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THE DYNAMICS OF THE FIELD OF LINEAR FRAMES AND GAUGE GRAVITATION

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Abstract. The paper is motivated by gauge theories of gravitation and condensed matter, tetrad models of gravitation and generalized Born-Infeld type nonlinearity. The main idea is that any generally-covariant and $\mathrm{GL}(n,\mathbb{R})$-invariant theory of the n-leg field (tetrad field when $n = 4$) must have the Born-Infeld structure. This means that Lagrangian is given by the square root of the determinant of some second-order twice covariant tensor built in a quadratic way of the field derivatives. It is shown that there exist interesting solutions of the group-theoretical structure. Some models of the interaction between gravitation and matter are suggested. It turns out that in a sense the space-time dimension $n = 4$, the normal-hyperbolic signature and velocity of light are integration constants of our differential equations.

1. Introduction

No doubt, the standard General Relativity based on the Hilbert-Einstein action functional, perhaps with an extra introduced cosmological term, is a most adequate relativistic theory of gravitational phenomena in the macroscopic and cosmic scale. It is both so-to speak intrinsically aesthetic and confirmed with an impressive accuracy by experimental data. Nevertheless, there are certain shortcomings when trying to apply it to the microscopic range of phenomena. It is a well-known historical fact that there is an intriguing discrepancy between General Relativity and quantum physics. There is still no good quantum version of this theory, and besides, it seems to be non-renormalizable in a rather notorious way. The problem did not exist before the advent of quantum theory, first of all before the theory of quantized fields was developed. There is also another, more modern circumstance which seemed to raise the idea of modifying General Relativity so as to make it suitable for describing microscopic gravitational phenomena. Namely, in a few last decades a new methodology or rather theoretical scheme appeared in physics, namely gauge theories. They are based on the idea of the local, i.e., coordinates-dependent internal invariance. The scheme is a generalization of the minimal coupling idea in electrodynamics and the related idea by H. Weyl of the local dilatational invariance. Besides, some primary ideas were formulated in XIX-th century by Kirchhoff in the theory of elastic rods. Nowadays the topic exploded and is basic for the modern theory of electroweak interactions and for the theory of strong interactions based on the idea of chromodynamics. Besides, the gauge ideology was successfully applied within the theory of condensed matter. This covers the theory of superfluids, superconductivity and to some extent the theory of defects in continua, first of all in elastic continua. Certain ideas and results concerning the gauge analogy between fundamental theories and condensed matter theory have been formulated among others by E. Kröner, F. Hehl, Y. Ne'emann, and the large crowd of other physicists. This scheme is theoretically very successful and seems to be a very natural and unifying idea in physics. This became particularly convincing after the proof that the gauge theory of electroweak interactions is renormalizable. But this again has opened the discussion concerning quantization of relativity. Namely, either gravitation is a completely different kind of phenomena than those described by all other fields theories, or should it be also reformulated in gauge terms. Many gauge models of gravitation did appear. Unlike in other, earlier gauge theories their gauge groups had originally spatio-temporal geometric origin, like the Lorentz or Poincare groups, the Lorentz-conformal group or projective group, but also the full linear or affine groups and even their complex extensions. To be honest, one uses the universal covering groups of the mentioned transformation groups, because it is just the inclusion of spin into the treatment that is an additional motivation for the gravitational gauge treatment. However, there is one very important point. Namely, the mentioned groups have the spatio-temporal origin and primarily they act in the flat space-time manifold. But in the gauge approaches to gravitation they change their meaning and become internal groups. Though the manifold treatment would be just incompatible with their original, literal meaning. Within the gauge approaches to gravitation and to the generally-relativistic spinor theory in manifolds, those groups are structural groups of appropriate fibre bundles over the space-time manifold. For example, the Lorentz group acts along the fibres of the principal bundle of the Lorentz-orthonormal frames, not as a transformation group of the a priori amorphous space-time. The connection forms describing gauge fields undergo the corresponding non-homogeneous transformation rule, while the matter fields, e.g., spinorial ones, transform in a homogeneous way. As mentioned, various groups were used as gauge groups of gravitation, or more precisely, of the spinorial geometrodynamics. Let me mention $\mathrm{SL}(2,\mathbb{C})$ as a covering of $\mathrm{SO}^{\uparrow}(1,3)$, $\mathrm{SL}(2,\mathbb{C}) \times \mathbb{R}^4$ as a covering of Poincare group, $\mathrm{SU}(2,2)$ as a covering of the Lorentz-conformal group, $\mathrm{GL}(4,\mathbb{R})$, $\mathrm{GAf}(4,\mathbb{R}) \simeq \mathrm{GL}(4,\mathbb{R}) \times \mathbb{R}^4$ [5,6,9–14], etc., e.g. $\mathrm{GL}(4,\mathbb{C})$, $\mathrm{GAf}(4,\mathbb{C}) \simeq \mathrm{GL}(4,\mathbb{C}) \times \mathbb{C}^4$. Let me mention that affine groups fail to be semisimple and their metric tensors are fairly non-unique (because their Killing tensors, being degenerate then, are non-suitable as internal metrics). And besides, there are some strange points concerning their translation subgroups which were often treated as a kind of "external gauge groups". From this point of view $\mathrm{SU}(2,2)$ seems to be more adequate.

In any case, roughly speaking, in gauge models it is in a sense a system of covector fields that is a proper gravitational potential. And here one returns to some very old idea in relativistic gravitation. Namely, many years ago it was just Einstein himself and Weyl who suggested the covector model of gravitation within a rather different framework of co-tetrads and tetrads. The idea was to remain as close to the specially-relativistic language as possible. Later on, Dirac formulated a generally-relativistic spinor theory which must use co-tetrads if we are not to modify too deeply the theory of spinors. But further on, Möller, Plebanski, Pellegrinni [7, 8] and many others have noticed that once we accept the idea that gravitation is to be described a generally-covariant variational principle for the cotetrad rather than by one for the metric tensor, the variety of a priori possible dynamical models becomes infinite. And many of them give theoretical predictions as compatible with experimental data as Einstein-Hilbert theory. Later on Toller [15] developed a more general theory based on covector and scalar fields. Obviously, all those models are not gauge theories, nevertheless they are also based on systems of covector fields as potentials. And all of them are generally-covariant. And many authors express the opinion that they are simplified, so-to-speak toy approaches to the gauge methodology. The main difference between them and gauge theories is that they are globally, not locally, invariant under the matrix transformations acting on the co-tetrad (or tetrad) legs.

But there is also some other motivation for those models intrinsically globally invariant under the action of $\mathrm{GL}(4,\mathbb{R})$. The first motivation is an attempt to formulate their higher-dimensional "Kaluza-Klein versions". It seems that in globally $\mathrm{GL}(n,\mathbb{R})$-invariant models in an n-dimensional $(n > 4)$ "Universe" there are solutions which from the four-dimensional point of view predict the existence of gauge as some special solutions. But there exists also another important motivation, perhaps more important one. It is based on the big-bang scenario. Namely, one can reasonably suppose that in the first moments of the expanding evolution of the Universe, the global $\mathrm{GL}(4,\mathbb{R})$-symmetry might be more essential than the local, i.e., x-dependent one. The local invariance under $\mathrm{SO}(1,3)$ might perhaps appear as an averaging effect on the later stage of the cosmic evolution. In any case, the globally $\mathrm{GL}(4,\mathbb{R})$- or $\mathrm{GL}(n,\mathbb{R})$-invariant term is at least a reasonable candidate for the generally-covariant gravitational Lagrangian. There are some ideas that perhaps the global $\mathrm{GL}(4, \mathbb{R})$-invariant term might be responsible for the dark matter/dark energy in our Universe [1].

2. Various n-Leg Models

Let M be a space-time manifold of yet non-specified dimension n. In the usual space-time we have $n = 4$, in some hypothetical "Kaluza-Klein Universe" $n > 4$. The bundle of linear frames, i.e., ordered bases will be denoted by FM and the natural projection by $\pi : FM \to M$. To be more precise

$$FM = \bigcup_{y\in M} F_y M \subset \bigcup_{y\in M} \underbrace{T_y M \times \cdots \times T_y M}_{n \text{ times}}$$

where $F_y M$ is the manifold of ordered linear frames in the tangent space $T_y M$. It is an open subset in

$$(T_y M)^n = \underbrace{T_y M \times \cdots \times T_y M}_{n \text{ times}}.$$

The manifold FM is canonically isomorphic with the dual manifold of co-frames

$$F^\star M = \bigcup_{y\in M} F^\star_y M \subset \bigcup_{y\in M} \underbrace{T^\star_y M \times \cdots \times T^\star_y M}_{n \text{ times}}$$

where obviously $T^\star_y M$ is the dual space of $T_y M$, i.e., the set of ordered dual bases. Projections $\pi : FM \to M$, $\pi^\star : F^\star M \to M$ are defined as

$$\pi(F_y M) = \pi^\star(F^\star_y M) = y.$$

The frame $\widetilde{e} = (\cdots, e^A, \cdots)$ dual to $e = (\cdots, e_A, \cdots)$ is defined according to the usual rule

$$e^A(e_B) = \langle e^A, e_B\rangle = \delta^A{}_B. \tag{1}$$

If x^i are local coordinates in M, then the corresponding local coordinates in FM, $F^\star M$ will be denoted by $(x^i, e^i{}_A)$, $(x^j, e^A{}_j)$, and then (1) may be written as

$$e^A{}_i e^i{}_B = \delta^A{}_B, \qquad e^i{}_A e^A{}_j = \delta^i{}_j.$$

FM, $F^\star M$ are principal fibre bundles of $\mathrm{GL}(n, \mathbb{R})$ in the sense of their standard action

$$e \mapsto eL = (\cdots, e_A, \cdots)L = (\cdots, e_B L^B{}_A, \cdots)$$

$$L \in \mathrm{GL}(n, \mathbb{R})$$

$$\widetilde{e} \mapsto \widetilde{e}L = (\cdots, e^A, \cdots)L = (\cdots, L^{-1A}{}_B e_B, \cdots).$$

In the gauge models of gravitation one considers the local action of $\mathrm{GL}(n, \mathbb{R})$ on the fields of frames and co-frames. So, any matrix field $L : M \to \mathrm{GL}(n, \mathbb{R})$ acts on the cross-sections fields

$$e : M \to FM, \qquad \widetilde{e} : M \to F^{\star}M$$

according to rule

$$(eL)_y = e_y L(y), \qquad (\widetilde{e}L)_y = \widetilde{e}_y L(y)$$

for any $y \in M$.

In the usual tetrad models of gravitation, especially when the spinor fields are used, e is used as a primary gravitational potential, and the metric tensor g is a merely its byproduct.

Let us now review certain algebraic and differential concomitants of the field of frame e.

The Dirac-Einstein metric tensor is given by

$$h[e, \eta] = \eta_{AB} e^A \otimes e^B, \qquad \text{i.e.,} \qquad h_{ij} = \eta_{AB} e^A{}_i e^B{}_j .$$

Here η is pseudo-Euclidean in $\mathbb{R}^n$, for example in the four-dimensional general relativity

$$[\eta_{AB}] = \mathrm{diag}(1, -1, \cdots, -1).$$

Let us observe that $h[e, \eta]$ is ocally invariant under $\mathrm{SO}(n, \eta)$, but is not invariant under one larger subgroup of $\mathrm{GL}(n, \mathbb{R})$.

Teleparallelism connection $\Gamma_{\mathrm{tel}}[e]$ is defined by the conditions

$$\overset{e}{\nabla} e_A = 0$$

and in local coordinates this means that

$$\overset{e}{\Gamma}{}^i{}_{jk} = e^i{}_A e^A{}_{j,k}$$

where the comma sign denotes the partial differentiation.

Teleparallelism torsion is just the torsion tensor of the above affine connection

$$S[e]^i{}_{jk} = \Gamma_{\mathrm{tel}}[e]^i{}_{[jk]} = \frac{1}{2} e^i{}_A \left(e^A{}_{j,k} - e^A{}_{k,j} \right).$$

Let us observe that this object is globally $\mathrm{GL}(n, \mathbb{R})$-invariant

$$S[eL] = S[e], \qquad L \in \mathrm{GL}(n, \mathbb{R})$$

but obviously it is not locally invariant, i.e., under coordinates-dependent L. This tensor is e-equivalent to the non-holonomy object of e, i.e.,

$$S^i{}_{jk} = \gamma^A{}_{BC} e^i{}_A e^B{}_j e^C{}_k, \qquad [e_A, e_B] = \gamma^C{}_{AB} e_C .$$

This quantity may be interpreted as the invariant tensorial derivative of the field of frames e.

Levi-Civita connection $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}_h$ built of the Dirac-Einstein metric tensor. It is globally $\mathbb{R}^{+}\mathrm{SO}(n,\eta)$-invariant. The manifold $(M, \Gamma[e], h[e,\eta])$ is a Riemann-Cartan space, because

$$\overset{e}{\nabla} h[e,\eta] = 0.$$

Therefore, we have

$$\Gamma^{i}{}_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + S^{i}{}_{jk} + S_{jk}{}^{i} - S_{k}{}^{i}{}_{j}$$

where the shift of indices is meant in the h-sense.

Scalar Weitzenböck invariants, i.e., basic $\mathrm{SO}(n,\eta)$-invariant quadratic forms of S

$$\mathcal{J}_1 = h_{ai} b^{bj} h^{ck} S^{a}{}_{bc} S^{i}{}_{jk}, \qquad \mathcal{J}_2 = h^{ij} S^{a}{}_{ib} S^{b}{}_{ja}, \qquad \mathcal{J}_3 = h^{ij} S^{a}{}_{ai} S^{b}{}_{bj}.$$

Globally $\mathrm{GL}(n,\mathbb{R})$-invariant tensors built of S. Let us quote a few types of them. The covariant Killing-Casimir tensors built of S

$$\gamma_i = 2S^{j}{}_{ij}, \qquad \gamma_{ij} = 4S^{k}{}_{im} S^{m}{}_{jk} - \text{“Killing tensor”}$$

$$\gamma_{i_1 \cdots i_k} = 2^k S^{j}_{i_1 l} S^{l}{}_{i_2 m} \cdots S^{p}{}_{i_k j}.$$

All of them are symmetric.

Mixed Γ-objects, i.e., the following ones

$$S^{i}{}_{jk} \quad \text{itself}, \quad \Gamma^{i}{}_{jmn} = 4S^{i}{}_{jk} S^{k}{}_{mn}, \quad \Gamma^{i}{}_{jkrs} = 8S^{i}{}_{mn} S^{m}{}_{jk} S^{n}{}_{rs}, \cdots, \quad \text{etc.}$$

Second order skew-symmetric object

$$\Gamma_{ij} = 4S^{k}{}_{lk} S^{l}{}_{ij} = -\Gamma_{ji}.$$

Finally let us define the most general second-order tensor quadratic in S

$$T_{ij} = \lambda\gamma_{ij} + \mu\gamma_i\gamma_j + \nu\Gamma_{ij} = 4\lambda S^{k}{}_{im} S^{m} jk + 4\mu S^{k}{}_{ik} S^{m}{}_{jm} + 4\nu S^{k}{}_{lk} S^{l}{}_{ij}.$$

Particularly interesting is the symmetric, in a sense metric-like part of the last tensor

$$g_{ij} = \lambda\gamma_{ij} + \mu\gamma_i\gamma_j = 4\lambda S^{k}{}_{im} S^{m} jk + 4\mu S^{k}{}_{ik} S^{m}{}_{jm}.$$

In particular, the dominant λ-term has the characteristic Killing structure. It is interesting that when trying to interpret g_{ij} as the metric tensor, we do not introduce signature “by hand”, as it is done when $h[e,\eta]$ is used.

Let us also quote a few globally $\mathrm{GL}(n,\mathbb{R})$-invariant scalars built of $(e, \partial e)$. Among them there are “Weitzenböck-like” quantities

$$I_1 = \gamma_{il} \gamma^{jm} \gamma^{kn} S^{i}{}_{jk} S^{l}{}_{mn}, \qquad I_3 = \gamma^{ij} S^{k}{}_{ik} S^{m}{}_{jm}$$

$$I_2 = \gamma^{ij} S^{m}{}_{in} S^{n}{}_{jm} = \frac{1}{4} - \text{trivial invariant.}$$

There are also other ones, e.g.,

$$\mathrm{Tr}(\hat{\Gamma}^p) = \underbrace{\Gamma^i{}_j \Gamma^j{}_k \cdots \Gamma^l{}_m \Gamma^m{}_i}_{p \text{ factors}}$$

where the indices are shifted with the help of γ_{ij}. Obviously, it was assumed here and in the previous formulas that the Killing tensor is non-degenerate.

One can easily prove that all $\mathrm{GL}(n, \mathbb{R})$-invariant scalars built intrinsically of S are zeroth order homogeneous functions of this quantity. The proof is not difficult and follows from the Noether theorems.

It must be stressed that even in Einstein general relativity the tetrad language is convenient because it enables one to reformulate the theory in such a way that its Lagrangian may be correctly identified with some weight-one Lagrangian density. Namely, after some easy calculations one can show that the scalar curvature density may be expanded as follows

$$R[h]\sqrt{|h|} = (\mathcal{J}_1 + 2\mathcal{J}_2 - 4\mathcal{J}_3)\sqrt{|h|} + 4\partial_i \left(S^a{}_{ab} h^{bi} \sqrt{|h|} \right). \tag{2}$$

But the last term here is a well-defined scalar density of weight one, containing all second derivatives. And being a divergence of the contravariant vector density of weight one it may be simply removed from the Lagrangian. The remaining term

$$L = L_1 + 2L_2 - 4L_3 \tag{3}$$

is a well-defined scalar density of weight one depending only on e and its first derivatives. But once written in this form. Lagrangian (3) suggests the possibility of various modifications like

$$L = c_1 L_1 + c_2 L_2 + c_3 4L_3 \tag{4}$$

where c_1, c_2, c_3 are to some extent arbitrary. It turns out that this does not violate predictions of the theory. There is however, one important point: (3) is locally Lorentz-invariant under the intrinsic action of $\mathrm{SO}(1, n-1)$, but (4) is not. Only global Lorentz invariance survives. But as mentioned above, one can also try to modify further the theory, for example by putting

$$L(S, h) = g(S, h)\sqrt{|h|} \tag{5}$$

where $g(S, h)$ need not any longer be linear in Weitzenböck invariants. This nonlinearity is stronger than the usual nonlinearity of the usual nonlinearity of the theory based on the Lagrangians in (3), (4). Because of this people were interested in such models when the cosmological singularities seemed to be not welcome.

Before going any further, let us summarize the above remarks and introduce two additional concepts, a kind of "field momenta", using electromagnetic terms. Our Lagrangians are scalar densities of weight one, built of two main variables: h and

S. Let us denote them by $L(S,h)$, however without assuming them to be given by (5). In analogy to electrodynamics, the quantity

$$H_i{}^{jk} := \frac{\partial L}{\partial S^i{}_{jk}}$$

will be referred to as a field momentum. Just like in electrodynamics the quantity

$$H^{ij} = \frac{\partial L_{\rm Maxwell}}{\partial F_{ij}}$$

i.e., the $(\bar{D},\bar{H})$-fields, is a field momentum conjugate to the electromagnetic field F_{ij}, i.e., to the $(\bar{E},\bar{B})$-fields.

In analogy to elasticity we introduce also the field which may be called Dirac-Einstein stress

$$Q^{ij} = \frac{\partial L}{\partial h_{ij}}\cdot$$

Then the resulting field equations may be symbolically written in the following rough form

$$K_i{}^j = \overset{e}{\nabla}_k H_i{}^{jk} + 2S^l{}_{lk}H_i{}^{jk} - 2h_{ik}Q^{kj} = 0. \tag{6}$$

3. Affinely-Invariant n-Leg Models

But now let us stress one important point. It is only Hilbert-Einstein model (2), (3) that is locally invariant under $\mathrm{SO}(n,\eta)$. But if we once decide to admit global $\mathrm{SO}(n,\eta)$-symmetry, there are no any reasons to insist on the global invariance under $\mathrm{SO}(n,\eta)$. Rather, one should seek models globally invariant under the total $\mathrm{GL}(n,\mathbb{R})$. Because if we do not describe explicitly spinors, this is the most natural symmetry group for the field of n-legs. And even if spinors are introduced, it seems to be a good candidate for the symmetry of at least some essential part of the gravitational Lagrangian. If we assume the global $\mathrm{GL}(n,\mathbb{R})$-invariance, then $Q^{ij} = 0$ and our field equations (6) reduce to

$$K_i{}^j = \overset{e}{\nabla}_k H_i{}^{jk} + 2S^l{}_{lk}H_i{}^{jk} = 0.$$

One can show that every generally-covariant and $\mathrm{GL}(n,\mathbb{R})$-invariant Lagrangian $L(S)$ is an n-th order homogeneous function of S

$$S^i{}_{jk}\frac{\partial L}{\partial S^i{}_{jk}} = S^i{}_{jk}H_i{}^{jk} = nL.$$

No doubt, the simplest scalar density of weight one built of S is given by

$$L = \sqrt{|\det[T_{ij}]|} = \sqrt{|\det[4\lambda S^k{}_{im}S^m{}_{jk} + 4\mu S^k{}_{ik}S^m{}_{jm} + 4\nu S^k{}_{lk}S^l{}_{ij}]|} \tag{7}$$

where λ,μ,ν are some constants. The most important is the λ-term.

It is seen that (7) is the special case of the following Lagrangian

$$L = \sqrt{|\det[L_{ij}(\Psi, \partial\Psi)]|} \tag{8}$$

where L_{ij}, the "Lagrange tensor" is particularly simple, because it is quadratic in field derivatives. This seems to be another pole of mathematical simplicity with respect to linear theories where Lagrangian itself is built in a quadratic way. The structure (8) is a generalized Born-Infeld nonlinearity. What is essential and very interesting is that this Born-Infeld structure is here a direct consequence of the assumed invariance group of the theory, $\mathrm{Diff}M \times \mathrm{GL}(n, \mathbb{R})$ (general covariance and the internal $\mathrm{GL}(n, \mathbb{R})$-symmetry).

The very strong, Born-Infeld-type nonlinearity prevents us from the general solving of the resulting equations. Moreover, it is not quite clear a priori if the theory is not empty (this may happen very easily in generally-covariant theories). It is even difficult to perform the Dirac analysis of constraints. let us only quote a few remarks concerning this question. Let us split our coordinates x^μ, $\mu = 0, 1, \cdots, (n-1)$ so that x^0 formally plays the role of "time". Let $\pi^\mu{}_\mathcal{K}$ denote the density of canonical momentum conjugate to $e^\mathcal{K}{}_\mu$ within the framework of the functional Hamiltonian formalism. One can show that primary constraints have the form

$$\pi^0{}_\mathcal{K} = 0, \qquad \mathcal{K} = 0, 1, \cdots, n-1.$$

Therefore, there are n redundant variables among the n^2 field quantities. One can try to eliminate them with the help of some coordinate conditions, e.g.,

$$e^\mu{}_\mathcal{K} = \delta^\mu{}_\mathcal{K} \qquad \text{for a fixed } \mathcal{K}$$

or

$$e^\mathcal{K}{}_\mu{}^{,\mu} = e^\mathcal{K}{}_{\mu,\nu} g^{\nu\mu} = 0, \qquad \text{for all values of } \mathcal{K}.$$

Further, one can show that secondary constraints, at least partially, are given by

$$\mathcal{K}_\alpha{}^0 = \overset{e}{\nabla}_\beta H_\alpha{}^{0\beta} + 2S^\mu{}_{\mu\beta} H^{0\beta} - 2h_{\alpha\beta} Q^{\beta 0} = 0$$

including the case of $\mathrm{SO}(n, \eta)$-invariant theory. The points is that $\mathcal{K}_\alpha{}^0$ do not involve second-order time derivatives $e^N_{,00}$.

In any case this resembles secondary constraints in the Hilbert-Einstein theory and in electrodynamics

$$R_\alpha{}^0 = 0, \qquad H^{0j}_{,j} = \mathrm{div}\bar{D} = 0.$$

But quite independently on this constraints argument, one can show that our equations are non-empty (non-contradictory). Namely, it is very easy to prove the following

Theorem 1. *Let us assume that the n-leg fields e_A, $A = 1, \cdots, n$ span a semisimple Lie algebra* (*in the Lie-bracket sense*)

$$[e_K, e_L] = \gamma^M{}_{KL} e_M, \qquad \gamma^M{}_{KL} = \mathrm{const}\,.$$

Then e solves our $\mathrm{GL}(n, \mathbb{R})$*-invariant field equations.*

It is very interesting that the signature of the Killing metric tensor is not introduced "by hand" or by experimental data, but is in a sense a system of integration constants for our equations.

Let us also mention about some reference to the Einstein-type general relativity. Namely, if G is a semisimple Lie group, γ its **Killing tensor** and $R_{\mu\nu}$ are components of the **Ricci tensor**, then [4]

$$R_{ij} - \frac{1}{2} R\gamma_{ij} = -\frac{1}{8}(n-2)\gamma_{ij}.$$

Rescaling the definition of the spatio-temporal metric tensor

$$g_{ij} = a\gamma_{ij}, \qquad a = \mathrm{const}$$

one obtains

$$R_{ij} - \frac{1}{2} R g_{ij} = \Lambda g_{ij}, \qquad \Lambda = -\frac{n-2}{8a}\cdot$$

But apparently the physical dimension $n = 4$ leads to some disappointment. Because the four-dimensional semisimple Lie algebras do not exist. One can try to answer the problem on a few independent ways.

$A)$ Perhaps physically $n > 4$, i.e., we are as a matter of fact in some "Kaluza-Klein Universe".

$B)$ Perhaps to include some matter in the Born-Infeld way, e.g., the complex scalar field Ψ

$$L_{\mu\nu} = (1 - a\bar{\Psi}\Psi)\gamma_{\mu\nu} + b\bar{\Psi}_{,\mu}\Psi_{,\nu}, \qquad L = \sqrt{|\det[L_{\mu\nu}]|}.$$

One can also admit a multicomponent scalar field [2, 3]

$$L_{\mu\nu} = \left(1 - a_{\bar{k}l}\bar{\Psi}^{\bar{k}}\Psi^l\right)\gamma_{\mu\nu} + b_{\bar{k}l}\bar{\Psi}^k_{,\mu}\Psi^l_{,\nu}$$

where the internal tensors (a, b) are hermitian.

$C)$ It turns out that there exist solutions of the form: deformed trivial central extensions of semisimple Lie groups.

Let us describe briefly the last idea. Take the field of frames $E = (\cdots, E_\Sigma, \cdots) = (E_0, \cdots, E_A, \cdots)$, and its dual $\widetilde{E} = (\cdots, E^\Sigma, \cdots) = (E^0, \cdots, E^A, \cdots)$, where

$\Sigma = 0, 1, \cdots, n-1$, $A = 1, \cdots, n-1$. Let us assume the following non-semisimple Lie-algebraic structure

$$[E_0, E_{\mathcal{K}}] = 0, \qquad [E_{\mathcal{K}}, E_L] = C^M{}_{\mathcal{K}L} E_M$$
$$\mathrm{d}E^0 = 0, \qquad \mathrm{d}E^{\mathcal{K}} = \frac{1}{2} C^{\mathcal{K}}{}_{LM} E^M \Lambda\, E^L$$
$$\det[C_{\mathcal{K}L}] = \det\left[C^M{}_{KN} C^N{}_{LM}\right] \neq 0.$$

Let us introduce the adapted system of coordinates in M: τ, x^i, $i = 1, \cdots, n-1$, such that

$$E_0 = \frac{\partial}{\partial \tau}, \qquad E_{\mathcal{K}} = E^i{}_{\mathcal{K}}(x) \frac{\partial}{\partial x^i}$$
$$E^0 = \mathrm{d}\tau, \qquad E^{\mathcal{K}} = E^{\mathcal{K}}{}_i(x) \mathrm{d}x^i$$
$$E^{\mathcal{K}}{}_i E^i{}_L = \delta^{\mathcal{K}}{}_L.$$

Therefore, τ is a parameter of integral curves of E_0, and $E_{\mathcal{K}}$ span an $(n-1)$-dimensional foliation; x^i are coordinates along this foliation. And now let us introduce the following field of frames

$$e = \varrho E, \qquad e_{\mathcal{K}} \varrho = E_{\mathcal{K}} \varrho = 0, \qquad \varrho : M \to \mathbb{R}. \tag{9}$$

Proposition 1. *If ϱ has no critical points, then the Killing field $\gamma[e]$ is non-degenerate, and e_0 is $\gamma[e]$-orthogonal to all of $e_{\mathcal{K}}$-s. We have then*

$$\gamma[e] = (n-1)\left(\frac{\mathrm{d}\varrho}{\mathrm{d}\tau}\right)^2 e^0 \otimes e^0 + \varrho^2 C_{\mathcal{K}L} e^{\mathcal{K}} \otimes e^L.$$

In adapted coordinates we have

$$\gamma[e] = (n-1)\left(\frac{\mathrm{d}\ln\varrho}{\mathrm{d}\tau}\right)^2 \mathrm{d}\tau \otimes \mathrm{d}\tau + \underset{(n-1)}{\gamma_{ij}}(x)\mathrm{d}x^i \otimes \mathrm{d}x^j$$
$$\underset{(n-1)}{\gamma_{ij}} = 4 S^k{}_{im} S^m{}_{jk}, \qquad i, j, k, m = 1, \cdots, n-1.$$

If the Lie algebra spanned by $(E_1, \cdots, E_{n-1})$ is compact, then $\gamma[e]$ is normal-hyperbolic $(+-- \cdots -)$. Therefore, the frame e expands, but $\gamma[e]$ is stationary and static. M becomes diffeomorphic to $\mathbb{R}_{(\mathrm{time})} \times G_{(\mathrm{space})}$. Introducing the quantity

$$x^0 := \pm\sqrt{n-1}\,\ln\frac{\varrho}{\delta}, \qquad \delta = \mathrm{const}$$

we obtain

$$\gamma[e] = \mathrm{d}x^0 \otimes \mathrm{d}x^0 + \underset{(n-1)}{\gamma_{ij}}(x^1, \cdots, x^{n-1})\mathrm{d}x^i \otimes \mathrm{d}x^j.$$

One can by a direct calculation prove the following

Theorem 2. *For any ϱ without critical points, the above e is a solution of our* $\mathrm{GL}(n,\mathbb{R})$*-invariant equations for frames.*

The quantity ϱ is a purely gauge variable. If we use the exponential gauge

$$\varrho = \delta \exp(\lambda\tau), \qquad \delta, \lambda = \mathrm{const}$$

then the time variable becomes proportional to the "parametric time" of E_0

$$\begin{aligned}\gamma[e] &= (n-1)\lambda^2 \mathrm{d}t \otimes \mathrm{d}t + \underset{(n-1)}{\gamma_{ij}}(x)\mathrm{d}x^i \otimes \mathrm{d}x^j \\ &= c^2 \mathrm{d}t \otimes \mathrm{d}t + \underset{(n-1)}{\gamma_{ij}}(x)\mathrm{d}x^i \otimes \mathrm{d}x^j.\end{aligned}$$

Let us assume that some test spinor matter is injected into M. According to the standard approach to spinors it will "feel" the Dirac-Einstein metric $h[e,\eta]$, where $\eta_{KL} = C_{KL}$ and $\eta_{00} = \beta^2$

$$h[e,\eta] = \frac{\beta^2}{\delta^2}\mathrm{d}T \otimes \mathrm{d}T + \frac{\lambda^2}{\delta^2}T^2 \underset{(n-1)}{\gamma}[e].$$

Here $\varrho = \delta\exp(\lambda t)$ and T is the absolute time of $h[e,\eta]$

$$T = \pm\frac{1}{\lambda}\exp(-\lambda t).$$

Therefore, although $\gamma[e]$ is stationary and static, the test spinor matter will witness about the cosmological expansion in the sense of the metric tensor $h[e,\eta]$.

If $\lambda < 0$ and $T = -\frac{1}{\lambda}\exp(-\lambda t)$, then $T \in [0,\infty]$ when $t \in [-\infty,\infty]$, and $(M, h[e,\eta])$ expands in spatial directions. If T is to run over $[-\infty,\infty]$, we must glue two situations: $\lambda = \pm p^2$, where p is real. The plus and minus signs correspond to the contraction and expansion phases $T < 0$, $T > 0$. At $T = 0$ there is a singularity.

Let us mention also about another approach to our expanding solutions. Namely, instead of (9) we might also use the following expansion

$$'e = ('e_0, \cdots, 'e_A, \cdots) = (e_0, \cdots, \varrho e_A, \cdots).$$

Now $\gamma['e]$ is the same as previously, $'e$ is a solution and the corresponding Dirac-Einstein metric is given by

$$h['e,\eta] = \beta^2 \mathrm{d}t \otimes \mathrm{d}t + \delta^{-2}\exp(-2\lambda t)\underset{(n-1)}{\gamma}['e].$$

Expansion in the h-sense will appear when $\lambda < 0$, and the contraction when $\lambda > 0$.

Let us finish with quoting examples of four-dimensional solutions

- $\mathbb{R} \times \mathrm{SU}(2)$ or $\mathbb{R} \times \mathrm{SO}(3,\mathbb{R})$, $(+---)$

- $\mathbb{R} \times \mathrm{SL}(2, \mathbb{R})$ $(+++-)$ – "expansion" in one spatial direction, closed time-like curves
- $\mathbb{R} \times \overline{\mathrm{SL}(2, \mathbb{R})}$ – as above, but there are no closed time-like curves. Obviously $\overline{\mathrm{SL}(2, \mathbb{R})}$ denotes the universal covering of $\mathrm{SL}(2, \mathbb{R})$.

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