ\varphi}$ take their values in $\operatorname{orth}(\ker T\varphi)$. We have

$$[X_{f\circ\varphi}, X_{g\circ\varphi}] = X_{\{f\circ\varphi, g\circ\varphi\}}.$$

Their bracket $[X_{f\circ\varphi}, X_{g\circ\varphi}]$ takes value in $\operatorname{orth}(\ker T\varphi)$ if and only if $\{f \circ \varphi, g \circ \varphi\}$ is constant on each fibre $\varphi^{-1}(y)$ of the submersion φ. The equivalence of Properties 1 and 2 easily follows.

Let us now assume that the equivalent properties 1 and 2 are satisfied. Since $\varphi : M \to P$ is a submersion with connected fibres, the map which associates to each function $f \in C^\infty(M_2, \mathbb{R})$ the function $f \circ \varphi$ is an isomorphism of $C^\infty(M_2, \mathbb{R})$ onto the subspace of $C^\infty(M_1, \mathbb{R})$ made by smooth functions which are constant on each fibre of φ. The existence and unicity of a Poisson structure on M_2 for which φ is a Poisson map follows. □

Remark 3. Poisson manifolds obtained as quotients of symplectic manifolds often come by pairs. Let us assume indeed that (M, ω) is a symplectic manifold and that the above Proposition can be applied to a smooth surjective submersion with connected fibres $\varphi : M \to P$, and defines a Poisson structure on P for which φ is a Poisson map. Since $\operatorname{orth}(\ker T\varphi)$ is integrable, it defines a foliation of M, which is said to be *simple* when the set of leaves Q of that foliation has a smooth manifold structure such that the map $\psi : M \to Q$, which associates to each point in M the leaf through this point, is a submersion. Then the maps $\varphi : M \to P$ and $\psi : M \to Q$ play similar parts, so there exists on Q a unique Poisson structure for which ψ is a Poisson map. Weinstein [44] has determined the links which exist between the local structures of the two Poisson manifolds P and Q at corresponding points (that means, at points which are the images of the same point in M by the maps φ and ψ).

Several kinds of remarkable submanifolds of a Poisson manifold can be defined [44]. The most important are the *coisotropic* submanifolds, defined below.

Definition 6. A submanifold N of a Poisson manifold (M, Λ) is said to be *coisotropic* if for any point $x \in N$ and any pair (f, g) of smooth functions defined on a neighbourhood U of x in M whose restrictions to $U \cap N$ are constants, the Poisson bracket $\{f, g\}$ vanishes on $U \cap N$.

3.4. Examples of Poisson Manifolds

3.4.1. Symplectic Manifolds

We have seen above that any symplectic manifold is a Poisson manifold.

3.4.2. Dual Spaces of Finite-Dimensional Lie Algebras

Let $\mathcal{G}$ be a finite-dimensional Lie algebra, and $\mathcal{G}^*$ its dual space. The Lie algebra $\mathcal{G}$ can be considered as the dual of $\mathcal{G}^*$, that means as the space of linear functions on $\mathcal{G}^*$, and the bracket of the Lie algebra $\mathcal{G}$ is a composition law on this space of linear functions. This composition law can be extended to the space $C^\infty(\mathcal{G}^*, \mathbb{R})$ by setting

$$\{f, g\}(x) = \big[\mathrm{d}f(x), \mathrm{d}g(x)\big], \qquad f \text{ and } g \in C^\infty(\mathcal{G}^*, \mathbb{R}), \qquad x \in \mathcal{G}^*.$$

This bracket on $C^\infty(\mathcal{G}^*, \mathbb{R})$ defines a Poisson structure on $\mathcal{G}^*$, called its **canonical Poisson structure**. It implicitly appears in the works of Lie, and was rediscovered by Kirillov [18], Kostant [23] and Souriau [38]. Its existence can be seen as an application of Proposition 8. Let indeed G be the connected and simply connected Lie group whose Lie algebra is $\mathcal{G}$. We know that the contangent bundle T^*G has a canonical symplectic structure. One can check easily that for this symplectic structure, the Poisson bracket of two smooth functions defined on T^*G and invariant with respect to the lift to T^*G of the action of G on itself by left translations, is too invariant with respect to that action. Application of Proposition 8, the submersion $\varphi : T^*G \to \mathcal{G}^*$ being the left translation which, for each $g \in G$, maps T^*_gG onto $T^*_eG \equiv \mathcal{G}^*$, yields the above defined Poisson structure on $\mathcal{G}^*$. If instead of translations on the left, we use translation on the right, we obtain on $\mathcal{G}^*$ the opposite Poisson structure. This illustrates Remark 3, since, as we will see later, each one of the tangent spaces at a point $\xi \in T^*G$ to the orbits of that point by the lifts to T^*G of the actions of G on itself by translations on the left and on the right, is the symplectic orthogonal of the other.

The symplectic leaves of $\mathcal{G}^*$ equipped with the above defined Poisson structure are the coadjoint orbits.

3.4.3. Symplectic Cocycles

A **symplectic cocycle** of the Lie algebra $\mathcal{G}$ is a skew-symmetric bilinear map $\Theta : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ which satisfies

$$\Theta\big([X,Y],Z\big) + \Theta\big([Y,Z],X\big) + \Theta\big([Z,X],Y\big) = 0.$$

The above defined canonical Poisson structure on $\mathcal{G}^*$ can be modified by means of a symplectic cocycle Θ by defining the new bracket (see for example [27])

$$\{f,g\}_\Theta(x) = \big[\mathrm{d}f(x), \mathrm{d}g(x)\big] - \Theta\big(\mathrm{d}f(x), \mathrm{d}g(x)), \quad f,g \in C^\infty(\mathcal{G}^*, \mathbb{R}), \quad x \in \mathcal{G}^*.$$

This Poisson structure is called the **modified canonical Poisson structure** by means of the symplectic cocycle Θ. The symplectic leaves of $\mathcal{G}^*$ equipped with this Poisson structures are the orbits of an affine action whose linear part is the coadjoint action, with an additional term determined by Θ.

4. Symplectic, Poisson and Hamiltonian Actions

4.1. Actions on a Smooth Manifold

Let us first recall some definitions and facts about actions of a Lie algebra or of a Lie group on a smooth manifold.

Definition 7. An *action on the left* (respectively an *action on the right*) of a Lie group G on a smooth manifold M is a smooth map $\Phi : G \times M \to M$ (respectively, $\Psi : M \times G \to M$) such that, for any $x \in M$, g_1 and $g_2 \in G$, $e \in G$ being the neutral element

- for an action on the left

$$\Phi\big(g_1, \Phi(g_2,x)\big) = \Phi(g_1 g_2, x), \qquad \Phi(e,x) = x$$

- for an action on the right

$$\Psi\big(\Psi(x,g_1), g_2\big) = \Psi(x, g_1 g_2), \qquad \Phi(x,e) = x.$$

4.1.1. Consequences

Let $\Phi : G \times M \to M$ be an action on the left of the Lie group G on the smooth manifold M. For each $g \in G$, we denote by $\Phi_g : M \to M$ the map

$$\Phi_g(x) = \Phi(g,x).$$

The map $g \mapsto \Phi_g$ is a groups homomorphism of G into the group of smooth diffeomorphisms of M. In other words, for each $g \in G$, Φ_g is a diffeomorphism of M, and we have

$$\Phi_g \circ \Phi_h = \Phi_{gh}, \qquad (\Phi_g)^{-1} = \Phi_{g^{-1}}, \qquad g \text{ and } h \in G.$$

Similarly, let $\Psi : M \times G \to M$ be an action on the right of the Lie group G on the smooth manifold M. For each $g \in G$, we denote by $\Psi_g : M \to M$ the map

$$\Psi_g(x) = \Psi(x, g).$$

The map $g \mapsto \Psi_g$ is a groups anti-homomorphism of G into the group of smooth diffeomorphisms of M. In other words, for each $g \in G$, Ψ_g is a diffeomorphism of M, and we have

$$\Psi_g \circ \Psi_h = \Psi_{hg}, \qquad (\Psi_g)^{-1} = \Psi_{g^{-1}}, \qquad g \text{ and } h \in G.$$

Definition 8. Let $\Phi : G \times M \to M$ be an action on the left (respectively let $\Psi : M \times G \to M$ be an action of the right) of the Lie group G on the smooth manifold M. With each element $X \in \mathcal{G} \equiv T_eG$ (the tangent space to the Lie group G at the neutral element) we associate the vector field X_M on M defined by

$$X_M(x) = \begin{cases} \left.\dfrac{\mathrm{d}\Phi\bigl(\exp(sX), x\bigr)}{\mathrm{d}s}\right|_{s=0} & \text{if } \Phi \text{ is an action on the left} \\ \left.\dfrac{\mathrm{d}\Psi\bigl(x, \exp(sX)\bigr)}{\mathrm{d}s}\right|_{s=0} & \text{if } \Psi \text{ is an action on the right} \end{cases}$$

The vector field X_M is called the **fundamental vector field** on M associated to X.

Definition 9. An *action* of a Lie algebra $\mathcal{G}$ on a smooth manifold M is a Lie algebras homomorphism φ of $\mathcal{G}$ into the Lie algebra $A^1(M)$ of smooth vector fields on M (with the Lie bracket of vector fields as composition law). In other words, it is a linear map $\varphi : \mathcal{G} \to A^1(M)$ such that for each pair $(X, Y) \in \mathcal{G} \times \mathcal{G}$,

$$\varphi\bigl([X, Y]\bigr) = \bigl[\varphi(X), \varphi(Y)\bigr].$$

Remark 4. Let G be a Lie group. There are two natural ways in which the tangent space $T_eG \equiv \mathcal{G}$ to a Lie group G at the neutral element e can be endowed with a Lie algebra structure.

In the first way, we associate with each element $X \in T_eG$ the *left invariant* vector field X^L on G such that $X^L(e) = X$; its value at a point $g \in G$ is $X^L(g) = TL_g(X)$, where $L_g : G \to G$ is the map $h \mapsto L_g(h) = gh$. We observe that for any pair (X, Y) of elements in $\mathcal{G}$ the Lie bracket $[X^L, Y^L]$ of the vector fields X^L and Y^L on G is left invariant, and we define the bracket $[X, Y]$ by setting $[X, Y] = [X^L, Y^L](e)$. This Lie algebra structure on $\mathcal{G} \equiv T_eG$ will be called the Lie algebra structure of *left invariant vector fields* on G.

In the second way, we choose the *right invariant* vector fields on G X^R and Y^R, instead of the left invariant vector fields X^L and Y^L. Since $[X^R, Y^R](e) = -[X^L, Y^L](e)$, the Lie algebra structure on $\mathcal{G} \equiv T_eG$ obtained in this way, called the Lie algebra structure of *right invariant vector fields*, is the *opposite* of that of left invariant vector fields. We have therefore on T_eG two opposite Lie algebras structures, both equally natural. Fortunately, the choice of one rather than the other as the Lie algebra $\mathcal{G}$ of G does not matter because the map $X \mapsto -X$ is a Lie algebras isomorphism between these two structures.

Proposition 9. *Let $\Phi : G \times M \to M$ be an action on the left (respectively let $\Psi : M \times G \to M$ be an action on the right) of a Lie group G on a smooth manifold M. We endow $\mathcal{G} \equiv T_eG$ with the Lie algebra structure of right invariant vector fields on G (resp, with the Lie algebra structure of left invariant vector fields on G). The map $\varphi : \mathcal{G} \to A^1(M)$ (respectively $\psi : \mathcal{G} \to A^1(M)$) which associates to each element X of the Lie algebra $\mathcal{G}$ of G the corresponding fundamental vector field X_M, is an action of the Lie algebra $\mathcal{G}$ on the manifold M. This Lie algebra action is said to be* associated *to the Lie group action Φ (respectively Ψ).*

Proof: Let us look at an action on the left Φ. Let $x \in M$, and let $\Phi^x : G \to M$ be the map $g \mapsto \Phi^x(g) = \Phi(g, x)$. For any $X \in T_eG$ and $g \in G$, we have

$$\begin{aligned} X_M\bigl(\Phi(g,x)\bigr) &= \frac{\mathrm{d}}{\mathrm{d}s}\,\Phi\bigl(\exp(sX), \Phi(g,x)\bigr)\,\Bigr|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\,\Phi\bigl(\exp(sX)g, x\bigr)\,\Bigr|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\,\Phi\Bigl(R_g\bigl(\exp(sX)\bigr), x\Bigr)\,\Bigr|_{s=0} = T\Phi^x \circ TR_g(X). \end{aligned}$$

We see that for each $X \in T_xG$, the right invariant vector field X^R on G and the fundamental vector field X_M on M are compatible with respect to the map $T\Phi^x : TG \to TM$. Therefore for any pair (X, Y) of elements in T_eG, we have $[X, Y]_M = [X_M, Y_M]$. In other words the map $X \mapsto X_M$ is an action of the Lie algebra $\mathcal{G} = T_eG$ (equipped with the Lie algebra structure of right invariant vector fields on G) on the manifold M.

For an action on the right Ψ, the proof is similar, $\mathcal{G} = T_eG$ being this time endowed with the Lie algebra structure of left invariant vector fields on G. □

Proposition 10. *Let $\Phi : G \times M \to M$ be an action on the left (respectively let $\Psi : M \times G \to M$ be an action on the right) of a Lie group G on a smooth manifold M. Let X_M be the fundamental vector field associated to an element $X \in \mathcal{G}$. For any $g \in G$, the direct image $(\Phi_g)_*(X_M)$ (respectively $(\Psi_g)_*(X_M)$) of the vector field X_M by the diffeomorphism $\Phi_g : M \to M$ (respectively $\Psi_g : M \to M$) is the fundamnetal vector field $(\mathrm{Ad}_g X)_M$ associated to $\mathrm{Ad}_g X$ (respectively the fundamental vector field $(\mathrm{Ad}_{g^{-1}} X)_M$ associated to $\mathrm{Ad}_{g^{-1}} X$).*

Proof: For each $x \in M$

$$\begin{aligned} (\Phi_g)_*(X_M)(x) &= T\Phi_g\Bigl(X_M\bigl(\Phi(g^{-1}, x)\bigr)\Bigr) = T\Phi_g\bigl(\frac{\mathrm{d}}{\mathrm{d}s}\Phi\bigl(\exp(sX)g^{-1}, x\bigr)\,\bigr|_{s=0}\bigr) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\Bigl(\Phi\bigl(g\exp(sX)g^{-1}, x\bigr)\Bigr)\,\Bigr|_{s=0} = (\mathrm{Ad}_g X)_M(x) \end{aligned}$$

since $g\exp(sX)g^{-1} = \exp(\mathrm{Ad}_g X)$. The proof for the action on the right Ψ is similar. $\square$

4.2. Poisson, Symplectic and Hamiltonian Actions

Definitions 6.

1. An action φ of a Lie algebra $\mathcal{G}$ on a Poisson manifold (M,Λ) is called a *Poisson action* if for any $X \in \mathcal{G}$ the corresponding vector field $\varphi(X)$ is a Poisson vector field. When the Poisson manifold is in fact a symplectic manifold (M,ω), Poisson vector fields on M are locally Hamiltonian vector fields and a Poisson action is called a **symplectic action**.

2. An action Φ (either on the left or on the right) of a Lie group G on a Poisson manifold (M,Λ) is called a *Poisson action* when for each $g \in G$

$$(\Phi_g)_*\Lambda = \Lambda.$$

When the Poisson manifold (M,Λ) is in fact a symplectic manifold (M,ω), a Poisson action is called a *symplectic action*; the fibre bundle isomorphism $\Lambda^\sharp : T^*M \to TM$ being the inverse of $\omega^\flat : TM \to T^*M$, we also can say that an action Φ of a Lie group G on a symplectic manifold (M,ω) is called a *symplectic action* when for each $g \in G$

$$(\Phi_g)^*\omega = \omega.$$

Proposition 11. *We assume that G is a connected Lie group which acts by an action Φ, either on the left or on the right, on a Poisson manifold (M,Λ), in such a way that the corresponding action of its Lie algebra $\mathcal{G}$ is a Poisson action. Then the action Φ itself is a Poisson action.*

Proof: Let $X \in \mathcal{G}$. For each $x \in M$, the parameterized curve $s \mapsto \Phi_{\exp(sX)}(x)$ is the integral curve of the fundamental vector field X_M which takes the value x for $s = 0$. In other words, the reduced flow of the vector field X_M is the map, defined on $\mathbb{R} \times M$ and taking its values in M

$$(s,x) \mapsto \Phi_{\exp(sX)}(x).$$

According to a formula which relates inverse images of multivectors or differential forms with respect to the flow of a vector field, with their Lie derivatives with respect to that vector field (see for example [27], Appendix 1, Section 3.4, page 351), for any $s_0 \in \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\big((\Phi_{\exp(sX)})^*(\Lambda)\big)(x)\Big)\Big|_{s=s_0} = \Big((\Phi_{\exp(s_0X)})^*\big(\mathcal{L}(X_M)\Lambda\big)\Big)(x) = 0$$

since $\mathcal{L}(X_M)\Lambda = 0$. Therefore for any $s \in \mathbb{R}$,

$$(\Phi_{\exp(sX)})^*\Lambda = (\Phi_{\exp(-sX)})_*\Lambda = \Lambda.$$

The Lie group G being connected, any $g \in G$ is the product of a finite number of exponentials, so $(\Phi_g)_*\Lambda = \Lambda$. $\square$

Exercise 2. Let Φ be an action, either on the left or on the right, of a Lie group G on a Poisson manifold (M, Λ). Prove that the following properties are equivalent. Conclude that any of these properties can be used as the definition of a Poisson action

- for each $g \in G$
$$(\Phi_g)_*\Lambda = \Lambda$$
- for each $g \in G$ and $f \in C^\infty(M, \mathbb{R})$
$$(\Phi_g)_*(X_f) = X_{(\Phi_g)_*(f)}$$
- for each $g \in G$, $\Phi_g : M \to M$ is a *Poisson map*, which means that for each pair (f_1, f_2) of smooth functions on M
$$\bigl\{(\Phi_g)^* f_1, (\Phi_g)^* f_2\bigr\} = (\Phi_g)^*\bigl(\{f_1, f_2\}\bigr)$$
- and in the special case when the Poisson manifold (M, Λ) is in fact a symplectic manifold (M, ω), for each $g \in G$
$$(\Phi_g)^*\omega = \omega.$$

Prove that when these equivalent properties are satisfied, the action of the Lie algebra $\mathcal{G}$ of G which associates, to each $X \in \mathcal{G}$, the fundamental vector field X_M on M, is a Poisson action.

Definitions 7.

1. An action φ of a Lie algebra $\mathcal{G}$ on a Poisson manifold (M, Λ) is called a *Hamiltonian action* if for every $X \in \mathcal{G}$ the corresponding vector field $\varphi(X)$ is a Hamiltonian vector field on M.

2. An action Φ (either on the left or on the right) of a Lie group G on a Poisson manifold (M, Λ) is called a **Hamiltonian action** if it is a Poisson action (or a symplectic action when the Poisson manifold (M, Λ) is in fact a symplectic manifold(M, ω)) and if, in addition, the associated action φ of its Lie algebra is a Hamiltonian action.

Remarks 4.

1. A Hamiltonian action of a Lie algebra on a Poisson manifold is automatically a Poisson action.

2. An action Φ of a connected Lie group G on a Poisson manifold such that the corresponding action of its Lie algebra is Hamiltonian, automatially is a Hamiltonian action.

Proposition 12. *Let φ be a Hamiltonian action of a Lie algebra $\mathcal{G}$ on a Poisson manifold (M, Λ). Let $\mathcal{G}^*$ be the dual space of $\mathcal{G}$. There exists a smooth map $J : M \to \mathcal{G}^*$ such that for each $X \in \mathcal{G}$ the corresponding Hamiltonian vector field X_M has the function $J_X : M \to \mathbb{R}$, defined by*

$$J_X(x) = \langle J(x), X \rangle, \qquad \textit{with } x \in M$$

as Hamiltonian.

Such a map $J : M \to \mathcal{G}^*$ is called a momentum map *for the Hamiltonian Lie algebra action φ. When φ is the Lie algebra action associated to a Hamiltonian action Φ of a Lie group G on the Poisson manifold (M, Λ), J is called a* momentum map *for the Hamiltonian Lie group action Φ.*

Proof: Let $(e_1, \ldots, e_p)$ be a basis of the Lie algebra $\mathcal{G}$ and$(\varepsilon^1, \ldots, \varepsilon^p)$ be the dual basis of $\mathcal{G}^*$. Since φ is Hamiltonian, for each i $(1 \leq i \leq p)$ there exists a Hamiltonian $J_{e_i} : M \to \mathbb{R}$ for the Hamiltonian vector field $\varphi(e_i)$. The map $J : M \to \mathcal{G}$ defined by

$$J(x) = \sum_{i=1}^{p} J_{e_i} \varepsilon^i, \qquad x \in M$$

is a momentum map for φ. □

The momentum map was introduced by Souriau [38] and in the Lagrangian formalism by Smale [39].

4.3. Some Properties of Momentum Maps

Proposition 13. *Let φ be a Hamiltonian action of a Lie algebra $\mathcal{G}$ on a Poisson manifold (M, Λ), and $J : M \to \mathcal{G}^*$ be a momentum map for that action. For any pair $(X, Y) \in \mathcal{G} \times \mathcal{G}$, the smooth function $\Theta(X, Y) : M \to \mathbb{R}$ defined by*

$$\Theta(X, Y) = \{J_X, J_Y\} - J_{[X,Y]}$$

is a Casimir of the Poisson algebra $C^\infty(M, \mathbb{R})$, which satisfies, for all X, Y and $Z \in \mathcal{G}$

$$\Theta\big([X, Y], Z\big) + \Theta\big([Y, Z], X\big) + \Theta\big([Z, X], Y\big) = 0. \tag{2}$$

When the Poisson manifold (M, Λ) is in fact a connected symplectic manifold (M, ω), for any pair $(X, Y) \in \mathcal{G} \times \mathcal{G}$ the function $\Theta(X, Y)$ is constant on M, and the map $\Theta : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ is a skew-symmetric bilinear form, which satisfies the above identity (2)*.*

Proof: Since J_X and J_Y are Hamiltonians for the Hamiltonian vector fields $\varphi(X)$ and $\varphi(Y)$, the Poisson bracket $\{J_X, J_Y\}$ is a Hamiltonian for $\big[\varphi(X), \varphi(Y)\big]$. Since $\varphi : \mathcal{G} \to A^1(M)$ is a Lie algebras homomorphism, $\big[\varphi(X), \varphi(Y)\big] = \varphi\big([X, Y]\big)$, and $J_{[X,Y]}$ is a Hamiltonian for this vector field. We have two different Hamiltonians for the same Hamiltonian vector field. Their difference $\Theta(X,Y)$ is therefore a Casimir of the Poisson algebra $C^\infty(M,\mathbb{R})$.

Let X, Y and Z be three elments in $\mathcal{G}$. We have

$$\begin{aligned}\Theta\bigl([X,Y],Z\bigr) &= \{J_{[X,Y]},J_Z\} - J_{\bigl[[X,Y],Z\bigr]}\\ &= \bigl\{\{J_X,J_Y\} - \Theta(X,Y),J_Z\bigr\} - J_{\bigl[[X,Y],Z\bigr]}\\ &= \bigl\{\{J_X,J_Y\},J_Z\bigr\} - J_{\bigl[[X,Y],Z\bigr]}\end{aligned}$$

since $\Theta(X,Y)$ is a Casimir of the Poisson algebra $C^\infty(M,\mathbb{R})$. Similarly

$$\begin{aligned}\Theta\bigl([Y,Z],X\bigr) &= \bigl\{\{J_Y,J_Z\},J_X\bigr\} - J_{\bigl[[Y,Z],X\bigr]}\\ \Theta\bigl([Z,X],Y\bigr) &= \bigl\{\{J_Z,J_X\},J_Y\bigr\} - J_{\bigl[[Z,X],Y\bigr]}\,.\end{aligned}$$

Adding these three terms and using the fact that the Poisson bracket of functions and the bracket in the Lie algebra $\mathcal{G}$ both satisfy the Jacobi identity, we see that Θ satisfies (2).

When (M,Λ) is in fact a connected symplectic manifold (M,ω), the only Casimirs of the Poisson algebra $C^\infty(M,\mathbb{R})$ are the constants, and Θ becomes a bilinear skew-symmetric form on $\mathcal{G}$. □

Definition 10. Under the assumptions of the above Proposition 13, the skew-symmetric bilinear map Θ, defined on $\mathcal{G}\times\mathcal{G}$ and taking its value in the space of Casimirs of the Poisson algebra $C^\infty(M,\mathbb{R})$ (real-valued when the Poisson manifold (M,Λ) is in fact a connected symplectic manifold (M,ω)), is called the *symplectic cocycle of the Lie algebra* $\mathcal{G}$ associated to the momentum map J.

Remark 5. Under the assumptions of Proposition 13, let us assume in addition that the Poisson manifold (M,Λ) is in fact a connected symplectic manifold (M,ω). The symplectic cocycle Θ is then a real-valued skew-symmetric bilinear form on $\mathcal{G}$. Therefore it is a symplectic cocycle in the sense of 3.4.3. Two different interpretations of this cocycle can be given.

- Let $\Theta^\flat:\mathcal{G}\to\mathcal{G}^*$ be the map such that, for all X and $Y\in\mathcal{G}$

  $$\bigl\langle\Theta^\flat(X),Y\bigr\rangle = \Theta(X,Y).$$

  The map $\Theta^\flat$ is a one-cocycle of the Lie algebra $\mathcal{G}$ for the coadjoint representation, in the sense of the cohomology theory of Lie algebras (see for example the book [16]).
- Let G be a Lie group whose Lie algebra is $\mathcal{G}$. The skew-symmetric bilinear form Θ on $\mathcal{G}=T_eG$ can be extended, either by left translations or by right

translations, into a left invariant (or a right invariant) closed differential two-form on G, since the above identity (2) means that its exterior differential $\mathrm{d}\Theta$ vanishes. In other words, Θ is a two-cocycle for the restriction of the de Rham cohomology of G to left (or right) invariant differential forms.

Theorem 2 (First Noether's theorem in Hamiltonian form)**.** *Let φ be a Hamiltonian action of a Lie algebra $\mathcal{G}$ on a Poisson manifold (M,Λ), $J : M \to \mathcal{G}^*$ be a momentum map for φ and $H : M \to \mathbb{R}$ be a smooth Hamiltonian. If the action φ leaves H invariant, that means if*

$$\mathcal{L}\bigl(\varphi(X)\bigr)H = 0, \qquad \textit{for any } X \in \mathcal{G}$$

the momentum map J keeps a constant value along each interal curve of the Hamiltonian vector field $\Lambda^\sharp(\mathrm{d}H)$.

Proof: For any $X \in \mathcal{G}$, let $J_X : M \to \mathbb{R}$ be the function $x \mapsto \bigl\langle J(x), X\bigr\rangle$. Let $t \mapsto \psi(t)$ be an integral curve of the Hamiltonian vector field $\Lambda^\sharp(\mathrm{d}H)$. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Bigl(J_X\bigl(\psi(t)\bigr)\Bigr) &= \mathcal{L}\bigl(\Lambda^\sharp(\mathrm{d}H)\bigr)\bigl(J_X\bigr)\bigl(\psi(t)\bigr) = \Lambda\bigl(\mathrm{d}H, \mathrm{d}J_X\bigr)\bigl(\psi(t)\bigr)\\ &= -\mathcal{L}\Bigl(\Lambda^\sharp\bigl(\mathrm{d}J_X\bigr)\Bigr)H = -\mathcal{L}\bigl(\varphi(X)\bigr)H = 0.\end{aligned}$$

Therefore, for any $X \in \mathcal{G}$, the derivative of $\langle J, X\rangle\bigl(\psi(t)\bigr)$ with respect to the parameter t of the parameterized curve $t \mapsto \psi(t)$ vanishes identically, which means that J keeps a constant value along that curve. □

The reader will find in the book by Kosmann-Schwarzbach [21] a very nice exposition of the history and scientific applications of the Noether's theorems.

Proposition 14. *Let φ be a Hamiltonian action of a Lie algebra $\mathcal{G}$ on a Poisson manifold (M,Λ) and $J : M \to \mathcal{G}^*$ be a momentum map for that action. Let S be a symplectic leaf of (M,Λ) and ω_S be its symplectic form.*

1. *For each $x \in S$, in the symplectic vector space $\bigl(T_xS, \omega_S(x)\bigr)$, each of the two vector subspaces $T_xS \cap \ker(T_xJ)$ and $\bigl\{\,\varphi(X)(x)\ ;\ X \in \mathcal{G}\,\bigr\}$ is the symplectic orthogonal of the other.*

2. *For each $x \in S$, $T_xJ(T_xS)$ is the annihilator of the isotropy subalgebra $\mathcal{G}_x = \bigl\{X \in \mathcal{G}\ ;\ \phi(X)(x) = 0\bigr\}$ of x.*

Proof: Let $v \in T_xS$. For each $X \in \mathcal{G}$ we have

$$\omega_S\bigl(v, \varphi(X)(x)\bigr) = \bigl\langle \mathrm{d}\langle J, X\rangle(x), v\bigr\rangle = \bigl\langle T_xJ(v), X\bigr\rangle.$$

Therefore a vector $v \in T_xS$ belongs to $\mathrm{orth}\bigl\{\,\varphi(X)(x)\ ;\ X \in \mathcal{G}\,\bigr\}$ if and only if $T_xJ(v) = 0$. In other words, in the symplectic vector space $\bigl(T_xS, \omega_S(x)\bigr)$, $T_xS \cap \ker(T_xJ)$ is the symplectic orthogonal of $\bigl\{\,\varphi(X)(x)\ ;\ X \in \mathcal{G}\,\bigr\}$. Of course, conversely $\bigl\{\,\varphi(X)(x)\ ;\ X \in \mathcal{G}\,\bigr\}$ is the symplectic orthogonal of $T_xS \cap \ker(T_xJ)$.

The same formula shows that $\langle T_x J(v), X \rangle = 0$ for all $v \in T_x S$ if and only if $X \in \mathcal{G}_x$. □

Remark 6. Under the assumptions of the above Proposition, when φ is the Lie algebra action associated to a the Hamiltonian action Φ of a Lie group G, the vector space $\{\, \varphi(X)(x) \;;\; X \in \mathcal{G} \,\}$ is the space tangent at x to the G-orbit of this point.

Corollary 1. *Let φ be a Hamiltonian action of a Lie algebra $\mathcal{G}$ on a symplectic manifold (M, ω) and $J : M \to \mathcal{G}^*$ be a momentum map for that action.*

1. *For each $x \in M$, in the symplectic vector space $\big(T_x M, \omega(x)\big)$ each of the two vector subspaces $\ker(T_x J)$ and $\{\, \varphi(X)(x) \;;\; X \in \mathcal{G} \,\}$ is the symplectic orthogonal of the other.*
2. *For each $x \in M$, $T_x J(T_x M)$ is the annihilator of the isotropy subalgebra $\mathcal{G}_x = \{X \in \mathcal{G} \;;\; \varphi(X)(x) = 0\}$ of x.*

Proof: These assertions both follow immediately from the above Proposition since the symplectic leaves of (M, ω) are its connected components. □

Proposition 15. *Let Φ be a Hamiltonian action of a Lie group G on a connected symplectic manifold (M, ω) and $J : M \to \mathcal{G}^*$ be a momentum map for that action. There exists a unique action a of the Lie group G on the dual $\mathcal{G}^*$ of its Lie algebra for which the momentum map J is equivariant, that means satisfies for each $x \in M$ and $g \in G$*

$$J\big(\Phi_g(x)\big) = a_g\big(J(x)\big).$$

The action a is an action on the left (respectively, on the right) if Φ is an action on the left (respectively, on the right), and its expression is

$$\begin{cases} a(g, \xi) = \mathrm{Ad}^*_{g^{-1}}(\xi) + \theta(g) & \text{if } \Phi \text{ is an action on the left,} \\ a(\xi, g) = \mathrm{Ad}^*_g(\xi) - \theta(g^{-1}) & \text{if } \Phi \text{ is an action on the right,} \end{cases} \qquad g \in G, \quad \xi \in \mathcal{G}^*.$$

The map $\theta : G \to \mathcal{G}^*$ is called the symplectic cocycle of the Lie group G *associated to the momentum map J.*

Proof: Let us first assume that Φ is an action on the left. For each $X \in \mathcal{G}$ the associated fundamental vector field X_M is Hamiltonian and the function $J_X : M \to \mathbb{R}$ defined by

$$J_X(x) = \langle J(x), X \rangle, \qquad x \in M$$

is a Hamiltonian for the vector field X_M. We know by Exercise 2 that its direct image $(\Phi_{g^{-1}})_*(X_M)$ by the diffeomorphism $\Phi_{g^{-1}} : M \to M$ is a Hamiltonian vector field for which the function $J_X \circ \Phi_g$ is a Hamiltonian. But Proposition 10 shows that $(\Phi_{g^{-1}})_*(X_M)$ is the fundamental vector field associated to $\mathrm{Ad}_{g^{-1}}(X)$, therefore has the function

$$x \mapsto \big\langle J(x), \mathrm{Ad}_{g^{-1}}(X)\big\rangle = \big\langle \mathrm{Ad}^*_{g^{-1}} \circ J(x), X\big\rangle$$

as a Hamiltonian. The difference between these two Hamiltonians for the same Hamiltonian vector field is a constant since M is assumed to be connected. Therefore the expression

$$\big\langle J \circ \Phi_g(x) - \mathrm{Ad}^*_{g^{-1}} \circ J(x), X\big\rangle$$

does not depend on $x \in M$, and depends linearly on $X \in \mathcal{G}$ (and of course smoothly depends on $g \in G$). We can therefore define a smooth map $\theta : G \to \mathcal{G}^*$ by setting

$$\theta(g) = J \circ \Phi_g - \mathrm{Ad}^*_{g^{-1}} \circ J, \qquad g \in G.$$

It follows that the map $a : G \times \mathcal{G}^* \to \mathcal{G}^*$

$$a(g, \xi) = \mathrm{Ad}^*_{g^{-1}}(\xi) + \theta(g)$$

is an action on the the left of the Lie group G on the dual $\mathcal{G}^*$ of its Lie algebra, which renders the momentum map J equivariant.

The case when Φ is an action on the right easily follows by observing that $(g, x) \mapsto \Phi(x, g^{-1})$ is a Hamiltonian action on the left whose momentum map is the opposite of that of Φ. □

Proposition 16. *Under the same assumptions as those of* Proposition 15*, the map $\theta : G \to \mathcal{G}^*$ satisfies, for all g and $h \in G$*

$$\theta(gh) = \theta(g) + \mathrm{Ad}^*_{g^{-1}}\big(\theta(h)\big).$$

Proof: In Proposition 15, the cocycle θ introduced for an action on the right $\Psi : M \times G \to M$ was the cocycle of the corresponding action on the left $\Phi : G \times M \to M$ defined by $\Phi(g, x) = \Psi(x, g^{-1})$. We can therefore consider only the case when Φ is an action on the left.

Let g and $h \in G$. We have

$$\begin{aligned}
\theta(gh) &= J\big(\Phi(gh, x)\big) - \mathrm{Ad}^*_{(gh)^{-1}} J(x) \\
&= J\Big(\Phi\big(g, \Phi(h, x)\big)\Big) - \mathrm{Ad}^*_{g^{-1}} \circ \mathrm{Ad}^*_{h^{-1}} J(x) \\
&= \theta(g) + \mathrm{Ad}^*_{g^{-1}}\Big(J\big(\Phi(h, x)\big) - \mathrm{Ad}^*_{h^{-1}} J(x)\Big) \\
&= \theta(g) + \mathrm{Ad}^*_{g^{-1}} \theta(h).
\end{aligned}$$

□

Proposition 17. *Let Φ be a Hamiltonian action of a Lie group G on a connected symplectic manifold (M, ω) and $J : M \to \mathcal{G}^*$ be a momentum map for that action. The symplectic cocycle $\theta : G \to \mathcal{G}^*$ of the Lie group G introduced in*

Proposition 15 *and the symplectic cocycle* $\Theta^\flat : \mathcal{G} \to \mathcal{G}^*$ *of its Lie algebra* $\mathcal{G}$ *introduced in* Definition 10 *and* Remark 5 *are related by*

$$\Theta^\flat = T_e\theta$$

where e *is the neutral element of* G*, the Lie algebra* $\mathcal{G}$ *being identified with* T_eG *and the tangent space at* $\mathcal{G}^*$ *at its origin being identified with* $\mathcal{G}^*$*. Moreover* J *is a Poisson map when* $\mathcal{G}^*$ *is endowed with*

- *its canonical Poisson structure modified by the symplectic cocycle* Θ (*defined in* 3.4.3) *if* Φ *is an action on the right,*
- *the opposite of this Poisson structure if* Φ *is an action on the left.*

Proof: As in the proof of Proposition 16, we have only to consider the case when Φ is an action on the left. The map which associates to each $X \in \mathcal{G}$ the fundamental vector field X_M is a Lie algebras homomorphism when $\mathcal{G}$ is endowed with the Lie algebra structure of *right invariant vector fields* on the Lie group G. We will follow here the more common convention, in which $\mathcal{G}$ is endowed with the Lie algebra structure of *left invariant vector fields* on G. With this convention the map $X \mapsto X_M$ is a Lie algebras antihomomorphism and we must change a sign in the definition of Θ given in Proposition 13 and take

$$\Theta(X, Y) = \langle \Theta^\flat(X), Y \rangle = \{J_X, J_Y\} + J_{[X,Y]}, \qquad X \text{ and } Y \in \mathcal{G}.$$

We have, for any $x \in M$

$$\begin{aligned}
\{J_X, J_Y\}(x) &= \omega(X_M, Y_M)(x) = \iota(X_M)\mathrm{d}\big(\langle J, Y\rangle\big)(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \Big\langle J\big(\Phi(\exp(tX), x\big), Y \Big\rangle\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \Big\langle \mathrm{Ad}^*_{\exp(-tX)} J(x) + \theta\big(\exp(tX)\big), Y \Big\rangle\Big|_{t=0} \\
&= \big\langle J(x), -[X, Y]\big\rangle + \big\langle T_e\theta(X), Y\big\rangle \\
&= -J_{[X,Y]}(x) + \big\langle T_e\theta(X), Y\big\rangle.
\end{aligned}$$

We see that $\Theta = T_e\theta$. Moreover, the elements X and Y in $\mathcal{G}$ can be considered as linear functions on $\mathcal{G}^*$. Their Poisson bracket, when $\mathcal{G}^*$ is equipped with its canonical Poisson structure modified by Θ, is

$$\{X, Y\}_\Theta(\xi) = \big\langle \xi, [X, Y]\big\rangle - \Theta(X, Y).$$

The formula $\{J_X, J_Y\}(x) = -J_{[X,Y]}(x) + \Theta(X, Y)$ can be read as

$$\{X \circ J, Y \circ J\}(x) = -\{X, Y\}_\Theta \circ J(x).$$

Since the value taken at a point by the Poisson bracket of two functions only depends on the values of the differentials of these two functions at that point, this result proves that J is a Poisson map when $\mathcal{G}^*$ is equipped with the opposite of the Poisson bracket $\{\ ,\ \}_\Theta$ □

4.3.1. Other Properties of the Momentum Map

The momentum map has several other very remarkable properties. Atiyah [5], Guillemin and Sternberg [13, 14] have shown that the image of the momentum map of a Hamiltonian action of a torus on a compact symplectic manifold is a convex polytope. Kirwan [20] adapted this result when the torus is replaced by any compact Lie group. Delzant [9] has shown that the convex polytope which is the image of a Hamiltonian action of a torus on a compact symplectic manifold determines this manifold.

4.4. Actions of a Lie Group on its Cotangent Bundle

In this section G is a Lie group, $\mathcal{G}$ is its Lie algebra and $\mathcal{G}^*$ is the dual space of $\mathcal{G}$. The Liouville one-form on T^*G is denoted by η_G.

The group composition law $m : G \times G \to G$, $m(g,h) = gh$, can be seen as an action of G on itself either on the left, or on the right. For each $g \in G$ we will denote by $L_g : G \to G$ and $R_g : G \to G$ the diffeomorphisms

$$L_g(h) = gh, \qquad R_g(h) = hg, \qquad h \in G$$

called, respectively, the *left translation* and the *right translation* of G by g.

Definitions 8. The *canonical lifts* to the tangent bundle TG of the actions of G on itself by left translations (respectively, by right translations) are, repectively, the maps $\overline{L} : G \times TG \to TG$ and $\overline{R} : TG \times G \to TG$

$$\overline{L}(g,v) = TL_g(v), \qquad \overline{R}(v,g) = TR_g(v), \qquad g \in G, \qquad v \in TG.$$

The *canonical lifts* to the cotangent bundle T^*G of the actions of G on itself by left translations (respectively, by right translations) are, respectively, the maps $\widehat{L} : G \times T^*G \to T^*G$ and $\widehat{R} : T^*G \times G \to T^*G$

$$\widehat{L}(g,\xi) = \bigl(TL_{g^{-1}}\bigr)^T(\xi), \qquad \widehat{R}(\xi,g) = \bigl(TR_{g^{-1}}\bigr)^T(\xi), \qquad g \in G, \quad \xi \in T^*G.$$

We have denoted by $\bigl(TL_{g^{-1}}\bigr)^T$ and $\bigl(TR_{g^{-1}}\bigr)^T$ the transposes of the vector bundles morphisms $TL_{g^{-1}}$ and $TR_{g^{-1}}$, respectively.

Proposition 18. *The canonical lifts to the tangent bundle and to the cotangent bundle of the actions of the Lie group G on itself by left translations* (*respectively, by right translations*) *are actions on the left* (*respectively, on the right*) *of G on its tangent bundle and on its cotangent bundle, which project onto the actions of G on* *itself by left translations (respectively, by right translations). It means that for all $g\in G$ and $v\in TG$*

$$\tau_G\bigl(\overline{L}(g,v)\bigr)=L_g\bigl(\tau_G(v)\bigr),\qquad \tau_G\bigl(\overline{R}(v,g)\bigr)=R_g\bigl(\tau_G(v)\bigr)$$

and that for all $g\in G$ and $\xi\in T^*G$

$$\pi_G\bigl(\widehat{L}(g,\xi)\bigr)=L_g\bigl(\pi_G(\xi)\bigr),\qquad \pi_G\bigl(\widehat{R}(\xi,g)\bigr)=R_g\bigl(\pi_G(\xi)\bigr).$$

Proof: It is an easy verification that the properties of actions are indeed satisfied by the maps $\overline{L}$, $\overline{R}$, $\widehat{L}$ and $\widehat{R}$, which is left to the reader. □

Proposition 19. *The canonical lifts to the cotangent bundle $\widehat{L}$ and $\widehat{R}$ of the actions of the Lie group G on itself by translations on the left and on the right are two Hamiltonian actions of G on the symplectic manifold $(T^*G,\mathrm{d}\eta_G)$. The two maps $J^L:T^*G\to\mathcal{G}^*$ and $J^R:T^*G\to\mathcal{G}^*$ defined, for each $\xi\in T^*G$, by*

$$J^L(\xi)=\widehat{R}\bigl(\xi,\pi_G(\xi)^{-1}\bigr),\qquad J^R(\xi)=\widehat{L}\bigl(\pi_G(\xi)^{-1},\xi\bigr)$$

are momentum maps for the actions $\widehat{L}$ and $\widehat{R}$, respectively.
Moreover, the map J^L is constant on each orbit of the action $\widehat{R}$, the map J^R is constant on each orbit of the action $\widehat{L}$ and for each $\xi\in T^*G$ each of the tangent spaces at ξ to the orbits $\widehat{L}(G,\xi)$ and $\widehat{R}(\xi,G)$ is the symplectic orthogonal of the other. The maps $J^L:T^*G\to\mathcal{G}^*$ and $J^R:T^*G\to\mathcal{G}^*$ are Poisson maps when T^*G is equipped with the Poisson structure associated to its canonical symplectic structure and when T^*G is equipped, respectively, with its canonical symplectic structure 3.4.2 *and with the opposite of its canonical symplectic structure.*

Proof: For each $X\in\mathcal{G}$, let X^L_G and X^R_G be the fundamental vector fields on G associated to X for the actions of G on itself, respectively by left and by right translations. Similarly, let $X^L_{T^*G}$ and $X^R_{T^*G}$ be the fundamental vector fields on T^*G associated to X for the actions $\widehat{L}$ and $\widehat{R}$ of G on T^*G defined in 8. The reduced flows of X^L and of X^R are the maps

$$\Phi^{X^L}(t,g)=\exp(TX)g,\qquad \Phi^{X^R}(t,g)=g\exp(tX),\quad t\in\mathbb{R},\qquad g\in G.$$

Therefore

$$X^L(g)=TR_g(X),\qquad X^R(g)=TL_g(X),\qquad g\in G$$

and we see that the fundamental vector fields $X^L_{T^*G}$ and $X^R_{T^*G}$ on T^*G are the canonical lifts to the cotangent bundle of the vector fields X^L_G and X^R_G on the Lie group G. Proposition 3 proves that $X^L_{T^*G}$ and $X^R_{T^*G}$ are Hamiltonian vector fields which admit as Hamiltonians, respectively, the maps

$$J^L(\xi)=\Bigl\langle\xi,X^L_G\bigl(\pi_G(\xi)\bigr)\Bigr\rangle,\qquad J^R(\xi)=\Bigl\langle\xi,X^R_G\bigl(\pi_G(\xi)\bigr)\Bigr\rangle,\qquad \xi\in T^*G.$$

Replacing X_G^L and X_G by their expressions given above and using the definitions of $\widehat{R}$ and $\widehat{L}$, we easily get the stated expressions for J^L and J^R. These expressions prove that J^L is constant on each orbit of the action $\widehat{R}$, and that J^R is constant on each orbit of the action $\widehat{L}$.

The actions $\widehat{L}$ and $\widehat{R}$ being free, each of their orbits is a smooth submanifold of T^*G of dimension $\dim G$. The ranks of the maps J^L and J^R are everywhere equal to $\dim G$ since their restrictions to each fibre of T^*G is a diffeomorphism of that fibre onto $\mathcal{G}$. Therefore, for each $\xi \in T^*G$

$$\ker T_\xi J^L = T_\xi\bigl(\widehat{R}(\xi, G)\bigr), \qquad \ker T_\xi J^R = T_\xi\bigl(\widehat{L}(\xi, G)\bigr).$$

Corollary 1 proves that for each $\xi \in T^*G$ each of the two vector subspaces of $T_\xi(T^*G)$

$$T_\xi\bigl(\widehat{L}(G, \xi)\bigr) \quad \text{and} \quad T_\xi\bigl(\widehat{R}(\xi, G)\bigr)$$

is the symplectic orthogonal of the other.

Finally, the fact that J^L and J^R are Poisson maps when $\mathcal{G}$ is equipped with its canonical Poisson structure or its opposite is an easy consequence of Proposition 8.

□

4.4.1. Generalization

Proposition 18 can be generalized by using a symplectic cocycle θ of the Lie group to modify the actions $\widehat{L}$ and $\widehat{R}$, and the associated symplectic cocycle Θ of the Lie algebra $\mathcal{G}$ to modify the symplectic structure of T^*G. The reader is referred to [27] for a proof of this generalization.

5. Reduction of Hamiltonian Systems with Symmetries

Very early, the first integrals were used by many scientists (Lagrange, Jacobi, Poincaré, …) to facilitate the determination of integral curves of Hamiltonian systems. It was observed that the knowledge of *one* real-valued first integral often allows the reduction by *two* units of the dimension of the phase space in which solutions are searched for.

Sniatycki and Tulczyjew [37] and, when first integrals come from the momentum map of a Lie group action, Meyer [34], Marsden and Weinstein [33], developed a geometric presentation of this reduction procedure, widely known now under the name "Marsden-Weinstein reduction".

Another way in which symmetries of a Hamiltonian system can be used to facilitate the determination of its integral curves was discovered around 1750 by Euler when he derived the equations of motion of a rigid body about a fixed point. In a short Note published in 1901 [36], Henri Poincaré formalized and generalized this reduction procedure, often called today, rather improperly, "Lagrangian reduction" while the equations obtained by its application are called the "Euler-Poincaré equations" [7, 8].

5.1. The Marsden-Weinstein Reduction Rrocedure

Theorem 3. *Let (M, ω) be a connected symplectic manifold on which a Lie group G acts by a Hamiltonian action Φ, with a momentum map $J : M \to \mathcal{G}^*$. Let $\xi \in J(M) \subset \mathcal{G}^*$ be a possible value of J. The subset G_ξ of elements $g \in G$ such that $\Phi_g\bigl(J^{-1}(\xi)\bigr) = J^{-1}(\xi)$ is a closed Lie subgroup of G.*

If in addition ξ is a weakly regular value of J in the sense of Bott [6]*, $J^{-1}(\xi)$ is a submanifold of M on which G_ξ acts, by the action Φ restricted to G_ξ and to $J^{-1}(\xi)$, in such a way that all orbits are of the same dimension. For each $x \in J^{-1}(\xi)$ the kernel of the two-form induced by ω on $J^{-1}(\xi)$ is the space tangent at this point to its G_ξ-orbit. Let $M_\xi = J^{-1}(\xi)/G_\xi$ be the set of all these orbits. When M_ξ has a structure of smooth manifold such that the canonical projection $\pi_\xi : J^{-1}(\xi) \to M_\xi$ is a submersion, there exists on M_ξ a unique symplectic form ω_ξ such that $\pi_\xi^*\omega_\xi$ is the two-form induced on $J^{-1}(\xi)$ by ω. The symplectic manifold (M_ξ, ω_ξ) is called the* reduced symplectic manifold (*in the sense of Marden and Weinstein*) *for the value ξ of the momentum map.*

Proof: Proposition 15 shows that there exists an affine action a of G on $\mathcal{G}^*$ for which the momentum map J is equivariant. The subset G_ξ of G is therefore the isotropy subgroup of ξ for the action a, which proves that it is indeed a closed subgroup of G. A well known theorem due to Cartan allows us to state that G_ξ is a Lie subgroup of G.

When ξ is a weakly regular value of J, $J^{-1}(\xi)$ is a submanifold of M and, for each $x \in J^{-1}(\xi)$, the tangent space at x to this submanifold is $\ker T_xJ$ (it is the definition of a weakly regular value in the sense of Bott). Let $N = J^{-1}(\xi)$ and let $i_N : N \to M$ be the canonical injection. For all $x \in N$, the vector spaces $\ker T_xJ$ all are of the same dimension $\dim N$, and $\dim\bigl(T_xJ(T_xM)\bigr) = \dim M - \dim N$. Corollary 1 shows that $T_xJ\bigl(T_xM\bigr)$ is the annihilator of $\mathcal{G}_x$. Therefore for all $x \in N$ the isotropy subalgebras $\mathcal{G}_x$ are of the same dimension $\dim G - \dim M + \dim N$. The G_ξ-orbits of all points $x \in N$ are all of the same dimension $\dim G_\xi - \dim G_x$. Corollary 1 also shows that $\mathrm{orth}(\ker T_xJ) = \mathrm{orth}(T_xN) = T_x\bigl(\Phi(G, x)\bigr)$. Therefore, for each $x \in N$

$$\ker(i_N^*\omega)(x) = T_xN \cap \mathrm{orth}(T_xN) = T_xN \cap T_x\bigl(\Phi(G, x)\bigr) = T_x\bigl(\Phi(G_\xi, x)\bigr).$$

It is indeed the space tangent at this point to its G_ξ-orbit. When $M_\xi = N/G_\xi$ has a smooth manifold structure such that the canonical projection $\pi_\xi : N \to M_\xi$ is a submersion, for each $x \in N$ the kernel of $T_x\pi_\xi$ is $\ker(i_N^*\omega)(x)$, and the existence on M_ξ of a symplectic form ω_ξ such that $\pi_\xi^*(\omega_\xi) = i_N^*\omega$ easily follows. $\square$

Proposition 20. *The assumptions made here are the strongest of those made in* Theorem 3: *the set $J^{-1}(\xi)/G_\xi$ has a smooth manifold structure such that the canonical projection $\pi_\xi : J^{-1}(\xi)/G_\xi$ is a submersion. Let $H : M \to \mathbb{R}$ be a smooth Hamiltonian, invariant under the action Φ. There exists an unique smooth function $H_\xi : M_\xi \to \mathbb{R}$ such that $H_\xi \circ \pi_\xi$ is equal to the restricton of H to $J^{-1}(\xi)$. Each integral curve $t \mapsto \varphi(t)$ of the Hamiltonian vector field X_H which meets $J^{-1}(\xi)$ is entirely contained in $J^{-1}(\xi)$, and in the reduced symplectic manifold (M_ξ, ω_ξ) the parameterized curve $t \mapsto \pi_\xi \circ \varphi(t)$ is an integral curve of X_{H_ξ}.*

Proof: As in the proof of Theorem 3, we set $N = J^{-1}(\xi)$ and denote by $i_N : N \to M$ the canonical injection. Let $\omega_N = i_N^*\omega$. Since H is invariant under the action Φ, it keeps a constant value on each orbit of G_ξ contained in N, so there exists on M_ξ an unique function H_ξ such that $H_\xi \circ \pi_\xi = H \circ i_N$. The projection π_ξ being a surjective submersion, H_ξ is smooth. Noether's theorem (2) proves that the momentum map J remains constant on each integral curve of the Hamiltonian vector field X_H. So if one of these integral curves meets N it is entirely contained in N, and we see that the Hamiltonian vector field X_H is tangent to N. We have, for each $x \in N$

$$\begin{aligned}\pi_\xi^*\Bigl(\iota\bigl(T_x\pi_\xi\bigl(X_H(x)\bigr)\bigr)\omega_\xi\bigl(\pi_\xi(x)\bigr)\Bigr) &= \iota\bigl(X_H(x)\bigr)\bigl(i_N^*\omega(x)\bigr) = -\mathrm{d}(i_N^*H)(x)\\ &= -\pi_\xi^*\bigl(\mathrm{d}H_\xi\bigr)(x) = \pi_\xi^*\bigl(\iota(X_{H_\xi})\omega_\xi\bigr)(x).\end{aligned}$$

Since π_ξ is a submersion and ω_ξ a non-degenerate two-form, this implies that for each $x \in N, T_x\pi_\xi\bigl(X_H(x)\bigr) = X_{H_\xi}\bigl(\pi_\xi(x)\bigr)$. The restriction of X_H to N and X_{H_ξ} are therefore two vector fields compatible with respect to the map $\pi_\xi : N \to M_\xi$, which implies the stated result. $\square$

Remark 7. Theorem 3 and Proposition 20 still hold when instead of the Lie group action Φ we have an action φ of a finite-dimensional Lie algebra. The proof of the fact that the G_ξ-orbits in $J^{-1}(\xi)$ all are of the same dimension can easily be adapted to prove that the vector spaces $\{\varphi(x); X \in \mathcal{G}_\xi\}$, with $x \in J^{-1}(\xi)$, all are of the same dimension and determine a foliation of $J^{-1}(\xi)$. We have then only to replace the G_ξ-orbits by the leaves of this foliation.

5.1.1. Use of the Marsden-Weinstein Reduction Procedure

Theorem 3 and Proposition 20 are used to determine the integral curves of the Hamiltonian vector field X_H contained in $J^{-1}(\xi)$ in two steps

- their projections on M_ξ are first determined: they are integral curves of the Hamiltonian vector field X_{H_ξ}. This step is often much easier than the full determination of the integral curves of X_H, since the dimension of the reduced symplectic manifold M_ξ is smaller than the dimension of M.

- Then these curves themselves are determined. This second step, called *reconstruction*, involves the resolution of a differential equation on the Lie group G_ξ.

Many scientists (T. Ratiu, R. Cushman, J. Sniatycki, L. Bates, J.-P. Ortega, ...) generalized this reduction procedure in several ways: when M is a Poisson manifold instead of a symplectic manifold, when ξ is not a weakly regular value of J, etc. The reader will find more results on the subject in the recent book by Ortega and Ratiu [35].

Reduced symplectic manifolds occur in many applications other than the determination of integral curves of Hamiltonian systems. The reader will find such applications in the book by Guillemin and Sternberg [15] and in the papers on the phase space of a particle in a Yang-Mills field [40, 43]).

5.2. The Euler-Poincaré Equation

In his Note [36], Poincaré writes the equations of motion of a Lagrangian mechanical system when there exists a locally transitive action of a finite-dimensional Lie algebra on its configuration space. Below we translate his results in the Hamiltonian formalism.

Proposition 21. *Let $\mathcal{G}$ be a finite-dimensional Lie algebra which acts, by an action $\varphi : \mathcal{G} \to A^1(N)$, on a smooth manifold N. The action φ is assumed to be locally transitive, which means that for each $x \in N$, $\bigl\{\varphi(X)(x)\,;X \in \mathcal{G}\bigr\} = T_xN$. Let $\widehat{\varphi} : \mathcal{G} \to A^1(T^*N)$ be the map which associates to each $X \in \mathcal{G}$ the canonical lift to T^*N of the vector field $\varphi(X)$ on N (4). The map $\widehat{\varphi}$ is a Hamiltonian action of $\mathcal{G}$ on $(T^*N, \mathrm{d}\eta_N)$ which admits the map $J : T^*N \to \mathcal{G}^*$, given by*

$$\bigl\langle J(\xi), X\bigr\rangle = \iota\bigl(\widehat{\varphi}(X)\bigr)\eta_N(\xi), \qquad X \in \mathcal{G}, \qquad \xi \in T^*N$$

as a momentum map. Let $H : T^*N \to \mathbb{R}$ be a smooth Hamiltonian, which comes from a hyper-regular Lagrangian $L : TN \to \mathbb{R}$ (hyper-regular means that the associated Legendre map $\mathcal{L} : TN \to T^*N$ is a diffeomorphism). Let $\psi : I \to T^*N$ be an integral curve of the Hamiltonian vector field X_H defined on an open interval I and let $V : I \to \mathcal{G}$ be a smooth parameterized curve in $\mathcal{G}$ which satisfies, for each $t \in I$

$$\varphi\bigl(\pi_N \circ \psi(t), V(t)\bigr) = \frac{\mathrm{d}\bigl(\pi_N \circ \psi(t)\bigr)}{\mathrm{d}t}. \tag{3}$$

The curve $J\circ\psi : I \to \mathcal{G}^*$, obtained by composition with J of the integral curve ψ of the Hamiltonian vector field X_H, satisfies the differential equation in $\mathcal{G}^*$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{ad}^*_{V(t)}\right)\bigl(J\circ\psi(t)\bigr) = J\Bigl(d_1\overline{L}\bigl(\pi_N\circ\psi(t), V(t)\bigr)\Bigr). \tag{4}$$

We have denoted by $\overline{L} : N\times\mathcal{G}\to\mathbb{R}$ the map

$$(x,X)\mapsto \overline{L}(x,X) = L\bigl(\varphi(X)(x)\bigr), \qquad x\in N, \qquad X\in\mathcal{G},$$

and by $d_1\overline{L} : N\times\mathcal{G}\to T^*N$ the partial differential of $\overline{L}$ with respect to its first variable.

Equation (4) *is called the* Euler-Poincaré equation*, while Equation* (3) *is called the* compatibility condition*.*

The reader is referred to [32] for the proof of this Proposition.

5.2.1. Use of the Euler-Poincaré Equation for Reduction.

Poincaré observes in his Note [36] that the Euler-Poincaré equation (4) can be useful mainly when its right hand side vanishes and when it reduces to an autonomous differential equation on $\mathcal{G}^*$ for the parameterized curve $t\mapsto J\circ\psi(t)$. The first condition is satisfied when the Hamiltonian system under consideration describes the motion of a rigid body around a fixed point in the absence of external forces (Euler-Poinsot problem). The second condition generally is not satisfied, since the Euler-Poincaré equation involves the parameterized curve $t\mapsto V(t)$ in $\mathcal{G}$, whose dependence on $J\circ\psi(t)$ is complicated.

However,, this simplification occurs when there exists a smooth function $h:\mathcal{G}^*\to\mathbb{R}$ such that

$$H = h\circ J$$

which implies that H is constant on each level set of J. Then it can be shown that the Euler-Poincaré equation becomes the Hamilton equation on $\mathcal{G}^*$ for the Hamiltonian h and its canonical Poisson structure.

If we assume that the manifold N is a Lie group G and that $\varphi:\mathcal{G}\to A^1(G)$ of its Lie algebra is the action associated to the action of G on itself by translations on the left (respectively, on the right), $\widehat{\varphi}$ is Lie algebra action associated to the canonical lift to T^*G of the canonical action of G on itself by translations on the left (respectively, on the right). The conditions under which the Euler-Poincaré equation can be used for reduction are exactly the same as those under which the Marsden-Weinstein reduction method can be applied, but for the canonical lift to T^*G of the action of G on itself by translations *on the right* (respectively, on the left). Moreover, applications of these two reduction methods lead to essentially the same equations: the only difference is that the Euler-Poincaré reduction method leads to a differential equation on $\mathcal{G}^*$, while the Marsden-Weinstein reduction method leads, for each value of the momentum map, to the same differential equation restricted to a coadjoint orbit of $\mathcal{G}^*$). The reader will find the proof of these assertions in [30, 32].

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