Geometry, Integrability and Quantization
September 1–10, 1999, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
Coral Press, Sofia 2000, pp 175-179

GENERALIZED ACTIONS

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Abstract. In this paper a generalization of the concept of action is considered. This notion is based on a new algebraic structure called generalized groups. An action is deduced by imposing an Abelian condition on a generalized group. Generalized actions on normal generalized groups are also considered.

1. Basic Notions

The theory of generalized groups was first introduced in [1]. A generalized group means a non-empty set G admitting an operation

$$\begin{aligned} G \times G &\to G \\ (a, b) &\mapsto ab \end{aligned}$$

called multiplication which satisfies the following conditions:

i) $(ab)c = a(bc)$ for all a, b, c in G;
ii) For each $a \in G$ there exists a unique $e(a) \in G$ such that $ae(a) = e(a)a = a$;
iii) For each $a \in G$ there exists $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e(a)$.

Theorem 1.1. [1] For each $a \in G$ there exists a unique $a^{-1} \in G$.

Theorem 1.2. [2] Let G be a generalized group and $ab = ba$ for all a, b in G. Then G is a group.

Example 1.1. Let $G = \mathbb{R} \times \mathbb{R}\backslash\{0\} \times \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Then G with the multiplication $(a_1, b_1, c_1)(a_2, b_2, c_2) = (b_1 a_1, b_1 b_2, b_1 c_2)$ is a generalized group.

In this paper we consider a generalized action of a generalized group on a set.

Definition 1.1. We say that a generalized group G acts on a set S if there exists a function

$$\begin{array}{rcl} G \times S & \rightarrow & S \\ (g,x) & \mapsto & gx \end{array}$$

which is called a generalized action such that:

- $(g_1 g_2)x = g_1(g_2 x)$ for all $g_1, g_2 \in G$, and $x \in S$;
- For all $x \in S$ there exists $e(g) \in G$ such that $e(g)x = x$.

Example 1.2. Let

$$G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}; \quad a, b, c \text{ and } d \text{ are real numbers} \right\}.$$

Then G with the product

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right) \mapsto \begin{bmatrix} a & f \\ g & d \end{bmatrix}$$

is a generalized group, and the function

$$\begin{array}{rcl} G \times \mathbb{R}^4 & \rightarrow & \mathbb{R}^4 \\ \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, (e,f,g,h) \right) & \mapsto & (a,f,g,d) \end{array}$$

is a generalized action of G on $\mathbb{R}^4$.

Theorem 1.3. Let $\tau : G \times S \rightarrow S$ be a genaralized action, and G be an Abelian generalized group. Then G is a group and τ is an action.

Proof: By theorem 1.2, G is a group. So τ is an action. □

2. Elementary Results on Generalized Actions

If G acts on a set S, then the relation $\sim$ defined by:

$$x_1 \sim x_2 \Leftrightarrow (g_1 x_1 = x_2 \text{ and } g_2 x_2 = x_1 \text{ for some } g_1, g_2 \in G)$$

is an equivalence relation.

Definition 2.1. If $x \in S$, then $O(x) = \{y \in S;\ x \sim y\}$ is called the generalized orbit of x.

Now we deduce a generalized subgroup by a generalized action.

Theorem 2.1. Let a generalized group G act on a set S. Then for every $x \in S$, the set $I_x = \{g \in G;\ gx = x\}$ is a generalized subgroup of G.

Proof: For $g \in I_x$ we have:

$$gx = x \Rightarrow (e(g)g)x = x \Rightarrow e(g)(gx) = x$$
$$\Rightarrow e(g)x = x \Rightarrow e(g) \in I_x\,,$$

and
$$g^{-1}x = g^{-1}(gx) = (g^{-1}g)x = e(g)x = x\,.$$

So
$$g^{-1} \in I_x\,.$$

If
$$g_1, g_2 \in I_x\,,$$

then
$$(g_1 g_2)x = g_1(g_2 x) = g_1 x = x\,.$$

Hence
$$g_1 g_2 \in I_x\,.$$

Thus I_x is a generalized subgroup of G. □

Theorem 2.2. Let $f : G \to E$ be a generalized group homomorphism. Then

$$\begin{array}{rcl} \tau : G \times E & \to & E \\ (g,h) & \mapsto & f(g)h \end{array}$$

is a generalized action.

Proof: Let $g, g' \in G$ and $h \in E$. Then:

$$\tau(g, \tau(g', h)) = f(g)\tau(g', h) = f(g)(f(g')h)$$
$$= f(gg')h = \tau(gg', h)\,.$$

Moreover if $g \in f^{-1}(\{e(h)\})$, then:

$$\tau(e(g), h) = f(e(g))h$$
$$= e(f(g))h = e(e(h))h$$
$$= e(h)h = h.$$

Thus τ is a generalized action. □

Example 2.1. Let $G = \mathbb{R} \times \mathbb{R}\backslash\{0\}$ with multiplication $(a,b)(c,d) = (bc, bd)$. Since

$$\begin{array}{rcl} f : G & \to & \mathbb{R} \\ (a,b) & \mapsto & \dfrac{a}{b} \end{array}$$

is a homomorphism, when the multiplication of $\mathbb{R}$ is $ab = b$, the function

$$\begin{array}{rcl} G \times \mathbb{R} & \to & \mathbb{R} \\ ((a,b),c) & \mapsto & \dfrac{ac}{b} \end{array}$$

is a generalized action.

3. Generalized Action of Normal Generalized Groups on a Set

A generalized group G is called a normal generalized group if $e(ab) = e(a)e(b)$ for all $a, b \in G$. In this section we assume that G is a normal generalized group.

Definition 3.1. [3] A generalized subgroup N of a generalized group G is called a generalized normal subgroup if there exist generalized group E and a homomorphism $f : G \to E$ such that for all $a \in G$,

$$N_a = \phi \ \text{ or } \ N_a = \text{kernel } f_a \,,$$

where $N_a = N \cap G_a$, $G_a = \{g \in G;\ e(g) = e(a)\}$, and $f_a = f|_{G_a}$.

Example 3.1. Let G be the generalized group of Example 2.1. Then $N = \{(a, b);\ a = b \text{ or } a = 3b\}$ is a generalized normal subgroup of G.

Theorem 3.1. Let G be a normal generalized group, and $f : G \to G$ be a generalized groups homomorphism. Moreover let $N = \ker\ f$. Then

$$\begin{array}{rcl} \tau : \dfrac{G}{N} \times f(G) & \to & f(G) \\ (gN_g, x) & \mapsto & f(g)x \end{array}$$

is a generalized action.

Proof:

i) If $(g_1N_{g_1}, x_1) = (g_2N_{g_2}, x_2)$, then $g_1N_{g_1} = g_2N_g$ and $x_1 = x_2$. So $N_{g_1} = N_{g_2}, x_1 = x_2$ and $g_1 = ng_2$ for some $n \in N_{g_1}$. Hence

$$f(g_1) = f(ng_2) = f(n)f(g_2) = f(e(g_2))f(g_2) = f(g_2)\,.$$

Therefore $f(g_1)x_1 = f(g_2)x_2$. Thus τ is well defind;

ii) Let $g_1N_{g_1},\ g_2N_{g_2} \in \dfrac{G}{N}$ and $x \in S$ are given. Then

$$\begin{aligned} \tau(g_1N_{g_1}, \tau(g_2N_{g_2}, x)) &= f(g_1)\tau(g_2N_{g_2}, x) = f(g_1)(f(g_2)x) \\ &= f(g_1g_2)x = \tau((g_1N_{g_1})(g_2N_{g_2}), x)\,; \end{aligned}$$

iii) If $x \in f(G)$, then $x = f(g)$ for some $g \in G$, and we have

$$\tau(e(g)N_g, x) = f(e(g))x = f(e(g))f(g) = f(g) = x\ .$$

□

Notation. We denote the set

$$\left\{ \begin{array}{c} \varphi_g : S \to S\ , \quad g \in G \\ x \mapsto gx \end{array} \right\}$$

by $H(S)$.
The following example shows that $H(S)$ with multiplication $\varphi_{g_1}\varphi_{g_2} := \varphi_{g_1} \circ \varphi_2$ is not a generalized group.

Example 3.2. Let $S = \mathbb{R} \times \mathbb{R}\backslash\{0\}$, and $G = \mathbb{R} \times \{1\}$ with multiplication: $(a, 1)(b, 1) = (b, 1)$. Then the function

$$\begin{array}{rcl} G \times S & \to & S \\ ((a,1),(c,d)) & \mapsto & (c,d) \end{array}$$

is a generalized action, but $H(S)$ is not a generalized group. Because the inverse of an element is not unique.

Theorem 3.2. Let a generalized group G act on a set S, and the function

$$\begin{array}{rcl} G & \to & H(S) \\ g & \mapsto & \varphi_g \end{array}$$

be a one-to-one mapping. Then $H(S)$ with the multiplication $\varphi_{g_1}\varphi_{g_2} := \varphi_{g_1} \circ \varphi_{g_2}$ is a generalized group. Moreovere if G be normal, then $H(S)$ is a normal generalized group.

Proof: Suppose that $\varphi_g \in H(S)$ is given. If $\varphi_g\varphi_h = \varphi_h\varphi_g = \varphi_g$, then $\varphi_{gh} = \varphi_{hg} = \varphi_g$. So $gh = hg = g$. Hence the identity of φ_g is $\varphi_{e(g)}$. Thus $h = e(g)$.
One can easily deduce other properties of generalized group. Now let G be a normal generalized group, and $\varphi_{g_1}, \varphi_{g_2} \in H(S)$. Then

$$e(\varphi_{g_1}\varphi_{g_2}) = e(\varphi_{g_1 g_2}) = \varphi_{e(g_1 g_2)} = \varphi_{e(g_1)e(g_2)} = e(\varphi(g_1))e(\varphi(g_2)).$$

□

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