

THE UNCERTAINTY WAY OF GENERALIZATION OF COHERENT STATES

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Abstract. The three ways of generalization of canonical coherent states are briefly reviewed and compared with the emphasis laid on the (minimum) uncertainty way. The characteristic uncertainty relations, which include the Schrödinger and Robertson inequalities, are extended to the case of several states. It is shown that the standard $SU(1,1)$ and $SU(2)$ coherent states are the unique states which minimize the second order characteristic inequality for the three generators. A set of states which minimize the Schrödinger inequality for the Hermitian components of the $su_q(1,1)$ ladder operator is also constructed. It is noted that the characteristic uncertainty relations can be written in the alternative complementary form.

1. Introduction

Coherent states (CS) introduced in 1963 in the pioneering works by Glauber and Klauder [1] pervade nearly all branches of quantum physics (see the reviews [1, 4]). This important overcomplete family of states $\{|\alpha\rangle\}$, $\alpha \in \mathbb{C}$, can be defined in three equivalent ways [3]:

- D1) As the set of eigenstates of boson destruction operator (the ladder operator) $a : a|\alpha\rangle = \alpha|\alpha\rangle$,
- D2) As the orbit of the ground state $|0\rangle$ ($a|0\rangle = 0$) under the action of the unitary displacement operators $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ (which realize ray representation of the Heisenberg–Weyl group H_1) : $|\alpha\rangle = D(\alpha)|0\rangle$.
- D3) As the set of states which minimize the Heisenberg uncertainty relation (UR) $(\Delta q)^2(\Delta p)^2 \geq 1/4$ for the Hermitian components q, p of a ($a = (q + ip)/\sqrt{2}$) with equal uncertainties: $(\Delta q)^2(\Delta p)^2 = 1/4$, $\Delta q = \Delta p$. Note that one requires the minimization plus the equality of the two variances.

The overcompleteness property reads ($d^2\alpha = d\operatorname{Re}\alpha d\operatorname{Im}\alpha$)

$$1 = \int |\alpha\rangle\langle\alpha| d\mu(\alpha), \quad d\mu(\alpha) = (1/\pi) d^2\alpha. \quad (1)$$

One says that the family $\{|\alpha\rangle\}$ resolves the unity operator with respect to the measure $d\mu(\alpha)$. The CS $|\alpha\rangle$ should be referred as *canonical* CS [1]. The resolution unity property (1) provides the important analytic representation (rep), known as canonical CS rep or Fock–Bargman analytic rep, in which $a = d/d\alpha$, $a^\dagger = \alpha$ and the state $|\Psi\rangle$ is represented by the function $\Psi(\alpha) = \exp(|\alpha|^2/2)\langle\alpha^*|\Psi\rangle$. In 1963–1964 Klauder [1] developed a general theory of the continuous reps and suggested the possibility to construct overcomplete sets of states using irreducible reps of Lie groups. Let us note that the resolution unity property (1) is not a defining one for the CS $|\alpha\rangle$.

Correspondingly to the definitions (D1)–(D3) there are three different ways (methods) of generalization of the canonical CS [3]: The diagonalization of non-Hermitian operators (the *eigenstate way*, or the ladder operator method [5]); The construction of Hilbert space orbit by means of unitary operators (*orbit way* or the displacement operator method [5]); The minimization of an appropriate UR (the *uncertainty way*). The first two methods and especially the second one (the orbit method) have enjoyed a considerable attention and vast applications to various fields of physics [1, 4], while the third method is receiving a significant attention only recently (see [7, 11, 13, 15]) and references therein. It is worth noting at the point that some authors were pessimistic about the possibility of effective generalization of the third defining property of canonical CS.

The aim of the present paper is to consider some of the new developments in the third way (the uncertainty way) and their relationship to the first two methods. We show that the Robertson [16] and other characteristic inequalities [14] are those uncertainty relations which are compatible with the generalizations of the ladder operator and displacement operator methods to the case of many observables.

In Section 2 we briefly review some of the main generalizations of the first two defining properties of the canonical CS and the relationship between the corresponding generalized CS. Some emphasis is laid on the family of squeezed states (SS) [2] and the Barut–Girardello CS (BG CS) [29] and their analytic reps. The canonical SS are the unique generalization of CS for which the three definitions (D1), (D2), (D3) are equivalently generalized.

Section 3 is devoted to the uncertainty way of generalization of CS. In Subsection 3.1 we consider the minimization of the Heisenberg and the Schrödinger UR [16] for two observables and the relation of the minimizing states to the corresponding group-related CS [1], on the examples of $SU(2)$, $SU(1, 1)$ and

$SU_q(1, 1)$. Here we note that the $SU(2)$ and $SU(1, 1)$ CS with lowest (highest) weight reference vector minimize the Schrödinger inequality for the first two generators, while the Heisenberg one is minimized in some subsets only. These group-related CS are particular cases of the corresponding minimizing states. A set of states which minimize the Schrödinger inequality for the Hermitian components of the $SU_q(1, 1)$ ladder operator is also constructed.

In Subsection 3.2 the minimization of the Robertson [17] and the other *characteristic UR* [14] for several observables is considered. In the case of the three generators (three observables) of $SU(1, 1)$ (and the $SU(2)$) we establish that the group-related CS with lowest (highest) weight reference vector are the unique states which minimize the second and the third characteristic UR for the three generators simultaneously. The characteristic UR, in particular the known Robertson and the Schrödinger ones, relate certain combinations of the second and first moment of the observables in one and the same quantum state. Here we extend these relations to the case of several states. States which minimize the characteristic UR are naturally called *characteristic uncertainty states* (characteristic US¹). The alternative names could be (characteristic) intelligent states and (characteristic) optimal US. The extended characteristic UR are also *invariant* under the linear nondegenerate transformation of the observables as the characteristic ones are. It is shown that the characteristic UR can be written in the *complementary form* [15] in terms of two positive quantities less than the unity. Finally it is noted that the positive definite characteristic uncertainty functionals (for several observables) can be used for the construction of distances between quantum states. In the Appendix the proofs of the Robertson relation (after Robertson) and of the uniqueness of the standard $SU(1, 1)$ CS minimization of the second (and third) order characteristic UR are provided.

2. The Eigenstate and Orbit Ways

Canonical CS $|\alpha\rangle = D(\alpha)|0\rangle$ diagonalize the boson destruction operator a , $[a, a^\dagger] = 1$. This was the first and seminal example of diagonalizing of a non-Hermitian operator. We stress that the eigenstates of a and other non-Hermitian operators in this paper are not orthogonal to each other — the term “diagonalization” is used for brevity and in analogy to the case of diagonalization of Hermitian operators. The second example was, to the best of our knowledge, the diagonalization of the complex combination of boson lowering and raising operators a, a^\dagger ($\alpha \in \mathbb{C}$), [18]

$$A(t)|\alpha; t\rangle = \alpha|\alpha; t\rangle, \quad A(t) = u(t)a + v(t)a^\dagger = A(u, v). \quad (2)$$

¹ Let us list the abbreviations used in the paper: CS = coherent state, SS = squeezed state, UR = uncertainty relation, US = uncertainty state, BG = Barut-Girardello, and rep = representation.

The operator $A(t)$ was constructed as a non-Hermitian invariant operator for the quantum varying frequency oscillator with Hamiltonian $H = \frac{p^2 + m^2\omega^2(t)q^2}{2m}$, i. e. $A(t)$ had to obey the equation $\partial A/\partial t - (i/\hbar)[A, H] = 0$ where, m is the mass, and $\omega(t)$ is the varying frequency; the case of varying mass $m(t)$ was reduced to that of constant mass by the time transformation $t \rightarrow t' = m \int^t d\tau/m(\tau)$. For that purpose the parameter $\epsilon = (u - v)/\sqrt{\omega_0}$ was introduced and subjected to obey the classical oscillator equation

$$\ddot{\epsilon} + \omega^2(t)\epsilon = 0. \quad (3)$$

The boson commutation relation $[A, A^\dagger] = 1$ was ensured by the Wronskian $\epsilon^*\dot{\epsilon} - \epsilon\dot{\epsilon}^* = 2i$. Then $\dot{\epsilon} = i(u + v)\sqrt{\omega_0}$, $|u|^2 - |v|^2 = 1$, and the invariant takes the form $A(t) = U(t)(u(0)a + v(0)a^\dagger)U^\dagger(t) \equiv U(t)A(0)U^\dagger(t)$, where $U(t)$ is the evolution operator, and the eigenstates $|\alpha; t\rangle \equiv |\alpha, u(t), v(t)\rangle$ satisfy the Schrödinger evolution equation. One has

$$|\alpha, u(t), v(t)\rangle = U(t)|\alpha, u_0, v_0\rangle, \quad (4)$$

where $A(0)|\alpha, u_0, v_0\rangle = \alpha|\alpha, u_0, v_0\rangle$ and $|u_0|^2 - |v_0|^2 = 1$. This shows that the set $\{|\alpha, u(t), v(t)\rangle\}$ is an orbit through $|\alpha, u_0, v_0\rangle$ of the evolution operator $U(t)$.

In the coordinate rep the wave functions $\langle q|\alpha, u(t), v(t)\rangle$ take the form of an exponential of a quadratic [18] (m is the mass parameter),

$$\langle q|\alpha, u, v\rangle = \frac{(m\omega_0/\pi\hbar)^{1/4}}{(u - v)^{1/2}} \exp \left[-\frac{m\omega_0}{2\hbar} \frac{v + u}{u - v} \left(q - \frac{2\hbar}{m} \frac{\alpha}{u + v} \right)^2 - \frac{1}{2} \left(-\frac{u^* + v^*}{u + v} \alpha^2 + |\alpha|^2 \right) \right]. \quad (5)$$

These wave packets are normalized but not orthogonal to each other. They are solutions to the wave equation for varying frequency oscillator if $u = (\epsilon\sqrt{\omega_0} - i\dot{\epsilon}/\sqrt{\omega_0})/2$, $v = -(\epsilon\sqrt{\omega_0} + i\dot{\epsilon}/\sqrt{\omega_0})/2$, and ϵ is any solution of (3). Note that the time dependence is embedded completely in u and v (or, equivalently, in ϵ and $\dot{\epsilon}$) which justifies the notation $|\alpha; t\rangle = |\alpha, u, v\rangle$. For other systems the invariant $A(t) = U(t)A(0)U^\dagger(t)$ is not linear in a and a^\dagger and its eigenstates are no more of the form $|\alpha, u, v\rangle$ [6]. Therefore the term ‘‘coherent states for the nonstationary oscillator’’ for $|\alpha; t\rangle = |\alpha, u, v\rangle$ [18] is indeed adequate. Time evolution of an initial $|\alpha, u_0, v_0\rangle$ for general quadratic Hamiltonian system was studied in greater detail in [19], where eigenstates of $ua + va^\dagger$ were denoted as $|\alpha\rangle_g$. The invariant $A(t)$ in [18] coincides with the boson operator $b(t)$ in [19].

The states (5) represent the time evolution of the canonical CS $|\alpha\rangle$ if the initial conditions [18] $\epsilon(0) = 1/\sqrt{\omega_0}$, $\dot{\epsilon}(0) = i\sqrt{\omega_0}$ are imposed (then $u(0) = 1$, $v(0) = 0$). Under these conditions $|\alpha, u(t), v(t)\rangle = U(t)|\alpha\rangle$, i. e. the set of $|\alpha, u(t), v(t)\rangle$ becomes an $SU(1, 1)$ orbit through the initial CS $|\alpha\rangle$, since the Hamiltonian of the varying frequency oscillator is an element of the $su(1, 1)$ algebra in the rep with Bargman index $k = 1/4, 3/4$. The $SU(1, 1)$ generators K_i in this rep read ($K_{\pm} = K_1 \pm iK_2$)

$$K_3 = \frac{a^\dagger a}{2} + \frac{1}{4}, \quad K_- = \frac{a^2}{2}, \quad K_+ = \frac{a^{\dagger 2}}{2}. \quad (6)$$

The parameters u, v are in a direct link to the $SU(1, 1)$ group parameters, and α — to the Heisenberg–Weyl group. The whole family of $|\alpha, u, v\rangle$, can be considered as an orbit through the ground state $|0\rangle$ of the unitary operators of the semidirect product $SU(1, 1) \wedge H_1$ [6]. Thus the two definitions (D1) and (D2) here are equivalently generalized. It has been shown [6] that the third definition is also equivalently generalized on the basis of the Schrödinger UR (see next section).

The set $\{|\alpha, u, v\rangle, u, v \text{ — fixed}\}$ resolves the unity operator with respect to the same measure as in the case (1) of canonical CS [18]: $1 = (1/\pi) \int d^2\alpha |\alpha, u, v\rangle\langle v, u, \alpha|$.

A second family of orthonormalized states $|n; t\rangle = |n, u, v\rangle$ was constructed in [18] as eigenstates of the quadratic in a and a^\dagger Hermitian invariant $A^\dagger(t)A(t) = (ua + va^\dagger)^\dagger(ua + va^\dagger)$ which is an element of the Lie algebra $su(1, 1)$. Note that any power of A and A^\dagger is also an invariant. $A^\dagger A$ coincides with the known Ermakov–Lewis invariant. For the N -dimensional quadratic system there are N linear in a_μ and a_μ^\dagger invariants $A_\mu(t) = u_{\mu\nu}a_\nu + v_{\mu\nu}a_\nu^\dagger \equiv A_\mu(u, v)$, which were simultaneously diagonalized [21],

$$A_\mu(u, v)|\vec{\alpha}, u, v\rangle = \alpha_\mu|\vec{\alpha}, u, v\rangle, \quad (7)$$

In different notations exact solutions to the Schrödinger equation for the non-stationary oscillator have been previously obtained e. g. by Husimi [20] and for nonstationary general N -dimensional Hamiltonian by Chernikov [20], but with no reference to the eigenvalue problem of the invariants $ua + va^\dagger$ and/or $(ua + va^\dagger)^\dagger(ua + va^\dagger)$. Eigenstates of other quadratic in a and a^\dagger operators were later considered in many papers, the general one-mode quadratic form being diagonalized by Brif (see [8] and references therein).

By means of the known BCH formula for the transformation $S(\zeta)aS^\dagger(\zeta)$ with $S(\zeta) = \exp[\zeta K_+ - \zeta^* K_-]$, $K_- = a^2/2$, $K_+ = a^{\dagger 2}/2$, the solutions $|\alpha, u, v\rangle$ are immediately brought, up to a phase factor, to the form of famous Stoler

states $|\alpha, \zeta\rangle = S(\zeta)|\alpha\rangle$ [22]:

$$|\alpha, u, v\rangle = e^{i \arg u} \exp(\zeta K_+ - \zeta^* K_-)|\alpha\rangle, \quad (8)$$

where $|\zeta| = \operatorname{arccosh}|u|$ and $\arg \zeta = \arg v - \arg u$. Yuen [19] called the eigenstates $|\alpha, u, v\rangle$ of $ua + va^\dagger$ *two photon CS* and suggested that the output radiation of an ideal monochromatic two photon laser is in a state $|\alpha, u, v\rangle$. In [24] these states were named *squeezed states (SS)* to reflect the property of this states to exhibit fluctuations in q or p less than those in CS $|\alpha\rangle$. They were intensively studied in quantum optics and are experimentally realized (see refs in [2, 3]). The eigenstates $|n, u, v\rangle$ of $(ua + va^\dagger)^\dagger(ua + va^\dagger)$ became known as *squeezed Fock states* ($|n = 0, u, v\rangle$ — *squeezed vacuum*) and the operator $S(\zeta)$ — (canonical) *squeeze operator* [2, 3]. Eigenstates $|\vec{\alpha}, u, v\rangle$, Eq. (7), became known as *multimode (canonical) SS*.

Noting that the variance $(\Delta X)^2$ of a Hermitian operator X in a state $|\Psi\rangle$ equals zero if $|\Psi\rangle$ is an eigenstate of X so it was suggested [7] to construct SS for *arbitrary two observables* X_1 and X_2 , in analogy to the canonical SS $|\alpha, u, v\rangle$, as eigenstates of their complex combination $\lambda X_1 + iX_2$, $\lambda \in \mathbb{C}$ (or equivalently $uA + vA^\dagger$, $A = (X_1 - iX_2)$), since if in such eigenstates $\lambda \rightarrow 0$ ($\lambda \rightarrow \infty$) then $\Delta X_2 \rightarrow 0$ ($\Delta X_1 \rightarrow 0$) [7].

Radcliffe [25] and Arecchi et al [26] introduced and studied the $SU(2)$ analog $|\theta, \varphi; j\rangle$ of the states $|\alpha=0, u, v\rangle$ in the similar form to that of Stoler states (8) (the displacement operator form) ($J_\pm = J_1 \pm iJ_2$),

$$|\theta, \varphi\rangle = \exp(\zeta J_+ - \zeta^* J_-)|j, -j\rangle = \left(\frac{-1}{1 + |\tau|^2} \right)^j e^{\tau J_+}|j, -j\rangle \equiv |\tau; j\rangle, \quad (9)$$

where $|j, m\rangle$ ($m = -j, -j+1, \dots, j$, $j = 1/2, 1, \dots$) are the standard Wigner–Dicke states, the operators J_1, J_2 and J_3 are the Hermitian generators of $SU(2)$, $\tau = \exp(-i\varphi) \tan(\theta/2)$, $\zeta = (\theta/2) \exp(-i\varphi)$ and φ and θ are the two angles in the spherical coordinate system. The system $\{|\theta, \varphi\rangle\}$ is overcomplete [26],

$$1 = [(2j + 1)/4\pi] \int d\Omega |\theta, \varphi\rangle \langle \varphi, \theta|, \quad (10)$$

where $d\Omega = \sin \theta d\theta d\varphi$. The states $|\theta, \varphi\rangle \equiv |\tau; j\rangle$ are known as *spin CS* [25] or *atomic CS* (Bloch states) [26].

The results of [25, 26] about the $SU(2)$ CS have been extended to the noncompact group $SU(1, 1)$ and to any Lie group G as well by Perelomov [23], who succeeded to prove the Klauder suggestion for construction of overcomplete families of states using unitary irreducible reps of a Lie group G . If $T(g)$ is an irreducible unitary rep of G , $|\Psi_0\rangle$ is a fixed vector in the rep space, H is stationary subgroup of $|\Psi_0\rangle$ (that is $T(h)|\Psi_0\rangle = \exp[i\alpha(h)]|\Psi_0\rangle$) then the

family of states $|x\rangle = T(s(x))|\Psi_0\rangle$, where $s(x)$ is a cross section in the group fiber bundle, $x \in \mathcal{X} = G/H$, is overcomplete, resolving the unity with respect to the G -invariant measure on \mathcal{X} ,

$$1 = \int |x\rangle\langle x| d\mu(x), \quad d\mu(g \cdot x) = d\mu(x). \tag{11}$$

Such states were called generalized CS and denoted as CS of the type $\{T(g), \Psi_0\}$ [23]. It is worth noting that an other type of “generalized CS” was previously introduced by Titulaer and Glauber (see the ref. in [1]) as the most general states which satisfy the Glauber field coherence condition. Therefore we adopt the notion “group-related CS” for the generalized CS of the type $\{T(g), \Psi_0\}$ [1]. The Perelomov $SU(1, 1)$ CS $|\zeta; k\rangle$ for the discrete series $D^+(k)$ with the reference vector $|\Psi_0\rangle = |k, k\rangle$ ($K_-|k, k\rangle = 0$, $K_3|k, k\rangle = k|k, k\rangle$) have quite similar form to that of spin CS (9) and Stoler states (8),

$$|\zeta, k\rangle = \exp(\zeta K_+ - \zeta^* K_-)|k, k\rangle = (1 - |\xi|^2)^k e^{\xi K_+}|k, k\rangle \equiv |\xi; k\rangle, \tag{12}$$

where $|\xi| = \tanh |\zeta|$, $\arg \xi = -\arg \zeta + \pi$. The $SU(1, 1)$ and $SU(2)$ invariant resolution unity measures for these sets of states are ($k \geq 1/2$) [23]

$$\begin{aligned} d\mu(\xi) &= [(2k - 1)/\pi] d^2\xi / (1 - |\xi|^2)^2, \\ d\mu(\tau) &= [(2j + 1)/\pi] d^2\tau / (1 + |\tau|^2)^2. \end{aligned} \tag{13}$$

The $SU(1, 1)$ reps with $k = 1/2$ and $k = 1/4$ are not square integrable against the invariant measure $d\mu(\xi)$. The whole family of canonical SS $|\alpha, u, v\rangle$, Eqs (2), (4), remains stable (up to a phase factor) under the action of unitary operators of the semidirect product $SU(1, 1) \wedge H_1$. However it does not resolve the identity operator with respect to the corresponding $SU(1, 1) \wedge H_1$ invariant measure [6]. Noninvariant resolution unity measures for the set of canonical SS were found in [6, 27]. The overcompleteness property of the CS $|\tau; j\rangle$ and $|\xi; k\rangle$ provide the analytic reps in the complex plain and in the unit disk respectively which were successfully used by Brif [9]) for diagonalization of the general complex combinations of the $SU(2)$ and $SU(1, 1)$ generators. The $SU(1, 1)$ analytic rep in the unit disk was also considered in [34, 36].

A lot of attention is paid in the physical literature, especially in quantum optics, to the group-related CS for $SU(2)$ and $SU(1, 1)$ in their one- and two-mode boson reps, such as the Schwinger two mode reps (see [1, 3, 33, 34] and references therein), and the one-mode Holstein–Primakoff reps (see e. g. [34, 35] and references therein).

An extension of the group-related CS, compatible with the resolution of the identity, can be obtained if the stationary subgroup $H \subset G$ in Gilmore–Perelomov scheme is replaced by other closed subgroup (references [1–8] in [4]). Significant progress is achieved recently [4] in the construction of more general type of continuous families of states (called also CS [4]) which satisfy the generalized overcompleteness relation $B = \int |x\rangle\langle x| d\mu(x)$, where B is a bounded, positive and invertible operator. When $B = 1$ the Klauder definition of general CS (overcomplete family of states) [1] is recovered.

Along the line of generalization of the eigenvalue property (D1) of the canonical CS the next step was made in 1971 by Barut and Girardello in [29], where the Weyl lowering generator K_- of $SU(1, 1)$ in the discrete series $D^\pm(k)$ was diagonalized explicitly,

$$K_-|z; k\rangle = z|z; k\rangle, \quad |z; k\rangle = N_{BG} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\Gamma(2k+n)}} |k, k+n\rangle. \quad (14)$$

The family $\{|z; k\rangle\}$ resolves the unity operator, $1 = \int |z; k\rangle\langle k, z| d\mu(z, k)$, the resolution unity measure being

$$d\mu(z, k) = \frac{2}{\pi} (N_{BG})^{-2} |z|^{2k-1} K_{2k-1}(2|z|) d^2z, \quad (15)$$

where $K_\nu(x)$ is the modified Bessel function of the third kind [31]. The identity operator resolution (15) provides a new analytic rep in Hilbert space [29]. The measure $d\mu(z, k)$, Eq. (15), is *not invariant* under the action of the $SU(1, 1)$ on $\mathbb{C} \ni z$. In the Barut–Girardello (BG) rep states $|\Psi\rangle$ are represented by functions $F_{BG}(z) = \langle k, z^*|\Psi\rangle/N_{BG}(|z|, k)$ which are of the growth $(1, 1)$. The orthonormalized states $|k, k+n\rangle$ are represented by monomials $z^n/\sqrt{n!(2k)_n}$, $(2k)_n = \Gamma(2k+n)/\Gamma(2k)$. The $SU(1, 1)$ generators K_\pm and K_3 act in the space \mathcal{H}_k of analytic functions $F_{BG}(z)$ as linear differential operators

$$K_+ = z, \quad K_- = 2k \frac{d}{dz} + z \frac{d^2}{dz^2}, \quad K_3 = k + z \frac{d}{dz}. \quad (16)$$

Originally established for the discrete series $D^+(k)$, $k = 1/2, 1, \dots$ the BG rep is in fact valid for any positive index k . Recently this rep has been used to diagonalize the complex combination $uK_- + vK_+$ of the Weyl operators K_\pm [7] and the general element of $su(1, 1)$ as well [30, 10, 8, 9]. The relations between BG rep and the Fock–Bargmann analytic rep (also called canonical CS rep) have been established in [28] (the case of $k = 1/4, 3/4$) and [11] (the cases of $k = 1/2, 1, 3/2, \dots$). The BG-type analytic rep was recently extended to the algebras $u(N, 1)$ [13] and $u(p, q)$ in their boson realizations [11]. The BG-type CS for these and any other (noncompact) semisimple Lie algebra are

defined [11] as common eigenstates of the mutually commuting Weyl ladder operators.

The BG CS $|z; k\rangle$ can be also defined according to the third definition (D3) on the basis of the Heisenberg relation for K_1 and K_2 . For this family the generalization of the definition (D2) does not exist [12].

The ladder operator method was extended to the deformed quantum oscillator in [37], where the q -deformed boson annihilation operator a_q ,

$$a_q a_q^\dagger - q a_q^\dagger a_q = q^{-\hat{n}}, \quad [\hat{n}, a_q^\dagger] = a_q^\dagger, \quad q > 0, \quad (17)$$

has been diagonalized, the eigenstates $|\alpha\rangle_q$ being called “ q -CS” or CS for the quantum Heisenberg–Weyl group $h_q(1)$,

$$|\alpha\rangle_q = \mathcal{N} \exp_q(\alpha a_q^\dagger) |0\rangle = \mathcal{N} \sum_n \frac{\alpha^n}{\sqrt{[n]_q!}} |n\rangle, \quad \mathcal{N} = \exp_q(-|\alpha|^2), \quad (18)$$

where $\exp_q(x) = \sum x^n/[n]_q!$, $[n]_q! = [1]_q \dots [n]_q$, $a^\dagger a |n\rangle = n |n\rangle$ (and $a^\dagger a_q |n\rangle = [n]_q |n\rangle$). The “classical limit” is obtained at $q = 1$: $a_{q=1} = a$. The q -SS have been constructed in the first paper of [39] as states $|v\rangle_q$ annihilated by the linear combination $a_q + v a_q^\dagger$, in analogy to the case of canonical squeezed vacuum states $|\alpha = 0, u, v\rangle : (a + v a^\dagger) |v\rangle = 0$. It was noted [39] that both q -CS and $|v\rangle_q$ can exhibit squeezing in the quadratures of the (ordinary) boson operator a . Group-related type CS associated with the q -deformed algebras $su_q(2)$, $[J_-(q), J_+(q)] = -[2J_3]_q$, $[J_3, J_\pm(q)] = \pm J_\pm(q)$, and $su_q(1, 1)$, $[K_-(q), K_+(q)] = [2K_3]_q$, $[K_3, K_\pm(q)] = \pm K_\pm(q)$, in their Holstein–Primakoff realizations in terms of a_q ,

$$J_-(q) = a_q \sqrt{[-\hat{n} + 2\kappa + 1]_q}, \quad J_3 = \hat{n} - \kappa, \quad (19)$$

$$J_+(q) = \sqrt{[-\hat{n} + 2\kappa + 1]_q} a_q^\dagger,$$

$$K_-(q) = a_q \sqrt{[\hat{n} + 2\kappa - 1]_q},$$

$$K_+(q) = \sqrt{[\hat{n} + 2\kappa - 1]_q} a_q^\dagger, \quad K_3 = \hat{n} + \kappa, \quad (20)$$

were constructed and discussed in [38, 39] ($\kappa = 1/2$ in [38] and any κ in [39]). Here $[x]_q \equiv (q^x - q^{-x})/(q - q^{-1})$. These $su(2)$ and $su(1, 1)$ q -CS are defined similarly to the ordinary group-related CS (9) and (12) with $J_i, K_i, n!$ and $(x)_n$ replaced by their q -generalizations [38, 39]. Their overcompleteness relations (in terms of the Jackson q -integral) can be found in [40], the corresponding

resolution unity measures being the q -deformed versions of $d^2\alpha$ and (13):
 $d\mu_q(\alpha) = d_q^2\alpha/\pi$,

$$d\mu_q(\tau) = \frac{[2j+1]_q}{{}_q\langle j; \tau \| \tau; j \rangle_q^2} d_q^2\tau, \quad d\mu_q(\xi) = \frac{[2k-1]_q}{{}_q\langle k; \xi \| \xi; k \rangle_q^{-2}} d_q^2\xi, \quad (21)$$

where $\|\tau; j\rangle_q = \exp_q(\tau J_+(q))|j, -j\rangle$, $\|\xi; j\rangle_q = \exp_q(\xi K_+(q))|k, k\rangle$. The Barut–Girardello q -CS (eigenstates of $K_-(q)$) are constructed in the first paper of [38]. The ladder operator formalism for several kinds of one- and two-mode boson states is considered recently in [42]. For further development in the field of q -deformed CS see e. g. [40, 41]. For CS related to supergroups (*super-CS*) see e. g. [43]. The canonical SS can be regarded as super-CS related to the orthosymplectic supergroup $OSp(1/2, R)$ [44].

3. The Uncertainty Way

3.1. The Heisenberg and the Schrödinger UR

Canonical CS $|\alpha\rangle$ (and only they) minimize the Heisenberg uncertainty relation with equal uncertainty of the two (dimensionless) canonical observables p and q : in $|\alpha\rangle$ the two variances are equal and α -independent, $(\Delta p)^2 = 1/2 = (\Delta q)^2$. $1/2$ is the lowest level at which the equality $(\Delta p)^2 = (\Delta q)^2$ can be maintained. Therefore the set of $|\alpha\rangle$ is *the set of p - q minimum uncertainty states*. The CS related to any other two (or more) *noncanonical observables* X_1 and X_2 are *not* with minimal and equal uncertainties — the lowest level of the equality $(\Delta X_1)^2 = (\Delta X_2)^2$ can be reached on some subsets only. For example, in the $SU(1, 1)$ CS $|\xi; k\rangle$ the variances of the generators K_1 and K_2 for $\xi \neq 0$ are always greater than their value in the lowest weight vector state $|k, k\rangle$: $\Delta K_{1,2}(\xi) > \Delta K_{1,2}(0) = \sqrt{k/2}$ [7]. The Heisenberg inequality for K_1 and K_2 is minimized in the subsets of states with $\text{Re } \xi = 0$ and/or $\text{Im } \xi = 0$ only, but the uncertainties $\Delta K_1(\xi)$ and $\Delta K_2(\xi)$ (calculated in [45]) are never equal unless $\xi = 0$. Similar is the uncertainty status of the spin CS ($SU(2)$ related CS) $|\tau; j\rangle$.

It turns out [7] that the above $SU(1, 1)$ and $SU(2)$ group related CS minimize, for any values of the parameters ξ and τ , the more precise uncertainty inequality of Schrödinger (called also Schrödinger–Robertson inequality) [16],

$$(\Delta X_1)^2(\Delta X_2)^2 \geq \frac{1}{4} |\langle [X_1, X_2] \rangle|^2 + (\Delta X_1 X_2)^2, \quad (22)$$

where $\langle X \rangle$ is the mean value of X , and $\Delta X_1 X_2 \equiv \langle X_1 X_2 + X_2 X_1 \rangle / 2 - \langle X_1 \rangle \langle X_2 \rangle$ is the covariance of X_1 and X_2 . However the sets of states which

minimize (22) for $K_{1,2}$ and $J_{1,2}$ are much larger than the sets of the corresponding group-related CS $|\xi; k\rangle$ and $|\tau; j\rangle$ — these larger sets have been constructed in [7] as eigenstates of the general *complex* combinations of the ladder operators K_{\pm} and J_{\pm} correspondingly since the necessary and sufficient condition for a state $|\Psi\rangle$ to minimize (22) was realized to be the eigenvalue equation

$$[u(X_1 - iX_2) + v(X_1 + iX_2)] |\Psi\rangle = z|\Psi\rangle. \tag{23}$$

The minimizing states should be denoted by $|z, u, v; X_1, X_2\rangle$ and called Schrödinger X_1 - X_2 *optimal uncertainty states* (optimal US). The other names already used in the literature are *generalized (or Schrödinger) intelligent states* [7, 30], *correlated CS* [49] and *Schrödinger minimum uncertainty states* [6]. The minimization of the inequality (22) for canonical p and q was considered in detail in [49], where the minimizing states were called *correlated CS*. The latter coincides with the canonical SS $|\alpha, u, v\rangle$ [6]. In the optimal US the uncertainties $\Delta X_1, \Delta X_2$ are minimal in the case of $X_1 = p, X_2 = q$ only. Therefore the frequently used term “minimum uncertainty states” [6, 8, 30, 32, 5, 33] is generally not in its direct meaning. The term *intelligent states* was introduced in [47] on the example of Heisenberg inequality for $J_{1,2}$. States $|\Psi\rangle$ for which the product functional $U[\Psi] \equiv (\Delta X_1)^2(\Delta X_2)^2$ is stationary under arbitrary variation of $|\Psi\rangle$ [46] were called by Jackiw *critical*. Obviously there is no commonly accepted name for the states which minimize an uncertainty inequality — the “optimal uncertainty states” is one more attempt in searching for more adequate name.

In the solutions $|z, u, v; X_1, X_2\rangle$ to (23) the three second moments of X_1 and X_2 are expressed in terms of the mean of their commutator [7] (note that in [7] λ, z' parameters were used instead of u, v, z : $\lambda = (v + u)/(v - u), z' = z/(v - u)$),

$$\begin{aligned} (\Delta X_1)^2 &= \frac{|u - v|^2}{|u|^2 - |v|^2} C_{12}, & (\Delta X_2)^2 &= \frac{|u + v|^2}{|u|^2 - |v|^2} C_{12}, \\ \Delta X_1 X_2 &= \frac{2 \operatorname{Im}(u^* v)}{|u|^2 - |v|^2} C_{12}, & C_{12} &= \frac{i}{2} \langle [X_1, X_2] \rangle. \end{aligned} \tag{24}$$

These moments satisfy the equality in (22) identically with respect to z, u, v . From $(\Delta X)^2 \geq 0$ and (24) it follows that if the commutator $i[X_1, X_2]$ is positive (negative) definite then normalized eigenstates of $u(X_1 - iX_2) + v(X_1 + iX_2)$ exist for $|u| > |v|$ ($|u| < |v|$) only [7]. In such cases one can rescale the parameters and put $|u|^2 - |v|^2 = 1$ ($|u|^2 - |v|^2 = -1$) as one normally does in the canonical case of $X_1 = p, X_2 = q$.

In order to establish the connection of K_1 - K_2 and J_1 - J_2 optimal US $|z, u, v; K_1, K_2\rangle \equiv |z, u, v; k\rangle$ and $|z, u, v; J_1, J_2\rangle \equiv |z, u, v; j\rangle$ with the dis-

placement operator method consider the operators

$$K'_3 = \frac{i}{2} \sqrt{uv} (uK_- + vK_+), \quad K'_\pm = iK_3 \mp \left(\sqrt{u/v} K_- - \sqrt{v/u} K_+ \right), \quad (25)$$

$$J'_3 = \frac{1}{2} \sqrt{uv} (uJ_- + vJ_+), \quad J'_\pm = J_3 \mp \left(\sqrt{u/v} J_- - \sqrt{v/u} J_+ \right), \quad (26)$$

which realize non-Hermitian reps of the algebras $su(1, 1)$ and $su(2)$ with the same indices k and j . Therefore $(K'_\pm)^n$ ($(J'_\pm)^n$) displace the eigenvalue z of $uK_- + vK_+$ ($uJ_- + vJ_+$) by $\pm n$. If one could properly define non-integer powers of K'_\pm (J'_\pm) (to be considered elsewhere) one might write $|z, u, v; k\rangle = \mathcal{N}_1 (K'_\pm)^z |0, u, v; k\rangle$ ($|z, u, v; j\rangle = \mathcal{N}_2 (J'_\pm)^z |0, u, v; j\rangle$), where $\mathcal{N}_{1,2}$ are normalization constants. In slightly different notations the operators J'_3, J'_\pm were introduced by Rashid [48].

An important physical property of the states $|z, u, v; X_1, X_2\rangle$ is that they can exhibit arbitrary strong squeezing of the variances of X_1 and X_2 when the parameter v tends to $\pm u$, i. e. $\Delta X_{1,2} \rightarrow 0$ when $v \rightarrow \pm u$ [7]. Therefore the families of $|z, u, v; X_1, X_2\rangle$ are the X_1 - X_2 ideal SS. The canonical SS $|\alpha, u, v\rangle$ are p - q ideal SS, while the group-related CS $|\tau; j\rangle$ and $|\xi; k\rangle$ are not. Explicitly the families of $|z, u, v; X_1, X_2\rangle$ are constructed for the generators K_i - K_j and J_i - J_j of $SU(1, 1)$ [7, 30, 9] and $SU(2)$ [47, 48, 9] (in [47, 48] with no reference to the inequality (22)). It is worth noting an important application of the K_i - K_j and J_i - J_j optimal US (intelligent states) in the quantum interferometry: the $SU(1, 1)$ and $SU(2)$ optimal US which are not group-related CS can greatly improve the sensitivity of the $SU(2)$ and $SU(1, 1)$ interferometers as shown by Brif and Mann [33]. Schemes for generation of $SU(1, 1)$ and $SU(2)$ optimal US of radiation field can be found e.g. in [12, 33].

Schrödinger optimal US can be constructed also for the two Hermitian quadratures $K_1(q), K_2(q)$ ($J_1(q), J_2(q)$) of the ladder operators of q -deformed $su_q(1, 1)$ ($su_q(2)$). Let us consider here the case of $su_q(1, 1)$. The $K_1(q)$ - $K_2(q)$ optimal US $|z, u, v; k\rangle_q$ have to obey (23) with $X_1 = K_1(q)$ and $X_2 = K_2(q)$. We put

$$|z, u, v; k\rangle_q = \mathcal{N}_q \|z, u, v; k\rangle_q = \mathcal{N}_q \sum_n g_n(z, u, v, q, k) |k, k+n\rangle, \quad (27)$$

and substitute this in (23). Using the actions $K_-(q)|k, k+n\rangle = \sqrt{[n][2k+n-1]}|k, k+n-1\rangle$, and $K_+(q)|k, k+n\rangle = \sqrt{[n+1][2k+n]}|k, k+n+1\rangle$ we get the recurrence relations for g_n ,

$$u\sqrt{[n+1][2k+n]}g_{n+1} + v\sqrt{[n+1][2k+n]}g_{n-1} = zg_n. \quad (28)$$

The solution $g_n(z, v, u, q, k)$ to these recurrence relations is a polynomial in z/u and v/u ,

$$g_n(z, u, v, q, k) = \sum_{m=0}^{\text{int}(n/2)} p_{n,m}(k, q) \left(\frac{z}{u}\right)^{n-2m} \left(-\frac{v}{u}\right)^m, \quad (29)$$

where $\text{int}(n/2)$ is the integer part of $n/2$. The particular case of $v = 0$ was solved in [38], $g_n(z, q, k) = z^n / \sqrt{[n]!([2k])_n}$. Here we write down the solution for the subset of $z = 0$,

$$g_{2n+1}(u, v, q) = 0, \quad g_{2n}(u, v, q) = \left(-\frac{v}{u}\right)^n \left(\frac{[2n-1]!!([2k])_{2n}}{[2n]!!([2k+1])_{2n}}\right)^{\frac{1}{2}}, \quad (30)$$

and for $q = 1$,

$$g_n(z, u, v, k) = \left(-\frac{l(u, v)}{2u}\right)^n \sqrt{\frac{(2k)_n}{n!}} {}_2F_1\left(k + \frac{z}{l(u, v)}, -n; 2k; 2\right), \quad (31)$$

where $l(u, v) = 2\sqrt{-uv}$, $(([x]))_{2n} = [x][x+2] \dots [x+2n-2]$ and ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function. The normalization condition is $|v| < |u|$. The BG CS are recovered at $v = 0, u = 1$. The construction of $g_n(z, u, v, q, k)$ in the general case is postponed until the next publication.

3.2. The Robertson Inequality and the Characteristic UR

Compared to the Heisenberg uncertainty relation the Schrödinger one, Eq. (22), has the important advantage to be *invariant* under nondegenerate linear transformations of the two observables involved. Indeed the relation (22) can be rewritten in the following invariant form [17] $\det \sigma(\vec{X}) \geq \det C(\vec{X})$, where \vec{X} is the column of X_1 and X_2 , $\vec{X} = (X_1, X_2)$, and

$$C(\vec{X}) = -\frac{i}{2} \begin{pmatrix} 0 & \langle [X_1, X_2] \rangle \\ \langle [X_2, X_1] \rangle & 0 \end{pmatrix}, \quad \sigma(\vec{X}) = \begin{pmatrix} \Delta X_1 X_1 & \Delta X_1 X_2 \\ \Delta X_2 X_1 & \Delta X_2 X_2 \end{pmatrix}. \quad (32)$$

$\sigma(\vec{X})$ is called the uncertainty (the dispersion) matrix for X_1 and X_2 . In order to symmetrize notations we have denoted in (32) the variance $(\Delta X_i)^2$ as $\Delta X_i X_j$. So $\sigma_{ij} = \Delta X_i X_j$ and $C_{kj} = -(i/2)\langle [X_k, X_j] \rangle$. Under linear transformations $\vec{X} \rightarrow \vec{X}' = \Lambda \vec{X}$, we have

$$\sigma' \equiv \sigma(\vec{X}') = \Lambda \sigma \Lambda^T, \quad C' \equiv C(\vec{X}') = \Lambda C \Lambda^T. \quad (33)$$

It is now seen that if the transformation is non-degenerate, $\det \Lambda \neq 0$, then the equality in the relation (22) remains invariant, i. e. $\det \sigma = \det C \rightarrow \det \sigma' =$

$\det C'$. This implies that in the canonical case of $X_1 = p$, $X_2 = q$ the equality in (22) is invariant under linear canonical transformations. The equality in the Heisenberg relation is not invariant under linear transformations.

In the Heisenberg and the Schrödinger inequalities the second moments of *two* observables $X_{1,2}$ are involved. However two operators never close an algebra [An exception is the Heisenberg–Weyl algebra h_1 due to the fact that the third operator closing the algebra is the identity operator: the equality in the p – q Schrödinger relation (but not in the Heisenberg one) is invariant under the linear canonical transformations]. Therefore the equality in these uncertainty relations is not invariant under the general transformations in the algebra to which $X_{1,2}$ may belong. For n generators of Lie algebras it is desirable to have uncertainty relations invariant under algebra automorphisms, in particular under the corresponding Lie group action in the algebra.

Such invariant uncertainty relations turned out to be those of Robertson [17] and of Trifonov and Donev [14]. The Robertson relation for n observables X_1, X_2, \dots, X_n reads ($i, j, k = 1, 2, \dots, n$)

$$\det \sigma(\vec{X}) \geq \det C(\vec{X}), \quad (34)$$

where $\sigma_{ij} = \Delta X_i X_j$, and $C_{kj} = -i\langle [X_k, X_j] \rangle / 2$. With minor changes the Robertson proof of (34) is provided in the Appendix. The minimization of (34) is considered in detail in [10], the minimizing states being called Robertson intelligent states or Robertson *optimal* US. A pure state minimize (34) if it is an eigenstate of a real combination of the observables. For odd n this is also a necessary condition. Robertson optimal US exist for a broad class of observables, the simplest example being given by the well known N -modes Glauber CS $|\vec{\alpha}\rangle = |\alpha_1\rangle |\alpha_2\rangle, \dots |\alpha_N\rangle$, and by the N -modes canonical SS $|\vec{\alpha}, u, v\rangle$ (constructed in [18, 21] with no reference to the Robertson relation). A more general example is given by the group-related CS $\{T(g), \Psi_0\}$ when $|\Psi_0\rangle$ is eigenstate of a (real) Lie algebra element [10]. If in addition $|\Psi_0\rangle$ is the lowest (highest) weight vector (the case of semisimple Lie groups [3]) then these CS minimize (34) for the Hermitian components of Weyl generators as well [10]. On the example of the $SU(2)$ and $SU(1, 1)$ CS, Eqs (9) and (12), the above minimization properties can be checked by direct calculations. In the case of one-mode and two-mode boson representations of $su(1, 1)$ the above properties mean that squeezed Fock states minimize (34) for the three generators K_i , but squeezed vacuum in addition minimizes (22) for K_1 and K_2 .

The number of the Hermitian components of Weyl generators (of a semisimple Lie group) is even. For the even number n of observables the Robertson inequality (34) is minimized in a state $|\Psi\rangle$ if the latter is an eigenstate of $n/2$ complex linear combinations of X_j . For these minimizing states the second

moments of X_i, X_j can be expressed in terms of the first moments of their commutators. In that purpose and keeping the analogy to the case of canonical SS (7) we define $\tilde{a}_\mu = X_\mu + iX_{\mu+N}$ and write down the $n/2 \equiv N$ complex combinations as $(\mu, \nu = 1, 2, \dots, N)$

$$A_\mu(u, v) := u_{\mu\nu}\tilde{a}_\nu + v_{\mu\nu}\tilde{a}_\nu^\dagger = \beta_{\mu j}X_j, \tag{35}$$

where $\beta_{\mu\nu} = u_{\mu\nu} + v_{\mu\nu}$, $\beta_{\mu, s+\nu} = i(u_{\mu\nu} - v_{\mu\nu})$. Then after some algebra we get that in the eigenstates $|\vec{z}, u, v\rangle$ of $A_\mu(\beta)$ the following general formula holds,

$$\sigma(\vec{X}; z, u, v) = \mathcal{B}^{-1} \begin{pmatrix} 0 & \tilde{C} \\ \tilde{C}^\top & 0 \end{pmatrix} \mathcal{B}^{-1\top}, \tag{36}$$

$$\mathcal{B} = \begin{pmatrix} u + v & i(u - v) \\ u^* + v^* & i(v^* - u^*) \end{pmatrix}, \quad \tilde{C}_{\mu\nu} = \frac{1}{2} \langle [A_\mu, A_\nu] \rangle.$$

Note that u, v are $N \times N$ matrices, β is an $N \times n$ matrix, while \mathcal{B} is $n \times n$. We suppose that \mathcal{B} is not singular. For two observables, $n = 2$, we have $\beta_{11} = u + v$, $\beta_{12} = i(u - v)$ and formula (36) recovers (24).

The Robertson inequality relates the determinants of two $n \times n$ matrices σ and C . These are the highest order *characteristic coefficients* of the two matrices [50] which are invariant under similarity transformations of the matrices. Then from (33) we see that $\det \sigma$ and $\det C$ are invariant under the orthogonal transformations of the observables. However, one can see, again from the transformation law (33), that the equality in (34) is invariant under *any* nondegenerate linear transformations of the n observables. Now we recall [50] that for an $n \times n$ matrix M there are n invariant characteristic coefficients $C_r^{(n)}$, $r = 1, 2, \dots, n$, defined by means of the secular equation

$$0 = \det(M - \lambda) = \sum_{r=0}^n C_r^{(n)}(M)(-\lambda)^{n-r}. \tag{37}$$

The characteristic coefficients $C_r^{(n)}$ are equal to the sum of all principle minors $\mathcal{M}(i_1, \dots, i_r; M)$ of order r . One has $C_0^{(n)} = 1$, $C_1^{(n)} = \text{Tr } M = \sum m_{ii}$ and $C_n^{(n)} = \det M$. For $n = 3$ we have, for example, three principle minors of order 2. In these notations Robertson inequality (34) reads $C_n^{(n)}(\sigma(\vec{X})) \geq C_n^{(n)}(C(\vec{X}))$. It is important to note now two points: (1) the uncertainty matrix $\sigma(\vec{X})$ and the mean commutator matrix $C(\vec{X})$ are nonnegative definite and such are all their principle minors; (2) The principle minors of $\sigma(\vec{X})$ and $C(\vec{X})$ of order r can be regarded as uncertainty matrix and mean commutator

matrix for r observables X_{i_1}, \dots, X_{i_r} correspondingly. Then all characteristic coefficients of the two matrices obey the inequalities [14]

$$C_r^{(n)}(\sigma(\vec{X})) \geq C_r^{(n)}(C(\vec{X})), \quad r = 1, 2, \dots, n. \quad (38)$$

These invariant relations can be called *characteristic uncertainty relations*. The Robertson relation (34) is one of them and can be called the n^{th} -order characteristic inequality.

The minimization of the *first order* inequality in (38), $\text{Tr } \sigma(\vec{X}) = \text{Tr } C(\vec{X})$, can occur in the case of commuting operators only since $\text{Tr } C(\vec{X}) \equiv 0$. Important examples of minimization of the *second order* inequality were pointed out in [14] — the spin and quasi spin CS $|\tau; j\rangle$ and $|\xi; k\rangle$ minimize the second order characteristic inequality for the three generators $J_{1,2,3}$ and $K_{1,2,3}$ correspondingly. We have already noted that these group-related CS minimize the *third order* inequalities too, so their characteristic minimization “ability” is maximal. The analysis of the solutions of the eigenvalue equation $[uK_- + vK_+ + wK_3]|\Psi\rangle = z|\Psi\rangle$ shows (see Appendix) that the CS $|\xi; k\rangle$ are *the unique states* which minimize simultaneously the second and the third order characteristic inequalities for $K_{1,2,3}$ and there are no states which minimize the second order inequality only. Thus the minimization of the characteristic inequalities (38) of order $r < n$ can be used for *finer classification* of group-related CS with symmetry. It turned out (see the Appendix) that the uniqueness of these states follows also from the requirement to minimize simultaneously (34) for the three generators and (22) for the Hermitian components of K_- .

All the above characteristic inequalities¹ relate combinations $C_r^{(n)}(\sigma(\vec{X}; \rho))$ of second moments of X_1, \dots, X_n in a (generally mixed) state ρ to the combinations $C_r^{(n)}(C(\vec{X}; \rho))$ of first moments of their commutators in the same state. It turned out that these relations can be extended to the case of *several state* in the following way. From the derivation of the characteristic inequalities (38) (see Appendix) one can deduce that they are valid for any nonnegative definite matrix $\mathcal{S} + i\mathcal{C}$ with \mathcal{S} nonnegative definite and symmetric and \mathcal{C} — antisymmetric. Well, the finite sum $\sum_m d_m \sigma_m$, $d_m \geq 0$, of nonnegative and symmetric matrices is nonnegative and symmetric, and the finite sum of antisymmetric matrices is again antisymmetric. And if $\sigma_m + iC_m \geq 0$ their finite sum is also nonnegative. Thus we obtain the *extended* characteristic uncertainty inequalities

$$C_r^{(n)}(\sum_m d_m \sigma_m) \geq C_r^{(n)}(\sum_m d_m C_m), \quad (39)$$

¹ Let us note that other types of uncertainty relations, e. g. the entropic and the parameter-based ones, are also considered in the literature [51].

where d_m are arbitrary real nonnegative parameters. Here σ_m and C_m , $m = 1, 2, \dots$, may be the uncertainty and the mean commutator matrices for \vec{X} in states ρ_m or the uncertainty and the mean commutator matrices of different sets of n observables $\vec{X}^{(m)}$ in the same state ρ . For $r = n$ in (39) we have the extension of the Robertson relation to the case of several states and/or several sets of n observables. In the first case the extension reads

$$\det \left(\sum_m d_m \sigma(\vec{X}, \rho_m) \right) \geq \det \left(\sum_m d_m C(\vec{X}, \rho_m) \right). \tag{40}$$

Since $\det \sum \sigma_m \neq \sum \det \sigma_m$ these are indeed *new* uncertainty inequalities, which extend the Robertson one to several states. We note that the extended relations (39), (40) are *invariant* under the nondegenerate linear transformations of the operators X_1, \dots, X_n . If the latter span a Lie algebra then we obtain the invariance of (39) under the Lie group action in the algebra. If for several states $|\psi_m\rangle$, $m = 1, 2, \dots$, the inequality (40) is minimized, then it is minimized also for the group-related CS $U(g)|\psi_m\rangle$ as well, $U(g)$ being the unitary rep of the group G . In the simplest case of two observables X, Y and two states $|\psi_{1,2}\rangle$ which minimize Schrödinger inequality (22) Eq. (40) produces

$$\begin{aligned} \frac{1}{2} [\sigma_{XX}(\psi_1)\sigma_{YY}(\psi_2) + \sigma_{XX}(\psi_2)\sigma_{YY}(\psi_1)] - \sigma_{XY}(\psi_1)\sigma_{XY}(\psi_2) \\ \geq -\frac{1}{4} \langle \psi_1 | [X, Y] | \psi_1 \rangle \langle \psi_2 | [X, Y] | \psi_2 \rangle, \end{aligned} \tag{41}$$

where, for convenience, $\sigma_{XX}(\psi)$ denotes the variance of X in $|\psi\rangle$ and $\sigma_{XY}(\psi)$ denotes the covariance. The more detailed analysis (to be presented elsewhere) shows that this uncertainty relation holds for every two states. For $\psi_1 = \psi_2$ the *new inequality* (41) recovers that of Schrödinger. One can easily verify (41) for p and q and any two Fock states $|n\rangle$ and/or Glauber CS $|\alpha\rangle$ for example. The relation is minimized in two squeezed states $|\alpha_1, u, v\rangle$ and $|\alpha_2, u, v\rangle$, $\Im(uv^*) = 0$. Looking at (41) and (22) one feels that, to complete the symmetry between states and observables, the third inequality is needed (for one observable and two states), namely

$$\begin{aligned} \sigma_{XX}(\psi_1)\sigma_{XX}(\psi_2) \geq |\langle \psi_2 | X^2 | \psi_1 \rangle|^2 \\ - \sigma_{XX}(\psi_1)\langle \psi_2 | X | \psi_2 \rangle^2 - \sigma_{XX}(\psi_2)\langle \psi_1 | X | \psi_1 \rangle^2. \end{aligned} \tag{42}$$

Relations (22) and (42) both follow from the Schwarz inequality, while (41) is different.

It is worth noting that every extended characteristic inequality can be written down in terms of two new positive quantities the sum of which is not greater

than unity. Indeed, let us put

$$C_r^{(n)}(\sigma(\vec{X}, \rho)) = \alpha_r(1 - P_r^2), \quad (43)$$

where $0 \leq P_r^2 \leq 1$ (i. e. $1 - P_r^2 \leq 1$) and $\alpha_r \neq 0$. For $r = n$ eq. (43) reads (omitting index $r = n$) $\det \sigma(\vec{X}, \rho) = \alpha(1 - P^2)$. α_r may be viewed as scaling parameters. Then we can put $C_r^{(n)}(C(\vec{X}, \rho)) = \alpha_r V_r^2$ and obtain from (38) the inequality for P_r and V_r

$$P_r^2(\vec{X}, \rho) + V_r^2(\vec{X}, \rho) \leq 1, \quad r = 1, \dots, n. \quad (44)$$

The equality in (44) corresponds to the equality in (38) (or (39)). For every set of observables X_1, \dots, X_n the nonnegative quantities P_r, V_r are functionals of the state ρ (or of ρ_1, ρ_2, \dots in the case of extended inequalities (39)). These can be called *complementary quantities* and the form (44) of the extended characteristic relations — *complementary form*. Let us note that P_r and V_r are not uniquely determined by the characteristic coefficients of σ and C . They depend on the choice of the scaling parameter α_r . In the case of bounded operators X_i (say spin components) the characteristic coefficients of σ and C are also bounded. In that case α_r can be taken as the inverse maximal value of $C_r^{(n)}(\sigma)$. In the very simple case of one state and two operators with only two eigenvalues each the complementary characteristic inequality (44) was recently considered in the important paper by Bjork et al [15]. In this particular case the meaning of the complementary quantities P and V was elucidated to be that of the *predictability* (P) and the *visibility* (V) in the *welcher weg* experiment [15].

Finally we note that as functionals of the states ρ the characteristic coefficients of positive definite uncertainty matrix $\sigma(\vec{X})$ (then the coefficients $C_r(\sigma(\vec{X}, \rho))$ are all positive), can be used for the construction of *distances* between quantum states. One possible series of such (Euclidean type) distances $D_r^2[\rho_1, \rho_2; \vec{X}]$ is [52]

$$D_r^2[\rho_1, \rho_2] = C_r(\sigma(\vec{X}, \rho_1)) + C_r(\sigma(\vec{X}, \rho_2)) - 2 \left(C_r(\sigma(\vec{X}, \rho_1)) C_r(\sigma(\vec{X}, \rho_2)) \right)^{\frac{1}{2}} g(\rho_1, \rho_2), \quad (45)$$

where $g(\rho_1, \rho_2)$ is any nonnegative functional of ρ_1, ρ_2 , such that $0 \leq g(\rho_1, \rho_2) \leq 1$ and $\rho_1 = \rho_2 \Leftrightarrow g = 1$. A known simple such functional (g -type functional) is $g(\rho_1, \rho_2) = \text{Tr}(\rho_1 \rho_2) / \sqrt{\text{Tr}(\rho_1^2) \text{Tr}(\rho_2^2)}$. By means of (42) with any observable X such that $X|\psi\rangle \neq 0$ (continuous or strictly positive X , for

example) we can construct a new g -type functional

$$g(\psi_1, \psi_2; X) = \frac{|\langle \psi_2 | X^2 | \psi_1 \rangle|}{\sqrt{\langle \psi_1 | X^2 | \psi_1 \rangle \langle \psi_2 | X^2 | \psi_2 \rangle}}, \quad (46)$$

which can be used for distance constructions, the simplest distance being $D^2 = 2(1 - g(\psi_1, \psi_2; X))$. Several other g -type functionals are also possible [52]. The uncertainty matrix $\sigma(\vec{X})$ is positive for examples in the case of X_i being the quadratures component of N q -deformed boson annihilation operators $a_{q,\mu}$ with positive q [10].

4. Conclusion

We have briefly reviewed and compared the three ways of generalization of canonical coherent states (CS) with the emphasis laid on the uncertainty (the third) way. The Robertson inequality and the other characteristic relations for several operators [14] are those uncertainty inequalities which bring together the three ways of generalization on the level of many observables. The equalities in these relations for the group generators are invariant under the group action in the Lie algebra. From the Robertson inequality minimization conditions [10] it follows that all group-related CS whose reference vector is eigenstate of an element of the corresponding Lie algebra do minimize the Robertson relation (34). The minimization of the other characteristic inequalities (38) can be used for *finer classification* of group-related CS with symmetry. Along these lines we have shown that $SU(1, 1)$ CS with lowest weight reference vector $|k, k\rangle$ are the unique states which minimize the second order characteristic inequality for the three $SU(1, 1)$ generators. Also, these are the unique states to minimize simultaneously the Robertson inequality for the three generators and the Schrödinger one for the Hermitian components of the ladder operator K_- . These statements are valid for the $SU(2)$ CS with the lowest (highest) reference vector $|j, \mp j\rangle$ as well. They can be extended to the case of semisimple Lie groups.

In all so far considered characteristic uncertainty inequalities (the Schrödinger and Robertson relations are characteristic ones) two or more observables and *one state* are involved. It turned out that these relations, for any n observables, are extendable to the case of two or more states. We also have shown that the (extended) characteristic inequalities can be written down in the *complementary form* in terms of two positive quantities less than unity. In the case of two observables with two eigenvalues each these *complementary quantities* were recently proved [15] to have the meaning of the predictability and visibility in the *welcher weg* experiment. The notion of “characteristic complementary

quantities” might be useful in treating complicated quantum systems. It was also noted that the characteristic coefficients of positive definite uncertainty matrices can be used for the construction of distances between quantum states.

Appendix

Robertson Proof of the Relation $\det \sigma \geq \det C$

Since the derivation of the characteristic (38) and the extended characteristic uncertainty inequalities (39) is based on the Robertson relation (34) here we provide the proof of (34) following Robertson’ paper [17] with some modern notations. Let X_1, X_2, \dots, X_n be Hermitian operators, and $|\psi\rangle$ be a pure state. Consider the squared norm of the composite state $|\psi'\rangle = \sum_j \alpha_j (X_j - \langle X_j \rangle)|\psi\rangle$, where α_j are arbitrary complex parameters. One has

$$\begin{aligned} \langle \psi' | \psi' \rangle &= \sum_{jk} \alpha_k^* \alpha_j \langle \psi | (X_k - \langle X_k \rangle)(X_j - \langle X_j \rangle) | \psi \rangle \\ &= \sum_{k,j} \alpha_k^* S_{kj} \alpha_j \equiv \mathcal{S}(\vec{\alpha}^*, \vec{\alpha}), \end{aligned} \quad (47)$$

where the matrix elements S_{kj} are $S_{kj} = \langle \psi | (X_k - \langle X_k \rangle)(X_j - \langle X_j \rangle) | \psi \rangle = \sigma_{jk} + iC_{jk}$. We see that $S = \sigma + iC$, where σ and C are the uncertainty and the mean commutator matrices of the operators X_1, \dots, X_n in the state $|\psi\rangle$ (see eq. (32)). In Hilbert space we have $\langle \psi' | \psi' \rangle = 0$ iff $|\psi'\rangle = \sum_j \alpha_j (X_j - \langle X_j \rangle)|\psi\rangle = 0$, which means that $|\psi\rangle$ is an eigenstate of the complex combination of X_j . Thus the form \mathcal{S} is nonnegative definite, which means that the $n \times n$ matrix $S = \sigma + iC$ is nonnegative: all its principle minors are nonnegative [50], in particular $\det S > 0$. For the case of two operators, $n = 2$, one can easily verify that

$$0 \leq \det S = \det(\sigma + iC) = \det \sigma - \det C, \quad (n = 2 \text{ only}). \quad (48)$$

This proves the Robertson relation for two observables which was also derived by Schrödinger [16] using the Schwarz inequality. The property (48) is due to the symmetricity of σ and antisymmetricity of C and is valid for $n = 2$ only.

For odd n , $n \geq 1$, the Robertson inequality $\det \sigma \geq \det C$ is trivial, since the determinant of an antisymmetric matrix of odd dimension vanishes identically. For even $n = 2N$ and $n > 2$ we follow the proof of Robertson [17], using however some notions from the present matrix theory [50]. One considers the regular sheaf (bundle) of the matrices σ and $\eta = iC$, $\eta - \lambda\sigma$, supposing $\sigma > 0$. There exist congruent transformation (by means of the so called sheaf principle matrix Z , $\det Z \neq 0$), which brings both matrices to the diagonal form – σ to the unit matrix, $\sigma' = Z^T \sigma T = 1$ and $\eta' = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_{2N}\}$, where λ_i are

the $2N$ roots of the secular equation $\det(\eta - \lambda\sigma) = 0$. The product of all roots equals $\det \eta / \det \sigma$. From $\det(\eta - \lambda\sigma) = \det(\eta - \lambda\sigma)^T = \det(\eta + \lambda\sigma)$ (since $\eta^T = -\eta$ and $n = 2N$) it follows that the polynomial $\det(\eta - \lambda\sigma)$ contains only even powers of λ , $\det(\eta - \lambda\sigma) = \det \eta + \dots + (-\lambda)^{2N} \det \sigma = 0$. This means that the $2N$ real roots λ_j are equal and opposite in pairs. Denoting positive roots as λ_μ , $\mu = 1, \dots, N$ and negative roots as $\lambda_{\mu+N} = -\lambda_\mu$ one writes

$$\det \eta = (-1)^N \det C = (-1)^N \prod_{\mu} \lambda_{\mu}^2 \det \sigma. \quad (49)$$

On the other hand the Hermitian matrix $\sigma + \eta = \sigma + iC$ is positive definite and after the diagonalization takes the form

$$\begin{aligned} \sigma' + \eta' &= \text{diag}\{1 + \lambda_1, \dots, 1 + \lambda_{2N}\} \\ &= \text{diag}\{1 + \lambda_1, 1 - \lambda_1, \dots, 1 + \lambda_N, 1 - \lambda_N\}. \end{aligned} \quad (50)$$

The diagonal matrix $\sigma' + \eta'$ is again nonnegative definite, i. e. all the elements on the diagonal are nonnegative, which implies that $\lambda_{\mu}^2 \leq 1$, $\mu = 1, \dots, N$. Then Eq. (49) yields the Robertson inequality $\det \sigma \geq \det C$. \square

Remarks:

- a) Robertson considered the case of pure states only. However one can see from the proof that his relation holds for mixed states as well;
- b) It is seen from the above proof that the inequality $\det \sigma \geq \det C$ holds for any two real matrices C and σ , one of which is antisymmetric (C), the other — symmetric and nonnegative definite and such that Hermitian matrix $\sigma + iC$ is again nonnegative;
- c) If the matrices σ_j and C_j , $j = 1, 2, \dots, m$, obey the requirements of (b) then $\det(\sigma_1 + \sigma_2 + \dots) \geq \det(C_1 + C_2 + \dots)$ since (as one can easily prove) the sum of nonnegative $\sigma_j + iC_j$ is again a nonnegative matrix.

These observations have been used in establishing the extended characteristic relations (39) for several states and in formulating the remark (a) as well.

The $SU(1, 1)$ CS $|\xi; k\rangle$ Are the Unique States Which Minimize the Characteristic Inequalities for the Three Generators

For the three generators K_i of $SU(1, 1)$ there are two nontrivial characteristic uncertainty inequalities corresponding to $r = n = 3$ and $r = n - 1 = 2$ in (38). The third order characteristic UR is minimized in a pure state $|\psi\rangle$ iff $|\psi\rangle$ is an eigenstate of a real combination of K_i , i. e. iff $|\psi\rangle = |z, u, v, w; k\rangle$ obey the equation

$$[uK_- + vK_+ + wK_3] |z, u, v, w; k\rangle = z |z, u, v, w; k\rangle \quad (51)$$

with real w and $v = u^*$. The second order characteristic UR is minimized iff $|\psi\rangle$ is an eigenstate of complex combinations of all three pair K_i - K_j simultaneously, i. e. iff

$$\begin{aligned} [u_1 K_- + v_1 K_+ + w_1 K_3] |\psi\rangle &= z_1 |\psi\rangle, & w_1 &= 0, \\ [u_2 K_- + v_2 K_+ + w_2 K_3] |\psi\rangle &= z_2 |\psi\rangle, & v_2 &= u_2, w_2 \neq 0, \\ [u_3 K_- + v_3 K_+ + w_3 K_3] |\psi\rangle &= z_3 |\psi\rangle, & v_3 &= -u_3, w_3 \neq 0, \end{aligned} \quad (52)$$

where the complex parameters u_1, v_1, u_2, w_2, u_3 , and w_3 shouldn't vanish and z_1, z_2, z_3 may be arbitrary. To solve this system it is convenient to use BG analytic rep (16). Let us start with the first equation in (52). Its normalizable solutions $|z_1, u_1, v_1; k\rangle$ for $k = 1/2, 1, \dots$ were found in [7]. They are normalizable for $|u_1| > |v_1|$ only and in BG rep have the form (up to the normalization constant)

$$\Phi_{z_1}(\eta; u_1, v_1) = e^{-\eta\sqrt{-v_1/u_1}} {}_1F_1\left(k + \frac{z_1/2}{\sqrt{-u_1 v_1}}; 2k; 2\eta\sqrt{-v_1/u_1}\right) \quad (53)$$

where (for any $|u_1| > |v_1|$) the eigenvalue z_1 is arbitrary complex number. Here the complex variable in the BG rep (16) is denoted by η . For $k < 1/2$ a second normalizable solution exist of the form

$$\begin{aligned} \Phi'_{z_1}(\eta; u_1, v_1) &= \eta^{1-2k} e^{-\eta\sqrt{-v_1/u_1}} \\ &\times {}_1F_1\left(\frac{z_1/2}{\sqrt{-u_1 v_1}} - k + 1; 2(1 - k); 2\eta\sqrt{-v_1/u_1}\right). \end{aligned} \quad (54)$$

In order to obtain second order $SU(1, 1)$ characteristic US we have to subject the solution (53) to obey the rest two equations in (52). Let us try to obey the second one. Since $u_1 \neq 0$ we can write $K_-|z_1, u_1, v_1; k\rangle = (z_1 - v_1 K_+)|z_1, u_1, v_1; k\rangle/u_1$ and substitute into the second equation to obtain

$$K_3|z_1, u_1, v_1; k\rangle = \frac{1}{w_2} \left[z_2 - \frac{u_2}{u_1} z_1 + (v_1 \frac{u_2}{u_1} - u_2) K_+ \right] |z_1, u_1, v_1; k\rangle. \quad (55)$$

In BG rep (16) this is a first order equation which the function (53) has to obey. By equating the coefficients of the terms proportional to η^n , $n = 0, 1, \dots$, we obtain after some manipulations the necessary conditions (a) $k + z_1/2\sqrt{-u_1 v_1} = 0$; (b) $k = z_2/w^2 - u_2 z_1/u_1 w_2$ and (c) $u_2(1 - v_1/u_1) = w_2\sqrt{-v_1/u_1}/2$. The first condition requires the relation between the parameters z_1, u_1, v_1 and reduces the "wave function" (53) to

$$\Phi_{z_1}(\eta; u_1, v_1) = \exp\left(-\eta\sqrt{-v_1/u_1}\right) \quad (56)$$

which is just the CS $|\xi; k\rangle$ in BG rep with $\xi = -\sqrt{-v_1/u_1}$. The second condition is always satisfied by $z_2 = kw_2 + u_2 z_1/u_1$, u_2, w_2 remaining arbitrary.

Thus *it is the CS* $|\xi; k\rangle$ *only*, $k = 1/2, 1, \dots$, which minimize simultaneously the Schrödinger inequality for K_1 , K_2 and K_1, K_3 .

Next it is a simple (but not short) exercise to check that $\exp(-\eta\sqrt{-v_1/u_1})$ satisfy the third equation in (52) with $w_3 = w_2(z_3u_1 - iu_3z_1)/(u_1z_2 - u_2z_1)$, $z_3 = i(u_3/u_2u_1)(u_1 + v_1z_1)/(u_1z_2 - u_2z_1) + iu_3z_1/u_1$, (u_2, z_2, u_3 being free) and the eigenvalue Eq. (51) with $v = u^*$ and real w , $w = (-uv_1 + u^*u_1)/\sqrt{-u_1v_1} = w(u_1, v, u)$. One can see that for every given $\xi = -\sqrt{-v_1/u_1}$ the equation $\Im[w(u_1, v_1, u)] = 0$ can be solved with respect to u , the solution being not unique: $u = |u| \exp[\pi/4 - \arg \xi/2]$, $|u|$ being arbitrary. So the family of CS $|\xi; k\rangle$ is the unique family of states which minimize the third and the second order characteristic US simultaneously. If we subject the function (53) directly to (51) we will get again (56).

In the case of $SU(1, 1)$ characteristic US for K_i in rep (6) ($k = 1/4, 3/4$) we have to consider the two solutions (53) and (54). The consideration gives no new result — again the Eqs (51) and (52) are satisfied by $\exp(-\eta\sqrt{-v_1/u_1})$ only.

Similar results can be obtained for the minimization of (34) in CS $|\tau; j\rangle$ using for example their own analytic representation and the results of paper [9].

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