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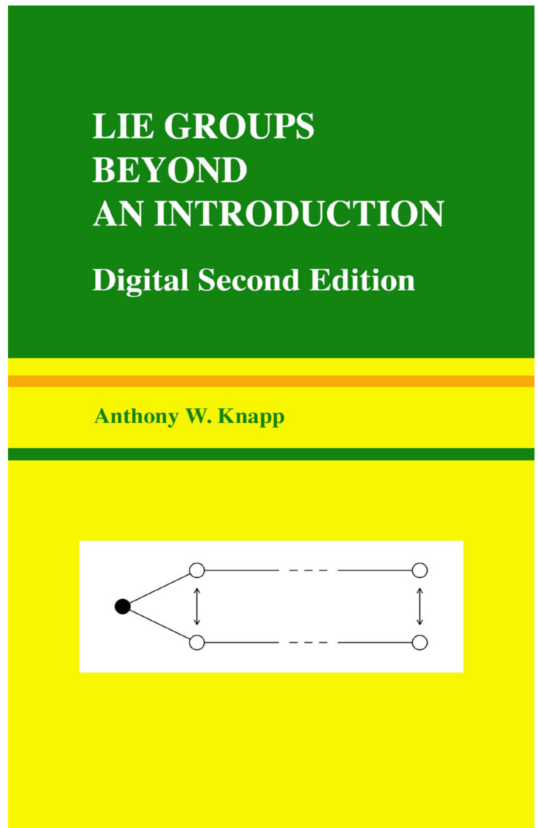

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Pages vii–xviii and 1–812 are the same in the digital and printed second editions. A list of corrections as of June 2023 has been included as pages 813–820 of the digital second edition. The corrections have not been implemented in the text.

Cover: Vogan diagram of $\mathfrak{sl}(2n, \mathbb{R})$. See page 399.

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CHAPTER VII

Advanced Structure Theory

Abstract. The first main results are that simply connected compact semisimple Lie groups are in one-one correspondence with abstract Cartan matrices and their associated Dynkin diagrams and that the outer automorphisms of such a group correspond exactly to automorphisms of the Dynkin diagram. The remainder of the first section prepares for the definition of a reductive Lie group: A compact connected Lie group has a complexification that is unique up to holomorphic isomorphism. A semisimple Lie group of matrices is topologically closed and has finite center.

Reductive Lie groups G are defined as 4-tuples (G, K, θ, B) satisfying certain compatibility conditions. Here G is a Lie group, K is a compact subgroup, θ is an involution of the Lie algebra $\mathfrak{g}_0$ of G, and B is a bilinear form on $\mathfrak{g}_0$. Examples include semisimple Lie groups with finite center, any connected closed linear group closed under conjugate transpose, and the centralizer in a reductive group of a θ stable abelian subalgebra of the Lie algebra. The involution θ, which is called the "Cartan involution" of the Lie algebra, is the differential of a global Cartan involution Θ of G. In terms of Θ, G has a global Cartan decomposition that generalizes the polar decomposition of matrices.

A number of properties of semisimple Lie groups with finite center generalize to reductive Lie groups. Among these are the conjugacy of the maximal abelian subspaces of the -1 eigenspace $\mathfrak{p}_0$ of θ, the theory of restricted roots, the Iwasawa decomposition, and properties of Cartan subalgebras. The chapter addresses also some properties not discussed in Chapter VI, such as the $KA_\mathfrak{p}K$ decomposition and the Bruhat decomposition. Here $A_\mathfrak{p}$ is the analytic subgroup corresponding to a maximal abelian subspace of $\mathfrak{p}_0$.

The degree of disconnectedness of the subgroup $M_\mathfrak{p} = Z_K(A_\mathfrak{p})$ controls the disconnectedness of many other subgroups of G. The most complete description of $M_\mathfrak{p}$ is in the case that G has a complexification, and then serious results from Chapter V about representation theory play a decisive role.

Parabolic subgroups are closed subgroups containing a conjugate of $M_\mathfrak{p}A_\mathfrak{p}N_\mathfrak{p}$. They are parametrized up to conjugacy by subsets of simple restricted roots. A Cartan subgroup is defined to be the centralizer of a Cartan subalgebra. It has only finitely many components, and each regular element of G lies in one and only one Cartan subgroup of G. When G has a complexification, the component structure of Cartan subgroups can be identified in terms of the elements that generate $M_\mathfrak{p}$.

A reductive Lie group G that is semisimple has the property that G/K admits a complex structure with G acting holomorphically if and only if the centralizer in $\mathfrak{g}_0$ of the center of the Lie algebra $\mathfrak{k}_0$ of K is just $\mathfrak{k}_0$. In this case, G/K may be realized as a bounded domain in some $\mathbb{C}^n$ by means of the Harish-Chandra decomposition. The proof of the Harish-Chandra

decomposition uses facts about parabolic subgroups. The spaces G/K of this kind may be classified easily by inspection of the classification of simple real Lie algebras in Chapter VI.

1. Further Properties of Compact Real Forms

Some aspects of compact real forms of complex semisimple Lie algebras were omitted in Chapter VI in order to move more quickly toward the classification of simple real Lie algebras. We take up these aspects now in order to prepare for the more advanced structure theory to be discussed in this chapter. The topics in this section are classification of compact semisimple Lie algebras and simply connected compact semisimple Lie groups, complex structures on semisimple Lie groups whose Lie algebras are complex, automorphisms of complex semisimple Lie algebras, and properties of connected linear groups with reductive Lie algebra. Toward the end of this section we discuss Weyl's unitary trick.

Proposition 7.1. The isomorphism classes of compact semisimple Lie algebras $\mathfrak{g}_0$ and the isomorphism classes of complex semisimple Lie algebras $\mathfrak{g}$ are in one-one correspondence, the correspondence being that $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0$ and $\mathfrak{g}_0$ is a compact real form of $\mathfrak{g}$. Under this correspondence simple Lie algebras correspond to simple Lie algebras.

REMARK. The proposition implies that the complexification of a compact simple Lie algebra is simple. It then follows from Theorem 6.94 that a compact simple Lie algebra is never complex.

PROOF. If a compact semisimple $\mathfrak{g}_0$ is given, we know that its complexification $\mathfrak{g}$ is complex semisimple. In the reverse direction Theorem 6.11 shows that any complex semisimple $\mathfrak{g}$ has a compact real form, and Corollary 6.20 shows that the compact real form is unique up to isomorphism. This proves the correspondence. If a complex $\mathfrak{g}$ is simple, then it is trivial that any real form is simple.

Conversely suppose that $\mathfrak{g}_0$ is compact simple. Arguing by contradiction, suppose that the complexification $\mathfrak{g}$ is semisimple but not simple. Write $\mathfrak{g}$ as the direct sum of simple ideals $\mathfrak{g}_i$ by Theorem 1.54, and let $(\mathfrak{g}_i)_0$ be a compact real form of $\mathfrak{g}_i$ as in Theorem 6.11. The Killing forms of distinct $\mathfrak{g}_i$'s are orthogonal, and it follows that the Killing form of the direct sum of the $(\mathfrak{g}_i)_0$'s is negative definite. By Proposition 4.27, the direct sum of the $(\mathfrak{g}_i)_0$'s is a compact real form of $\mathfrak{g}$. By Corollary 6.20 the direct sum of the $(\mathfrak{g}_i)_0$'s is isomorphic to $\mathfrak{g}_0$ and exhibits $\mathfrak{g}_0$ as semisimple but not simple, contradiction.

Proposition 7.2. The isomorphism classes of simply connected compact semisimple Lie groups are in one-one correspondence with the isomorphism classes of compact semisimple Lie algebras by passage from a Lie group to its Lie algebra.

PROOF. The Lie algebra of a compact semisimple group is compact semisimple by Proposition 4.23. Conversely if a compact semisimple Lie algebra $\mathfrak{g}_0$ is given, then the Killing form of $\mathfrak{g}_0$ is negative definite by Corollary 4.26 and Cartan's Criterion for Semisimplicity (Theorem 1.45). Consequently $\operatorname{Int}\mathfrak{g}_0$ is a subgroup of a compact orthogonal group. On the other hand, Propositions 1.120 and 1.121 show that $\operatorname{Int}\mathfrak{g}_0 \cong (\operatorname{Aut}\mathfrak{g}_0)_0$ and hence that $\operatorname{Int}\mathfrak{g}_0$ is closed. Thus $\operatorname{Int}\mathfrak{g}_0$ is a compact connected Lie group with Lie algebra $\operatorname{ad}\mathfrak{g}_0 \cong \mathfrak{g}_0$. By Weyl's Theorem (Theorem 4.69) a universal covering group of $\operatorname{Int}\mathfrak{g}_0$ is a simply connected compact semisimple group with Lie algebra $\mathfrak{g}_0$. Since two simply connected analytic groups with isomorphic Lie algebras are isomorphic, the proposition follows.

Corollary 7.3. The isomorphism classes of

(a) simply connected compact semisimple Lie groups,
(b) compact semisimple Lie algebras,
(c) complex semisimple Lie algebras,
(d) reduced abstract root systems, and
(e) abstract Cartan matrices and their associated Dynkin diagrams

are in one-one correspondence by passage from a Lie group to its Lie algebra, then to the complexification of the Lie algebra, and then to the underlying root system.

PROOF. The correspondence of (a) to (b) is addressed by Proposition 7.2, that of (b) to (c) is addressed by Proposition 7.1, and that of (c) to (d) to (e) is addressed by Chapter II.

Proposition 7.4. A semisimple Lie group G whose Lie algebra $\mathfrak{g}$ is complex admits uniquely the structure of a complex Lie group in such a way that the exponential mapping is holomorphic.

REMARK. The proof will invoke Proposition 1.110, which in the general case made use of the complex form of Ado's Theorem (Theorem B.8). For semisimple G, the use of Ado's Theorem is not necessary. One has only to invoke the matrix-group form of Proposition 1.110 for the matrix group $\operatorname{Ad}(G)$ and then lift the complex structure from $\operatorname{Ad}(G)$ to the covering group G. As a result of this proposition, we may speak unambiguously of a **complex semisimple Lie group** as being a semisimple Lie group whose Lie algebra is complex.

PROOF. For existence, suppose that $\mathfrak{g}$ is complex. Then the converse part of Proposition 1.110 shows that G admits the structure of a complex Lie group compatibly with the multiplication-by-i mapping within $\mathfrak{g}$, and the direct part of Proposition 1.110 says that the exponential mapping is holomorphic. For uniqueness, suppose that G is complex with a holomorphic exponential mapping. Since exp is invertible as a smooth function on some open neighborhood V of the identity, $(V, \exp^{-1})$ is a chart for the complex structure on G, and the left translates $(L_g V, \exp^{-1} \circ L_g^{-1})$ form an atlas. This atlas does not depend on what complex structure makes G into a complex Lie group with holomorphic exponential mapping, and thus the complex structure is unique.

Proposition 7.5. A complex semisimple Lie group necessarily has finite center. Let G and G' be complex semisimple Lie groups, and let K and K' be the subgroups fixed by the respective global Cartan involutions of G and G'. Then K and K' are compact, and a homomorphism of K into K' as Lie groups induces a holomorphic homomorphism of G into G'. If the homomorphism $K \to K'$ is an isomorphism, then the holomorphic homomorphism $G \to G'$ is a holomorphic isomorphism.

PROOF. If G has Lie algebra $\mathfrak{g}$, then the most general Cartan decomposition of $\mathfrak{g}^{\mathbb{R}}$ is $\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$, where $\mathfrak{g}_0$ is a compact real form of $\mathfrak{g}$ by Proposition 6.14 and Corollary 6.19. The Lie algebra $\mathfrak{g}_0$ is compact semisimple, and Weyl's Theorem (Theorem 4.69) shows that the corresponding analytic subgroup K is compact. Theorem 6.31f then shows that G has finite center.

In a similar fashion let $\mathfrak{g}'$ be the Lie algebra of G'. We may suppose that there is a Cartan decomposition $\mathfrak{g}'^{\mathbb{R}} = \mathfrak{g}'_0 \oplus i\mathfrak{g}'_0$ of $\mathfrak{g}'^{\mathbb{R}}$ such that K' is the analytic subgroup of G' with Lie algebra $\mathfrak{g}'_0$. As with K, K' is compact.

A homomorphism φ of K into K' yields a homomorphism $d\varphi$ of $\mathfrak{g}_0$ into $\mathfrak{g}'_0$, and this extends uniquely to a complex-linear homomorphism, also denoted $d\varphi$, of $\mathfrak{g}$ into $\mathfrak{g}'$. Let $\widetilde{G}$ be a universal covering group of G, let $e : \widetilde{G} \to G$ be the covering homomorphism, and let $\widetilde{K}$ be the analytic subgroup of $\widetilde{G}$ with Lie algebra $\mathfrak{g}_0$. Since $\widetilde{G}$ is simply connected, $d\varphi$ lifts to a smooth homomorphism $\widetilde{\varphi}$ of $\widetilde{G}$ into G'.

We want to see that $\widetilde{\varphi}$ descends to a homomorphism of G into G'. To see this, we show that $\widetilde{\varphi}$ is 1 on the kernel of e. The restriction $\widetilde{\varphi}|_{\widetilde{K}}$ and the composition $\varphi \circ (e|_{\widetilde{K}})$ both have $d\varphi$ as differential. Therefore they are equal, and $\widetilde{\varphi}$ is 1 on the kernel of $e|_{\widetilde{K}}$. Theorem 6.31e shows that the kernel of e in $\widetilde{G}$ is contained in $\widetilde{K}$, and it follows that $\widetilde{\varphi}$ descends to a homomorphism of G into G' with differential $d\varphi$. Let us call this homomorphism φ. Then φ is a homomorphism between complex Lie groups, and its differential is complex linear. By Proposition 1.110, φ is holomorphic.

If the given homomorphism is an isomorphism, then we can reverse the roles of G and G', obtaining a holomorphic homomorphism $\psi : G' \to G$ whose differential is the inverse of $d\varphi$. Since $\psi \circ \varphi$ and $\varphi \circ \psi$ have differential the identity, φ and ψ are inverses. Therefore φ is a holomorphic isomorphism.

Corollary 7.6. If G is a complex semisimple Lie group, then G is holomorphically isomorphic to a complex Lie group of matrices.

PROOF. Let $\mathfrak{g}$ be the Lie algebra of G, let $\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ be a Cartan decomposition of $\mathfrak{g}^{\mathbb{R}}$, and let K be the analytic subgroup of G with Lie algebra $\mathfrak{g}_0$. The group K is compact by Proposition 7.5. By Corollary 4.22, K is isomorphic to a closed linear group, say K', and there is no loss of generality in assuming that the members of K' are in $GL(V)$ for a real vector space V. Let $\mathfrak{g}_0'$ be the linear Lie algebra of K', and write the complexification $\mathfrak{g}'$ of $\mathfrak{g}_0'$ as a Lie algebra of complex endomorphisms of $V^{\mathbb{C}}$. If G' is the analytic subgroup of $GL(V^{\mathbb{C}})$ with Lie algebra $\mathfrak{g}'$, then G' is a complex Lie group by Corollary 1.116 since $GL(V^{\mathbb{C}})$ is complex and $\mathfrak{g}'$ is closed under multiplication by i. Applying Proposition 7.5, we can extend the isomorphism of K onto K' to a holomorphic isomorphism of G onto G'. Thus G' provides the required complex Lie group of matrices.

Let G be a semisimple Lie group, and suppose that $G^{\mathbb{C}}$ is a complex semisimple Lie group such that G is an analytic subgroup of $G^{\mathbb{C}}$ and the Lie algebra of $G^{\mathbb{C}}$ is the complexification of the Lie algebra of G. Then we say that $G^{\mathbb{C}}$ is a **complexification** of G and that G has a complexification $G^{\mathbb{C}}$. For example, $SU(n)$ and $SL(n,\mathbb{R})$ both have $SL(n,\mathbb{C})$ as complexification. Because of Corollary 7.6 it will follow from Proposition 7.9 below that if G has a complexification $G^{\mathbb{C}}$, then G is necessarily closed in $G^{\mathbb{C}}$. Not every semisimple Lie group has a complexification; because of Corollary 7.6, the example at the end of §VI.3 shows that a double cover of $SL(2,\mathbb{R})$ has no complexification. If G has a complexification, then the complexification is not necessarily unique up to isomorphism. However, Proposition 7.5 shows that the complexification is unique if G is compact.

We now use the correspondence of Corollary 7.3 to investigate automorphisms of complex semisimple Lie algebras.

Lemma 7.7. Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\Delta^+(\mathfrak{g},\mathfrak{h})$ be a positive system for the roots. If H denotes the analytic subgroup of G with Lie algebra $\mathfrak{h}$, then any member of $\operatorname{Int}\mathfrak{g}$ carrying $\mathfrak{h}$ to itself and $\Delta^+(\mathfrak{g},\mathfrak{h})$ to itself is in $\operatorname{Ad}_{\mathfrak{g}}(H)$.

PROOF. The construction of Theorem 6.11 produces a compact real form $\mathfrak{g}_0$ of $\mathfrak{g}$ such that $\mathfrak{g}_0 \cap \mathfrak{h} = \mathfrak{h}_0$ is a maximal abelian subspace of $\mathfrak{g}_0$. The decomposition $\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ is a Cartan decomposition of $\mathfrak{g}^{\mathbb{R}}$ by Proposition 6.14, and we let θ be the Cartan involution. Let K be the analytic subgroup of G with Lie algebra $\mathfrak{g}_0$. The subgroup K is compact by Proposition 7.5. If T is the analytic subgroup of K with Lie algebra $\mathfrak{h}_0$, then T is a maximal torus of K.

Let g be in G, and suppose that $\operatorname{Ad}(g)$ carries $\mathfrak{h}$ to itself and $\Delta^+(\mathfrak{g},\mathfrak{h})$ to itself. By Theorem 6.31 we can write $g = k\exp X$ with $k \in K$ and $X \in i\mathfrak{g}_0$. The map $\operatorname{Ad}(\Theta g)$ is the differential at 1 of $g \mapsto (\Theta g)x(\Theta g)^{-1} = \Theta(g(\Theta x)g^{-1})$, hence is $\theta\operatorname{Ad}(g)\theta$. Since $\theta\mathfrak{h} = \mathfrak{h}$, $\operatorname{Ad}(\Theta g)$ carries $\mathfrak{h}$ to itself. Therefore so does $\operatorname{Ad}((\Theta g)^{-1}g) = \operatorname{Ad}(\exp 2X)$.

The linear transformation $\operatorname{Ad}(\exp 2X)$ is diagonable on $\mathfrak{g}^{\mathbb{R}}$ with positive eigenvalues. Since it carries $\mathfrak{h}$ to $\mathfrak{h}$, there exists a real subspace $\mathfrak{h}'$ of $\mathfrak{g}^{\mathbb{R}}$ carried to itself by $\operatorname{Ad}(\exp 2X)$ such that $\mathfrak{g}^{\mathbb{R}} = \mathfrak{h} \oplus \mathfrak{h}'$. The transformation $\operatorname{Ad}(\exp 2X)$ has a unique diagonable logarithm with real eigenvalues, and there are two candidates for this logarithm. One is $\operatorname{ad} 2X$, and the other is the sum of the logarithms on $\mathfrak{h}$ and $\mathfrak{h}'$ separately. By uniqueness we conclude that $\operatorname{ad} 2X$ carries $\mathfrak{h}$ to $\mathfrak{h}$. By Proposition 2.7, X is in $\mathfrak{h}$.

Therefore $\exp X$ is in H, and it is enough to show that k is in T. Here k is a member of K such that $\operatorname{Ad}(k)$ leaves $\mathfrak{h}_0$ stable and $\Delta^+(\mathfrak{g},\mathfrak{h})$ stable. Since $\operatorname{Ad}(k)$ leaves $\mathfrak{h}_0$ stable, Theorem 4.54 says that $\operatorname{Ad}(k)$ is in the Weyl group $W(\mathfrak{g},\mathfrak{h})$. Since $\operatorname{Ad}(k)$ leaves $\Delta^+(\mathfrak{g},\mathfrak{h})$ stable, Theorem 2.63 says that $\operatorname{Ad}(k)$ yields the identity element in $W(\mathfrak{g},\mathfrak{h})$. Therefore $\operatorname{Ad}(k)$ is 1 on $\mathfrak{h}$, and k commutes with T. By Corollary 4.52, k is in T.

Theorem 7.8. If $\mathfrak{g}_0$ is a compact semisimple Lie algebra and $\mathfrak{g}$ is its complexification, then the following three groups are canonically isomorphic:

(a) $\operatorname{Aut}_{\mathbb{R}}\mathfrak{g}_0/\operatorname{Int}\mathfrak{g}_0$,
(b) $\operatorname{Aut}_{\mathbb{C}}\mathfrak{g}/\operatorname{Int}\mathfrak{g}$, and
(c) the group of automorphisms of the Dynkin diagram of $\mathfrak{g}$.

PROOF. By Proposition 7.4 let G be a simply connected complex Lie group with Lie algebra $\mathfrak{g}$, for example a universal covering group of $\operatorname{Int}\mathfrak{g}$.

The analytic subgroup K with Lie algebra $\mathfrak{g}_0$ is simply connected by Theorem 6.31, and K is compact by Proposition 7.5.

Fix a maximal abelian subspace $\mathfrak{h}_0$ of $\mathfrak{g}_0$, let $\Delta^+(\mathfrak{g}, \mathfrak{h})$ be a positive system of roots, and let T be the maximal torus of K with Lie algebra $\mathfrak{h}_0$. Let D be the Dynkin diagram of $\mathfrak{g}$, and let $\operatorname{Aut} D$ be the group of automorphisms of D. Any member of $\operatorname{Aut}_{\mathbb{R}} \mathfrak{g}_0$ extends by complexifying to a member of $\operatorname{Aut}_{\mathbb{C}} \mathfrak{g}$, and members of $\operatorname{Int} \mathfrak{g}_0$ yield members of $\operatorname{Int} \mathfrak{g}$. Thus we obtain a group homomorphism $\Phi : \operatorname{Aut}_{\mathbb{R}} \mathfrak{g}_0 / \operatorname{Int} \mathfrak{g}_0 \to \operatorname{Aut}_{\mathbb{C}} \mathfrak{g} / \operatorname{Int} \mathfrak{g}$.

Let us observe that Φ is onto. In fact, if a member φ of $\operatorname{Aut}_{\mathbb{C}} \mathfrak{g}$ is given, then $\varphi(\mathfrak{g}_0)$ is a compact real form of $\mathfrak{g}$. By Corollary 6.20 we can adjust φ by a member of $\operatorname{Int} \mathfrak{g}$ so that φ carries $\mathfrak{g}_0$ into itself. Thus some automorphism of $\mathfrak{g}_0$ is carried to the coset of φ under Φ.

We shall construct a group homomorphism $\Psi : \operatorname{Aut}_{\mathbb{C}} \mathfrak{g} / \operatorname{Int} \mathfrak{g} \to \operatorname{Aut} D$. Let $\varphi \in \operatorname{Aut}_{\mathbb{C}} \mathfrak{g}$ be given. Since $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ (by Proposition 2.13), $\varphi(\mathfrak{h})$ is another Cartan subalgebra. By Theorem 2.15 there exists $\psi_1 \in \operatorname{Int} \mathfrak{g}$ with $\psi_1 \varphi(\mathfrak{h}) = \mathfrak{h}$. Then $\psi_1 \varphi$ maps $\Delta(\mathfrak{g}, \mathfrak{h})$ to itself and carries $\Delta^+(\mathfrak{g}, \mathfrak{h})$ to another positive system $(\Delta^+)'(\mathfrak{g}, \mathfrak{h})$. By Theorem 2.63 there exists a unique member w of the Weyl group $W(\mathfrak{g}, \mathfrak{h})$ carrying $(\Delta^+)'(\mathfrak{g}, \mathfrak{h})$ to $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Theorem 4.54 shows that w is implemented by a member of $\operatorname{Ad}(K)$, hence by a member ψ_2 of $\operatorname{Int} \mathfrak{g}$. Then $\psi_2 \psi_1 \varphi$ maps $\Delta^+(\mathfrak{g}, \mathfrak{h})$ to itself and yields an automorphism of the Dynkin diagram.

Let us see the effect of the choices we have made. With different choices, we would be led to some $\psi_2' \psi_1' \varphi$ mapping $\Delta^+(\mathfrak{g}, \mathfrak{h})$ to itself, and the claim is that we get the same member of $\operatorname{Aut} D$. In fact the composition $\psi = (\psi_2' \psi_1' \varphi) \circ (\psi_2 \psi_1 \varphi)^{-1}$ is in $\operatorname{Int} \mathfrak{g}$. Lemma 7.7 shows that ψ acts as the identity on $\mathfrak{h}$, and hence the automorphism of the Dynkin diagram corresponding to ψ is the identity. Therefore $\psi_2 \psi_1 \varphi$ and $\psi_2' \psi_1' \varphi$ lead to the same member of $\operatorname{Aut} D$.

Consequently the above construction yields a well defined function Ψ from $\operatorname{Aut}_{\mathbb{C}} \mathfrak{g} / \operatorname{Int} \mathfrak{g}$ into $\operatorname{Aut} D$. Since we can adjust any $\varphi \in \operatorname{Aut}_{\mathbb{C}} \mathfrak{g}$ by a member of $\operatorname{Int} \mathfrak{g}$ so that $\mathfrak{h}$ maps to itself and $\Delta^+(\mathfrak{g}, \mathfrak{h})$ maps to itself, it follows that Ψ is a homomorphism.

Let us prove that $\Psi \circ \Phi$ is one-one. Thus let $\varphi \in \operatorname{Aut}_{\mathbb{R}} \mathfrak{g}_0$ lead to the identity element of $\operatorname{Aut} D$. Write φ also for the corresponding complex-linear automorphism on $\mathfrak{g}$. Theorem 4.34 shows that we may adjust φ by a member of $\operatorname{Int} \mathfrak{g}_0$ so that φ carries $\mathfrak{h}_0$ to itself, and Theorems 2.63 and 4.54 show that we may adjust φ further by a member of $\operatorname{Int} \mathfrak{g}_0$ so that φ carries $\Delta^+(\mathfrak{g}, \mathfrak{h})$ to itself. Let E_{α_i} be root vectors for the simple roots $\alpha_1, \dots, \alpha_l$ of $\mathfrak{g}$. Since φ is the identity on $\mathfrak{h}$, $\varphi(E_{\alpha_i}) = c_i E_{\alpha_i}$ for nonzero constants

$c_1, \ldots, c_l$. For each j, let x_j be any complex number with $e^{x_j} = c_j$. Choose, for $1 \le i \le l$, members h_j of $\mathfrak{h}$ with $\alpha_i(h_j) = \delta_{ij}$, and put $g = \exp\left(\sum_{j=1}^{l} x_j h_j\right)$. The element g is in H. Then $\operatorname{Ad}(g)(E_{\alpha_i}) = c_i E_{\alpha_i}$ for each i. Consequently $\operatorname{Ad}(g)$ is a member of $\operatorname{Int}\mathfrak{g}$ that agrees with φ on $\mathfrak{h}$ and on each E_{α_i}. By the Isomorphism Theorem (Theorem 2.108), $\varphi = \operatorname{Ad}(g)$.

To complete the proof that $\Psi \circ \Phi$ is one-one, we show that g is in T. We need to see that $|c_j| = 1$ for all j, so that x_j can be chosen purely imaginary. First we show that $\overline{E}_{\alpha_j}$ is a root vector for $-\alpha_j$ if bar denotes the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. In fact, write $E_{\alpha_j} = X_j + iY_j$ with X_j and Y_j in $\mathfrak{g}_0$. If h is in $\mathfrak{h}_0$, then $\alpha_j(h)$ is purely imaginary. Since $[\mathfrak{h}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0$, it follows from the equality

$$[h, X_j] + i[h, Y_j] = [h, E_{\alpha_j}] = \alpha_j(h) E_{\alpha_j} = i\alpha_j(h) Y_j + \alpha_j(h) X_j$$

that $[h, X_j] = i\alpha_j(h)Y_j$ and $i[h, Y_j] = \alpha_j(h)X_j$. Subtracting these two formulas gives

$$[h, X_j - iY_j] = i\alpha_j(h)Y_j - \alpha_j(h)X_j = -\alpha_j(h)(X_j - iY_j)$$

and shows that $\overline{E}_{\alpha_j}$ is indeed a root vector for $-\alpha_j$. Hence we find that $[E_{\alpha_j}, \overline{E}_{\alpha_j}]$ is in $\mathfrak{h}$. Since φ is complex linear and carries $\mathfrak{g}_0$ to itself, φ respects bar. Therefore $\varphi(\overline{E}_{\alpha_j}) = \bar{c}_j \overline{E}_{\alpha_j}$. Since φ fixes every element of $\mathfrak{h}$, φ fixes $[E_{\alpha_j}, \overline{E}_{\alpha_j}]$, and it follows that $c_j \bar{c}_j = 1$. We conclude that g is in T and that $\Psi \circ \Phi$ is one-one.

Since Φ is onto and $\Psi \circ \Phi$ is one-one, both Φ and Ψ are one-one. The fact that Ψ is onto is a consequence of the Isomorphism Theorem (Theorem 2.108) and is worked out in detail in the second example at the end of §II.10. This completes the proof of the theorem.

Now we take up some properties of Lie groups of matrices to prepare for the definition of "reductive Lie group" in the next section.

Proposition 7.9. Let G be an analytic subgroup of real or complex matrices whose Lie algebra $\mathfrak{g}_0$ is semisimple. Then G has finite center and is a closed linear group.

PROOF. Without loss of generality we may assume that G is an analytic subgroup of $GL(V)$ for a real vector space V. Let $\mathfrak{g}_0$ be the linear Lie algebra of G, and write the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ as a Lie algebra of complex endomorphisms of $V^{\mathbb{C}}$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition, and let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}_0$. The Lie subalgebra $\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ of $\operatorname{End}_{\mathbb{C}} V$ is a compact semisimple Lie algebra, and we let U be the analytic subgroup of $GL(V^{\mathbb{C}})$ with Lie algebra $\mathfrak{u}_0$. Proposition 7.2 implies that the universal covering group $\widetilde{U}$ of U is compact, and it follows that U is compact. Since U has discrete center, the center Z_U of U must be finite.

The center Z_G of G is contained in K by Theorem 6.31e, and $K \subseteq U$ since $\mathfrak{k}_0 \subseteq \mathfrak{u}_0$. Since $\operatorname{Ad}_{\mathfrak{g}}(Z_G)$ acts as 1 on $\mathfrak{u}_0$, we conclude that $Z_G \subseteq Z_U$. Therefore Z_G is finite. This proves the first conclusion. By Theorem 6.31f, K is compact.

Since U is compact, Proposition 4.6 shows that $V^{\mathbb{C}}$ has a Hermitian inner product preserved by U. Then U is contained in the unitary group $U(V^{\mathbb{C}})$. Let $\mathfrak{p}(V^{\mathbb{C}})$ be the vector space of Hermitian transformations of $V^{\mathbb{C}}$ so that $GL(V^{\mathbb{C}})$ has the polar decomposition $GL(V^{\mathbb{C}}) = U(V^{\mathbb{C}}) \exp \mathfrak{p}(V^{\mathbb{C}})$. The members of $\mathfrak{u}_0$ are skew Hermitian, and hence the members of $\mathfrak{k}_0$ are skew Hermitian and the members of $\mathfrak{p}_0$ are Hermitian. Therefore the global Cartan decomposition $G = K \exp \mathfrak{p}_0$ of G that is given in Theorem 6.31c is compatible with the polar decomposition of $GL(V^{\mathbb{C}})$.

We are to prove that G is closed in $GL(V^{\mathbb{C}})$. Let $g_n = k_n \exp X_n$ tend to $g \in GL(V^{\mathbb{C}})$. Using the compactness of K and passing to a subsequence, we may assume that k_n tends to $k \in K$. Therefore $\exp X_n$ converges. Since the polar decomposition of $GL(V^{\mathbb{C}})$ is a homeomorphism, it follows that $\exp X_n$ has limit $\exp X$ for some $X \in \mathfrak{p}(V^{\mathbb{C}})$. Since $\mathfrak{p}_0$ is closed in $\mathfrak{p}(V^{\mathbb{C}})$, X is in $\mathfrak{p}_0$. Therefore $g = k \exp X$ exhibits g as in G, and G is closed.

Corollary 7.10. Let G be an analytic subgroup of real or complex matrices whose Lie algebra $\mathfrak{g}_0$ is reductive, and suppose that the identity component of the center of G is compact. Then G is a closed linear group.

REMARK. In this result and some to follow, we shall work with analytic groups whose Lie algebras are direct sums. If G is an analytic group whose Lie algebra $\mathfrak{g}_0$ is a direct sum $\mathfrak{g}_0 = \mathfrak{a}_0 \oplus \mathfrak{b}_0$ of ideals and if A and B are the analytic subgroups corresponding to $\mathfrak{a}_0$ and $\mathfrak{b}_0$, then G is a commuting product $G = AB$. This fact follows from Proposition 1.122 or may be derived directly, as in the proof of Theorem 4.29.

PROOF. Write $\mathfrak{g}_0 = Z_{\mathfrak{g}_0} \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$ by Corollary 1.56. The analytic subgroup of G corresponding to $Z_{\mathfrak{g}_0}$ is $(Z_G)_0$, and we let G_{ss} be the analytic subgroup corresponding to $[\mathfrak{g}_0, \mathfrak{g}_0]$. By the remarks before the proof, G is the commuting product $(Z_G)_0 G_{ss}$. The group G_{ss} is closed as a group of matrices by Proposition 7.9, and $(Z_G)_0$ is compact by assumption. Hence the set of products, which is G, is closed.

Corollary 7.11. Let G be a connected closed linear group whose Lie algebra $\mathfrak{g}_0$ is reductive. Then the analytic subgroup G_{ss} of G with Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ is closed, and G is the commuting product $G = (Z_G)_0 G_{ss}$.

PROOF. The subgroup G_{ss} is closed by Proposition 7.9, and G is the commuting product $(Z_G)_0 G_{ss}$ by the remarks with Corollary 7.10.

Proposition 7.12. Let G be a compact connected linear Lie group, and let $\mathfrak{g}_0$ be its linear Lie algebra. Then the complex analytic group $G^{\mathbb{C}}$ of matrices with linear Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ is a closed linear group.

REMARKS. If G is a compact connected Lie group, then Corollary 4.22 implies that G is isomorphic to a closed linear group. If G is realized as a closed linear group in two different ways, then this proposition in principle produces two different groups $G^{\mathbb{C}}$. However, Proposition 7.5 shows that the two groups $G^{\mathbb{C}}$ are isomorphic. Therefore with no reference to linear groups, we can speak of the complexification $G^{\mathbb{C}}$ of a compact connected Lie group G, and $G^{\mathbb{C}}$ is unique up to isomorphism. Proposition 7.5 shows that a homomorphism between two such groups G and G' induces a holomorphic homomorphism between their complexifications.

PROOF. By Theorem 4.29 let us write $G = (Z_G)_0 G_{ss}$ with G_{ss} compact semisimple. Proposition 4.6 shows that we may assume without loss of generality that G is a connected closed subgroup of a unitary group $U(n)$ for some n, and Corollary 4.7 shows that we may take $(Z_G)_0$ to be diagonal.

Let us complexify the decomposition $\mathfrak{g}_0 = Z_{\mathfrak{g}_0} \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$ to obtain $\mathfrak{g}^{\mathbb{R}} = Z_{\mathfrak{g}_0} \oplus iZ_{\mathfrak{g}_0} \oplus [\mathfrak{g}, \mathfrak{g}]$. The analytic subgroup corresponding to $Z_{\mathfrak{g}_0}$ is $G_1 = (Z_G)_0$ and is compact. Since $iZ_{\mathfrak{g}_0}$ consists of real diagonal matrices, Corollary 1.134 shows that its corresponding analytic subgroup G_2 is closed. In addition the analytic subgroup G_3 with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is closed by Proposition 7.9. By the remarks with Corollary 7.10, the group $G^{\mathbb{C}}$ is the commuting product of these three subgroups, and we are to show that the product is closed.

For G_3, negative conjugate transpose is a Cartan involution of its Lie algebra, and therefore conjugate transpose inverse is a global Cartan involution of G_3. Consequently G_3 has a global Cartan decomposition $G_3 = G_{ss} \exp(\mathfrak{p}_3)_0$, where $(\mathfrak{p}_3)_0 = i[\mathfrak{g}_0, \mathfrak{g}_0]$. Since $iZ_{\mathfrak{g}_0}$ commutes with $(\mathfrak{p}_3)_0$ and since the polar decomposition of all matrices is a homeomorphism, it follows that the product G_2G_3 is closed. Since G_1 is compact, $G^{\mathbb{C}} = G_1G_2G_3$ is closed.

Lemma 7.13. On matrices let Θ be conjugate transpose inverse, and let θ be negative conjugate transpose. Let G be a connected abelian closed linear group that is stable under Θ, and let $\mathfrak{g}_0$ be its linear Lie algebra, stable under θ. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the decomposition of $\mathfrak{g}_0$ into $+1$ and -1 eigenspaces under θ, and let $K = \{x \in G \mid \Theta x = x\}$. Then the map $K \times \mathfrak{p}_0 \to G$ given by $(k, X) \mapsto k \exp X$ is a Lie group isomorphism.

PROOF. The group K is a closed subgroup of the unitary group and is compact with Lie algebra $\mathfrak{k}_0$. Since $\mathfrak{p}_0$ is abelian, $\exp \mathfrak{p}_0$ is the analytic subgroup of G with Lie algebra $\mathfrak{p}_0$. By the remarks following the statement of Corollary 7.10, $G = K \exp \mathfrak{p}_0$. The smooth map $K \times \mathfrak{p}_0 \to G$ is compatible with the polar decomposition of matrices and is therefore one-one. It is a Lie group homomorphism since G and $\mathfrak{p}_0$ are abelian. Its inverse is smooth since the inverse of the polar decomposition of matrices is smooth (by an argument in the proof of Theorem 6.31).

Proposition 7.14. On matrices let Θ be conjugate transpose inverse, and let θ be negative conjugate transpose. Let G be a connected closed linear group that is stable under Θ, and let $\mathfrak{g}_0$ be its linear Lie algebra, stable under θ. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the decomposition of $\mathfrak{g}_0$ into $+1$ and -1 eigenspaces under θ, and let $K = \{x \in G \mid \Theta x = x\}$. Then the map $K \times \mathfrak{p}_0 \to G$ given by $(k, X) \mapsto k \exp X$ is a diffeomorphism onto.

PROOF. By Proposition 1.59, $\mathfrak{g}_0$ is reductive. Therefore Corollary 1.56 allows us to write $\mathfrak{g}_0 = Z_{\mathfrak{g}_0} \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$ with $[\mathfrak{g}_0, \mathfrak{g}_0]$ semisimple. The analytic subgroup of G with Lie algebra $Z_{\mathfrak{g}_0}$ is $(Z_G)_0$, and we let G_{ss} be the analytic subgroup of G with Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$. By Corollary 7.11, $(Z_G)_0$ and G_{ss} are closed, and $G = (Z_G)_0 G_{ss}$. It is clear that $Z_{\mathfrak{g}_0}$ and $[\mathfrak{g}_0, \mathfrak{g}_0]$ are stable under θ, and hence $(Z_G)_0$ and G_{ss} are stable under Θ.

Because of the polar decomposition of matrices, the map $K \times \mathfrak{p}_0 \to G$ is smooth and one-one. The parts of this map associated with $(Z_G)_0$ and G_{ss} are onto by Lemma 7.13 and Theorem 6.31, respectively. Since $(Z_G)_0$ and G_{ss} commute with each other, it follows that $K \times \mathfrak{p}_0 \to G$ is onto. The inverse is smooth since the inverse of the polar decomposition of matrices is smooth (by an argument in the proof of Theorem 6.31).

Proposition 7.15 (Weyl's unitary trick). Let G be an analytic subgroup of complex matrices whose linear Lie algebra $\mathfrak{g}_0$ is semisimple and is stable under the map θ given by negative conjugate transpose. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition of $\mathfrak{g}_0$ defined by θ, and suppose that $\mathfrak{k}_0 \cap i\mathfrak{p}_0 = 0$. Let U and $G^{\mathbb{C}}$ be the analytic subgroups of matrices with respective Lie algebras $\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ and $\mathfrak{g} = (\mathfrak{k}_0 \oplus \mathfrak{p}_0)^{\mathbb{C}}$. The group U is compact. Suppose that U is simply connected. If V is any finite-dimensional complex vector space, then a representation of any of the following kinds on V leads, via the formula

$$\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0 = \mathfrak{u}_0 \oplus i\mathfrak{u}_0, \tag{7.16}$$

to a representation of each of the other kinds. Under this correspondence invariant subspaces and equivalences are preserved:

(a) a representation of G on V,
(b) a representation of U on V,
(c) a holomorphic representation of $G^{\mathbb{C}}$ on V,
(d) a representation of $\mathfrak{g}_0$ on V,
(e) a representation of $\mathfrak{u}_0$ on V,
(f) a complex-linear representation of $\mathfrak{g}$ on V.

PROOF. The groups G, U, and $G^{\mathbb{C}}$ are closed linear groups by Proposition 7.9, and U is compact, being a closed subgroup of the unitary group. Since U is simply connected and its Lie algebra is a compact real form of $\mathfrak{g}$, $G^{\mathbb{C}}$ is simply connected.

We can pass from (c) to (a) or (b) by restriction. Since continuous homomorphisms between Lie groups are smooth, we can pass from (a) or (b) to (d) or (e) by taking differentials. Formula (7.16) allows us to pass from (d) or (e) to (f). Since $G^{\mathbb{C}}$ is simply connected, a Lie algebra homomorphism as in (f) lifts to a group homomorphism, and the group homomorphism must be holomorphic since the Lie algebra homomorphism is assumed complex linear (Proposition 1.110). Thus we can pass from (f) to (c). If we follow the steps all the way around, starting from (c), we end up with the original representation, since the differential at the identity uniquely determines a homomorphism of connected Lie groups. Thus invariant subspaces and equivalence are preserved.

EXAMPLE. Weyl's unitary trick gives us a new proof of the fact that finite-dimensional complex-linear representations of complex semisimple Lie algebras are completely reducible (Theorem 5.29); the crux of the new proof is the existence of a compact real form (Theorem 6.11). For the argument let the Lie algebra $\mathfrak{g}$ be given, and let G be a simply connected complex semisimple group with Lie algebra $\mathfrak{g}$. Corollary 7.6 allows us to regard G as a subgroup of $GL(V^{\mathbb{C}})$ for some finite-dimensional complex vector space $V^{\mathbb{C}}$. Let $\mathfrak{u}_0$ be a compact real form of $\mathfrak{g}$, so that $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u}_0 \oplus i\mathfrak{u}_0$, and let U be the analytic subgroup of G with Lie algebra $\mathfrak{u}_0$. Proposition 7.15 notes that U is compact. By Proposition 4.6 we can introduce a Hermitian inner product into $V^{\mathbb{C}}$ so that U is a subgroup of the unitary group. If a complex-linear representation of $\mathfrak{g}$ is given, we can use the passage (f) to (b) in Proposition 7.15 to obtain a representation of U. This is completely reducible by Corollary 4.7, and the complete reducibility of the given representation of $\mathfrak{g}$ follows.

The final proposition shows how to recognize a Cartan decomposition of a real semisimple Lie algebra in terms of a bilinear form other than the Killing form.

Proposition 7.17. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, let θ be an involution of $\mathfrak{g}_0$, and let B be a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}_0$ such that $B(\theta X, \theta Y) = B(X, Y)$ for all X and Y in $\mathfrak{g}_0$. If the form $B_\theta(X, Y) = -B(X, \theta Y)$ is positive definite, then θ is a Cartan involution of $\mathfrak{g}_0$.

PROOF. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the decomposition of $\mathfrak{g}_0$ into $+1$ and -1 eigenspaces under θ, and extend B to be complex bilinear on the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$. Since θ is an involution, $\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ is a Lie subalgebra of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$, necessarily a real form. Here $\mathfrak{g}$ is semisimple, and then so is $\mathfrak{u}_0$. Since B_θ is positive definite, B is negative definite on $\mathfrak{k}_0$ and on $i\mathfrak{p}_0$. Also $\mathfrak{k}_0$ and $i\mathfrak{p}_0$ are orthogonal since $X \in \mathfrak{k}_0$ and $Y \in i\mathfrak{p}_0$ implies

$$B(X, Y) = B(\theta X, \theta Y) = B(X, -Y) = -B(X, Y).$$

Hence B is real valued and negative definite on $\mathfrak{u}_0$.

By Propositions 1.120 and 1.121, $\operatorname{Int}\mathfrak{u}_0 = (\operatorname{Aut}_{\mathbb{R}}\mathfrak{u}_0)_0$. Consequently $\operatorname{Int}\mathfrak{u}_0$ is a closed subgroup of $GL(\mathfrak{u}_0)$. On the other hand, we have just seen that $-B$ is an inner product on $\mathfrak{u}_0$, and in this inner product every member of $\operatorname{ad}\mathfrak{u}_0$ is skew symmetric. Therefore the corresponding analytic subgroup $\operatorname{Int}\mathfrak{u}_0$ of $GL(\mathfrak{u}_0)$ acts by orthogonal transformations. Since $\operatorname{Int}\mathfrak{u}_0$ is then exhibited as a closed subgroup of the orthogonal group, $\operatorname{Int}\mathfrak{u}_0$ is compact. Hence $\mathfrak{u}_0$ is a compact real form of $\mathfrak{g}$. By the remarks preceding Lemma 6.27, θ is a Cartan involution of $\mathfrak{g}_0$.

2. Reductive Lie Groups

We are ready to define the class of groups that will be the objects of study in this chapter. The intention is to study semisimple groups, but, as was already the case in Chapters IV and VI, we shall often have to work with centralizers of abelian analytic subgroups invariant under a Cartan involution, and these centralizers may be disconnected and may have positive-dimensional center. To be able to use arguments that take advantage of such subgroups and proceed by induction on the dimension, we are forced to enlarge the class of groups under study. Groups in the enlarged class are always called "reductive," but their characterizing properties vary from author to author. We shall use the following definition.

A **reductive Lie group** is actually a 4-tuple (G, K, θ, B) consisting of a Lie group G, a compact subgroup K of G, a Lie algebra involution θ of the Lie algebra $\mathfrak{g}_0$ of G, and a nondegenerate, $\mathrm{Ad}(G)$ invariant, θ invariant, bilinear form B on $\mathfrak{g}_0$ such that

(i) $\mathfrak{g}_0$ is a reductive Lie algebra,
(ii) the decomposition of $\mathfrak{g}_0$ into $+1$ and -1 eigenspaces under θ is $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, where $\mathfrak{k}_0$ is the Lie algebra of K,
(iii) $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are orthogonal under B, and B is positive definite on $\mathfrak{p}_0$ and negative definite on $\mathfrak{k}_0$,
(iv) multiplication, as a map from $K \times \exp \mathfrak{p}_0$ into G, is a diffeomorphism onto, and
(v) every automorphism $\mathrm{Ad}(g)$ of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ is **inner** for $g \in G$, i.e., is given by some x in $\mathrm{Int}\,\mathfrak{g}$.

When informality permits, we shall refer to the reductive Lie group simply as G. Then θ will be called the **Cartan involution**, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ will be called the **Cartan decomposition** of $\mathfrak{g}_0$, K will be called the associated **maximal compact subgroup** (a name justified by Proposition 7.19a below), and B will be called the **invariant bilinear form**.

The idea is that a reductive Lie group G is a Lie group whose Lie algebra is reductive, whose center is not too wild, and whose disconnectedness is not too wild. The various properties make precise the notion "not too wild." In particular, property (iv) and the compactness of K say that G has only finitely many components.

We write G_{ss} for the semisimple analytic subgroup of G with Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$. The decomposition of G in property (iv) is called the **global**

Cartan decomposition. Sometimes one assumes about a reductive Lie group that also

(vi) G_{ss} has finite center.

In this case the reductive group will be said to be in the **Harish-Chandra class** because of the use of axioms equivalent with (i) through (vi) by Harish-Chandra. Reductive groups in the Harish-Chandra class have often been the groups studied in representation theory.

EXAMPLES.

1) G is any semisimple Lie group with finite center, θ is a Cartan involution, K is the analytic subgroup with Lie algebra $\mathfrak{k}_0$, and B is the Killing form. Property (iv) and the compactness of K follow from Theorem 6.31. Property (v) is automatic since G connected makes $\operatorname{Ad}(G) = \operatorname{Int}\mathfrak{g}_0 \subseteq \operatorname{Int}\mathfrak{g}$. Property (vi) has been built into the definition for this example.

2) G is any connected closed linear group of real or complex matrices closed under conjugate transpose inverse, θ is negative conjugate transpose, K is the intersection of G with the unitary group, and $B(X, Y)$ is $\operatorname{Re}\operatorname{Tr}(XY)$. The compactness of K follows since K is the intersection of the unitary group with the closed group of matrices G. Property (iv) follows from Proposition 7.14, and property (v) is automatic since G is connected. Property (vi) is automatic for any linear group by Proposition 7.9.

3) G is any compact Lie group satisfying property (v). Then $K = G$, $\theta = 1$, and B is the negative of an inner product constructed as in Proposition 4.24. Properties (i) through (iv) are trivial, and property (vi) follows from Theorem 4.21. Every finite group G is trivially an example where property (v) holds. Property (v) is satisfied by the orthogonal group $O(n)$ if n is odd but not by $O(n)$ if n is even.

4) G is any closed linear group of real or complex matrices closed under conjugate transpose inverse, given as the common zero locus of some set of real-valued polynomials in the real and imaginary parts of the matrix entries, and satisfying property (v). Here θ is negative conjugate transpose, K is the intersection of G with the unitary group, and $B(X, Y)$ is $\operatorname{Re}\operatorname{Tr}(XY)$. The compactness of K follows since K is the intersection of the unitary group with the closed group of matrices G. Properties (iv) and (vi) follow from Propositions 1.143 and 7.9, respectively. The closed linear group of real matrices of determinant ± 1 satisfies property (v) since

$$\operatorname{Ad}(\operatorname{diag}(-1, 1, \ldots, 1)) = \operatorname{Ad}(\operatorname{diag}(e^{i\pi(n-1)/n}, e^{-i\pi/n}, \ldots, e^{-i\pi/n})).$$

But as noted in Example 3, the orthogonal group $O(n)$ does not satisfy property (v) if n is even.

5) G is the centralizer in a reductive group $\widetilde{G}$ of a θ stable abelian subalgebra of the Lie algebra of $\widetilde{G}$. Here K is obtained by intersection, and θ and B are obtained by restriction. The verification that G is a reductive Lie group will be given below in Proposition 7.25.

If G is semisimple with finite center and if K, θ, and B are specified so that G is considered as a reductive group, then θ is forced to be a Cartan involution in the sense of Chapter VI. This is the content of Proposition 7.17. Hence the new terms "Cartan involution" and "Cartan decomposition" are consistent with the terminology of Chapter VI in the case that G is semisimple.

An alternative way of saying (iii) is that the symmetric bilinear form

$$B_\theta(X, Y) = -B(X, \theta Y) \tag{7.18}$$

is positive definite on $\mathfrak{g}_0$.

We use the notation $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$, etc., to denote the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$, etc. Using complex linearity, we extend θ from $\mathfrak{g}_0$ to $\mathfrak{g}$ and B from $\mathfrak{g}_0 \times \mathfrak{g}_0$ to $\mathfrak{g} \times \mathfrak{g}$.

Proposition 7.19. If G is a reductive Lie group, then

(a) K is a maximal compact subgroup of G,
(b) K meets every component of G, i.e., $G = KG_0$,
(c) each member of $\operatorname{Ad}(K)$ leaves $\mathfrak{k}_0$ and $\mathfrak{p}_0$ stable and therefore commutes with θ,
(d) $(\operatorname{ad} X)^* = -\operatorname{ad}\theta X$ relative to B_θ if X is in $\mathfrak{g}_0$,
(e) θ leaves $Z_{\mathfrak{g}_0}$ and $[\mathfrak{g}_0, \mathfrak{g}_0]$ stable, and the restriction of θ to $[\mathfrak{g}_0, \mathfrak{g}_0]$ is a Cartan involution,
(f) the identity component G_0 is a reductive Lie group (with maximal compact subgroup obtained by intersection and with Cartan involution and invariant form unchanged).

PROOF. For (a) assume the contrary, and let K_1 be a compact subgroup of G properly containing K. If k_1 is in K_1 but not K, write $k_1 = k \exp X$ according to (iv). Then $\exp X$ is in K_1. By compactness of K_1, $(\exp X)^n = \exp nX$ has a convergent subsequence in G, but this contradicts the homeomorphism in (iv).

Conclusion (b) is clear from (iv). In (c), $\text{Ad}(K)(\mathfrak{k}_0) \subseteq \mathfrak{k}_0$ since K has Lie algebra $\mathfrak{k}_0$. Since B is $\text{Ad}(K)$ invariant, $\text{Ad}(K)$ leaves stable the subspace of $\mathfrak{g}_0$ orthogonal to $\mathfrak{k}_0$, and this is just $\mathfrak{p}_0$.

For (d) we have

$$\begin{aligned} B_\theta((\text{ad}\, X)Y, Z) &= -B((\text{ad}\, X)Y, \theta Z) = B(Y, [X, \theta Z]) \\ &= B(Y, \theta[\theta X, Z]) = B_\theta(Y, -(\text{ad}\, \theta X)Z), \end{aligned}$$

and (d) is proved. Conclusion (e) follows from the facts that θ is an involution and B_θ is positive definite, and conclusion (f) is trivial.

Proposition 7.20. If G is a reductive Lie group in the Harish-Chandra class, then

(a) G_{ss} is a closed subgroup,
(b) any semisimple analytic subgroup of G_{ss} has finite center.

REMARK. Because of (b), in checking whether a particular subgroup of G is reductive in the Harish-Chandra class, property (vi) is automatic for the subgroup if it holds for G.

PROOF.

(a) Write the global Cartan decomposition of Theorem 6.31c for G_{ss} as $G_{ss} = K_{ss} \exp(\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0])$. This is compatible with the decomposition in (iv). By (vi) and Theorem 6.31f, K_{ss} is compact. Hence $K_{ss} \times (\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0])$ is closed in $K \times \mathfrak{p}_0$, and (iv) implies that G_{ss} is closed in G.

(b) Let S be a semisimple analytic subgroup of G_{ss} with Lie algebra $\mathfrak{s}_0$. The group $\text{Ad}_\mathfrak{g}(S)$ is a semisimple analytic subgroup of the linear group $GL(\mathfrak{g})$ and has finite center by Proposition 7.9. Under $\text{Ad}_\mathfrak{g}$, Z_S maps into the center of $\text{Ad}_\mathfrak{g}(S)$. Hence the image of Z_S is finite. The kernel of $\text{Ad}_\mathfrak{g}$ on S consists of certain members x of G_{ss} for which $\text{Ad}_\mathfrak{g}(x) = 1$. These x's are in $Z_{G_{ss}}$, and the kernel is then finite by property (vi) for G. Consequently Z_S is finite.

Proposition 7.21. If G is a reductive Lie group, then the function $\Theta : G \to G$ defined by

$$\Theta(k \exp X) = k \exp(-X) \qquad \text{for } k \in K \text{ and } X \in \mathfrak{p}_0$$

is an automorphism of G and its differential is θ.

REMARK. As in the semisimple case, Θ is called the **global Cartan involution**.

PROOF. The function Θ is a well defined diffeomorphism by (iv). First consider its restriction to the analytic subgroup G_{ss} with Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$. By Proposition 7.19e the Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ has a Cartan decomposition

$$[\mathfrak{g}_0, \mathfrak{g}_0] = ([\mathfrak{g}_0, \mathfrak{g}_0] \cap \mathfrak{k}_0) \oplus ([\mathfrak{g}_0, \mathfrak{g}_0] \cap \mathfrak{p}_0).$$

If K_{ss} denotes the analytic subgroup of G_{ss} whose Lie algebra is the first summand on the right side, then Theorem 6.31 shows that G_{ss} consists exactly of the elements in $K_{ss} \exp([\mathfrak{g}_0, \mathfrak{g}_0] \cap \mathfrak{p}_0)$ and that Θ is an automorphism on G_{ss} with differential θ.

Next consider the restriction of Θ to the analytic subgroup $(Z_{G_0})_0$. By Proposition 7.19e the Lie algebra of this abelian group decomposes as

$$Z_{\mathfrak{g}_0} = (Z_{\mathfrak{g}_0} \cap \mathfrak{k}_0) \oplus (Z_{\mathfrak{g}_0} \cap \mathfrak{p}_0).$$

Since all the subalgebras in question are abelian, the exponential mappings in question are onto, and $(Z_{G_0})_0$ is a commuting product

$$(Z_{G_0})_0 = \exp(Z_{\mathfrak{g}_0} \cap \mathfrak{k}_0) \exp(Z_{\mathfrak{g}_0} \cap \mathfrak{p}_0)$$

contained in $K \exp \mathfrak{p}_0$. Thus Θ on $(Z_{G_0})_0$ is the lift to the group of θ on the Lie algebra and hence is an automorphism of the subgroup $(Z_{G_0})_0$.

The subgroups G_{ss} and $(Z_{G_0})_0$ commute, and hence Θ is an automorphism of their commuting product, which is G_0 by the remarks with Corollary 7.10.

Now consider Θ on all of G, where it is given consistently by $\Theta(kg_0) = k\Theta(g_0)$ for $k \in K$ and $g_0 \in G_0$. By Proposition 7.19c we have $\theta \operatorname{Ad}(k) = \operatorname{Ad}(k)\theta$ on $\mathfrak{g}_0$, from which we obtain $\Theta(k \exp X\, k^{-1}) = k\Theta(\exp X)k^{-1}$ for $k \in K$ and $X \in \mathfrak{g}_0$. Therefore

$$\Theta(kg_0k^{-1}) = k\Theta(g_0)k^{-1} \qquad \text{for } k \in K \text{ and } g \in G_0.$$

On the product of two general elements kg_0 and $k'g_0'$ of G, we therefore have

$$\begin{aligned}\Theta(kg_0k'g_0') &= \Theta(kk'k'^{-1}g_0k'g_0') = kk'\Theta(k'^{-1}g_0k'g_0') \\ &= kk'\Theta(k'^{-1}g_0k')\Theta(g_0') = k\Theta(g_0)k'\Theta(g_0') = \Theta(kg_0)\Theta(k'g_0'),\end{aligned}$$

as required.

Lemma 7.22. Let G be a reductive Lie group, and let $g = k \exp X$ be the global Cartan decomposition of an element g of G. If $\mathfrak{s}_0$ is a θ stable subspace of $\mathfrak{g}_0$ such that $\operatorname{Ad}(g)$ normalizes $\mathfrak{s}_0$, then $\operatorname{Ad}(k)$ and $\operatorname{ad} X$ each normalize $\mathfrak{s}_0$. If $\operatorname{Ad}(g)$ centralizes $\mathfrak{s}_0$, then $\operatorname{Ad}(k)$ and $\operatorname{ad} X$ each centralize $\mathfrak{s}_0$.

PROOF. For $x \in G$, we have $(\Theta g)x(\Theta g)^{-1} = \Theta(g(\Theta x)g^{-1})$. Differentiating at $x = 1$, we obtain

$$\operatorname{Ad}(\Theta g) = \theta \operatorname{Ad}(g)\theta. \tag{7.23}$$

Therefore $\operatorname{Ad}(\Theta g)$ normalizes $\mathfrak{s}_0$. Since $\Theta g = k \exp(-X)$, it follows that Ad of $(\Theta g)^{-1} g = \exp 2X$ normalizes $\mathfrak{s}_0$. Because of Proposition 7.19d, $\operatorname{Ad}(\exp 2X)$ is positive definite relative to B_θ, hence diagonable. Then there exists a vector subspace $\mathfrak{s}_0'$ of $\mathfrak{g}_0$ invariant under $\operatorname{Ad}(\exp 2X)$ such that $\mathfrak{g}_0 = \mathfrak{s}_0 \oplus \mathfrak{s}_0'$. The transformation $\operatorname{Ad}(\exp 2X)$ has a unique logarithm with real eigenvalues, and $\operatorname{ad} 2X$ is a candidate for it. Another candidate is the logarithm on each subspace, which normalizes $\mathfrak{s}_0$ and $\mathfrak{s}_0'$. These two candidates must be equal, and therefore $\operatorname{ad} 2X$ normalizes $\mathfrak{s}_0$ and $\mathfrak{s}_0'$. Hence the same thing is true of $\operatorname{ad} X$. Then $\operatorname{Ad}(\exp X)$ and $\operatorname{Ad}(g)$ both normalize $\mathfrak{s}_0$ and $\mathfrak{s}_0'$, and the same thing must be true of $\operatorname{Ad}(k)$.

If $\operatorname{Ad}(g)$ centralizes $\mathfrak{s}_0$, we can go over the above argument to see that $\operatorname{Ad}(k)$ and $\operatorname{ad} X$ each centralize $\mathfrak{s}_0$. In fact, $\operatorname{Ad}(\exp 2X)$ must centralize $\mathfrak{s}_0$, the unique real logarithm must be 0 on $\mathfrak{s}_0$, and $\operatorname{ad} X$ must be 0 on $\mathfrak{s}_0$. The lemma follows.

Lemma 7.24. Let G be a reductive Lie group, and let $\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$. Then $\operatorname{Ad}_{\mathfrak{g}}(K)$ is contained in $\operatorname{Int}_{\mathfrak{g}}(\mathfrak{u}_0)$.

PROOF. The group $\operatorname{Int} \mathfrak{g}$ is complex semisimple with Lie algebra $\operatorname{ad}_{\mathfrak{g}}(\mathfrak{g})$. If bar denotes the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$, then the extension $B_\theta(Z_1, Z_2) = -B(Z_1, \theta \overline{Z_2})$ is a Hermitian inner product on $\mathfrak{g}$, and the compact real form $\operatorname{ad}_{\mathfrak{g}}(\mathfrak{u}_0)$ of $\operatorname{ad}_{\mathfrak{g}}(\mathfrak{g})$ consists of skew Hermitian transformations. Hence $\operatorname{Int}_{\mathfrak{g}}(\mathfrak{u}_0)$ consists of unitary transformations and $\operatorname{ad}_{\mathfrak{g}}(i\mathfrak{u}_0)$ consists of Hermitian transformations. Therefore the global Cartan decomposition of $\operatorname{Int} \mathfrak{g}$ given in Theorem 6.31c is compatible with the polar decomposition relative to B_θ, and every unitary member of $\operatorname{Int} \mathfrak{g}$ is in the compact real form $\operatorname{Int}_{\mathfrak{g}}(\mathfrak{u}_0)$.

Let k be in K. The transformation $\operatorname{Ad}_{\mathfrak{g}}(k)$ is in $\operatorname{Int} \mathfrak{g}$ by property (v) for G, and $\operatorname{Ad}_{\mathfrak{g}}(k)$ is unitary since B is $\operatorname{Ad}(k)$ invariant and since $\operatorname{Ad}(k)$ commutes with bar and θ (Proposition 7.19c). From the result of the previous paragraph, we conclude that $\operatorname{Ad}_{\mathfrak{g}}(k)$ is in $\operatorname{Int}_{\mathfrak{g}}(\mathfrak{u}_0)$.

Proposition 7.25. If G is a reductive Lie group and $\mathfrak{h}_0$ is a θ stable abelian subalgebra of its Lie algebra, then $Z_G(\mathfrak{h}_0)$ is a reductive Lie group. Here the maximal compact subgroup of $Z_G(\mathfrak{h}_0)$ is given by intersection, and the Cartan involution and invariant form are given by restriction.

REMARK. The hypothesis "abelian" will be used only in the proof of property (v) for $Z_G(\mathfrak{h}_0)$, and we shall make use of this fact in Corollary 7.26 below.

PROOF. The group $Z_G(\mathfrak{h}_0)$ is closed, hence Lie. Its Lie algebra is $Z_{\mathfrak{g}_0}(\mathfrak{h}_0)$, which is θ stable. Then it follows, just as in the proof of Corollary 6.29, that $Z_{\mathfrak{g}_0}(\mathfrak{h}_0)$ is reductive. This proves property (i) of a reductive Lie group. Since $Z_{\mathfrak{g}_0}(\mathfrak{h}_0)$ is θ stable, we have

$$Z_{\mathfrak{g}_0}(\mathfrak{h}_0) = (Z_{\mathfrak{g}_0}(\mathfrak{h}_0) \cap \mathfrak{k}_0) \oplus (Z_{\mathfrak{g}_0}(\mathfrak{h}_0) \cap \mathfrak{p}_0),$$

and the first summand on the right side is the Lie algebra of $Z_G(\mathfrak{h}_0) \cap K$. This proves property (ii), and property (iii) is trivial.

In view of property (iv) for G, what needs proof in (iv) for $Z_G(\mathfrak{h}_0)$ is that $Z_K(\mathfrak{h}_0) \times (Z_{\mathfrak{g}_0}(\mathfrak{h}_0) \cap \mathfrak{p}_0)$ maps onto $Z_G(\mathfrak{h}_0)$. That is, we need to see that if $g = k \exp X$ is the global Cartan decomposition of a member g of $Z_G(\mathfrak{h}_0)$, then k is in $Z_G(\mathfrak{h}_0)$ and X is in $Z_{\mathfrak{g}_0}(\mathfrak{h}_0)$. But this is immediate from Lemma 7.22, and (iv) follows.

For property (v) we are to show that $\mathrm{Ad}_{Z_\mathfrak{g}(\mathfrak{h})}$ carries $Z_G(\mathfrak{h}_0)$ into $\mathrm{Int}\, Z_\mathfrak{g}(\mathfrak{h})$. If $x \in Z_G(\mathfrak{h}_0)$ is given, then property (iv) allows us to write $x = k \exp X$ with $k \in Z_K(\mathfrak{h}_0)$ and $X \in Z_{\mathfrak{g}_0}(\mathfrak{h}_0) \cap \mathfrak{p}_0$. Then $\mathrm{Ad}_{Z_\mathfrak{g}(\mathfrak{h})}(\exp X)$ is in $\mathrm{Int}\, Z_\mathfrak{g}(\mathfrak{h})$, and it is enough to treat k. By Lemma 7.24, $\mathrm{Ad}_\mathfrak{g}(k)$ is in the subgroup $\mathrm{Int}_\mathfrak{g}(\mathfrak{u}_0)$, which is compact by Proposition 7.9.

The element $\mathrm{Ad}_\mathfrak{g}(k)$ centralizes $\mathfrak{h}_0$ and hence centralizes the variant $(\mathfrak{h}_0 \cap \mathfrak{k}_0) \oplus i(\mathfrak{h}_0 \cap \mathfrak{p}_0)$. Since $(\mathfrak{h}_0 \cap \mathfrak{k}_0) \oplus i(\mathfrak{h}_0 \cap \mathfrak{p}_0)$ is an abelian subalgebra of $\mathfrak{g}$, the centralizer of $\mathfrak{h}_0$ in $\mathrm{Int}_\mathfrak{g}(\mathfrak{u}_0)$ is the centralizer of a torus, which is connected by Corollary 4.51. Therefore $\mathrm{Ad}_\mathfrak{g}(k)$ is in the analytic subgroup of $\mathrm{Int}\,\mathfrak{g}$ with Lie algebra $Z_{\mathfrak{u}_0}((\mathfrak{h}_0 \cap \mathfrak{k}_0) \oplus i(\mathfrak{h}_0 \cap \mathfrak{p}_0))$. By Corollary 4.48 we can write $\mathrm{Ad}_\mathfrak{g}(k) = \exp \mathrm{ad}_\mathfrak{g} Y$ with Y in this Lie algebra. Then $\mathrm{Ad}_{Z_\mathfrak{g}(\mathfrak{h})}(k) = \exp \mathrm{ad}_{Z_\mathfrak{g}(\mathfrak{h})} Y$, and Y is in $Z_\mathfrak{g}(\mathfrak{h})$. Then $\mathrm{Ad}_{Z_\mathfrak{g}(\mathfrak{h})}(k)$ is in $\mathrm{Int}\, Z_\mathfrak{g}(\mathfrak{h})$, and (v) is proved.

Corollary 7.26. If G is a reductive Lie group, then

(a) $(Z_{G_0})_0 \subseteq Z_G$,

(b) Z_G is a reductive Lie group (with maximal compact subgroup given by intersection and with Cartan involution and invariant form given by restriction).

PROOF. Property (v) for G gives $\mathrm{Ad}_{\mathfrak{g}}(G) \subseteq \mathrm{Int}\,\mathfrak{g}$, and $\mathrm{Int}\,\mathfrak{g}$ acts trivially on $Z_{\mathfrak{g}}$. Hence $\mathrm{Ad}(G)$ acts trivially on $Z_{\mathfrak{g}_0}$, and G centralizes $(Z_{G_0})_0$. This proves (a).

From (a) it follows that Z_G has Lie algebra $Z_{\mathfrak{g}_0}$, which is also the Lie algebra of $Z_G(\mathfrak{g}_0)$. Therefore property (v) is trivial for both Z_G and $Z_G(\mathfrak{g}_0)$. Proposition 7.25 and its remark show that $Z_G(\mathfrak{g}_0)$ is reductive, and consequently only property (iv) needs proof for Z_G. We need to see that if $z \in Z_G$ decomposes in G under (iv) as $z = k \exp X$, then k is in $Z_G \cap K$ and X is in $Z_{\mathfrak{g}_0}$. By Lemma 7.22 we know that k is in $Z_G(\mathfrak{g}_0)$ and X is in $Z_{\mathfrak{g}_0}$. Then $\exp X$ is in $(Z_{G_0})_0$, and (a) shows that $\exp X$ is in Z_G. Since z and $\exp X$ are in Z_G, so is k. This completes the proof of (iv), and (b) follows.

Let G be reductive. Since $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{g}$ carries $[\mathfrak{g}, \mathfrak{g}]$ to itself, $\mathrm{Int}\,\mathfrak{g}$ carries $[\mathfrak{g}, \mathfrak{g}]$ to itself. By (v), $\mathrm{Ad}(G)$ normalizes $[\mathfrak{g}_0, \mathfrak{g}_0]$. Consequently ${}^0G = K G_{ss}$ is a subgroup of G.

The vector subspace $\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0}$ is an abelian subspace of $\mathfrak{g}_0$, and therefore $Z_{vec} = \exp(\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0})$ is an analytic subgroup of G.

Proposition 7.27. If G is a reductive Lie group, then

(a) ${}^0G = K \exp(\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0])$, and ${}^0G$ is a closed subgroup,
(b) the Lie algebra ${}^0\mathfrak{g}_0$ of ${}^0G$ is $\mathfrak{k}_0 \oplus (\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0])$,
(c) ${}^0G$ is reductive (with maximal compact subgroup K and with Cartan involution and invariant form given by restriction),
(d) the center of ${}^0G$ is a compact subgroup of K,
(e) Z_{vec} is closed, is isomorphic to the additive group of a Euclidean space, and is contained in the center of G,
(f) the multiplication map exhibits ${}^0G \times Z_{vec}$ as isomorphic to G.

REMARK. The closed subgroup Z_{vec} is called the **split component** of G.

PROOF.

(a) If we write the global Cartan decomposition of G_{ss} as $G_{ss} = K_{ss} \exp(\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0])$, then ${}^0G = K \exp(\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0])$, and we see from property (iv) that ${}^0G$ is closed.

(b) Because of (a), ${}^0G$ is a Lie subgroup. Since ${}^0G$ contains K and G_{ss}, its Lie algebra must contain $\mathfrak{k}_0 \oplus (\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0])$. From property (iv) for G, the formula ${}^0G = K \exp(\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0])$ shows that $\dim {}^0\mathfrak{g}_0 = \dim \mathfrak{k}_0 + \dim(\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0])$. So ${}^0\mathfrak{g}_0 = \mathfrak{k}_0 \oplus (\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0])$.

(c) From (b) we see that ${}^0\mathfrak{g}_0$ is θ stable. From this fact all the properties of a reductive group are clear except properties (iv) and (v). Property (iv) follows from (a). For property (v) we know that any $\mathrm{Ad}_{\mathfrak{g}}(g)$ for $g \in {}^0G$ is in $\mathrm{Int}\,\mathfrak{g}$. Therefore we can write $\mathrm{Ad}_{\mathfrak{g}}(g)$ as a product of elements $\exp \mathrm{ad}_{\mathfrak{g}}(X_j)$ with X_j in $[\mathfrak{g}, \mathfrak{g}]$ or $Z_{\mathfrak{g}}$. When X_j is in $Z_{\mathfrak{g}}$, $\exp \mathrm{ad}_{\mathfrak{g}}(X_j)$ is trivial. Therefore $\mathrm{Ad}_{\mathfrak{g}}(g)$ agrees with a product of elements $\exp \mathrm{ad}_{\mathfrak{g}}(X_j)$ with X_j in $[\mathfrak{g}, \mathfrak{g}]$. Restricting the action to $[\mathfrak{g}, \mathfrak{g}]$, we see that $\mathrm{Ad}_{[\mathfrak{g},\mathfrak{g}]}(g)$ is in $\mathrm{Int}\,[\mathfrak{g}, \mathfrak{g}]$.

(d) Conclusion (c) and Corollary 7.26 show that the center of ${}^0G$ is reductive. The intersection of the Lie algebra of the center with $\mathfrak{p}_0$ is 0, and hence property (iv) shows that the center is contained in K.

(e) Since $\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0}$ is a closed subspace of $\mathfrak{p}_0$, property (iv) implies that Z_{vec} is closed and that Z_{vec} is isomorphic to the additive group of a Euclidean space. Since $\mathrm{Int}\,\mathfrak{g}$ acts trivially on $Z_{\mathfrak{g}}$, property (v) implies that $\mathrm{Ad}(g) = 1$ on $\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0}$ for every $g \in G$. Hence Z_{vec} is contained in the center of G.

(f) Multiplication is a diffeomorphism, as we see by combining (a), property (iv), and the formula $\exp(X + Y) = \exp X \exp Y$ for X in $\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ and Y in $\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0}$. Multiplication is a homomorphism since, by (e), Z_{vec} is contained in the center of G.

Reductive Lie groups are supposed to have all the essential structure-theoretic properties of semisimple groups and to be closed under various operations that allow us to prove theorems by induction on the dimension of the group. The remainder of this section will be occupied with reviewing the structure theory developed in Chapter VI to describe how the results should be interpreted for reductive Lie groups.

The first remarks concern the Cartan decomposition. The decomposition on the Lie algebra level is built into the definition of reductive Lie group, and the properties of the global Cartan decomposition (generalizing Theorem 6.31) are given partly in property (iv) of the definition and partly in Proposition 7.21.

It might look as if property (iv) would be a hard thing to check for a particular candidate for a reductive group. It is possible to substitute various axioms concerning the component structure of G that are easier to state, but it is often true that ones gets at the component structure by first proving (iv). Proposition 1.143 and Lemma 7.22 provide examples of this order of events; the global Cartan decomposition in those cases implies that the number of components of the group under study is finite. Thus property (iv) is the natural property to include in the definition even though its statement is complicated.

The other two general structure-theoretic topics in Chapter VI are the

Iwasawa decomposition and Cartan subalgebras. Let us first extend the notion of an Iwasawa decomposition to the context of reductive Lie groups. Let a reductive Lie group G be given, and write its Lie algebra as $\mathfrak{g}_0 = Z_{\mathfrak{g}_0} \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$. Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}_0$. Certainly $\mathfrak{a}_0$ contains $\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0}$, and therefore $\mathfrak{a}_0$ is of the form

$$\mathfrak{a}_0 = (\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0}) \oplus (\mathfrak{a}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0]), \tag{7.28}$$

where $\mathfrak{a}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ is a maximal abelian subspace of $\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0]$. Theorem 6.51 shows that any two maximal abelian subspaces of $\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ are conjugate via $\mathrm{Ad}(K)$, and it follows from (7.28) that this result extends to our reductive $\mathfrak{g}_0$.

Proposition 7.29. Let G be a reductive Lie group. If $\mathfrak{a}_0$ and $\mathfrak{a}_0'$ are two maximal abelian subspaces of $\mathfrak{p}_0$, then there is a member k of K with $\mathrm{Ad}(k)\mathfrak{a}_0' = \mathfrak{a}_0$. The member k of K can be taken to be in $K \cap G_{ss}$. Hence $\mathfrak{p}_0 = \bigcup_{k \in K_{ss}} \mathrm{Ad}(k)\mathfrak{a}_0$.

Relative to $\mathfrak{a}_0$, we can form restricted roots just as in §VI.4. A **restricted root** of $\mathfrak{g}_0$, also called a **root** of $(\mathfrak{g}_0, \mathfrak{a}_0)$, is a nonzero $\lambda \in \mathfrak{a}_0^*$ such that the space

$$(\mathfrak{g}_0)_\lambda = \{X \in \mathfrak{g}_0 \mid (\mathrm{ad}\, H)X = \lambda(H)X \text{ for all } H \in \mathfrak{a}_0\}$$

is nonzero. It is apparent that such a restricted root is obtained by taking a restricted root for $[\mathfrak{g}_0, \mathfrak{g}_0]$ and extending it from $\mathfrak{a}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ to $\mathfrak{a}_0$ by making it be 0 on $\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0}$. The restricted-root space decomposition for $[\mathfrak{g}_0, \mathfrak{g}_0]$ gives us a restricted-root space decomposition for $\mathfrak{g}_0$. We define $\mathfrak{m}_0 = Z_{\mathfrak{k}_0}(\mathfrak{a}_0)$, so that the centralizer of $\mathfrak{a}_0$ in $\mathfrak{g}_0$ is $\mathfrak{m}_0 \oplus \mathfrak{a}_0$.

The set of restricted roots is denoted Σ. Choose a notion of positivity for $\mathfrak{a}_0^*$ in the manner of §II.5, as for example by using a lexicographic ordering. Let Σ^+ be the set of positive restricted roots, and define $\mathfrak{n}_0 = \bigoplus_{\lambda \in \Sigma^+} (\mathfrak{g}_0)_\lambda$. Then $\mathfrak{n}_0$ is a nilpotent Lie subalgebra of $\mathfrak{g}_0$, and we have an Iwasawa decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0 \tag{7.30}$$

with all the properties in Proposition 6.43.

Proposition 7.31. Let G be a reductive Lie group, let (7.30) be an Iwasawa decomposition of the Lie algebra $\mathfrak{g}_0$ of G, and let A and N be the analytic subgroups of G with Lie algebras $\mathfrak{a}_0$ and $\mathfrak{n}_0$. Then the multiplication map $K \times A \times N \to G$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism onto. The groups A and N are simply connected.

PROOF. Multiplication is certainly smooth, and it is regular by Lemma 6.44. To see that it is one-one, it is enough, as in the proof of Theorem 6.46, to see that we cannot have $kan = 1$ nontrivially. The identity $kan = 1$ would force the orthogonal transformation $\mathrm{Ad}(k)$ to be upper triangular with positive diagonal entries in the matrix realization of Lemma 6.45, and consequently we may assume that $\mathrm{Ad}(k) = \mathrm{Ad}(a) = \mathrm{Ad}(n) = 1$. Thus k, a, and n are in $Z_G(\mathfrak{g}_0)$. By Lemma 7.22, a is the exponential of something in $Z_{\mathfrak{g}_0}(\mathfrak{g}_0) = Z_{\mathfrak{g}_0}$. Hence a is in Z_{vec}. By construction n is in G_{ss}, and hence k and n are in ${}^0G$. By Proposition 7.27f, $a = 1$ and $kn = 1$. But then the identity $kn = 1$ is valid in G_{ss}, and Theorem 6.46 implies that $k = n = 1$.

To see that multiplication is onto G, we observe from Theorem 6.46 that $\exp(\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0])$ is in the image. By Proposition 7.27a, the image contains ${}^0G$. Also Z_{vec} is in the image (of $1 \times A \times 1$), and Z_{vec} commutes with ${}^0G$. Hence the image contains ${}^0G Z_{vec}$. This is all of G by Proposition 7.27f.

We define $\mathfrak{n}_0^- = \bigoplus_{\lambda \in \Sigma^+} (\mathfrak{g}_0)_{-\lambda}$. Then $\mathfrak{n}_0^-$ is a nilpotent Lie subalgebra of $\mathfrak{g}_0$, and we let N^- be the corresponding analytic subgroup. Since $-\Sigma^+$ is the set of positive restricted roots for another notion of positivity on $\mathfrak{a}_0^*$, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0^-$ is another Iwasawa decomposition of $\mathfrak{g}_0$ and $G = KAN^-$ is another Iwasawa decomposition of G. The identity $\theta(\mathfrak{g}_0)_\lambda = (\mathfrak{g}_0)_{-\lambda}$ given in Proposition 6.40c implies that $\theta\mathfrak{n}_0 = \mathfrak{n}_0^-$. By Proposition 7.21, $\Theta N = N^-$.

We write M for the group $Z_K(\mathfrak{a}_0)$. This is a compact subgroup since it is closed in K, and its Lie algebra is $Z_{\mathfrak{k}_0}(\mathfrak{a}_0)$. This subgroup normalizes each $(\mathfrak{g}_0)_\lambda$ since

$$\begin{aligned}\mathrm{ad}(H)(\mathrm{Ad}(m)X_\lambda) &= \mathrm{Ad}(m)\mathrm{ad}(\mathrm{Ad}(m)^{-1}H)X_\lambda \\ &= \mathrm{Ad}(m)\mathrm{ad}(H)X_\lambda = \lambda(H)\mathrm{Ad}(m)X_\lambda\end{aligned}$$

for $m \in M$, $H \in \mathfrak{a}_0$, and $X_\lambda \in (\mathfrak{g}_0)_\lambda$. Consequently M normalizes $\mathfrak{n}_0$. Thus M centralizes A and normalizes N. Since M is compact and AN is closed, MAN is a closed subgroup.

Reflections in the restricted roots generate a group $W(\Sigma)$, which we call the **Weyl group** of Σ. The elements of $W(\Sigma)$ are nothing more than the elements of the Weyl group for the restricted roots of $[\mathfrak{g}_0, \mathfrak{g}_0]$, with each element extended to $\mathfrak{a}_0^*$ by being defined to be the identity on $\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0}$.

We define $W(G, A) = N_K(\mathfrak{a}_0)/Z_K(\mathfrak{a}_0)$. By the same proof as for Lemma 6.56, the Lie algebra of $N_K(\mathfrak{a}_0)$ is $\mathfrak{m}_0$. Therefore $W(G, A)$ is a finite group.

Proposition 7.32. If G is a reductive Lie group, then the group $W(G, A)$ coincides with $W(\Sigma)$.

PROOF. Just as with the corresponding result in the semisimple case (Theorem 6.57), we know that $W(\Sigma) \subseteq W(G, A)$. Fix a simple system Σ^+ for Σ. As in the proof of Theorem 6.57, it suffices to show that if $k \in N_K(\mathfrak{a}_0)$ has $\mathrm{Ad}(k)\Sigma^+ = \Sigma^+$, then k is in $Z_K(\mathfrak{a}_0)$. By Lemma 7.24, $\mathrm{Ad}_{\mathfrak{g}}(k)$ is in the compact semisimple Lie group $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{u}_0)$, where $\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$. The connectedness of $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{u}_0)$ is the key, and the remainder of the proof of Theorem 6.57 is applicable to this situation.

Proposition 7.33. If G is a reductive Lie group, then M meets every component of K, hence every component of G.

PROOF. Let $k \in K$ be given. Since $\mathrm{Ad}(k)^{-1}(\mathfrak{a}_0)$ is maximal abelian in $\mathfrak{p}_0$, Proposition 7.28 gives us $k_0 \in K_0$ with $\mathrm{Ad}(k_0^{-1}k^{-1})(\mathfrak{a}_0) = \mathfrak{a}_0$. Thus $k_0^{-1}k^{-1}$ normalizes $\mathfrak{a}_0$. Comparison of Proposition 7.32 and Theorem 6.57 produces $k_1^{-1} \in K_0$ so that $k_1^{-1}k_0^{-1}k^{-1}$ centralizes $\mathfrak{a}_0$. Then kk_0k_1 is in M, and k is in MK_0.

Next let us extend the notion of Cartan subalgebras to the context of reductive Lie groups. We recall from §IV.5 that a Lie subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ is a **Cartan subalgebra** if its complexification $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$. Since $\mathfrak{h}$ must equal its own normalizer (Proposition 2.7), it follows that $Z_{\mathfrak{g}} \subseteq \mathfrak{h}$. Therefore $\mathfrak{h}_0$ must be of the form

$$\mathfrak{h}_0 = Z_{\mathfrak{g}_0} \oplus (\mathfrak{h}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0]), \tag{7.34}$$

where $\mathfrak{h}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ is a Cartan subalgebra of the semisimple Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$. By Proposition 2.13 a sufficient condition for $\mathfrak{h}_0$ to be a Cartan subalgebra of $\mathfrak{g}_0$ is that $\mathfrak{h}_0$ is maximal abelian in $\mathfrak{g}_0$ and $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{h}_0$ is simultaneously diagonable.

As in the special case (4.31), we can form a set of roots $\Delta(\mathfrak{g}, \mathfrak{h})$, which amount to the roots of $[\mathfrak{g}, \mathfrak{g}]$ with respect to $\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$, extended to $\mathfrak{h}$ by being defined to be 0 on $Z_{\mathfrak{g}}$. We can form also a Weyl group $W(\mathfrak{g}, \mathfrak{h})$ generated by the reflections in the members of Δ; $W(\mathfrak{g}, \mathfrak{h})$ consists of the members of $W([\mathfrak{g}, \mathfrak{g}], \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}])$ extended to $\mathfrak{g}$ by being defined to be the identity on $Z_{\mathfrak{g}}$.

Because of the form (7.34) of Cartan subalgebras of $\mathfrak{g}_0$, Proposition 6.59 implies that any Cartan subalgebra is conjugate via $\mathrm{Int}\,\mathfrak{g}_0$ to a θ stable Cartan subalgebra. There are only finitely many conjugacy classes (Proposition 6.64), and these can be related by Cayley transforms.

The maximally noncompact θ stable Cartan subalgebras are obtained by adjoining to an Iwasawa $\mathfrak{a}_0$ a maximal abelian subspace of $\mathfrak{m}_0$. As in Proposition 6.61, all such Cartan subalgebras are conjugate via K. The restricted roots relative to $\mathfrak{a}_0$ are the nonzero restrictions to $\mathfrak{a}_0$ of the roots relative to this Cartan subalgebra.

Any maximally compact θ stable Cartan subalgebra is obtained as the centralizer of a maximal abelian subspace of $\mathfrak{k}_0$. As in Proposition 6.61, all such Cartan subalgebras are conjugate via K.

Proposition 7.35. Let G be a reductive Lie group. If two θ stable Cartan subalgebras of $\mathfrak{g}_0$ are conjugate via G, then they are conjugate via G_{ss} and in fact by $K \cap G_{ss}$.

PROOF. Let $\mathfrak{h}_0$ and $\mathfrak{h}'_0$ be θ stable Cartan subalgebras, and suppose that $\operatorname{Ad}(g)(\mathfrak{h}_0) = \mathfrak{h}'_0$. By (7.23), $\operatorname{Ad}(\Theta g)(\mathfrak{h}_0) = \mathfrak{h}'_0$. If $g = k \exp X$ with $k \in K$ and $X \in \mathfrak{p}_0$, then it follows that Ad of $(\Theta g)^{-1} g = \exp 2X$ normalizes $\mathfrak{h}_0$. Applying Lemma 7.22 to $\exp 2X$, we see that $[X, \mathfrak{h}_0] \subseteq \mathfrak{h}_0$. Therefore $\exp X$ normalizes $\mathfrak{h}_0$, and $\operatorname{Ad}(k)$ carries $\mathfrak{h}_0$ to $\mathfrak{h}'_0$.

Since $\operatorname{Ad}(k)$ commutes with θ, $\operatorname{Ad}(k)$ carries $\mathfrak{h}_0 \cap \mathfrak{p}_0$ to $\mathfrak{h}'_0 \cap \mathfrak{p}_0$. Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}_0$ containing $\mathfrak{h}_0 \cap \mathfrak{p}_0$, and choose $k_0 \in K_0$ by Proposition 7.29 so that $\operatorname{Ad}(k_0 k)(\mathfrak{a}_0) = \mathfrak{a}_0$. Comparing Proposition 7.32 and Theorem 6.57, we can find $k_1 \in K_0$ so that $k_1 k_0 k$ centralizes $\mathfrak{a}_0$. Then $\operatorname{Ad}(k)|_{\mathfrak{a}_0} = \operatorname{Ad}(k_0^{-1} k_1^{-1})|_{\mathfrak{a}_0}$, and the element $k' = k_0^{-1} k_1^{-1}$ of K_0 has the property that $\operatorname{Ad}(k')(\mathfrak{h}_0 \cap \mathfrak{p}_0) = \mathfrak{h}'_0 \cap \mathfrak{p}_0$. The θ stable Cartan subalgebras $\mathfrak{h}_0$ and $\operatorname{Ad}(k')^{-1}(\mathfrak{h}'_0)$ therefore have the same $\mathfrak{p}_0$ part, and Lemma 6.62 shows that they are conjugate via $K \cap G_{ss}$.

3. KAK Decomposition

Throughout this section we let G be a reductive Lie group, and we let other notation be as in §2.

From the global Cartan decomposition $G = K \exp \mathfrak{p}_0$ and from the equality $\mathfrak{p}_0 = \bigcup_{k \in K} \operatorname{Ad}(k)\mathfrak{a}_0$ of Proposition 7.29, it is immediate that $G = KAK$ in the sense that every element of G can be decomposed as a product of an element of K, an element of A, and a second element of K. In this section we shall examine the degree of nonuniqueness of this decomposition.

Lemma 7.36. If X is in $\mathfrak{p}_0$, then $Z_G(\exp X) = Z_G(\mathbb{R}X)$.

PROOF. Certainly $Z_G(\mathbb{R}X) \subseteq Z_G(\exp X)$. In the reverse direction if g is in $Z_G(\exp X)$, then $\mathrm{Ad}(g)\mathrm{Ad}(\exp X) = \mathrm{Ad}(\exp X)\mathrm{Ad}(g)$. By Proposition 7.19d, $\mathrm{Ad}(\exp X)$ is positive definite on $\mathfrak{g}_0$, thus diagonable. Consequently $\mathrm{Ad}(g)$ carries each eigenspace of $\mathrm{Ad}(\exp X)$ to itself, and it follows that $\mathrm{Ad}(g)\mathrm{ad}(X) = \mathrm{ad}(X)\mathrm{Ad}(g)$. By Lemma 1.118,

$$\mathrm{ad}(\mathrm{Ad}(g)X) = \mathrm{ad}(X). \tag{7.37}$$

Write $X = Y + Z$ with $Y \in Z_{\mathfrak{g}_0}$ and $Z \in [\mathfrak{g}_0, \mathfrak{g}_0]$. By property (v) of a reductive group, $\mathrm{Ad}(g)Y = Y$. Comparing this equality with (7.37), we see that $\mathrm{ad}(\mathrm{Ad}(g)Z) = \mathrm{ad}(Z)$, hence that $\mathrm{Ad}(g)Z - Z$ is in the center of $\mathfrak{g}_0$. Since it is in $[\mathfrak{g}_0, \mathfrak{g}_0]$ also, it is 0. Therefore $\mathrm{Ad}(g)X = X$, and g is in the centralizer of $\mathbb{R}X$.

Lemma 7.38. If k is in K and if a and a' are in A with $kak^{-1} = a'$, then there exists k_0 in $N_K(\mathfrak{a}_0)$ with $k_0ak_0^{-1} = a'$.

PROOF. The subgroup $Z_G(a')$ is reductive by Lemma 7.36 and Proposition 7.25, and its Lie algebra is $Z_{\mathfrak{g}_0}(a') = \{X \in \mathfrak{g}_0 \mid \mathrm{Ad}(a')X = X\}$. Now $\mathfrak{a}_0$ and $\mathrm{Ad}(k)\mathfrak{a}_0$ are two maximal abelian subspaces of $Z_{\mathfrak{g}_0}(a') \cap \mathfrak{p}_0$ since $kak^{-1} = a'$. By Proposition 7.29 there exists k_1 in $K \cap Z_G(a')$ with $\mathrm{Ad}(k_1)\mathrm{Ad}(k)\mathfrak{a}_0 = \mathfrak{a}_0$. Then $k_0 = k_1k$ is in $N_K(\mathfrak{a}_0)$, and

$$k_0ak_0^{-1} = k_1(kak^{-1})k_1^{-1} = k_1a'k_1^{-1} = a'.$$

Theorem 7.39 (KAK decomposition). Every element in G has a decomposition as k_1ak_2 with $k_1, k_2 \in K$ and $a \in A$. In this decomposition, a is uniquely determined up to conjugation by a member of $W(G, A)$. If a is fixed as $\exp H$ with $H \in \mathfrak{a}_0$ and if $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$, then k_1 is unique up to right multiplication by a member of M.

PROOF. Existence of the decomposition was noted at the beginning of the section. For uniqueness suppose $k_1'a'k_2' = k_1''ak_2''$. If $k' = k_1''^{-1}k_1'$ and $k = k_2'k_2''^{-1}$, then $k'a'k = a$ and hence $(k'k)(k^{-1}a'k) = a$. By the uniqueness of the global Cartan decomposition, $k'k = 1$ and $k^{-1}a'k = a$. Lemma 7.38 then shows that a' and a are conjugate via $N_K(\mathfrak{a}_0)$.

Now let $a = a' = \exp H$ with $H \in \mathfrak{a}_0$ and $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$. We have seen that $k^{-1}ak = a$. By Lemma 7.36, $\mathrm{Ad}(k)^{-1}H = H$. Since $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$, Lemma 6.50 shows that $Z_{\mathfrak{g}_0}(H) = \mathfrak{a}_0 \oplus \mathfrak{m}_0$. Hence the centralizer of H in $\mathfrak{p}_0$ is $\mathfrak{a}_0$, and the centralizer of $\mathrm{Ad}(k)^{-1}H$ in $\mathfrak{p}_0$ is $\mathrm{Ad}(k)^{-1}\mathfrak{a}_0$. But $\mathrm{Ad}(k)^{-1}H = H$ implies that these centralizers are the same: $\mathrm{Ad}(k)^{-1}\mathfrak{a}_0 = \mathfrak{a}_0$. Thus k is in $N_K(\mathfrak{a}_0)$.

By Proposition 7.32, $\mathrm{Ad}(k)$ is given by an element w of the Weyl group $W(\Sigma)$. Since $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$, we can define a lexicographic ordering so that the positive restricted roots are positive on H. Then $\mathrm{Ad}(k)H = H$ says that w permutes the positive restricted roots. By Theorem 2.63, $w = 1$. Therefore $\mathrm{Ad}(k)$ centralizes $\mathfrak{a}_0$, and k is in M.

From $k'k = 1$, we see that k' is in M. Then $k' = k_1''^{-1}k_1'$ shows that k_1' and k_1'' differ by an element of M on the right.

4. Bruhat Decomposition

We continue to assume that G is a reductive Lie group and that other notation is as in §2.

We know that the subgroup $M = Z_K(\mathfrak{a}_0)$ of K is compact, and we saw in §2 that MAN is a closed subgroup of G. It follows from the Iwasawa decomposition that the multiplication map $M \times A \times N \to MAN$ is a diffeomorphism onto.

The Bruhat decomposition describes the double-coset decomposition $MAN\backslash G/MAN$ of G with respect to MAN. Here is an example.

EXAMPLE. Let $G = SL(2, \mathbb{R})$. Here $MAN = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\right\}$. The normalizer $N_K(\mathfrak{a}_0)$ consists of the four matrices $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$ and $\pm\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$, while the centralizer $Z_K(\mathfrak{a}_0)$ consists of the two matrices $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$. Thus $|W(G, A)| = 2$, and $\widetilde{w} = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$ is a representative of the nontrivial element of $W(G, A)$. Let $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ be given in G. If $c = 0$, then g is in MAN. If $c \neq 0$, then

$$\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} c & d \\ -a & -b\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -ac^{-1} & 1\end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1}\end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1}\end{pmatrix}.$$

Hence

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1}\end{pmatrix}$$

exhibits $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ as in $MAN\widetilde{w}MAN$. Thus the double-coset space $MAN\backslash G/MAN$ consists of two elements, with 1 and $\widetilde{w}$ as representatives.

Theorem 7.40 (Bruhat decomposition). The set of double cosets of $MAN\backslash G/MAN$ is parametrized in a one-one fashion by $W(G, A)$, the double coset corresponding to $w \in W(G, A)$ being $MAN\widetilde{w}MAN$, where $\widetilde{w}$ is any representative of w in $N_K(\mathfrak{a}_0)$.

PROOF OF UNIQUENESS. Suppose that w_1 and w_2 are in $W(G, A)$, with $\widetilde{w}_1$ and $\widetilde{w}_2$ as representatives, and that x_1 and x_2 in MAN have

$$x_1\widetilde{w}_1 = \widetilde{w}_2 x_2. \tag{7.41}$$

Now $\mathrm{Ad}(N) = \exp(\mathrm{ad}(\mathfrak{n}_0))$ by Theorem 1.127, and hence $\mathrm{Ad}(N)$ carries $\mathfrak{a}_0$ to $\mathfrak{a}_0 \oplus \mathfrak{n}_0$ while leaving the $\mathfrak{a}_0$ component unchanged. Meanwhile under Ad, $N_K(\mathfrak{a}_0)$ permutes the restricted-root spaces and thus carries $\mathfrak{m}_0 \oplus \bigoplus_{\lambda\in\Sigma}(\mathfrak{g}_0)_\lambda$ to itself. Apply Ad of both sides of (7.41) to an element $H \in \mathfrak{a}_0$ and project to $\mathfrak{a}_0$ along $\mathfrak{m}_0 \oplus \bigoplus_{\lambda\in\Sigma}(\mathfrak{g}_0)_\lambda$. The resulting left side is in $\mathfrak{a}_0 \oplus \mathfrak{n}_0$ with $\mathfrak{a}_0$ component $\mathrm{Ad}(\widetilde{w}_1)H$, while the right side is in $\mathrm{Ad}(\widetilde{w}_2)H + \mathrm{Ad}(\widetilde{w}_2)(\mathfrak{m}_0 \oplus \mathfrak{n}_0)$. Hence $\mathrm{Ad}(\widetilde{w}_1)H = \mathrm{Ad}(\widetilde{w}_2)H$. Since H is arbitrary, $\widetilde{w}_2^{-1}\widetilde{w}_1$ centralizes $\mathfrak{a}_0$. Therefore $w_1 = w_2$.

The proof of existence in Theorem 7.40 will be preceded by three lemmas.

Lemma 7.42. Let $H \in \mathfrak{a}_0$ be such that $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$. Then the mapping $\varphi : N \to \mathfrak{g}_0$ given by $n \mapsto \mathrm{Ad}(n)H - H$ carries N onto $\mathfrak{n}_0$.

PROOF. Write $\mathfrak{n}_0 = \bigoplus(\mathfrak{g}_0)_\lambda$ as a sum of restricted-root spaces, and regard the restricted roots as ordered lexicographically. For any restricted root α, the subspace $\mathfrak{n}_\alpha = \bigoplus_{\lambda\geq\alpha}(\mathfrak{g}_0)_\lambda$ is an ideal, and we prove by induction downward on α that φ carries $N_\alpha = \exp\mathfrak{n}_\alpha$ onto $\mathfrak{n}_\alpha$. This conclusion for α equal to the smallest positive restricted root gives the lemma.

If α is given, we can write $\mathfrak{n}_\alpha = (\mathfrak{g}_0)_\alpha \oplus \mathfrak{n}_\beta$ with $\beta > \alpha$. Let X be given in $\mathfrak{n}_\alpha$, and write X as $X_1 + X_2$ with $X_1 \in (\mathfrak{g}_0)_\alpha$ and $X_2 \in \mathfrak{n}_\beta$. Since $\alpha(H) \neq 0$, we can choose $Y_1 \in (\mathfrak{g}_0)_\alpha$ with $[H, Y_1] = X_1$. Then

$$\begin{aligned}\mathrm{Ad}(\exp Y_1)H - H &= (H + [Y_1, H] + \tfrac{1}{2}(\mathrm{ad}\,Y_1)^2 H + \cdots) - H \\ &= -X_1 + (\mathfrak{n}_\beta \text{ terms}),\end{aligned}$$

and hence $\mathrm{Ad}(\exp Y_1)(H + X) - H$ is in $\mathfrak{n}_\beta$. By inductive hypothesis we can find $n \in N_\beta$ with

$$\mathrm{Ad}(n)H - H = \mathrm{Ad}(\exp Y_1)(H + X) - H.$$

Then $\mathrm{Ad}((\exp Y_1)^{-1}n)H - H = X$, and the element $(\exp Y_1)^{-1}n$ of N_α is the required element to complete the induction.

Lemma 7.43. Let $\mathfrak{s}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$. Then

(a) $\mathfrak{n}_0 \oplus Z_{\mathfrak{g}_0} = \{Z \in \mathfrak{s}_0 \mid \operatorname{ad}_{\mathfrak{g}}(Z) \text{ is nilpotent}\}$ and

(b) $\mathfrak{a}_0 \oplus \mathfrak{n}_0 \oplus (\mathfrak{m}_0 \cap Z_{\mathfrak{g}_0}) = \{Z \in \mathfrak{s}_0 \mid \operatorname{ad}_{\mathfrak{g}}(Z) \text{ has all eigenvalues real}\}$.

PROOF. Certainly the left sides in (a) and (b) are contained in the right sides. For the reverse containments write $Z \in \mathfrak{s}_0$ as $Z = X_0 + H + X$ with $X_0 \in \mathfrak{m}_0$, $H \in \mathfrak{a}_0$, and $X \in \mathfrak{n}_0$. Extend $\mathbb{R}X_0$ to a maximal abelian subspace $\mathfrak{t}_0$ of $\mathfrak{m}_0$, so that $\mathfrak{a}_0 \oplus \mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$. Extending the ordering of $\mathfrak{a}_0$ to one of $\mathfrak{a}_0 \oplus i\mathfrak{t}_0$ so that $\mathfrak{a}_0$ is taken before $i\mathfrak{t}_0$, we obtain a positive system Δ^+ for $\Delta(\mathfrak{g}, (\mathfrak{a} \oplus \mathfrak{t}))$ such that Σ^+ arises as the set of nonzero restrictions of members of Δ^+. Arrange the members of Δ^+ in decreasing order and form the matrix of $\operatorname{ad} Z$ in a corresponding basis of root vectors (with vectors from $\mathfrak{a} \oplus \mathfrak{t}$ used at the appropriate place in the middle). The matrix is upper triangular. The diagonal entries in the positions corresponding to the root vectors are $\alpha(X_0+H) = \alpha(X_0)+\alpha(H)$ for $\alpha \in \Delta$, and the diagonal entries are 0 in the positions corresponding to basis vectors in $\mathfrak{a} \oplus \mathfrak{t}$. Here $\alpha(X_0)$ is imaginary , and $\alpha(H)$ is real. To have $\operatorname{ad} Z$ nilpotent, we must get 0 for all α. Thus the component of $X_0 + H$ in $[\mathfrak{g}_0, \mathfrak{g}_0]$ is 0. This proves (a). To have $\operatorname{ad} Z$ have real eigenvalues, we must have $\alpha(X_0) = 0$ for all $X \in \Delta$. Thus the component of X_0 in $[\mathfrak{g}_0, \mathfrak{g}_0]$ is 0. This proves (b).

Lemma 7.44. For each $g \in G$, put $\mathfrak{s}_0^g = \mathfrak{s}_0 \cap \operatorname{Ad}(g)\mathfrak{s}_0$. Then

$$\mathfrak{s}_0 = \mathfrak{s}_0^g + \mathfrak{n}_0.$$

PROOF. Certainly $\mathfrak{s}_0 \supseteq \mathfrak{s}_0^g + \mathfrak{n}_0$, and therefore it is enough to show that $\dim(\mathfrak{s}_0^g + \mathfrak{n}_0) = \dim \mathfrak{s}_0$. Since $G = KAN$, there is no loss of generality in assuming that g is in K. Write $k = g$. Let $(\,\cdot\,)^\perp$ denote orthogonal complement within $\mathfrak{g}_0$ relative to B_θ. From $\theta(\mathfrak{g}_0)_\lambda = (\mathfrak{g}_0)_{-\lambda}$, we have $\mathfrak{s}_0^\perp = \theta\mathfrak{n}_0$. Since $\operatorname{Ad}(k)$ acts in an orthogonal fashion,

$$\begin{aligned}(\mathfrak{s}_0 + \operatorname{Ad}(k)\mathfrak{s}_0)^\perp &= \mathfrak{s}_0^\perp \cap (\operatorname{Ad}(k)\mathfrak{s}_0)^\perp = \theta\mathfrak{n}_0 \cap \operatorname{Ad}(k)\mathfrak{s}_0^\perp \\ &= \theta\mathfrak{n}_0 \cap \operatorname{Ad}(k)\theta\mathfrak{n}_0 = \theta(\mathfrak{n}_0 \cap \operatorname{Ad}(k)\mathfrak{n}_0).\end{aligned} \tag{7.45}$$

Let X be in $\mathfrak{s}_0 \cap \operatorname{Ad}(k)\mathfrak{s}_0$ and in $\mathfrak{n}_0$. Then $\operatorname{ad}_{\mathfrak{g}}(X)$ is nilpotent by Lemma 7.43a. Since $\operatorname{ad}_{\mathfrak{g}}(\operatorname{Ad}(k)^{-1}X)$ and $\operatorname{ad}_{\mathfrak{g}}(X)$ have the same eigenvalues, $\operatorname{ad}_{\mathfrak{g}}(\operatorname{Ad}(k)^{-1}X)$ is nilpotent. By Lemma 7.43a, $\operatorname{Ad}(k)^{-1}X$ is in $\mathfrak{n}_0 \oplus Z_{\mathfrak{g}_0}$. Since $\operatorname{Ad}(k)$ fixes $Z_{\mathfrak{g}_0}$ (by property (v)), $\operatorname{Ad}(k)^{-1}X$ is in $\mathfrak{n}_0$. Therefore X is in $\operatorname{Ad}(k)\mathfrak{n}_0$, and we obtain

$$\mathfrak{n}_0 \cap \operatorname{Ad}(k)\mathfrak{n}_0 = \mathfrak{n}_0 \cap (\mathfrak{s}_0 \cap \operatorname{Ad}(k)\mathfrak{s}_0) = \mathfrak{n}_0 \cap \mathfrak{s}_0^k. \tag{7.46}$$

Consequently

$$\begin{aligned} 2\dim \mathfrak{s}_0 - \dim \mathfrak{s}_0^k &= \dim(\mathfrak{s}_0 + \mathrm{Ad}(k)\mathfrak{s}_0) \\ &= \dim \mathfrak{g}_0 - \dim(\mathfrak{n}_0 \cap \mathrm{Ad}(k)\mathfrak{n}_0) && \text{by (7.45)} \\ &= \dim \mathfrak{g}_0 - \dim(\mathfrak{n}_0 \cap \mathfrak{s}_0^k) && \text{by (7.46)} \\ &= \dim \mathfrak{g}_0 + \dim(\mathfrak{n}_0 + \mathfrak{s}_0^k) - \dim \mathfrak{n}_0 - \dim \mathfrak{s}_0^k, \end{aligned}$$

and we conclude that

$$\dim \mathfrak{g}_0 + \dim(\mathfrak{n}_0 + \mathfrak{s}_0^k) - \dim \mathfrak{n}_0 = 2\dim \mathfrak{s}_0.$$

Since $\dim \mathfrak{n}_0 + \dim \mathfrak{s}_0 = \dim \mathfrak{g}_0$, we obtain $\dim(\mathfrak{n}_0 + \mathfrak{s}_0^k) = \dim \mathfrak{s}_0$, as required.

PROOF OF EXISTENCE IN THEOREM 7.40. Fix $H \in \mathfrak{a}_0$ with $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$. Let $x \in G$ be given. Since $\mathfrak{a}_0 \subseteq \mathfrak{s}_0$, Lemma 7.44 allows us to write $H = X + Y$ with $X \in \mathfrak{n}_0$ and $Y \in \mathfrak{s}_0^x$. By Lemma 7.42 we can choose $n_1 \in N$ with $\mathrm{Ad}(n_1)H - H = -X$. Then

$$\mathrm{Ad}(n_1)H = H - X = Y \in \mathfrak{s}_0^x \subseteq \mathrm{Ad}(x)\mathfrak{s}_0.$$

So $Z = \mathrm{Ad}(x^{-1}n_1)H$ is in $\mathfrak{s}_0$. Since $\mathrm{ad}_{\mathfrak{g}} Z$ and $\mathrm{ad}_{\mathfrak{g}} H$ have the same eigenvalues, Lemma 7.43b shows that Z is in $\mathfrak{a}_0 \oplus \mathfrak{n}_0 \oplus (\mathfrak{m}_0 \cap Z_{\mathfrak{g}_0})$. Since $\mathrm{Ad}(x^{-1}n_1)^{-1}$ fixes $Z_{\mathfrak{g}_0}$ (by property (v)), Z is in $\mathfrak{a}_0 \oplus \mathfrak{m}_0$. Write $Z = H' + X'$ correspondingly. Here $\mathrm{ad}\, H$ and $\mathrm{ad}\, H'$ have the same eigenvalues, so that $\lambda(H') \neq 0$ for all $\lambda \in \Sigma$. By Lemma 7.42 there exists $n_2 \in N$ with $\mathrm{Ad}(n_2)^{-1}H' - H' = X'$. Then $\mathrm{Ad}(n_2)^{-1}H' = H' + X' = Z$, and

$$H' = \mathrm{Ad}(n_2)Z = \mathrm{Ad}(n_2 x^{-1} n_1)H.$$

The centralizers of H' and H are both $\mathfrak{a}_0 \oplus \mathfrak{m}_0$ by Lemma 6.50. Thus

$$\mathrm{Ad}(n_2 x^{-1} n_1)(\mathfrak{a}_0 \oplus \mathfrak{m}_0) = \mathfrak{a}_0 \oplus \mathfrak{m}_0. \tag{7.47}$$

If X is in $\mathfrak{a}_0$, then $\mathrm{ad}_{\mathfrak{g}}(X)$ has real eigenvalues by Lemma 7.43b. Since $\mathrm{ad}_{\mathfrak{g}}(\mathrm{Ad}(n_2x^{-1}n_1)X)$ and $\mathrm{ad}_{\mathfrak{g}}(X)$ have the same eigenvalues, Lemma 7.43b shows that $\mathrm{Ad}(n_2x^{-1}n_1)X$ is in $\mathfrak{a}_0 \oplus (\mathfrak{m}_0 \cap Z_{\mathfrak{g}_0})$. Since $\mathrm{Ad}(n_2x^{-1}n_1)^{-1}$ fixes $Z_{\mathfrak{g}_0}$ (by property (v)), $\mathrm{Ad}(n_2x^{-1}n_1)X$ is in $\mathfrak{a}_0$. We conclude that $n_2x^{-1}n_1$ is in $N_G(\mathfrak{a}_0)$.

Let $n_2x^{-1}n_1 = u \exp X_0$ be the global Cartan decomposition of $n_2x^{-1}n_1$. By Lemma 7.22, u is in $N_K(\mathfrak{a}_0)$ and X_0 is in $N_{\mathfrak{g}_0}(\mathfrak{a}_0)$. By the same argument as in Lemma 6.56, $N_{\mathfrak{g}_0}(\mathfrak{a}_0) = \mathfrak{a}_0 \oplus \mathfrak{m}_0$. Since X_0 is in $\mathfrak{p}_0$, X_0 is in $\mathfrak{a}_0$. Therefore u is in $N_K(\mathfrak{a}_0)$ and $\exp X_0$ is in A. In other words, $n_2x^{-1}n_1$ is in uA, and x is in the same MAN double coset as the member u^{-1} of $N_K(\mathfrak{a}_0)$.

5. Structure of M

We continue to assume that G is a reductive Lie group and that other notation is as in §2. The fundamental source of disconnectedness in the structure theory of semisimple groups is the behavior of the subgroup $M = Z_K(\mathfrak{a}_0)$. We shall examine M in this section, paying particular attention to its component structure. For the first time we shall make serious use of results from Chapter V.

Proposition 7.48. M is a reductive Lie group.

PROOF. Proposition 7.25 shows that $Z_G(\mathfrak{a}_0)$ is a reductive Lie group, necessarily of the form $Z_K(\mathfrak{a}_0)\exp(Z_{\mathfrak{g}_0}(\mathfrak{a}_0) \cap \mathfrak{p}_0) = MA$. By Proposition 7.27, ${}^0(MA) = M$ is a reductive Lie group.

Proposition 7.33 already tells us that M meets every component of G. But M can be disconnected even when G is connected. (Recall from the examples in §VI.5 that M is disconnected when $G = SL(n, \mathbb{R})$.) Choose and fix a maximal abelian subspace $\mathfrak{t}_0$ of $\mathfrak{m}_0$. Then $\mathfrak{a}_0 \oplus \mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$.

Proposition 7.49. Every component of M contains a member of M that centralizes $\mathfrak{t}_0$, so that $M = Z_M(\mathfrak{t}_0)M_0$.

REMARK. The proposition says that we may focus our attention on $Z_M(\mathfrak{t}_0)$. After this proof we shall study $Z_M(\mathfrak{t}_0)$ by considering it as a subgroup of $Z_K(\mathfrak{t}_0)$.

PROOF. If $m \in M$ is given, then $\mathrm{Ad}(m)\mathfrak{t}_0$ is a maximal abelian subspace of $\mathfrak{m}_0$. By Theorem 4.34 (applied to M_0), there exists $m_0 \in M_0$ such that $\mathrm{Ad}(m_0)\mathrm{Ad}(m)\mathfrak{t}_0 = \mathfrak{t}_0$. Then $m_0 m$ is in $N_M(\mathfrak{m}_0)$. Introduce a positive system Δ^+ for the root system $\Delta = \Delta(\mathfrak{m}, \mathfrak{t})$. Then $\mathrm{Ad}(m_0 m)\Delta^+$ is a positive system for Δ, and Theorems 4.54 and 2.63 together say that we can find $m_1 \in M_0$ such that $\mathrm{Ad}(m_1 m_0 m)$ maps Δ^+ to itself. By Proposition 7.48, M satisfies property (v) of reductive Lie groups. Therefore $\mathrm{Ad}_{\mathfrak{m}}(m_1 m_0 m)$ is in $\mathrm{Int}\,\mathfrak{m}$. Then $\mathrm{Ad}_{\mathfrak{m}}(m_1 m_0 m)$ must be induced by an element in $\mathrm{Int}_{\mathfrak{m}}\,[\mathfrak{m}, \mathfrak{m}]$, and Theorem 7.8 says that this element fixes each member of Δ^+. Therefore $m_1 m_0 m$ centralizes $\mathfrak{t}_0$, and the result follows.

Suppose that the root α in $\Delta(\mathfrak{g}, \mathfrak{a} \oplus \mathfrak{t})$ is real, i.e., α vanishes on $\mathfrak{t}$. As in the discussion following (6.66), the root space $\mathfrak{g}_\alpha$ in $\mathfrak{g}$ is invariant under the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Since $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = 1$, $\mathfrak{g}_\alpha$ contains a nonzero root vector E_α that is in $\mathfrak{g}_0$. Also as in the discussion following (6.66), we may normalize E_α by a real constant so that $B(E_\alpha, \theta E_\alpha) = -2/|\alpha|^2$. Put $H'_\alpha = 2|\alpha|^{-2}H_\alpha$. Then $\{H'_\alpha, E_\alpha, \theta E_\alpha\}$ spans a copy of $\mathfrak{sl}(2,\mathbb{R})$ with

$$H'_\alpha \leftrightarrow h, \qquad E_\alpha \leftrightarrow e, \qquad \theta E_\alpha \leftrightarrow -f. \tag{7.50}$$

Let us write $(\mathfrak{g}_0)_\alpha$ for $\mathbb{R}E_\alpha$ and $(\mathfrak{g}_0)_{-\alpha}$ for $\mathbb{R}\theta E_\alpha$.

Proposition 7.51. The subgroup $Z_G(\mathfrak{t}_0)$ of G

(a) is reductive with global Cartan decomposition

$$Z_G(\mathfrak{t}_0) = Z_K(\mathfrak{t}_0)\exp(\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0}(\mathfrak{t}_0)),$$

(b) has Lie algebra

$$Z_{\mathfrak{g}_0}(\mathfrak{t}_0) = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \oplus \bigoplus_{\substack{\alpha\in\Delta(\mathfrak{g},\mathfrak{a}\oplus\mathfrak{t}),\\ \alpha \text{ real}}} (\mathfrak{g}_0)_\alpha,$$

which is the direct sum of its center with a real semisimple Lie algebra that is a split real form of its complexification,

(c) is such that the component groups of G, K, $Z_G(\mathfrak{t}_0)$, and $Z_K(\mathfrak{t}_0)$ are all isomorphic.

PROOF. Conclusion (a) is immediate from Proposition 7.25. For (b) it is clear that

$$Z_{\mathfrak{g}}(\mathfrak{t}_0) = \mathfrak{t} \oplus \mathfrak{a} \oplus \bigoplus_{\substack{\alpha\in\Delta(\mathfrak{g},\mathfrak{a}\oplus\mathfrak{t}),\\ \alpha \text{ real}}} \mathfrak{g}_\alpha.$$

The conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ carries every term of the right side into itself, and therefore we obtain the formula of (b). Here $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0}(\mathfrak{t}_0)$, and therefore this decomposition is the restricted-root space decomposition of $\mathfrak{g}_0$. Applying Corollary 6.49 to $[\mathfrak{g}_0, \mathfrak{g}_0]$, we obtain (b). In (c), G and K have isomorphic component groups as a consequence of the global Cartan decomposition, and $Z_G(\mathfrak{t}_0)$ and $Z_K(\mathfrak{t}_0)$ have the same component groups as a consequence of (a). Consider the natural homomorphism

$$Z_K(\mathfrak{t}_0)/Z_K(\mathfrak{t}_0)_0 \to K/K_0$$

induced by inclusion. Propositions 7.49 and 7.33 show that this map is onto, and Corollary 4.51 shows that it is one-one. This proves (c).

We cannot expect to say much about the disconnectedness of M that results from the disconnectedness of G. Thus we shall assume for the remainder of this section that G is connected. Proposition 7.51c notes that $Z_G(\mathfrak{t}_0)$ is connected. To study $Z_G(\mathfrak{t}_0)$, we shall work with the analytic subgroup of $Z_G(\mathfrak{t}_0)$ whose Lie algebra is $[Z_{\mathfrak{g}_0}(\mathfrak{t}_0), Z_{\mathfrak{g}_0}(\mathfrak{t}_0)]$. This is the subgroup that could be called $Z_G(\mathfrak{t}_0)_{ss}$ in the notation of §2. It is semisimple, and its Lie algebra is a split real form. We call the subgroup the **associated split semisimple subgroup**, and we introduce the notation G_{split} for it in order to emphasize that its Lie algebra is split.

Let T be the maximal torus of M_0 with Lie algebra $\mathfrak{t}_0$. Under the assumption that G is connected, it follows from Proposition 7.51b that $Z_G(\mathfrak{t}_0)$ is a commuting product

$$Z_G(\mathfrak{t}_0) = TAG_{\text{split}}.$$

By Proposition 7.27,

$${}^0Z_G(\mathfrak{t}_0) = TG_{\text{split}}$$

is a reductive Lie group.

The group G_{split} need not have finite center, but the structure theory of Chapter VI is available to describe it. Let K_{split} and A_{split} be the analytic subgroups with Lie algebras given as the intersections of $\mathfrak{k}_0$ and $\mathfrak{a}_0$ with $[Z_{\mathfrak{g}_0}(\mathfrak{t}_0), Z_{\mathfrak{g}_0}(\mathfrak{t}_0)]$. Let $F = M_{\text{split}}$ be the centralizer of A_{split} in K_{split}. The subgroup F will play a key role in the analysis of M. It centralizes both T and A.

Corollary 7.52. The subgroup F normalizes M_0, and $M = FM_0$.

PROOF. Since F centralizes A and is a subgroup of K, it is a subgroup of M. Therefore F normalizes M_0, and FM_0 is a group. We know from Proposition 7.49 that $M = Z_M(\mathfrak{t}_0)M_0$. Since $T \subseteq M_0$, it is enough to prove that $Z_M(\mathfrak{t}_0) = TF$. The subgroup $Z_M(\mathfrak{t}_0)$ is contained in $Z_K(\mathfrak{t}_0)$, which in turn is contained in ${}^0Z_G(\mathfrak{t}_0) = TG_{\text{split}}$. Since $Z_M(\mathfrak{t}_0)$ is contained in K, it is therefore contained in TK_{split}. Decompose a member m of $Z_M(\mathfrak{t}_0)$ in a corresponding fashion as $m = tk$. Since m and t centralize A, so does k. Therefore k is in $F = M_{\text{split}}$, and the result follows.

Without additional hypotheses we cannot obtain further nontrivial results about F, and accordingly we recall the following definition from §1.

A semisimple group G has a **complexification** $G^{\mathbb{C}}$ if $G^{\mathbb{C}}$ is a connected complex Lie group with Lie algebra $\mathfrak{g}$ such that G is the analytic subgroup corresponding to the real form $\mathfrak{g}_0$ of $\mathfrak{g}$. By Corollary 7.6, $G^{\mathbb{C}}$ is isomorphic to a matrix group, and hence the same thing is true of G and G_{split}. By Proposition 7.9, each of G and G_{split} has finite center. Therefore we may consider G and G_{split} in the context of reductive Lie groups.

Fix K, θ, and B for G. If the Cartan decomposition of $\mathfrak{g}_0$ is $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, then

$$\mathfrak{g} = (\mathfrak{k}_0 \oplus i\mathfrak{p}_0) \oplus (\mathfrak{p}_0 \oplus i\mathfrak{k}_0)$$

is a Cartan decomposition of $\mathfrak{g}$, and the corresponding Cartan involution of $\mathfrak{g}$ is bar $\circ\, \theta$, where bar is the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. The Lie algebra $\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ is compact semisimple, and it follows from Proposition 7.9 that the corresponding analytic subgroup U of $G^{\mathbb{C}}$ is compact. Then the tuple $(G^{\mathbb{C}}, U, \text{bar} \circ \theta, B)$ makes $G^{\mathbb{C}}$ into a reductive Lie group. Whenever a semisimple Lie group G has a complexification $G^{\mathbb{C}}$ and we consider G as a reductive Lie group (G, K, θ, B), we may consider $G^{\mathbb{C}}$ as the reductive Lie group $(G^{\mathbb{C}}, U, \text{bar} \circ \theta, B)$.

Under the assumption that the semisimple group G has a complexification $G^{\mathbb{C}}$, $\exp i\mathfrak{a}_0$ is well defined as an analytic subgroup of U.

Theorem 7.53. Suppose that the reductive Lie group G is semisimple and has a complexification $G^{\mathbb{C}}$. Then

(a) $F = K_{\text{split}} \cap \exp i\mathfrak{a}_0$,
(b) F is contained in the center of M,
(c) M is the commuting product $M = FM_0$,
(d) F is finite abelian, and every element $f \neq 1$ in F has order 2.

PROOF.

(a) Every member of $K_{\text{split}} \cap \exp i\mathfrak{a}_0$ centralizes $\mathfrak{a}_0$ and lies in K_{split}, hence lies in F. For the reverse inclusion we have $F \subseteq K_{\text{split}}$ by definition. To see that $F \subseteq \exp i\mathfrak{a}_0$, let U_{split} be the analytic subgroup of $G^{\mathbb{C}}$ with Lie algebra the intersection of $\mathfrak{u}_0$ with the Lie algebra $[Z_{\mathfrak{g}}(\mathfrak{t}_0), Z_{\mathfrak{g}}(\mathfrak{t}_0)]$. Then U_{split} is compact, and $i\mathfrak{a}_0 \cap [Z_{\mathfrak{g}}(\mathfrak{t}_0), Z_{\mathfrak{g}}(\mathfrak{t}_0)]$ is a maximal abelian subspace of its Lie algebra. By Corollary 4.52 the corresponding torus is its own centralizer. Hence the centralizer of $\mathfrak{a}_0$ in U_{split} is contained in $\exp i\mathfrak{a}_0$. Since $K_{\text{split}} \subseteq U_{\text{split}}$, it follows that $F \subseteq \exp i\mathfrak{a}_0$.

(b, c) Corollary 7.52 says that $M = FM_0$. By (a), every element of F commutes with any element that centralizes $\mathfrak{a}_0$. Hence F is central in M, and (b) and (c) follow.

(d) Since G_{split} has finite center, F is compact. Its Lie algebra is 0, and thus it is finite. By (b), F is abelian. We still have to prove that every element $f \neq 1$ in F has order 2.

Since G has a complexification, so does G_{split}. Call this group $G^{\mathbb{C}}_{\text{split}}$, let $\widetilde{G}^{\mathbb{C}}_{\text{split}}$ be a simply connected covering group, and let φ be the covering map. Let $\widetilde{G}_{\text{split}}$ be the analytic subgroup with the same Lie algebra as for G_{split}, and form the subgroups $\widetilde{K}_{\text{split}}$ and $\widetilde{F}$ of $\widetilde{G}_{\text{split}}$. The subgroup $\widetilde{F}$ is the complete inverse image of F under φ. Let $\widetilde{U}_{\text{split}}$ play the same role for $\widetilde{G}^{\mathbb{C}}_{\text{split}}$ that U plays for $G^{\mathbb{C}}$. The automorphism θ of the Lie algebra of G_{split} complexifies and lifts to an automorphism $\widetilde{\theta}$ of $\widetilde{G}^{\mathbb{C}}_{\text{split}}$ that carries $\widetilde{U}_{\text{split}}$ into itself. The automorphism $\widetilde{\theta}$ acts as $x \mapsto x^{-1}$ on $\exp i\mathfrak{a}_0$ and as the identity on $\widetilde{K}_{\text{split}}$. The elements of $\widetilde{F}$ are the elements of the intersection, by (a), and hence $\widetilde{f}^{-1} = \widetilde{f}$ for every element $\widetilde{f}$ of $\widetilde{F}$. That is $\widetilde{f}^2 = 1$. Applying φ and using the fact that φ maps $\widetilde{F}$ onto F, we conclude that every element $f \neq 1$ in F has order 2.

EXAMPLE. When G does not have a complexification, the subgroup F need not be abelian. For an example we observe that the group K for $SL(3,\mathbb{R})$ is $SO(3)$, which has $SU(2)$ as a 2-sheeted simply connected covering group. Thus $SL(3,\mathbb{R})$ has a 2-sheeted simply connected covering group, and we take this covering group as G. We already noted in §VI.5 that the group M for $SL(3,\mathbb{R})$ consists of the diagonal matrices with diagonal entries ± 1 and determinant 1. Thus M is the direct sum of two 2-element groups. The subgroup F of G is the complete inverse image of M under the covering map and thus has order 8. Moreover it is a subgroup of $SU(2)$, which has only one element of order 2. Thus F is a group of order 8 with only one element of order 2 and no element of order 8. Of the five abstract groups of order 8, only the 8-element subgroup $\{\pm 1, \pm i, \pm j, \pm k\}$ of the quaternions has this property. This group is nonabelian, and hence F is nonabelian.

Let α be a real root of $\Delta(\mathfrak{g}, \mathfrak{a} \oplus \mathfrak{t})$. From (7.50) we obtain a one-one homomorphism $\mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}_0$ whose only ambiguity is a sign in the definition of E_α. This homomorphism carries $\mathfrak{so}(2)$ to $\mathfrak{k}_0$ and complexifies to a homomorphism $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$. Under the assumption that G is semisimple and has a complexification $G^{\mathbb{C}}$, we can form the analytic subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. This will be a homomorphic image of $SL(2,\mathbb{C})$ since $SL(2,\mathbb{C})$ is simply connected. We let γ_α be the image of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. This element is evidently in the image of $SO(2) \subseteq SL(2,\mathbb{R})$ and hence lies in K_{split}. Clearly it does not depend upon the choice of the ambiguous sign in the definition of E_α. A formula for γ_α is

$$\gamma_\alpha = \exp 2\pi i |\alpha|^{-2} H_\alpha. \tag{7.54}$$

Theorem 7.55. Suppose that the reductive Lie group G is semisimple and has a complexification $G^{\mathbb{C}}$. Then F is generated by all elements γ_α for all real roots α.

PROOF. Our construction of γ_α shows that γ_α is in both K_{split} and $\exp i\mathfrak{a}_0$. By Theorem 7.53a, γ_α is in F. In the reverse direction we use the construction in the proof of Theorem 7.53d, forming a simply connected cover $\widetilde{G}^{\mathbb{C}}_{\text{split}}$ of the complexification $G^{\mathbb{C}}_{\text{split}}$ of G_{split}. We form also the groups $\widetilde{K}_{\text{split}}$, $\widetilde{F}$, and $\widetilde{U}_{\text{split}}$. The elements γ_α are well defined in $\widetilde{F}$ via (7.54), and we show that they generate $\widetilde{F}$. Then the theorem will follow by applying the covering map $\widetilde{G}^{\mathbb{C}}_{\text{split}} \to G^{\mathbb{C}}_{\text{split}}$, since $\widetilde{F}$ maps onto F.

Let $\widetilde{H}$ be the maximal torus of $\widetilde{U}_{\text{split}}$ with Lie algebra $i\mathfrak{a}_0$. We know from Theorem 7.53 that $\widetilde{F}$ is a finite subgroup of $\widetilde{H}$. Arguing by contradiction, suppose that the elements γ_α generate a proper subgroup $\widetilde{F}_0$ of $\widetilde{F}$. Let $\widetilde{f}$ be an element of $\widetilde{F}$ not in $\widetilde{F}_0$. Applying the Peter–Weyl Theorem (Theorem 4.20) to $\widetilde{H}/\widetilde{F}_0$, we can obtain a multiplicative character χ_ν of $\widetilde{H}$ that is 1 on $\widetilde{F}_0$ and is $\neq 1$ on $\widetilde{f}$. Here ν is the imaginary-valued linear functional on $i\mathfrak{a}_0$ such that $\chi_\nu(\exp ih) = e^{\nu(ih)}$ for $h \in \mathfrak{a}_0$. The roots for $\widetilde{U}_{\text{split}}$ are the real roots for $\mathfrak{g}_0$, and our assumption is that each such real root α has

$$1 = \chi_\nu(\gamma_\alpha) = \chi(\exp 2\pi i|\alpha|^{-2}H_\alpha) = e^{\nu(2\pi i|\alpha|^{-2}H_\alpha)} = e^{\pi i(2\langle\nu,\alpha\rangle/|\alpha|^2)}.$$

That is $2\langle\nu,\alpha\rangle/|\alpha|^2$ is an even integer for all α. Hence $\frac{1}{2}\nu$ is algebraically integral.

Since $\widetilde{U}_{\text{split}}$ is simply connected, Theorem 5.107 shows that $\frac{1}{2}\nu$ is analytically integral. Thus the multiplicative character $\chi_{\frac{1}{2}\nu}$ of $\widetilde{H}$ given by $\chi_{\frac{1}{2}\nu}(\exp ih) = e^{\frac{1}{2}\nu(ih)}$ is well defined. Theorem 7.53d says that $\widetilde{f}^2 = 1$, and therefore $\chi_{\frac{1}{2}\nu}(\widetilde{f}) = \pm 1$. Since $\chi_\nu = (\chi_{\frac{1}{2}\nu})^2$, we obtain $\chi_\nu(\widetilde{f}) = 1$, contradiction. We conclude that $\widetilde{F}_0$ equals $\widetilde{F}$, and the proof is complete.

It is sometimes handy to enlarge the collection of elements γ_α. Let β be any restricted root, and let X_β be any restricted-root vector corresponding to β. Then θX_β is a restricted-root vector for the restricted root $-\beta$ by Proposition 6.40c. Proposition 6.52 shows that we can normalize X_β so that $[X_\beta, \theta X_\beta] = -2|\beta|^{-2}H_\beta$, and then the correspondence

$$h \leftrightarrow 2|\beta|^{-2}H_\beta, \qquad e \leftrightarrow X_\beta, \qquad f \leftrightarrow -\theta X_\beta \tag{7.56}$$

is an isomorphism of $\mathfrak{sl}(2,\mathbb{R})$ with the real span of $H_\beta, X_\beta, \theta X_\beta$ in $\mathfrak{g}_0$. Once again this homomorphism carries $\mathfrak{so}(2) = \mathbb{R}(e - f)$ to $\mathfrak{k}_0$ and complexifies to a homomorphism $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$. Under the assumption that G is semisimple and has a complexification $G^{\mathbb{C}}$, we can form the analytic subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. This will be a homomorphic image of $SL(2,\mathbb{C})$ since $SL(2,\mathbb{C})$ is simply connected. We let γ_β be the image of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, namely

$$\gamma_\beta = \exp 2\pi i |\beta|^{-2} H_\beta. \tag{7.57}$$

This element is evidently in the image of $SO(2) \subseteq SL(2,\mathbb{R})$ and hence lies in K. Formula (7.57) makes it clear that γ_β does not depend on the choice of X_β, except for the normalization, and also (7.57) shows that γ_β commutes with $\mathfrak{a}_0$. Hence

$$\gamma_\beta \text{ is in } M \text{ for each restricted root } \beta. \tag{7.58}$$

Since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ has square the identity, it follows that

$$\gamma_\beta^2 = 1 \qquad \text{for each restricted root } \beta. \tag{7.59}$$

In the special case that β extends to a real root α of $\Delta(\mathfrak{g}, \mathfrak{a} \oplus \mathfrak{t})$ when set equal to 0 on $\mathfrak{t}$, γ_β equals the element γ_α defined in (7.54). The more general elements (7.57) are not needed for the description of F in Theorem 7.55, but they will play a role in Chapter VIII.

6. Real-rank-one Subgroups

We continue to assume that G is a reductive Lie group, and we use the other notation of §2. In addition, we use the notation F of §5.

The **real rank** of G is the dimension of a maximal abelian subspace of $\mathfrak{p}_0$. Proposition 7.29 shows that real rank is well defined. Since any maximal abelian subspace of $\mathfrak{p}_0$ contains $\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0}$, it follows that

$$\text{real rank}(G) = \text{real rank}({}^0G) + \dim Z_{vec}. \tag{7.60}$$

Our objective in this section is to identify some subgroups of G of real rank one and illustrate how information about these subgroups can give information about G.

"Real rank" is meaningful for a real semisimple Lie algebra outside the context of reductive Lie groups (G, K, θ, B), since Cartan decompositions exist and all are conjugate. But it is not meaningful for a reductive Lie algebra by itself, since the splitting of $Z_{\mathfrak{g}_0}$ into its $\mathfrak{k}_0$ part and its $\mathfrak{p}_0$ part depends upon the choice of θ.

The Lie subalgebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ of $\mathfrak{g}_0$, being semisimple, is uniquely the sum of simple ideals. These ideals are orthogonal with respect to B, since if $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are distinct ideals, then

$$(7.61) \qquad B(\mathfrak{g}_i, \mathfrak{g}_j) = B([\mathfrak{g}_i, \mathfrak{g}_i], \mathfrak{g}_j) = B(\mathfrak{g}_i, [\mathfrak{g}_i, \mathfrak{g}_j]) = B(\mathfrak{g}_i, 0) = 0.$$

Since $[\mathfrak{g}_0, \mathfrak{g}_0]$ is invariant under θ, θ permutes these simple ideals, necessarily in orbits of one or two ideals. But actually there are no 2-ideal orbits since if X and θX are nonzero elements of distinct ideals, then (7.61) gives

$$0 < B_\theta(X, X) = -B(X, \theta X) = 0,$$

contradiction. Hence each simple ideal is invariant under θ, and it follows that $\mathfrak{p}_0$ is the direct sum of its components in each simple ideal and its component in $Z_{\mathfrak{g}_0}$.

We would like to conclude that the real rank of G is the sum of the real ranks from the components and from the center. But to do so, we need either to define real rank for triples $(\mathfrak{g}_0, \theta, B)$ or to lift the setting from Lie algebras to Lie groups. Following the latter procedure, assume that G is in the Harish-Chandra class; this condition is satisfied automatically if G is semisimple. If G_i is the analytic subgroup of G whose Lie algebra is one of the various simple ideals of G, then Proposition 7.20b shows that G_i has finite center. Consequently G_i is a reductive group. Also in this case the subgroup K_i of G_i fixed by Θ is compact, and it follows from property (iv) that G_i is closed in G. We summarize as follows.

Proposition 7.62. Let the reductive Lie group G be in the Harish-Chandra class, and let $G_1, \ldots, G_n$ be the analytic subgroups of G whose Lie algebra are the simple ideals of $\mathfrak{g}_0$. Then $G_1, \ldots, G_n$ are reductive Lie groups, they are closed in G, and the sum of the real ranks of the G_i's, together with the dimension of Z_{vec}, equals the real rank of $\mathfrak{g}_0$.

With the maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}_0$ fixed, let λ be a restricted root. Denote by $H_\lambda^\perp$ the orthogonal complement of $\mathbb{R}H_\lambda$ in $\mathfrak{a}_0$ relative to B_θ. Propositions 7.25 and 7.27 show that $Z_G(H_\lambda^\perp)$ and ${}^0Z_G(H_\lambda^\perp)$ are reductive Lie groups. All of $\mathfrak{a}_0$ is in $Z_G(H_\lambda^\perp)$, and therefore $Z_G(H_\lambda^\perp)$ has the same real rank as G. The split component of $Z_G(H_\lambda^\perp)$ is $H_\lambda^\perp$, and it follows from (7.60) that ${}^0Z_G(H_\lambda^\perp)$ is a reductive Lie group of real rank one.

The subgroup ${}^0Z_G(H_\lambda^\perp)$ is what is meant by the real-rank-one reductive subgroup of G corresponding to the restricted root λ. A maximal abelian subspace of the $\mathfrak{p}_0$ for ${}^0Z_G(H_\lambda^\perp)$ is $\mathbb{R}H_\lambda$, and the restricted roots for this group are those nonzero multiples of λ that provide restricted roots for $\mathfrak{g}_0$. In other words the restricted-root space decomposition of the Lie algebra of ${}^0Z_G(H_\lambda^\perp)$ is

$$\mathbb{R}H_\lambda \oplus \mathfrak{m}_0 \oplus \bigoplus_{c\neq 0} (\mathfrak{g}_0)_{c\lambda}. \tag{7.63}$$

Sometimes it is desirable to associate to λ a real-rank-one subgroup whose Lie algebra is simple. To do so, let us assume that G is in the Harish-Chandra class. Then so is ${}^0Z_G(H_\lambda^\perp)$. Since this group has compact center, Proposition 7.62 shows that the sum of the real ranks of the subgroups G_i of ${}^0Z_G(H_\lambda^\perp)$ corresponding to the simple ideals of the Lie algebra is 1. Hence exactly one G_i has real rank one, and that is the real-rank-one reductive subgroup that we can use. The part of (7.63) that is being dropped to get a simple Lie algebra is contained in $\mathfrak{m}_0$.

In the case that the reductive group G is semisimple and has a complexification, the extent of the disconnectedness of M can be investigated with the help of the real-rank-one subgroups ${}^0Z_G(H_\lambda^\perp)$. The result that we use about the real-rank-one case is given in Theorem 7.66 below.

Lemma 7.64. $N^- \cap MAN = \{1\}$.

PROOF. Let $x \neq 1$ be in $N^- = \Theta N$. By Theorem 1.127 write $x = \exp X$ with X in $\mathfrak{n}_0^- = \theta\mathfrak{n}_0$. Recall from Proposition 6.40c that $\theta(\mathfrak{g}_0)_\lambda = (\mathfrak{g}_0)_{-\lambda}$, let $X = \sum_{\mu\in\Sigma} X_\mu$ be the decomposition of X into restricted-root vectors, and choose $\mu = \mu_0$ as large as possible so that $X_\mu \neq 0$. If we take any $H \in \mathfrak{a}_0$ such that $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$, then

$$\begin{aligned}\operatorname{Ad}(x)H - H &= e^{\operatorname{ad} X}H - H\\ &= [X, H] + \tfrac{1}{2}[X, [X, H]] + \cdots\\ &= [X_{\mu_0}, H] + \text{terms for lower restricted roots.}\end{aligned}$$

In particular, $\operatorname{Ad}(x)H - H$ is in $\mathfrak{n}_0^-$ and is not 0. On the other hand, if x is in MAN, then $\operatorname{Ad}(x)H - H$ is in $\mathfrak{n}_0$. Since $\mathfrak{n}_0^- \cap \mathfrak{n}_0 = 0$, we must have $N^- \cap MAN = \{1\}$.

Lemma 7.65. The map $K/M \to G/MAN$ induced by inclusion is a diffeomorphism.

PROOF. The given map is certainly smooth. If $\kappa(g)$ denotes the K component of g in the Iwasawa decomposition $G = KAN$ of Proposition 7.31, then $g \mapsto \kappa(g)$ is smooth, and the map $gMAN \mapsto \kappa(g)M$ is a two-sided inverse to the given map.

Theorem 7.66. Suppose that the reductive Lie group G is semisimple, is of real rank one, and has a complexification $G^{\mathbb{C}}$. Then M is connected unless $\dim \mathfrak{n}_0 = 1$.

REMARKS. Since G is semisimple, it is in the Harish-Chandra class. The above remarks about simple components are therefore applicable. The condition $\dim \mathfrak{n}_0 = 1$ is the same as the condition that the simple component of $\mathfrak{g}_0$ containing $\mathfrak{a}_0$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. In fact, if $\dim \mathfrak{n}_0 = 1$, then $\mathfrak{n}_0$ is of the form $\mathbb{R}X$ for some X. Then X, θX, and $[X, \theta X]$ span a copy of $\mathfrak{sl}(2, \mathbb{R})$, and we obtain $\mathfrak{g}_0 \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{m}_0$. The Lie subalgebra $\mathfrak{m}_0$ must centralize X, θX, and $[X, \theta X]$ and hence must be an ideal in $\mathfrak{g}_0$. The complementary ideal is $\mathfrak{sl}(2, \mathbb{R})$, as asserted.

PROOF. The multiplication map $N^- \times M_0AN \to G$ is smooth and everywhere regular by Lemma 6.44. Hence the map $N^- \to G/M_0AN$ induced by inclusion is smooth and regular, and so is the map

$$N^- \to G/MAN, \tag{7.67}$$

which is the composition of $N^- \to G/M_0AN$ and a covering map. Also the map (7.67) is one-one by Lemma 7.64. Therefore (7.67) is a diffeomorphism onto an open set.

Since G is semisimple and has real rank 1, the Weyl group $W(\Sigma)$ has two elements. By Proposition 7.32, $W(G, A)$ has two elements. Let $\widetilde{w} \in N_K(\mathfrak{a}_0)$ represent the nontrivial element of $W(G, A)$. By the Bruhat decomposition (Theorem 7.40),

$$G = MAN \cup MAN\widetilde{w}MAN = MAN \cup N\widetilde{w}MAN. \tag{7.68}$$

Since $\mathrm{Ad}(\widetilde{w})^{-1}$ acts as -1 on $\mathfrak{a}_0$, it sends the positive restricted roots to the negative restricted roots, and it follows from Proposition 6.40c that $\mathrm{Ad}(\widetilde{w})^{-1}\mathfrak{n}_0 = \mathfrak{n}_0^-$. Therefore $\widetilde{w}^{-1}N\widetilde{w} = N^-$. Multiplying (7.68) on the left by $\widetilde{w}^{-1}$, we obtain

$$G = \widetilde{w}MAN \cup N^-MAN.$$

Hence G/MAN is the disjoint union of the single point $\widetilde{w}MAN$ and the image of the map (7.67).

We have seen that (7.67) is a diffeomorphism onto an open subset of G/MAN. Lemma 7.65 shows that G/MAN is diffeomorphic to K/M. Since Theorem 1.127 shows that N^- is diffeomorphic to Euclidean space, K/M is a one-point compactification of a Euclidean space, hence a sphere. Since K is connected, M must be connected whenever K/M is simply connected, i.e., whenever $\dim K/M > 1$. Since $\dim K/M = \dim \mathfrak{n}_0$, M is connected unless $\dim \mathfrak{n}_0 = 1$.

Corollary 7.69. Suppose that the reductive Lie group G is semisimple and has a complexification $G^{\mathbb{C}}$. Let $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a} \oplus \mathfrak{t})$ be a real root. If the positive multiples of the restricted root $\alpha|_{\mathfrak{a}_0}$ have combined restricted-root multiplicity greater than one, then γ_α is in M_0.

PROOF. The element γ_α is in the homomorphic image of $SL(2, \mathbb{R})$ associated to the root α, hence is in the subgroup $G' = {}^0Z_G(H_\alpha^\perp)_0$. Consequently it is in the M subgroup of G'. The subgroup G' satisfies the hypotheses of Theorem 7.66, and its $\mathfrak{n}_0$ has dimension >1 by hypothesis. By Theorem 7.66 its M subgroup is connected. Hence γ_α is in the identity component of the M subgroup for G.

7. Parabolic Subgroups

In this section G will denote a reductive Lie group, and we shall use the other notation of §2 concerning the Cartan decomposition. But we shall abandon the use of $\mathfrak{a}_0$ as a maximal abelian subspace of $\mathfrak{p}_0$, as well as the other notation connected with the Iwasawa decomposition. Instead of using the symbols $\mathfrak{a}_0$, $\mathfrak{n}_0$, $\mathfrak{m}_0$, $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{m}$, A, N, and M for these objects, we shall use the symbols $\mathfrak{a}_{\mathfrak{p},0}$, $\mathfrak{n}_{\mathfrak{p},0}$, $\mathfrak{m}_{\mathfrak{p},0}$, $\mathfrak{a}_{\mathfrak{p}}$, $\mathfrak{n}_{\mathfrak{p}}$, $\mathfrak{m}_{\mathfrak{p}}$, $A_{\mathfrak{p}}$, $N_{\mathfrak{p}}$, and $M_{\mathfrak{p}}$.

Our objective is to define and characterize "parabolic subgroups" of G, first working with "parabolic subalgebras" of $\mathfrak{g}_0$. Each parabolic subgroup Q will have a canonical decomposition in the form $Q = MAN$, known as the "Langlands decomposition" of Q. As we suggested at the start of §2, a number of arguments with reductive Lie groups are carried out by induction on the dimension of the group. One way of implementing this idea is to reduce proofs from G to the M of some parabolic subgroup. For such a procedure to succeed, we build into the definition of M the fact that M is a reductive Lie group.

In developing our theory, one approach would be to define a parabolic subalgebra of $\mathfrak{g}_0$ to be a subalgebra whose complexification is a parabolic subalgebra of $\mathfrak{g}$. Then we could deduce properties of parabolic subalgebras of $\mathfrak{g}_0$ from the theory in §V.7. But it will be more convenient to work with parabolic subalgebras of $\mathfrak{g}_0$ directly, proving results by imitating the theory of §V.7, rather than by applying it.

A **minimal parabolic subalgebra** of $\mathfrak{g}_0$ is any subalgebra of $\mathfrak{g}_0$ that is conjugate to $\mathfrak{q}_{\mathfrak{p},0} = \mathfrak{m}_{\mathfrak{p},0} \oplus \mathfrak{a}_{\mathfrak{p},0} \oplus \mathfrak{n}_{\mathfrak{p},0}$ via $\mathrm{Ad}(G)$. Because of the Iwasawa decomposition $G = KA_{\mathfrak{p}}N_{\mathfrak{p}}$, we may as well assume that the conjugacy is via $\mathrm{Ad}(K)$. The subalgebra $\mathfrak{q}_{\mathfrak{p},0}$ contains the maximally noncompact θ stable Cartan subalgebra $\mathfrak{a}_{\mathfrak{p},0} \oplus \mathfrak{t}_{\mathfrak{p},0}$, where $\mathfrak{t}_{\mathfrak{p},0}$ is any maximal abelian subspace of $\mathfrak{m}_{\mathfrak{p},0}$, and $\mathrm{Ad}(k)$ sends any such Cartan subalgebra into another such Cartan subalgebra if k is in K. Hence every minimal parabolic subalgebra of $\mathfrak{g}_0$ contains a maximally noncompact θ stable Cartan subalgebra of $\mathfrak{g}_0$. A **parabolic subalgebra** $\mathfrak{q}_0$ of $\mathfrak{g}_0$ is a Lie subalgebra containing some minimal parabolic subalgebra. A parabolic subalgebra must contain a maximally noncompact θ stable Cartan subalgebra of $\mathfrak{g}_0$.

Therefore there is no loss of generality in assuming that $\mathfrak{q}_0$ contains a minimal parabolic subalgebra of the form $\mathfrak{m}_{\mathfrak{p},0} \oplus \mathfrak{a}_{\mathfrak{p},0} \oplus \mathfrak{n}_{\mathfrak{p},0}$, where $\mathfrak{a}_{\mathfrak{p},0}$ is maximal abelian in $\mathfrak{p}_0$, and $\mathfrak{m}_{\mathfrak{p},0}$ and $\mathfrak{n}_{\mathfrak{p},0}$ are constructed are usual. Let Σ denote the set of restricted roots of $\mathfrak{g}_0$ relative to $\mathfrak{a}_{\mathfrak{p},0}$. The restricted roots contributing to $\mathfrak{n}_{\mathfrak{p},0}$ are taken to be the positive ones.

We can obtain examples of parabolic subalgebras as follows. Let Π be the set of simple restricted roots, fix a subset Π' of Π, and let

$$\Gamma = \Sigma^+ \cup \{\beta \in \Sigma \mid \beta \in \mathrm{span}(\Pi')\}. \tag{7.70}$$

Then

$$\mathfrak{q}_0 = \mathfrak{a}_{\mathfrak{p},0} \oplus \mathfrak{m}_{\mathfrak{p},0} \oplus \bigoplus_{\beta\in\Gamma} (\mathfrak{g}_0)_\beta \tag{7.71}$$

is a parabolic subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{m}_{\mathfrak{p},0} \oplus \mathfrak{a}_{\mathfrak{p},0} \oplus \mathfrak{n}_{\mathfrak{p},0}$. This construction is an analog of the corresponding construction of parabolic subalgebras of $\mathfrak{g}$ given in (5.88) and (5.89), and Proposition 7.76 will show that every parabolic subalgebra of $\mathfrak{g}_0$ is of the form given in (7.70) and (7.71). But the proof requires more preparation than in the situation with (5.88) and (5.89).

EXAMPLES.

1) Let $G = SL(n, \mathbb{K})$, where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. When $\mathfrak{g}_0$ is realized as matrices, the Lie subalgebra of upper-triangular matrices is a minimal parabolic subalgebra $\mathfrak{q}_{\mathfrak{p},0}$. The other examples of parabolic subalgebras $\mathfrak{q}_0$ containing $\mathfrak{q}_{\mathfrak{p},0}$ and written as in (7.70) and (7.71) are the full Lie subalgebras of block upper-triangular matrices, one subalgebra for each arrangement of blocks.

2) Let G have compact center and be of real rank one. The examples as in (7.70) and (7.71) are the minimal parabolic subalgebras and $\mathfrak{g}_0$ itself.

We shall work with a vector X in the restricted-root space $(\mathfrak{g}_0)_\gamma$. Proposition 6.40c shows that θX is in $(\mathfrak{g}_0)_{-\gamma}$, and Proposition 6.52 shows that $B(X, \theta X)H_\gamma$ is a negative multiple of H_γ. Normalizing, we may assume that $B(X, \theta X) = -2/|\gamma|^2$. Put $H'_\gamma = 2|\gamma|^{-2}H_\gamma$. Then the linear span $\mathfrak{sl}_X$ of $\{X, \theta X, H'_\gamma\}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ under the isomorphism

$$H'_\gamma \leftrightarrow h, \qquad X \leftrightarrow e, \qquad \theta X \leftrightarrow -f. \tag{7.72}$$

We shall make use of the copy $\mathfrak{sl}_X$ of $\mathfrak{sl}(2, \mathbb{R})$ in the same way as in the proof of Corollary 6.53. This subalgebra of $\mathfrak{g}_0$ acts by ad on $\mathfrak{g}_0$ and hence acts on $\mathfrak{g}$. We know from Theorem 1.67 that the resulting representation of $\mathfrak{sl}_X$ is completely reducible, and we know the structure of each irreducible subspace from Theorem 1.66.

Lemma 7.73. Let γ be a restricted root, and let $X \neq 0$ be in $(\mathfrak{g}_0)_\gamma$. Then

(a) $\operatorname{ad} X$ carries $(\mathfrak{g}_0)_\gamma$ onto $(\mathfrak{g}_0)_{2\gamma}$,
(b) $(\operatorname{ad} \theta X)^2$ carries $(\mathfrak{g}_0)_\gamma$ onto $(\mathfrak{g}_0)_{-\gamma}$,
(c) $(\operatorname{ad} \theta X)^4$ carries $(\mathfrak{g}_0)_{2\gamma}$ onto $(\mathfrak{g}_0)_{-2\gamma}$.

PROOF. Without loss of generality, we may assume that X is normalized as in (7.72). The complexification of $\bigoplus_{c\in\mathbb{Z}} (\mathfrak{g}_0)_{c\gamma}$ is an invariant subspace of $\mathfrak{g}$ under the representation ad of $\mathfrak{sl}_X$. Using Theorem 1.67, we decompose it as the direct sum of irreducible representations. Each member of $(\mathfrak{g}_0)_{c\gamma}$ is an eigenvector for $\operatorname{ad} H'_\gamma$ with eigenvalue $2c$, and H'_γ corresponds to the member h of $\mathfrak{sl}(2, \mathbb{R})$. From Theorem 1.66 we see that the only possibilities for irreducible subspaces are 5-dimensional subspaces consisting of one dimension each from

$$(\mathfrak{g}_0)_{2\gamma},\ (\mathfrak{g}_0)_\gamma,\ \mathfrak{m}_0,\ (\mathfrak{g}_0)_{-\gamma},\ (\mathfrak{g}_0)_{-2\gamma};$$

3-dimensional subspaces consisting of one dimension each from

$$(\mathfrak{g}_0)_\gamma,\ \mathfrak{m}_0,\ (\mathfrak{g}_0)_{-\gamma};$$

and 1-dimensional subspaces consisting of one dimension each from $\mathfrak{m}_0$. In any 5-dimensional such subspace, $\operatorname{ad} X$ carries a nonzero vector of eigenvalue 2 to a nonzero vector of eigenvalue 4. This proves (a). Also in any 5-dimensional such subspace, $(\operatorname{ad}\theta X)^4$ carries a nonzero vector of eigenvalue 4 to a nonzero vector of eigenvalue -4. This proves (c). Finally in any 5-dimensional such subspace or 3-dimensional such subspace, $(\operatorname{ad}\theta X)^2$ carries a nonzero vector of eigenvalue 2 to a nonzero vector of eigenvalue -2. This proves (b).

Lemma 7.74. Every parabolic subalgebra $\mathfrak{q}_0$ of $\mathfrak{g}_0$ containing the minimal parabolic subalgebra $\mathfrak{m}_{\mathfrak{p},0} \oplus \mathfrak{a}_{\mathfrak{p},0} \oplus \mathfrak{n}_{\mathfrak{p},0}$ is of the form

$$\mathfrak{q}_0 = \mathfrak{a}_{\mathfrak{p},0} \oplus \mathfrak{m}_{\mathfrak{p},0} \oplus \bigoplus_{\beta\in\Gamma} (\mathfrak{g}_0)_\beta$$

for some subset Γ of Σ that contains Σ^+.

PROOF. Since $\mathfrak{q}_0$ contains $\mathfrak{a}_{\mathfrak{p},0} \oplus \mathfrak{m}_{\mathfrak{p},0}$ and is invariant under $\operatorname{ad}(\mathfrak{a}_{\mathfrak{p},0})$, it is of the form

$$\mathfrak{q}_0 = \mathfrak{a}_{\mathfrak{p},0} \oplus \mathfrak{m}_{\mathfrak{p},0} \oplus \bigoplus_{\beta\in\Sigma} ((\mathfrak{g}_0)_\beta \cap \mathfrak{q}_0).$$

Thus we are to show that if $\mathfrak{q}_0$ contains one nonzero vector Y of $(\mathfrak{g}_0)_\beta$, then it contains all of $(\mathfrak{g}_0)_\beta$. Since $\mathfrak{q}_0$ contains $\mathfrak{n}_{\mathfrak{p},0}$, we may assume that β is negative. We apply Lemma 7.73b with $X = \theta Y$ and $\gamma = -\beta$. The lemma says that $(\operatorname{ad} Y)^2$ carries $(\mathfrak{g}_0)_{-\beta}$ onto $(\mathfrak{g}_0)_\beta$. Since Y and $(\mathfrak{g}_0)_{-\beta}$ are contained in $\mathfrak{q}_0$, so is $(\mathfrak{g}_0)_\beta$.

Lemma 7.75. If β, γ, and $\beta+\gamma$ are restricted roots and X is a nonzero member of $(\mathfrak{g}_0)_\gamma$, then $[X, (\mathfrak{g}_0)_\beta]$ is a nonzero subspace of $(\mathfrak{g}_0)_{\beta+\gamma}$.

PROOF. Without loss of generality, we may assume that X is normalized as in (7.72). The complexification of $\bigoplus_{c\in\mathbb{Z}} (\mathfrak{g}_0)_{\beta+c\gamma}$ is an invariant subspace of $\mathfrak{g}$ under the representation ad of $\mathfrak{sl}_X$. Using Theorem 1.67, we decompose it as the direct sum of irreducible representations. Each member of $(\mathfrak{g}_0)_{\beta+c\gamma}$ is an eigenvector for $\operatorname{ad} H'_\gamma$ with eigenvalue $\frac{2\langle\beta,\gamma\rangle}{|\gamma|^2} + 2c$, and H'_γ corresponds to the member h of $\mathfrak{sl}(2,\mathbb{R})$. We apply Theorem 1.66 and divide matters into cases according to the sign of $\frac{2\langle\beta,\gamma\rangle}{|\gamma|^2}$. If the sign is < 0, then $\operatorname{ad} X$ is one-one on $(\mathfrak{g}_0)_\beta$, and the lemma follows. If the sign is ≥ 0, then $\operatorname{ad}\theta X$ and $\operatorname{ad} X \operatorname{ad}\theta X$ are one-one on $(\mathfrak{g}_0)_\beta$, and hence $\operatorname{ad} X$ is nonzero on the member $[\theta X, Y]$ if Y is nonzero in $(\mathfrak{g}_0)_{\beta+\gamma}$.

Proposition 7.76. The parabolic subalgebras $\mathfrak{q}_0$ containing the minimal parabolic subalgebra $\mathfrak{m}_{\mathfrak{p},0} \oplus \mathfrak{a}_{\mathfrak{p},0} \oplus \mathfrak{n}_{\mathfrak{p},0}$ are parametrized by the set of subsets of simple restricted roots; the one corresponding to a subset Π' is of the form (7.71) with Γ as in (7.70).

PROOF. Lemma 7.74 establishes that any $\mathfrak{q}_0$ is of the form (7.71) for some subset Γ. We can now go over the proof of Proposition 5.90 to see that it applies. What is needed is a substitute for Corollary 2.35, which says that $[\mathfrak{g}_\beta, \mathfrak{g}_\gamma] = \mathfrak{g}_{\beta+\gamma}$ if β, γ, and $\beta+\gamma$ are all roots. Lemma 7.75 provides the appropriate substitute, and the proposition follows.

In the notation of the proposition, $\Gamma \cap -\Gamma$ consists of all restricted roots in the span of Π', and the other members of Γ are all positive and have expansions in terms of simple restricted roots that involve a simple restricted root not in Π'. Define

$$\begin{aligned}
\mathfrak{a}_0 &= \bigcap_{\beta\in\Gamma\cap-\Gamma} \ker\beta \subseteq \mathfrak{a}_{\mathfrak{p},0}\\
\mathfrak{a}_{M,0} &= \mathfrak{a}_0^\perp \subseteq \mathfrak{a}_{\mathfrak{p},0}\\
\mathfrak{m}_0 &= \mathfrak{a}_{M,0} \oplus \mathfrak{m}_{\mathfrak{p},0} \oplus \bigoplus_{\beta\in\Gamma\cap-\Gamma} (\mathfrak{g}_0)_\beta\\
\mathfrak{n}_0 &= \bigoplus_{\substack{\beta\in\Gamma,\\ \beta\notin-\Gamma}} (\mathfrak{g}_0)_\beta\\
\mathfrak{n}_{M,0} &= \mathfrak{n}_{\mathfrak{p},0} \cap \mathfrak{m}_0,
\end{aligned} \tag{7.77a}$$

so that

$$\mathfrak{q}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0. \tag{7.77b}$$

The decomposition (7.77b) is called the **Langlands decomposition** of $\mathfrak{q}_0$.

EXAMPLE. Let $G = SU(2,2)$. The Lie algebra $\mathfrak{g}_0$ consists of all 4-by-4 complex matrices of the block form

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix}$$

with X_{11} and X_{22} skew Hermitian and the total trace equal to 0. We take the Cartan involution to be negative conjugate transpose, so that

$$\mathfrak{k}_0 = \left\{ \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} \right\} \qquad \text{and} \qquad \mathfrak{p}_0 = \left\{ \begin{pmatrix} 0 & X_{12} \\ X_{12}^* & 0 \end{pmatrix} \right\}.$$

Let us take

$$\mathfrak{a}_{\mathfrak{p},0} = \left\{ \begin{pmatrix} 0 & 0 & s & 0 \\ 0 & 0 & 0 & t \\ s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \end{pmatrix} \middle|\ s \text{ and } t \text{ in } \mathbb{R} \right\}.$$

Define linear functionals f_1 and f_2 on $\mathfrak{a}_{\mathfrak{p},0}$ by saying that f_1 of the above matrix is s and f_2 of the matrix is t. Then

$$\Sigma = \{\pm f_1 \pm f_2,\ \pm 2f_1,\ \pm 2f_2\},$$

which is a root system of type C_2. Here $\pm f_1 \pm f_2$ have multiplicity 2, and the others have multiplicity one. In the obvious ordering, Σ^+ consists of $f_1 \pm f_2$ and $2f_1$ and $2f_2$, and the simple restricted roots are $f_1 - f_2$ and $2f_2$. Then

$$\begin{aligned} \mathfrak{m}_{\mathfrak{p},0} &= \{\mathrm{diag}(ir, -ir, ir, -ir)\} \\ \mathfrak{n}_{\mathfrak{p},0} &= \bigoplus_{\beta\in\Sigma^+} (\mathfrak{g}_0)_\beta \quad \text{with } \dim \mathfrak{n}_{\mathfrak{p},0} = 6. \end{aligned}$$

Our minimal parabolic subalgebra is $\mathfrak{q}_{\mathfrak{p},0} = \mathfrak{m}_{\mathfrak{p},0} \oplus \mathfrak{a}_{\mathfrak{p},0} \oplus \mathfrak{n}_{\mathfrak{p},0}$, and this is reproduced as $\mathfrak{q}_0$ by (7.70) and (7.71) with $\Pi' = \emptyset$. When $\Pi' = \{f_1 - f_2,\ 2f_2\}$, then $\mathfrak{q}_0 = \mathfrak{g}_0$. The two intermediate cases are as follows. If $\Pi' = \{f_1 - f_2\}$, then

$$\begin{aligned} \mathfrak{a}_0 &= \{H \in \mathfrak{a}_{\mathfrak{p},0} \mid (f_1 - f_2)(H) = 0\} \qquad (s = t \text{ in } \mathfrak{a}_{\mathfrak{p},0}) \\ \mathfrak{m}_0 &= \left\{ \left. \begin{pmatrix} ir & w & x & z \\ -\bar{w} & -ir & \bar{z} & -x \\ x & z & ir & w \\ \bar{z} & -x & -\bar{w} & -ir \end{pmatrix} \right| x, r \in \mathbb{R} \text{ and } w, z \in \mathbb{C} \right\} \\ \mathfrak{n}_0 &= (\mathfrak{g}_0)_{2f_1} \oplus (\mathfrak{g}_0)_{f_1+f_2} \oplus (\mathfrak{g}_0)_{2f_2}. \end{aligned}$$

If $\Pi' = \{2f_2\}$, then

$$\begin{aligned} \mathfrak{a}_0 &= \{H \in \mathfrak{a}_{\mathfrak{p},0} \mid 2f_2(H) = 0\} \qquad (t = 0 \text{ in } \mathfrak{a}_{\mathfrak{p},0}) \\ \mathfrak{m}_0 &= \mathfrak{m}_{\mathfrak{p},0} \oplus \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & is & 0 & z \\ 0 & 0 & 0 & 0 \\ 0 & \bar{z} & 0 & -is \end{pmatrix} \right| s \in \mathbb{R} \text{ and } z \in \mathbb{C} \right\} \\ \mathfrak{n}_0 &= (\mathfrak{g}_0)_{2f_1} \oplus (\mathfrak{g}_0)_{f_1+f_2} \oplus (\mathfrak{g}_0)_{f_1-f_2}. \end{aligned}$$

Proposition 7.76 says that there are no other parabolic subalgebras $\mathfrak{q}_0$ containing $\mathfrak{q}_{\mathfrak{p},0}$.

Proposition 7.78. A parabolic subalgebra $\mathfrak{q}_0$ containing the minimal parabolic subalgebra $\mathfrak{m}_{\mathfrak{p},0} \oplus \mathfrak{a}_{\mathfrak{p},0} \oplus \mathfrak{n}_{\mathfrak{p},0}$ has the properties that

(a) $\mathfrak{m}_0$, $\mathfrak{a}_0$, and $\mathfrak{n}_0$ are Lie subalgebras, and $\mathfrak{n}_0$ is an ideal in $\mathfrak{q}_0$,
(b) $\mathfrak{a}_0$ is abelian, and $\mathfrak{n}_0$ is nilpotent,
(c) $\mathfrak{a}_0 \oplus \mathfrak{m}_0$ is the centralizer of $\mathfrak{a}_0$ in $\mathfrak{g}_0$,
(d) $\mathfrak{q}_0 \cap \theta\mathfrak{q}_0 = \mathfrak{a}_0 \oplus \mathfrak{m}_0$, and $\mathfrak{a}_0 \oplus \mathfrak{m}_0$ is reductive,
(e) $\mathfrak{a}_{\mathfrak{p},0} = \mathfrak{a}_0 \oplus \mathfrak{a}_{M,0}$,
(f) $\mathfrak{n}_{\mathfrak{p},0} = \mathfrak{n}_0 \oplus \mathfrak{n}_{M,0}$ as vector spaces,
(g) $\mathfrak{g}_0 = \mathfrak{a}_0 \oplus \mathfrak{m}_0 \oplus \mathfrak{n}_0 \oplus \theta\mathfrak{n}_0$ orthogonally with respect to θ,
(h) $\mathfrak{m}_0 = \mathfrak{m}_{\mathfrak{p},0} \oplus \mathfrak{a}_{M,0} \oplus \mathfrak{n}_{M,0} \oplus \theta\mathfrak{n}_{M,0}$.

PROOF.
(a, b, e, f) All parts of these are clear.
(c) The centralizer of $\mathfrak{a}_0$ is spanned by $\mathfrak{a}_{\mathfrak{p},0}$, $\mathfrak{m}_{\mathfrak{p},0}$, and all the restricted root spaces for restricted roots vanishing on $\mathfrak{a}_0$. The sum of these is $\mathfrak{a}_0 \oplus \mathfrak{m}_0$.
(d) Since $\theta(\mathfrak{g}_0)_\beta = (\mathfrak{g}_0)_{-\beta}$ by Proposition 6.40c, $\mathfrak{q}_0 \cap \theta\mathfrak{q}_0 = \mathfrak{a}_0 \oplus \mathfrak{m}_0$. Then $\mathfrak{a}_0 \oplus \mathfrak{m}_0$ is reductive by Corollary 6.29.
(g, h) These follow from Proposition 6.40.

Proposition 7.79. Among the parabolic subalgebras containing $\mathfrak{q}_{\mathfrak{p},0}$, let $\mathfrak{q}_0$ be the one corresponding to the subset Π' of simple restricted roots. For $\eta \neq 0$ in $\mathfrak{a}_0^*$, let

$$(\mathfrak{g}_0)_{(\eta)} = \bigoplus_{\substack{\beta \in \mathfrak{a}_{\mathfrak{p},0}^*, \\ \beta|_{\mathfrak{a}_0} = \eta}} (\mathfrak{g}_0)_\beta .$$

Then $(\mathfrak{g}_0)_{(\eta)} \subseteq \mathfrak{n}_0$ or $(\mathfrak{g}_0)_{(\eta)} \subseteq \theta\mathfrak{n}_0$.

PROOF. We have

$$\mathfrak{a}_{M,0} = \mathfrak{a}_0^\perp = \Big(\bigcap_{\beta \in \Gamma \cap -\Gamma} \ker \beta \Big)^\perp = \Big(\bigcap_{\beta \in \Gamma \cap -\Gamma} H_\beta^\perp \Big)^\perp = \sum_{\beta \in \Gamma \cap -\Gamma} \mathbb{R} H_\beta = \sum_{\beta \in \Pi'} \mathbb{R} H_\beta .$$

Let β and β' be restricted roots with a common nonzero restriction η to members of $\mathfrak{a}_0$. Then $\beta - \beta'$ is 0 on $\mathfrak{a}_0$, and $H_\beta - H_{\beta'}$ is in $\mathfrak{a}_{M,0}$. From the formula for $\mathfrak{a}_{M,0}$, the expansion of $\beta - \beta'$ in terms of simple restricted roots involves only the members of Π'. Since $\eta \neq 0$, the individual expansions of β and β' involve nonzero coefficients for at least one simple restricted root other than the ones in Π'. The coefficients for this other simple restricted root must be equal and in particular of the same sign. By Proposition 2.49, β and β' are both positive or both negative, and the result follows.

Motivated by Proposition 7.79, we define, for $\eta \in \mathfrak{a}_0^*$,

$$(\mathfrak{g}_0)_{(\eta)} = \{X \in \mathfrak{g}_0 \mid [H, X] = \eta(H)X \text{ for all } H \in \mathfrak{a}_0\}. \tag{7.80}$$

We say that η is an $\mathfrak{a}_0$ **root**, or root of $(\mathfrak{g}_0, \mathfrak{a}_0)$, if $\eta \neq 0$ and $(\mathfrak{g}_0)_{(\eta)} \neq 0$. In this case we call $(\mathfrak{g}_0)_{(\eta)}$ the corresponding $\mathfrak{a}_0$ **root space**. The proposition says that $\mathfrak{n}_0$ is the sum of $\mathfrak{a}_0$ root spaces, and so is $\theta\mathfrak{n}_0$. We call an $\mathfrak{a}_0$ root **positive** if it contributes to $\mathfrak{n}_0$, otherwise **negative**. The set of $\mathfrak{a}_0$ roots does not necessarily form an abstract root system, but the notion of an $\mathfrak{a}_0$ root is still helpful.

Corollary 7.81. The normalizer of $\mathfrak{a}_0$ in $\mathfrak{g}_0$ is $\mathfrak{a}_0 \oplus \mathfrak{m}_0$.

PROOF. The normalizer contains $\mathfrak{a}_0 \oplus \mathfrak{m}_0$ by Proposition 7.78c. In the reverse direction let X be in the normalizer, and write

$$X = H_0 + X_0 + \sum_{\substack{\eta \neq 0, \\ \eta \in \mathfrak{a}_0^*}} X_\eta \qquad \text{with } H_0 \in \mathfrak{a}_0,\ X_0 \in \mathfrak{m}_0,\ X_\eta \in (\mathfrak{g}_0)_{(\eta)}.$$

If H is in $\mathfrak{a}_0$, then $[X, H] = -\sum_\eta \eta(H)X_\eta$, and this can be in $\mathfrak{a}_0$ for all such H only if $X_\eta = 0$ for all η. Therefore $X = H_0 + X_0$ is in $\mathfrak{a}_0 \oplus \mathfrak{m}_0$.

Now let A and N be the analytic subgroups of G with Lie algebras $\mathfrak{a}_0$ and $\mathfrak{n}_0$, and define $M = {}^0Z_G(\mathfrak{a}_0)$. We shall see in Proposition 7.83 below that $Q = MAN$ is the normalizer of $\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ in G, and we define it to be the **parabolic subgroup** associated to the parabolic subalgebra $\mathfrak{q}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$. The decomposition of elements of Q according to MAN will be seen to be unique, and $Q = MAN$ is called the **Langlands decomposition** of Q. When $\mathfrak{q}_0$ is a minimal parabolic subalgebra, the corresponding Q is called a **minimal parabolic subgroup**. We write $N^- = \Theta N$.

Let A_M and N_M be the analytic subgroups of $\mathfrak{g}_0$ with Lie algebras $\mathfrak{a}_{M,0}$ and $\mathfrak{n}_{M,0}$, and let $M_M = Z_{K \cap M}(\mathfrak{a}_{M,0})$. Define $K_M = K \cap M$. Recall the subgroup F of G that is the subject of Corollary 7.52.

Proposition 7.82. The subgroups M, A, N, K_M, M_M, A_M, and N_M have the properties that

(a) $MA = Z_G(\mathfrak{a}_0)$ is reductive, $M = {}^0(MA)$ is reductive, and A is Z_{vec} for MA,
(b) M has Lie algebra $\mathfrak{m}_0$,

(c) $M_M = M_{\mathfrak{p}}$, $M_{\mathfrak{p},0}A_M N_M$ is a minimal parabolic subgroup of M, and $M = K_M A_M N_M$,
(d) $M = FM_0$ if G is connected,
(e) $A_{\mathfrak{p}} = AA_M$ as a direct product,
(f) $N_{\mathfrak{p}} = NN_M$ as a semidirect product with N normal.

PROOF.
(a, b) The subgroups $Z_G(\mathfrak{a}_0)$ and ${}^0Z_G(\mathfrak{a}_0)$ are reductive by Propositions 7.25 and 7.27. By Proposition 7.78, $Z_{\mathfrak{g}_0}(\mathfrak{a}_0) = \mathfrak{a}_0 \oplus \mathfrak{m}_0$. Thus the space Z_{vec} for the group $Z_G(\mathfrak{a}_0)$ is the analytic subgroup corresponding to the intersection of $\mathfrak{p}_0$ with the center of $\mathfrak{a}_0 \oplus \mathfrak{m}_0$. From the definition of $\mathfrak{m}_0$, the center of $Z_{\mathfrak{g}_0}(\mathfrak{a}_0)$ has to be contained in $\mathfrak{a}_{\mathfrak{p},0} \oplus \mathfrak{m}_{\mathfrak{p},0}$, and the $\mathfrak{p}_0$ part of this is $\mathfrak{a}_{\mathfrak{p},0}$. The part of $\mathfrak{a}_{\mathfrak{p},0}$ that commutes with $\mathfrak{m}_0$ is $\mathfrak{a}_0$ by definition of $\mathfrak{m}_0$. Therefore $Z_{vec} = \exp \mathfrak{a}_0 = A$, and $Z_G(\mathfrak{a}_0) = ({}^0Z_G(\mathfrak{a}_0))A$ by Proposition 7.27. Then (a) and (b) follow.

(c) By (a), M is reductive. It is clear that $\mathfrak{a}_{M,0}$ is a maximal abelian subspace of $\mathfrak{p}_0 \cap \mathfrak{m}_0$, since $\mathfrak{m}_0 \cap \mathfrak{a}_0 = 0$. The restricted roots of $\mathfrak{m}_0$ relative to $\mathfrak{a}_{M,0}$ are then the members of $\Gamma \cap -\Gamma$, and the sum of the restricted-root spaces for the positive such restricted roots is $\mathfrak{n}_{M,0}$. Therefore the minimal parabolic subgroup in question for M is $M_M A_M N_M$. The computation

$$\begin{aligned} M_M = Z_{K\cap M}(\mathfrak{a}_{M,0}) &= MA \cap Z_K(\mathfrak{a}_{M,0}) \\ &= Z_G(\mathfrak{a}_0) \cap Z_K(\mathfrak{a}_{M,0}) = Z_K(\mathfrak{a}_{\mathfrak{p},0}) = M_{\mathfrak{p}} \end{aligned}$$

identifies M_M, and $M = K_M A_M N_M$ by the Iwasawa decomposition for M (Proposition 7.31).

(d) By (a), M is reductive. Hence $M = M_M M_0$ by Proposition 7.33. But (c) shows that $M_M = M_{\mathfrak{p}}$, and Corollary 7.52 shows that $M_{\mathfrak{p}} = F(M_{\mathfrak{p}})_0$. Hence $M = FM_0$.

(e) This follows from Proposition 7.78e and the simple connectivity of $A_{\mathfrak{p}}$.

(f) This follows from Proposition 7.78f, Theorem 1.125, and the simple connectivity of $N_{\mathfrak{p}}$.

Proposition 7.83. The subgroups M, A, and N have the properties that
(a) MA normalizes N, so that $Q = MAN$ is a group,
(b) $Q = N_G(\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0)$, and hence Q is a closed subgroup,
(c) Q has Lie algebra $\mathfrak{q}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$,
(d) multiplication $M \times A \times N \to Q$ is a diffeomorphism,
(e) $N^- \cap Q = \{1\}$,
(f) $G = KQ$.

PROOF.

(a) Let z be in $MA = Z_G(\mathfrak{a}_0)$, and fix $(\mathfrak{g}_0)_{(\eta)} \subseteq \mathfrak{n}_0$ as in (7.80). If X is in $(\mathfrak{g}_0)_{(\eta)}$ and H is in $\mathfrak{a}_0$, then

$$[H, \mathrm{Ad}(z)X] = [\mathrm{Ad}(z)H, \mathrm{Ad}(z)X] = \mathrm{Ad}(z)[H, X] = \eta(H)\mathrm{Ad}(z)X.$$

Hence $\mathrm{Ad}(z)X$ is in $(\mathfrak{g}_0)_{(\eta)}$, and $\mathrm{Ad}(z)$ maps $(\mathfrak{g}_0)_{(\eta)}$ into itself. Since $\mathfrak{n}_0$ is the sum of such spaces, $\mathrm{Ad}(z)\mathfrak{n}_0 \subseteq \mathfrak{n}_0$. Therefore MA normalizes N.

(b) The subgroup MA normalizes its Lie algebra $\mathfrak{m}_0 \oplus \mathfrak{a}_0$, and it normalizes $\mathfrak{n}_0$ by (a). The subgroup N normalizes $\mathfrak{q}_0$ because it is connected with a Lie algebra that normalizes $\mathfrak{q}_0$ by Proposition 7.78a. Hence MAN normalizes $\mathfrak{q}_0$. In the reverse direction let x be in $N_G(\mathfrak{q}_0)$. We are to prove that x is in MAN. Let us write x in terms of the Iwasawa decomposition $G = KA_{\mathfrak{p}}N_{\mathfrak{p}}$. Here $A_{\mathfrak{p}} = AA_M$ by Proposition 7.82e, and A and A_M are both contained in MA. Also $N_{\mathfrak{p}} = NN_M$ by Proposition 7.82f, and N and N_M are both contained in MN. Thus we may assume that x is in $N_K(\mathfrak{q}_0)$. By (7.23), $\mathrm{Ad}(\Theta x) = \theta\mathrm{Ad}(x)\theta$, and thus $\mathrm{Ad}(\Theta x)$ normalizes $\theta\mathfrak{q}_0$. But $\Theta x = x$ since x is in K, and therefore $\mathrm{Ad}(x)$ normalizes both $\mathfrak{q}_0$ and $\theta\mathfrak{q}_0$. By Proposition 7.78d, $\mathrm{Ad}(x)$ normalizes $\mathfrak{a}_0 \oplus \mathfrak{m}_0$. Since $\mathfrak{a}_0$ is the $\mathfrak{p}_0$ part of the center of $\mathfrak{a}_0 \oplus \mathfrak{m}_0$, $\mathrm{Ad}(x)$ normalizes $\mathfrak{a}_0$ and $\mathfrak{m}_0$ individually. Let η be an $\mathfrak{a}_0$ root contributing to $\mathfrak{n}_0$. If X is in $(\mathfrak{g}_0)_\eta$ and H is in $\mathfrak{a}_0$, then

$$\begin{aligned}[H, \mathrm{Ad}(x)X] &= \mathrm{Ad}(x)[\mathrm{Ad}(x)^{-1}H, X]\\ &= \eta(\mathrm{Ad}(x)^{-1}H)\mathrm{Ad}(x)X = (\mathrm{Ad}(x)\eta)(H)\mathrm{Ad}(x)X.\end{aligned}$$

In other words, $\mathrm{Ad}(x)$ carries $(\mathfrak{g}_0)_{(\eta)}$ to $(\mathfrak{g}_0)_{(\mathrm{Ad}(x)\eta)}$. So whenever η is the restriction to $\mathfrak{a}_0$ of a positive restricted root, so is $\mathrm{Ad}(x)\eta$. Meanwhile, $\mathrm{Ad}(x)$ carries $\mathfrak{a}_{M,0}$ to a maximal abelian subspace of $\mathfrak{p}_0 \cap \mathfrak{m}_0$, and Proposition 7.29 allows us to adjust it by some $\mathrm{Ad}(k) \in \mathrm{Ad}(K \cap M)$ so that $\mathrm{Ad}(kx)\mathfrak{a}_{M,0} = \mathfrak{a}_{M,0}$. Taking Proposition 7.32 and Theorem 2.63 into account, we can choose $k' \in K \cap M$ so that $\mathrm{Ad}(k'kx)$ is the identity on $\mathfrak{a}_{M,0}$. Then $\mathrm{Ad}(k'kx)$ sends Σ^+ to itself. By Proposition 7.32 and Theorem 2.63, $\mathrm{Ad}(k'kx)$ is the identity on $\mathfrak{a}_{\mathfrak{p},0}$ and in particular on $\mathfrak{a}_0$. Hence $k'kx$ is in M, and so is x. We conclude that $MAN = N_G(\mathfrak{q}_0)$, and consequently MAN is closed.

(c) By (b), Q is closed, hence Lie. The Lie algebra of Q is $N_{\mathfrak{g}_0}(\mathfrak{q}_0)$, which certainly contains $\mathfrak{q}_0$. In the reverse direction let $X \in \mathfrak{g}_0$ normalize $\mathfrak{q}_0$. Since $\mathfrak{a}_{\mathfrak{p},0}$ and $\mathfrak{n}_{\mathfrak{p},0}$ are contained in $\mathfrak{q}_0$, the Iwasawa decomposition on the Lie algebra level allows us to assume that X is in $\mathfrak{k}_0$. Since X normalizes $\mathfrak{q}_0$, θX normalizes $\theta\mathfrak{q}_0$. But $X = \theta X$, and hence X normalizes $\mathfrak{q}_0 \cap \theta\mathfrak{q}_0$, which is $\mathfrak{a}_0 \oplus \mathfrak{m}_0$ by Proposition 7.78d. Since $\mathfrak{a}_0$ is the $\mathfrak{p}_0$ part of the center of $\mathfrak{a}_0 \oplus \mathfrak{m}_0$, X normalizes $\mathfrak{a}_0$ and $\mathfrak{m}_0$ individually. By Corollary 7.81, X is in $\mathfrak{a}_0 \oplus \mathfrak{m}_0$.

(d) Use of Lemma 6.44 twice shows that multiplication from $M \times A \times N$ into Q is regular on $M_0 \times A \times N$, and translation to M shows that it is regular everywhere. We are left with showing that it is one-one. Since $A \subseteq A_{\mathfrak{p}}$ and $N \subseteq N_{\mathfrak{p}}$, the uniqueness for the Iwasawa decomposition of G (Proposition 7.31) shows that it is enough to prove that $M \cap AN = \{1\}$. Given $m \in M$, let the Iwasawa decomposition of m according to $M = K_M A_M N_M$ be $m = k_M a_M n_M$. If this element is to be in AN, then $k_M = 1$, a_M is in $A_M \cap A$, and n_M is in $N_M \cap N$, by uniqueness of the Iwasawa decomposition in G. But $A_M \cap A = \{1\}$ and $N_M \cap N = \{1\}$ by (e) and (f) of Proposition 7.82. Therefore $m = 1$, and we conclude that $M \cap AN = \{1\}$.

(e) This is proved in the same way as Lemma 7.64, which is stated for a minimal parabolic subgroup.

(f) Since $Q \supseteq A_{\mathfrak{p}} N_{\mathfrak{p}}$, $G = KQ$ by the Iwasawa decomposition for G (Proposition 7.31).

Although the set of $\mathfrak{a}_0$ roots does not necessarily form an abstract root system, it is still meaningful to define

(7.84a) $$W(G, A) = N_K(\mathfrak{a}_0)/Z_K(\mathfrak{a}_0),$$

just as we did in the case that $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{p}_0$. Corollary 7.81 and Proposition 7.78c show that $N_K(\mathfrak{a}_0)$ and $Z_K(\mathfrak{a}_0)$ both have $\mathfrak{k}_0 \cap \mathfrak{m}_0$ as Lie algebra. Hence $W(G, A)$ is a compact 0-dimensional group, and we conclude that $W(G, A)$ is finite. An alternative formula for $W(G, A)$ is

(7.84b) $$W(G, A) = N_G(\mathfrak{a}_0)/Z_G(\mathfrak{a}_0).$$

The equality of the right sides of (7.84a) and (7.84b) is an immediate consequence of Lemma 7.22 and Corollary 7.81. To compute $N_K(\mathfrak{a}_0)$, it is sometimes handy to use the following proposition.

Proposition 7.85. Every element of $N_K(\mathfrak{a}_0)$ decomposes as a product zn, where n is in $N_K(\mathfrak{a}_{\mathfrak{p},0})$ and z is in $Z_K(\mathfrak{a}_0)$.

PROOF. Let k be in $N_K(\mathfrak{a}_0)$ and form $\mathrm{Ad}(k)\mathfrak{a}_{M,0}$. Since $\mathfrak{a}_{M,0}$ commutes with $\mathfrak{a}_0$, $\mathrm{Ad}(k)\mathfrak{a}_{M,0}$ commutes with $\mathrm{Ad}(k)\mathfrak{a}_0 = \mathfrak{a}_0$. By Proposition 7.78c, $\mathrm{Ad}(k)\mathfrak{a}_{M,0}$ is contained in $\mathfrak{a}_0 \oplus \mathfrak{m}_0$. Since $\mathfrak{a}_{M,0}$ is orthogonal to $\mathfrak{a}_0$ under B_θ, $\mathrm{Ad}(k)\mathfrak{a}_{M,0}$ is orthogonal to $\mathrm{Ad}(k)\mathfrak{a}_0 = \mathfrak{a}_0$. Hence $\mathrm{Ad}(k)\mathfrak{a}_{M,0}$ is contained in $\mathfrak{m}_0$ and therefore in $\mathfrak{p}_0 \cap \mathfrak{m}_0$. By Proposition 7.29 there exists z in $K \cap M$ with $\mathrm{Ad}(z)^{-1}\mathrm{Ad}(k)\mathfrak{a}_{M,0} = \mathfrak{a}_{M,0}$. Then $n = z^{-1}k$ is in $N_K(\mathfrak{a}_0)$ and in $N_K(\mathfrak{a}_{M,0})$, hence in $N_K(\mathfrak{a}_{\mathfrak{p},0})$.

EXAMPLE. Let $G = SL(3, \mathbb{R})$. Take $\mathfrak{a}_{\mathfrak{p},0}$ to be the diagonal subalgebra, and let $\Sigma^+ = \{f_1 - f_2,\ f_2 - f_3,\ f_1 - f_3\}$ in the notation of Example 1 of §VI.4. Define a parabolic subalgebra $\mathfrak{q}_0$ by using $\Pi' = \{f_1 - f_2\}$. The corresponding parabolic subgroup is the block upper-triangular group with blocks of sizes 2 and 1, respectively. The subalgebra $\mathfrak{a}_0$ equals $\{\text{diag}(r, r, -2r)\}$. Suppose that w is in $W(G, A)$. Proposition 7.85 says that w extends to a member of $W(G, A_\mathfrak{p})$ leaving $\mathfrak{a}_0$ and $\mathfrak{a}_{M,0}$ individually stable. Here $W(G, A_\mathfrak{p}) = W(\Sigma)$, and the only member of $W(\Sigma)$ sending $\mathfrak{a}_0$ to itself is the identity. So $W(G, A) = \{1\}$.

The members of $W(G, A)$ act on set of the $\mathfrak{a}_0$ roots, and we have the following substitute for Theorem 2.63.

Proposition 7.86. The only member of $W(G, A)$ that leaves stable the set of positive $\mathfrak{a}_0$ roots is the identity.

PROOF. Let k be in $N_K(\mathfrak{a}_0)$. By assumption $\text{Ad}(k)\mathfrak{n}_0 = \mathfrak{n}_0$. The centralizer of $\mathfrak{a}_0$ in $\mathfrak{g}_0$ is $\mathfrak{a}_0 \oplus \mathfrak{m}_0$ by Proposition 7.78c. If X is in this centralizer and if H is arbitrary in $\mathfrak{a}_0$, then

$$[H, \text{Ad}(k)X] = \text{Ad}(k)[\text{Ad}(k)^{-1}H, X] = 0$$

shows that $\text{Ad}(k)X$ is in the centralizer. Hence $\text{Ad}(k)(\mathfrak{a}_0 \oplus \mathfrak{m}_0) = \mathfrak{a}_0 \oplus \mathfrak{m}_0$. By Proposition 7.83b, k is in MAN. By Proposition 7.82c and the uniqueness of the Iwasawa decomposition for G, k is in M. Therefore k is in $Z_K(\mathfrak{a}_0)$.

A parabolic subalgebra $\mathfrak{q}_0$ of $\mathfrak{g}_0$ and the corresponding parabolic subgroup $Q = MAN$ of G are said to be **cuspidal** if $\mathfrak{m}_0$ has a θ stable compact Cartan subalgebra, say $\mathfrak{t}_0$. In this case, $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ is a θ stable Cartan subalgebra of $\mathfrak{g}_0$. The restriction of a root in $\Delta(\mathfrak{g}, \mathfrak{h})$ to $\mathfrak{a}_0$ is an $\mathfrak{a}_0$ root if it is not 0, and we can identify $\Delta(\mathfrak{m}, \mathfrak{t})$ with the set of roots in $\Delta(\mathfrak{g}, \mathfrak{h})$ that vanish on $\mathfrak{a}$. Let us choose a positive system $\Delta^+(\mathfrak{m}, \mathfrak{t})$ for $\mathfrak{m}$ and extend it to a positive system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ by saying that a root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ with nonzero restriction to $\mathfrak{a}_0$ is positive if $\alpha|_{\mathfrak{a}_0}$ is a positive $\mathfrak{a}_0$ root. Let us decompose members α of $\mathfrak{h}^*$ according to their projections on $\mathfrak{a}^*$ and $\mathfrak{t}^*$ as $\alpha = \alpha_\mathfrak{a} + \alpha_\mathfrak{t}$. Now $\theta\alpha = -\alpha_\mathfrak{a} + \alpha_\mathfrak{t}$, and θ carries roots to roots. Hence if $\alpha_\mathfrak{a} + \alpha_\mathfrak{t}$ is a root, so is $\alpha_\mathfrak{a} - \alpha_\mathfrak{t}$.

The positive system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ just defined is given by a lexicographic ordering that takes $\mathfrak{a}_0$ before $i\mathfrak{t}_0$. In fact, write the half sum of positive roots as $\delta = \delta_\mathfrak{a} + \delta_\mathfrak{t}$. The claim is that positivity is determined by inner products with the ordered set $\{\delta_{\mathfrak{a}}, \delta_{\mathfrak{t}}\}$ and that $\delta_{\mathfrak{t}}$ is equal to the half sum of the members of $\Delta^+(\mathfrak{m}, \mathfrak{t})$. To see this, let $\alpha = \alpha_{\mathfrak{a}} + \alpha_{\mathfrak{t}}$ be in $\Delta^+(\mathfrak{g}, \mathfrak{h})$. If $\alpha_{\mathfrak{a}} \neq 0$, then $\alpha_{\mathfrak{a}} - \alpha_{\mathfrak{t}}$ is in $\Delta^+(\mathfrak{g}, \mathfrak{h})$, and

$$\langle \alpha, \delta_{\mathfrak{a}} \rangle = \langle \alpha_{\mathfrak{a}}, \delta_{\mathfrak{a}} \rangle = \langle \alpha_{\mathfrak{a}}, \delta \rangle = \tfrac{1}{2}\langle \alpha_{\mathfrak{a}} + \alpha_{\mathfrak{t}}, \delta \rangle + \tfrac{1}{2}\langle \alpha_{\mathfrak{a}} - \alpha_{\mathfrak{t}}, \delta \rangle > 0.$$

Since the positive roots with nonzero restriction to $\mathfrak{a}$ cancel in pairs when added, we see that $\delta_{\mathfrak{t}}$ equals half the sum of the members of $\Delta^+(\mathfrak{m}, \mathfrak{t})$. Finally if $\alpha_{\mathfrak{a}} = 0$, then $\langle \alpha, \delta_{\mathfrak{a}} \rangle = 0$ and $\langle \alpha, \delta_{\mathfrak{t}} \rangle > 0$. Hence $\Delta^+(\mathfrak{g}, \mathfrak{h})$ is indeed given by a lexicographic ordering of the type described.

The next proposition gives a converse that tells a useful way to construct cuspidal parabolic subalgebras of $\mathfrak{g}_0$ directly.

Proposition 7.87. Let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be the decomposition of a θ stable Cartan subalgebra according to θ, and suppose that a lexicographic ordering taking $\mathfrak{a}_0$ before $i\mathfrak{t}_0$ is used to define a positive system $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Define

$$\mathfrak{m}_0 = \mathfrak{g}_0 \cap \Big(\mathfrak{t} \oplus \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}), \\ \alpha|_{\mathfrak{a}} = 0}} \mathfrak{g}_\alpha\Big)$$

and

$$\mathfrak{n}_0 = \mathfrak{g}_0 \cap \Big(\bigoplus_{\substack{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}), \\ \alpha|_{\mathfrak{a}} \neq 0}} \mathfrak{g}_\alpha\Big).$$

Then $\mathfrak{q}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ is the Langlands decomposition of a cuspidal parabolic subalgebra of $\mathfrak{g}_0$.

PROOF. In view of the definitions, we have to relate $\mathfrak{q}_0$ to a minimal parabolic subalgebra. Let bar denote conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. If $\alpha = \alpha_{\mathfrak{a}} + \alpha_{\mathfrak{t}}$ is a root, let $\bar{\alpha} = -\theta\alpha = \alpha_{\mathfrak{a}} - \alpha_{\mathfrak{t}}$. Then $\overline{\mathfrak{g}_\alpha} = \mathfrak{g}_{\bar{\alpha}}$, and it follows that

$$\mathfrak{m} = \mathfrak{t} \oplus \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}), \\ \alpha|_{\mathfrak{a}} = 0}} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{n} = \bigoplus_{\substack{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}), \\ \alpha|_{\mathfrak{a}} \neq 0}} \mathfrak{g}_\alpha. \tag{7.88}$$

In particular, $\mathfrak{m}_0$ is θ stable, hence reductive. Let $\mathfrak{h}_{M,0} = \mathfrak{t}_{M,0} \oplus \mathfrak{a}_{M,0}$ be the decomposition of a maximally noncompact θ stable Cartan subalgebra of $\mathfrak{m}_0$ according to θ. Since Theorem 2.15 shows that $\mathfrak{h}_M$ is conjugate to $\mathfrak{t}$ via $\operatorname{Int}\mathfrak{m}$, $\mathfrak{h}' = \mathfrak{a} \oplus \mathfrak{h}_M$ is conjugate to $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ via a member of $\operatorname{Int}\mathfrak{g}$ that fixes $\mathfrak{a}_0$. In particular, $\mathfrak{h}'_0 = \mathfrak{a}_0 \oplus \mathfrak{h}_{M,0}$ is a Cartan subalgebra of $\mathfrak{g}_0$. Applying our constructed member of $\operatorname{Int}\mathfrak{g}$ to (7.88), we obtain

$$\mathfrak{m} = \mathfrak{h}_M \oplus \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}'), \\ \alpha|_{\mathfrak{a}} = 0}} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{n} = \bigoplus_{\substack{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}'), \\ \alpha|_{\mathfrak{a}} \neq 0}} \mathfrak{g}_\alpha \tag{7.89}$$

for the positive system $\Delta^+(\mathfrak{g}, \mathfrak{h}')$ obtained by transferring positivity from $\Delta^+(\mathfrak{g}, \mathfrak{h})$.

Let us observe that $\mathfrak{a}_{\mathfrak{p},0} = \mathfrak{a}_0 \oplus \mathfrak{a}_{M,0}$ is a maximal abelian subspace of $\mathfrak{p}_0$. In fact, the centralizer of $\mathfrak{a}_0$ in $\mathfrak{g}_0$ is $\mathfrak{a}_0 \oplus \mathfrak{m}_0$, and $\mathfrak{a}_{M,0}$ is maximal abelian in $\mathfrak{m}_0 \cap \mathfrak{p}_0$; hence the assertion follows. We introduce a lexicographic ordering for $\mathfrak{h}'_0$ that is as before on $\mathfrak{a}_0$, takes $\mathfrak{a}_0$ before $\mathfrak{a}_{M,0}$, and takes $\mathfrak{a}_{M,0}$ before $i\mathfrak{t}_{M,0}$. Then we obtain a positive system $\Delta^{+\prime}(\mathfrak{g}, \mathfrak{h}')$ with the property that a root α with $\alpha|_{\mathfrak{a}_0} \neq 0$ is positive if and only if $\alpha|_{\mathfrak{a}_0}$ is the restriction to $\mathfrak{a}_0$ of a member of $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Consequently we can replace $\Delta^+(\mathfrak{g}, \mathfrak{h}')$ in (7.89) by $\Delta^{+\prime}(\mathfrak{g}, \mathfrak{h}')$. Then it is apparent that $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ contains $\mathfrak{m}_{\mathfrak{p}} \oplus \mathfrak{a}_{\mathfrak{p}} \oplus \mathfrak{n}_{\mathfrak{p}}$ defined relative to the positive restricted roots obtained from $\Delta^{+\prime}(\mathfrak{g}, \mathfrak{h}')$, and hence $\mathfrak{q}_0$ is a parabolic subalgebra. Referring to (7.77), we see that $\mathfrak{q}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ is the Langlands decomposition. Finally $\mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{m}_0$ by Proposition 2.13, and hence $\mathfrak{q}_0$ is cuspidal.

8. Cartan Subgroups

We continue to assume that G is a reductive Lie group and to use the notation of §2 concerning the Cartan decomposition. A **Cartan subgroup** of G is the centralizer in G of a Cartan subalgebra. We know from §§VI.6 and VII.2 that any Cartan subalgebra is conjugate via $\operatorname{Int}\mathfrak{g}_0$ to a θ stable Cartan subalgebra and that there are only finitely many conjugacy classes of Cartan subalgebras. Consequently any Cartan subgroup of G is conjugate via G to a Θ stable Cartan subgroup, and there are only finitely many conjugacy classes of Cartan subgroups. A Θ stable Cartan subgroup is a reductive Lie group by Proposition 7.25.

When G is compact connected and T is a maximal torus, every element of G is conjugate to a member of T, according to Theorem 4.36. In particular every member of G lies in a Cartan subgroup. This statement does not extend to noncompact groups, as the following example shows.

EXAMPLE. Let $G = SL(2,\mathbb{R})$. We saw in §VI.6 that every Cartan subalgebra is conjugate to one of

$$\left\{\begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}\right\} \qquad \text{and} \qquad \left\{\begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix}\right\},$$

and the corresponding Cartan subgroups are

$$\left\{\pm\begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix}\right\} \qquad \text{and} \qquad \left\{\begin{pmatrix} \cos r & \sin r \\ -\sin r & \cos r \end{pmatrix}\right\}.$$

Some features of these subgroups are worth noting. The first Cartan subgroup is disconnected; disconnectedness is common among Cartan subgroups for general G. Also every member of either Cartan subgroup is diagonable over $\mathbb{C}$. Hence $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ lies in no Cartan subgroup.

Although the union of the Cartan subgroups of G need not exhaust G, it turns out that the union exhausts almost all of G. This fact is the most important conclusion about Cartan subgroups to be derived in this section and appears below as Theorem 7.108. When we treat integration in Chapter VIII, this fact will permit integration of functions on G by integrating over the conjugates of a finite set of Cartan subgroups; the resulting formula, known as the "Weyl Integration Formula," is an important tool for harmonic analysis on G.

Before coming to this main result, we give a proposition about the component structure of Cartan subgroups and we introduce a finite group $W(G, H)$ for each Cartan subgroup analogous to the groups $W(G, A)$ considered in §7.

Proposition 7.90. Let H be a Cartan subgroup of G.

(a) If H is maximally noncompact, then H meets every component of G.

(b) If H is maximally compact and if G is connected, then H is connected.

REMARKS. The modifiers "maximally noncompact" and "maximally compact" are to be interpreted in terms of the Lie algebras. If $\mathfrak{h}_0$ is a Cartan subalgebra, $\mathfrak{h}_0$ is conjugate to a θ stable Cartan subalgebra $\mathfrak{h}_0'$, and we defined "maximally noncompact" and "maximally compact" for $\mathfrak{h}_0'$ in §§VI.6 and VII.2. Proposition 7.35 says that any two candidates for $\mathfrak{h}_0'$ are conjugate via K, and hence it is meaningful to say that $\mathfrak{h}_0$ is maximally noncompact or maximally compact if $\mathfrak{h}_0'$ is.

PROOF. Let $\mathfrak{h}_0$ be the Lie algebra of H. We may assume that $\mathfrak{h}_0$ is θ stable. Let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be the decomposition of $\mathfrak{h}_0$ into $+1$ and -1 eigenspaces under θ.

(a) If $\mathfrak{h}_0$ is maximally noncompact, then $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{p}_0$. The group H contains the subgroup F introduced before Corollary 7.52, and Corollary 7.52 and Proposition 7.33 show that F meets every component of G.

(b) If $\mathfrak{h}_0$ is maximally compact, then $\mathfrak{t}_0$ is a maximal abelian subspace of $\mathfrak{k}_0$. Since K is connected, the subgroup $Z_K(\mathfrak{t}_0)$ is connected by Corollary

4.51, and $Z_K(\mathfrak{t}_0)\exp\mathfrak{a}_0$ is therefore a connected closed subgroup of G with Lie algebra $\mathfrak{h}_0$. On the other hand, Proposition 7.25 implies that

$$H = Z_K(\mathfrak{h}_0)\exp\mathfrak{a}_0 \subseteq Z_K(\mathfrak{t}_0)\exp\mathfrak{a}_0.$$

Since H and $Z_K(\mathfrak{t}_0)\exp\mathfrak{a}_0$ are closed subgroups with the same Lie algebra and since $Z_K(\mathfrak{t}_0)\exp\mathfrak{a}_0$ is connected, it follows that $H = Z_K(\mathfrak{t}_0)\exp\mathfrak{a}_0$.

Corollary 7.91. If a maximally noncompact Cartan subgroup H of G is abelian, then $Z_{G_0} \subseteq Z_G$.

PROOF. By Proposition 7.90a, $G = G_0 H$. If z is in Z_{G_0}, then $\mathrm{Ad}(z) = 1$ on $\mathfrak{h}_0$, and hence z is in $Z_G(\mathfrak{h}_0) = H$. Let $g \in G$ be given, and write $g = g_0 h$ with $g \in G_0$ and $h \in H$. Then $zg_0 = g_0 z$ since z commutes with members of G_0, and $zh = hz$ since z is in H and H is abelian. Hence $zg = gz$, and z is in Z_G.

If H is a Cartan subgroup of G with Lie algebra $\mathfrak{h}_0$, we define

$$W(G, H) = N_G(\mathfrak{h}_0)/Z_G(\mathfrak{h}_0). \tag{7.92a}$$

Here $Z_G(\mathfrak{h}_0)$ is nothing more than H itself, by definition. When $\mathfrak{h}_0$ is θ stable, an alternative formula for $W(G, H)$ is

$$W(G, H) = N_K(\mathfrak{h}_0)/Z_K(\mathfrak{h}_0). \tag{7.92b}$$

The equality of the right sides of (7.92a) and (7.92b) is an immediate consequence of Lemma 7.22 and Proposition 2.7. Proposition 2.7 shows that $N_K(\mathfrak{h}_0)$ and $Z_K(\mathfrak{h}_0)$ both have $\mathfrak{k}_0 \cap \mathfrak{h}_0 = \mathfrak{t}_0$ as Lie algebra. Hence $W(G, H)$ is a compact 0-dimensional group, and we conclude that $W(G, H)$ is finite.

Each member of $N_G(\mathfrak{h}_0)$ sends roots of $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ to roots, and the action of $N_G(\mathfrak{h}_0)$ on Δ descends to $W(G, H)$. It is clear that only the identity in $W(G, H)$ acts as the identity on Δ. Since $\mathrm{Ad}_{\mathfrak{g}}(G) \subseteq \mathrm{Int}\,\mathfrak{g}$, it follows from Theorem 7.8 that

$$W(G, H) \subseteq W(\Delta(\mathfrak{g}, \mathfrak{h})). \tag{7.93}$$

EXAMPLE. Let $G = SL(2, \mathbb{R})$. For any $\mathfrak{h}$, $W(\mathfrak{g}, \mathfrak{h})$ has order 2. When $\mathfrak{h}_0 = \left\{\begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}\right\}$, $W(G, H)$ has order 2, a representative of the nontrivial coset being $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. When $\mathfrak{h}_0 = \left\{\begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix}\right\}$, $W(G, H)$ has order 1.

Now we begin to work toward the main result of this section, that the union of all Cartan subgroups of G exhausts almost all of G. We shall use the notion of a "regular element" of G. Recall that in Chapter II we introduced regular elements in the complexified Lie algebra $\mathfrak{g}$. Let $\dim\mathfrak{g} = n$. For $X \in \mathfrak{g}$, we formed the characteristic polynomial

$$\det(\lambda 1 - \operatorname{ad} X) = \lambda^n + \sum_{j=0}^{n-1} d_j(X)\lambda^j. \tag{7.94}$$

Here each d_j is a holomorphic polynomial function on $\mathfrak{g}$. The **rank** of $\mathfrak{g}$ is the minimum index l such that $d_l(X) \not\equiv 0$, and the **regular elements** of $\mathfrak{g}$ are those elements X such that $d_l(X) \neq 0$. For such an X, Theorem 2.9′ shows that the generalized eigenspace of $\operatorname{ad} X$ for eigenvalue 0 is a Cartan subalgebra of $\mathfrak{g}$. Because $\mathfrak{g}$ is reductive, the Cartan subalgebra acts completely reducibly on $\mathfrak{g}$, and hence the generalized eigenspace of $\operatorname{ad} X$ for eigenvalue 0 is nothing more than the centralizer of X in $\mathfrak{g}$.

Within $\mathfrak{g}$, let $\mathfrak{h}$ be a Cartan subalgebra, and let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$. For $X \in \mathfrak{h}$, $d_l(X) = \prod_{\alpha\in\Delta}\alpha(X)$, so that $X \in \mathfrak{h}$ is regular if and only if no root vanishes on X. If $\mathfrak{h}_0$ is a Cartan subalgebra of our real form $\mathfrak{g}_0$, then we can find $X \in \mathfrak{h}_0$ so that $\alpha(X) \neq 0$ for all $\alpha \in \Delta$.

On the level of Lie algebras, we have concentrated on eigenvalue 0 for $\operatorname{ad} X$. On the level of reductive Lie groups, the analogous procedure is to concentrate on eigenvalue 1 for $\operatorname{Ad}(x)$. Thus for $x \in G$, we define

$$D(x, \lambda) = \det((\lambda + 1)1 - \operatorname{Ad}(x)) = \lambda^n + \sum_{j=0}^{n-1} D_j(x)\lambda^j.$$

Here each $D_j(x)$ is real analytic on G and descends to a real analytic function on $\operatorname{Ad}(G)$. But $\operatorname{Ad}(G) \subseteq \operatorname{Int}\mathfrak{g}$ by property (v) for reductive Lie groups, and the formula for $D_j(x)$ extends to be valid on $\operatorname{Int}\mathfrak{g}$ and to define a holomorphic function on $\operatorname{Int}\mathfrak{g}$. Let l' be the minimum index such that $D_{l'}(x) \not\equiv 0$ (on G or equivalently on $\operatorname{Int}\mathfrak{g}$). We shall observe shortly that $l' = l$. With this understanding the **regular elements** of G are those elements x such that $D_l(x) \neq 0$. Elements that are not regular are **singular**. The set of regular elements is denoted G'. The function D satisfies

$$D(yxy^{-1}, \lambda) = D(x, \lambda), \tag{7.95}$$

and it follows that G' is stable under group conjugation. It is almost but not quite true that the centralizer of a regular element of G is a Cartan subgroup. Here is an example of how close things get in a complex group.

EXAMPLE. Let $G = SL(2,\mathbb{C})/\{\pm 1\}$. We work with elements of G as 2-by-2 matrices identified when they differ only by a sign. The element $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$, with $z \neq 0$, is regular if $z \neq \pm 1$. For most values of z other than ± 1, the centralizer of $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ is the diagonal subgroup, which is a Cartan subgroup. But for $z = \pm i$, the centralizer is generated by the diagonal subgroup and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; thus the Cartan subgroup has index 2 in the centralizer.

Now, as promised, we prove that $l = l'$, i.e., the minimum index l such that $d_l(X) \not\equiv 0$ equals the minimum index l' such that $D_{l'}(x) \not\equiv 0$. Let ad X have generalized eigenvalue 0 exactly l times. For sufficiently small r, ad X has all eigenvalues $< 2\pi$ in absolute value, and it follows for such X that $\operatorname{Ad}(\exp X)$ has generalized eigenvalue 1 exactly l times. Thus $l' \leq l$. In the reverse direction suppose $D_{l'}(x) \not\equiv 0$. Since $D_{l'}$ extends holomorphically to the connected complex group $\operatorname{Int}\mathfrak{g}$, $D_{l'}$ cannot be identically 0 in any neighborhood of the identity in $\operatorname{Int}\mathfrak{g}$. Hence $D_{l'}(x)$ cannot be identically 0 in any neighborhood of $x = 1$ in G. Choose a neighborhood U of X's in $\mathfrak{g}_0$ about 0 such that all ad X have all eigenvalues $< 2\pi$ in absolute value and such that exp is a diffeomorphism onto a neighborhood of 1 in G. Under these conditions the multiplicity of 0 as a generalized eigenvalue for ad X equals the multiplicity of 1 as a generalized eigenvalue for $\operatorname{Ad}(\exp X)$. Thus if $D_{l'}(x)$ is somewhere nonzero on $\exp U$, then $d_l(X)$ is somewhere nonzero on U. Thus $l \leq l'$, and we conclude that $l = l'$.

To understand the relationship between regular elements and Cartan subgroups, we shall first study the case of a complex group (which in practice will usually be $\operatorname{Int}\mathfrak{g}$). The result in this case is Theorem 7.101 below. We establish notation for this theorem after proving three lemmas.

Lemma 7.96. Let Z be a connected complex manifold, and let $f : Z \to \mathbb{C}^n$ be a holomorphic function not identically 0. Then the subset of Z where f is not 0 is connected.

PROOF. Lemma 2.14 proves this result for the case that $Z = \mathbb{C}^m$ and f is a polynomial. But the same proof works if Z is a bounded polydisc $\Pi_{j=1}^m \{|z_j| < r_j\}$ and f is a holomorphic function on a neighborhood of the closure of the polydisc. We shall piece together local results of this kind to handle general Z.

Thus let the manifold structure of Z be specified by compatible charts $(V_\alpha, \varphi_\alpha)$ with $\varphi_\alpha : V_\alpha \to \mathbb{C}^m$ holomorphic onto a bounded polydisc. There is no loss of generality in assuming that there are open subsets U_α covering Z such that $\varphi_\alpha(U_\alpha)$ is an open polydisc whose closure is contained in $\varphi_\alpha(V_\alpha)$. For any subset S of Z, let S' denote the subset of S where f is not 0. The result of the previous paragraph implies that U'_α is connected for each α, and we are to prove that Z' is connected. Also U'_α is dense in U_α since the subset of a connected open set where a nonzero holomorphic function takes on nonzero values is dense.

Fix $U = U_0$. To each point $z \in Z$, we can find a chain of U_α's of the form $U = U_0, U_1, \dots, U_k$ such that z is in U_k and $U_{i-1} \cap U_i \neq \emptyset$ for $1 \leq i \leq k$. In fact, the set of z's for which this assertion is true is nonempty open closed and hence is all of Z.

Now let $z \in Z'$ be given, and form the chain $U = U_0, U_1, \dots, U_k$. Here z is in U'_k. We readily see by induction on $m \leq k$ that $U'_0 \cup \cdots \cup U'_m$ is connected, hence that $U'_0 \cup \cdots \cup U'_k$ is connected. Thus each $z \in Z'$ lies in a connected open set containing U'_0, and it follows that the union of these connected open sets is connected. The union is Z', and hence Z' is connected.

Lemma 7.97. Let N be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}_0$, and let $\mathfrak{n}'_0$ be an ideal in $\mathfrak{n}_0$. If X is in $\mathfrak{n}_0$ and Y is in $\mathfrak{n}'_0$, then $\exp(X+Y) = \exp X \, \exp Y'$ for some Y' in $\mathfrak{n}'_0$.

PROOF. If N' is the analytic subgroup corresponding to $\mathfrak{n}'_0$, then N' is certainly normal, and N' is closed as a consequence of Theorem 1.127. Let $\varphi : N \to N/N'$ be the quotient homomorphism, and let $d\varphi$ be its differential. Since $d\varphi(Y) = 0$, we have

$$\begin{aligned}\varphi((\exp(X+Y))(\exp X)^{-1}) &= \varphi(\exp(X+Y))\varphi(\exp X)^{-1} \\ &= \exp(d\varphi(X) + d\varphi(Y))(\exp d\varphi(X))^{-1} \\ &= \exp(d\varphi(X))(\exp d\varphi(X))^{-1} = 1.\end{aligned}$$

Therefore $(\exp(X+Y))(\exp X)^{-1}$ is in N', and Theorem 1.127 shows that it is of the form $\exp Y'$ for some $Y' \in \mathfrak{n}'_0$.

Lemma 7.98. Let $G = KAN$ be an Iwasawa decomposition of the reductive group G, let $M = Z_K(A)$, and let $\mathfrak{n}_0$ be the Lie algebra of N. If $h \in MA$ has the property that $\operatorname{Ad}(h)$ acts as a scalar on each restricted-root space and $\operatorname{Ad}(h)^{-1} - 1$ is nonsingular on $\mathfrak{n}_0$, then the map $\varphi : N \to N$ given by $\varphi(n) = h^{-1}nhn^{-1}$ is onto N.

REMARK. This lemma may be regarded as a Lie group version of the Lie algebra result given as Lemma 7.42.

PROOF. Write $\mathfrak{n}_0 = \bigoplus(\mathfrak{g}_0)_\lambda$ as a sum of restricted-root spaces, and regard the restricted roots as ordered lexicographically. For any restricted root α, the subspace $\mathfrak{n}_\alpha = \bigoplus_{\lambda\geq\alpha}(\mathfrak{g}_0)_\lambda$ is an ideal, and we prove by induction downward on α that φ carries $\exp\mathfrak{n}_\alpha$ onto itself. This conclusion when α is equal to the smallest positive restricted root gives the lemma since exp carries $\mathfrak{n}_0$ onto N (Theorem 1.127).

If α is given, we can write $\mathfrak{n}_\alpha = (\mathfrak{g}_0)_\alpha \oplus \mathfrak{n}_\beta$ with $\beta > \alpha$. Let X be given in $\mathfrak{n}_\alpha$, and write X as $X_1 + X_2$ with $X_1 \in (\mathfrak{g}_0)_\alpha$ and $X_2 \in \mathfrak{n}_\beta$. Since $\operatorname{Ad}(h)^{-1} - 1$ is nonsingular on $(\mathfrak{g}_0)_\alpha$, we can choose $Y_1 \in (\mathfrak{g}_0)_\alpha$ with $X_1 = (\operatorname{Ad}(h)^{-1} - 1)Y_1$. Put $n_1 = \exp Y_1$. Since $\operatorname{Ad}(h)^{-1}Y_1$ is a multiple of Y_1, $\operatorname{Ad}(h)^{-1}Y_1$ commutes with Y_1. Therefore

$$(7.99)\quad h^{-1}n_1hn_1^{-1} = (\exp\operatorname{Ad}(h)^{-1}Y_1)(\exp Y_1)^{-1} = \exp((\operatorname{Ad}(h)^{-1} - 1)Y_1) = \exp X_1.$$

Thus

$$\begin{aligned}\exp X &= \exp(X_1 + X_2)\\ &= \exp X_1\ \exp X_2' && \text{by Lemma 7.97}\\ &= h^{-1}n_1hn_1^{-1}\exp X_2' && \text{by (7.99)}\\ &= h^{-1}n_1h\exp X_2''\,n_1^{-1} && \text{with } X_2'' \in \mathfrak{n}_\beta.\end{aligned}$$

By induction $\exp X_2'' = h^{-1}n_2hn_2^{-1}$. Hence $\exp X = h^{-1}(n_1n_2)h(n_1n_2)^{-1}$, and the induction is complete.

Now we are ready for the main result about Cartan subgroups in the complex case. Let G_c be a complex semisimple Lie group (which will usually be $\operatorname{Int}\mathfrak{g}$ when we return to our reductive Lie group G). Proposition 7.5 shows that G_c is a reductive Lie group. Let $G_c = UAN$ be an Iwasawa decomposition of G_c, and let $M = Z_U(A)$. We denote by $\mathfrak{g}$, $\mathfrak{u}_0$, $\mathfrak{a}_0$, $\mathfrak{n}_0$, and $\mathfrak{m}_0$ the respective Lie algebras. Here $\mathfrak{m}_0 = i\mathfrak{a}_0$, $\mathfrak{m}_0$ is maximal abelian in $\mathfrak{u}_0$, and $\mathfrak{h} = \mathfrak{a}_0 \oplus \mathfrak{m}_0$ is a Cartan subalgebra of $\mathfrak{g}$. The corresponding Cartan subgroup of G_c is of the form $H_c = MA$ since Proposition 7.25 shows that H_c is a reductive Lie group. Since

$$M = Z_U(\mathfrak{a}_0) = Z_U(i\mathfrak{a}_0) = Z_U(\mathfrak{m}_0),$$

Corollary 4.52 shows that M is connected. Therefore

$$(7.100)\qquad H_c \text{ is connected.}$$

Let G_c' denote the regular set in G_c.

Theorem 7.101. For the complex semisimple Lie group G_c, the regular set G'_c is connected and satisfies $G'_c \subseteq \bigcup_{x \in G_c} xH_cx^{-1}$. If X_0 is any regular element in $\mathfrak{h}$, then $Z_{G_c}(X_0) = H_c$.

PROOF. We may regard $D_l(x)$ as a holomorphic function on G_c. The regular set G'_c is the set where $D_l(x) \neq 0$, and Lemma 7.96 shows that G'_c is connected.

Let $H'_c = H_c \cap G'_c$, and define $V' = \bigcup_{x \in G_c} xH'_cx^{-1}$. Then $V' \subseteq G'_c$ by (7.95). If $X_0 \in \mathfrak{h}$ is chosen so that no root in $\Delta(\mathfrak{g}, \mathfrak{h})$ vanishes on X_0, then we have seen that $\exp rX_0$ is in H'_c for all sufficiently small $r > 0$. Hence V' is nonempty. We shall prove that V' is open and closed in G'_c, and then it follows that $G'_c = V'$, hence that $G'_c \subseteq \bigcup_{x \in G_c} xH_cx^{-1}$.

To prove that V' is closed in G'_c, we observe that H_cN is closed in G_c, being the minimal parabolic subgroup MAN. Since U is compact, it follows that

$$V = \bigcup_{u \in U} uH_cNu^{-1}$$

is closed in G_c. By (7.95),

$$V \cap G'_c = \bigcup_{u \in U} u(H_cN)'u^{-1},$$

where $(H_cN)' = H_cN \cap G'_c$. If h is in H_c and n is in N, then $\operatorname{Ad}(hn)$ has the same generalized eigenvalues as $\operatorname{Ad}(h)$. Hence $(H_cN)' = H'_cN$. If h is in H'_c, then $\operatorname{Ad}(h)$ is scalar on each restricted root space contributing to $\mathfrak{n}_0$, and $\operatorname{Ad}(h) - 1$ is nonsingular on $\mathfrak{n}_0$. By Lemma 7.98 such an h has the property that $n \mapsto h^{-1}nhn^{-1}$ carries N onto N. Let $n_0 \in N$ be given, and write $n_0 = h^{-1}nhn^{-1}$. Then $hn_0 = nhn^{-1}$, and we see that every element of hN is an N conjugate of h. Since every N conjugate of h is certainly in hN, we obtain

$$H'_cN = \bigcup_{n \in N} nH'_cn^{-1}.$$

Therefore

$$V \cap G'_c = \bigcup_{u \in U} \bigcup_{n \in N} (un)H'_c(un)^{-1}.$$

Since $aH'_ca^{-1} = H'_c$ for $a \in A$ and since $G_c = UAN = UNA$, we obtain $V \cap G'_c = V'$. Thus V' is exhibited as the intersection of G'_c with the closed set V, and V' is therefore closed in G'_c.

To prove that V' is open in G'_c, it is enough to prove that the map $\psi : G_c \times H_c \to G_c$ given by $\psi(y, x) = yxy^{-1}$ has differential mapping onto at every point of $G_c \times H'_c$. The argument imitates part of the proof of Theorem 4.36. Let us abbreviate yxy^{-1} as x^y. Fix $y \in G_c$ and $x \in H'_c$. We identify the tangent spaces at y, x, and x^y with $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{g}$ by left translation. First let Y be in $\mathfrak{g}$. To compute $(d\psi)_{(y,x)}(Y, 0)$, we observe from (1.88) that

$$x^{y \exp rY} = x^y \exp(r\mathrm{Ad}(yx^{-1})Y)\exp(-r\mathrm{Ad}(y)Y). \tag{7.102}$$

We know from Lemma 1.90a that

$$\exp rX' \exp rY' = \exp\{r(X' + Y') + O(r^2)\} \qquad \text{as } r \to 0.$$

Hence the right side of (7.102) is

$$= x^y \exp(r\mathrm{Ad}(y)(\mathrm{Ad}(x^{-1}) - 1)Y + O(r^2)),$$

and

$$d\psi(Y, 0) = \mathrm{Ad}(y)(\mathrm{Ad}(x^{-1}) - 1)Y. \tag{7.103}$$

Next if X is in $\mathfrak{h}$, then (1.88) gives

$$(x \exp rX)^y = x^y \exp(r\mathrm{Ad}(y)X),$$

and hence

$$d\psi(0, X) = \mathrm{Ad}(y)X. \tag{7.104}$$

Combining (7.103) and (7.104), we obtain

$$d\psi(Y, X) = \mathrm{Ad}(y)((\mathrm{Ad}(x^{-1}) - 1)Y + X). \tag{7.105}$$

Since x is in H'_c, $\mathrm{Ad}(x^{-1}) - 1$ is invertible on the sum of the restricted-root spaces, and thus the set of all $(\mathrm{Ad}(x^{-1}) - 1)Y$ contains this sum. Since X is arbitrary in $\mathfrak{h}$, the set of all $(\mathrm{Ad}(x^{-1}) - 1)Y + X$ is all of $\mathfrak{g}$. But $\mathrm{Ad}(y)$ is invertible, and thus (7.105) shows that $d\psi$ is onto $\mathfrak{g}$. This completes the proof that V' is open in G'_c.

We are left with proving that any regular element X_0 of $\mathfrak{h}$ has $Z_{G_c}(X_0) = H_c$. Let $x \in G_c$ satisfy $\mathrm{Ad}(x)X_0 = X_0$. Since the centralizer of X_0 in $\mathfrak{g}$ is $\mathfrak{h}$, $\mathrm{Ad}(x)\mathfrak{h} = \mathfrak{h}$. If $x = u \exp X$ is the global Cartan decomposition of x, then Lemma 7.22 shows that $\mathrm{Ad}(u)\mathfrak{h} = \mathfrak{h}$ and $(\mathrm{ad}\, X)\mathfrak{h} = \mathfrak{h}$. By Proposition 2.7, X is in $\mathfrak{h}$. Thus $\mathrm{Ad}(u)X_0 = X_0$, and it is enough to prove that u is in M. Write $X_0 = X_1 + iX_2$ with X_1 and X_2 in $\mathfrak{m}_0$. Since $\mathrm{Ad}(u)\mathfrak{u}_0 = \mathfrak{u}_0$, we must have $\mathrm{Ad}(u)X_1 = X_1$. The centralizer of the torus $\overline{\exp \mathbb{R}X_1}$ in U is connected, by Corollary 4.51, and must lie in the analytic subgroup of U with Lie algebra $Z_{\mathfrak{u}_0}(X_1)$. Since X_1 is regular, Lemma 4.33 shows that $Z_{\mathfrak{u}_0}(X_1) = \mathfrak{m}_0$. Therefore u is in M, and the proof is complete.

Corollary 7.106. For the complex semisimple Lie group G_c, let H_x denote the centralizer in G_c of a regular element x of G_c. Then the identity component of H_x is a Cartan subgroup $(H_x)_0$ of G_c, and H_x lies in the normalizer $N_{G_c}((H_x)_0)$. Consequently H_x has only a finite number of connected components.

REMARK. Compare this conclusion with the example of $SL(2,\mathbb{C})/\{\pm 1\}$ given after (7.95).

PROOF. Theorem 7.101 shows that we can choose y in G_c with $h = y^{-1}xy$ in H_c. Since x is regular, so is h. Therefore $\mathrm{Ad}(h)$ has 1 as a generalized eigenvalue with multiplicity $l = \dim_{\mathbb{C}} \mathfrak{h}$. Since $\mathrm{Ad}(h)$ acts as the identity on $\mathfrak{h}$, it follows that $\mathfrak{h}$ is the centralizer of h in $\mathfrak{g}$. Hence $\mathrm{Ad}(y)\mathfrak{h}$ is the centralizer of $x = yhy^{-1}$ in $\mathfrak{g}$, and $\mathrm{Ad}(y)\mathfrak{h}$ is therefore the Lie algebra of H_x. Then $(H_x)_0 = yH_cy^{-1}$ is a Cartan subgroup of G_c by (7.100).

Next any element of a Lie group normalizes its identity component, and hence H_x lies in the normalizer $N_{G_c}((H_x)_0)$. By (7.93), H_x has a finite number of components.

Corollary 7.107. For the complex semisimple Lie group G_c, the centralizer in $\mathfrak{g}$ of a regular element of G_c is a Cartan subalgebra of $\mathfrak{g}$.

PROOF. This follows from the first conclusion of Corollary 7.106.

We return to the general reductive Lie group G. The relationship between the regular set in G and the Cartan subgroups of G follows quickly from Corollary 7.107.

Theorem 7.108. For the reductive Lie group G, let $(\mathfrak{h}_1)_0, \ldots, (\mathfrak{h}_r)_0$ be a maximal set of nonconjugate θ stable Cartan subalgebras of $\mathfrak{g}_0$, and let $H_1, \ldots, H_r$ be the corresponding Cartan subgroups of G. Then

(a) $G' \subseteq \bigcup_{i=1}^r \bigcup_{x\in G} xH_ix^{-1}$,
(b) each member of G' lies in just one Cartan subgroup of G,
(c) each H_i is abelian if G is semisimple and has a complexification.

PROOF.
(a) We apply Corollary 7.107 with $G_c = \mathrm{Int}\,\mathfrak{g}$. Property (v) of reductive Lie groups says that $\mathrm{Ad}(G) \subseteq G_c$, and the regular elements of G are exactly the elements x of G for which $\mathrm{Ad}(x)$ is regular in G_c. If x is in G', then Corollary 7.107 shows that $Z_{\mathfrak{g}}(x)$ is a Cartan subalgebra of $\mathfrak{g}$. Since x is in G, $Z_{\mathfrak{g}}(x)$ is the complexification of $Z_{\mathfrak{g}_0}(x)$, and hence $Z_{\mathfrak{g}_0}(x)$ is a

Cartan subalgebra of $\mathfrak{g}_0$. Therefore $Z_{\mathfrak{g}_0}(x) = \mathrm{Ad}(y)(\mathfrak{h}_i)_0$ for some $y \in G$ and some i with $1 \leq i \leq r$. Write $\widetilde{\mathfrak{h}}_0$ for $Z_{\mathfrak{g}_0}(x)$, and let $\widetilde{H} = Z_G(\widetilde{\mathfrak{h}}_0)$ be the corresponding Cartan subgroup. By definition, x is in $\widetilde{H}$. Since $\widetilde{\mathfrak{h}}_0 = \mathrm{Ad}(y)(\mathfrak{h}_i)_0$, it follows that $\widetilde{H} = yH_iy^{-1}$. Therefore x is in yH_iy^{-1}, and (a) is proved.

(b) We again apply Corollary 7.107 with $G_c = \mathrm{Int}\,\mathfrak{g}$. If $x \in G'$ lies in two distinct Cartan subgroups, then it centralizes two distinct Cartan subalgebras of $\mathfrak{g}_0$ and also their complexifications in $\mathfrak{g}$. Hence the centralizer of x in $\mathfrak{g}$ contains the sum of the two Cartan subalgebras in $\mathfrak{g}$, in contradiction with Corollary 7.107.

(c) This time we regard G_c as the complexification of G. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$, and let H be the corresponding Cartan subgroup of G. The centralizer H_c of $\mathfrak{h}$ in G_c is connected by (7.100), and H is a subgroup of this group. Since H_c has abelian Lie algebra, it is abelian. Hence H is abelian.

Now we return to the component structure of Cartan subgroups, but we shall restrict attention to the case that the reductive Lie group G is semisimple and has a complexification $G^{\mathbb{C}}$. Let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be the decomposition into $+1$ and -1 eigenspaces under θ of a θ stable Cartan subalgebra $\mathfrak{h}_0$. Let H be the Cartan subgroup $Z_G(\mathfrak{h}_0)$, let $T = \exp \mathfrak{t}_0$, and let $A = \exp \mathfrak{a}_0$. Here T is closed in K since otherwise the Lie algebra of its closure would form with $\mathfrak{a}_0$ an abelian subspace larger than $\mathfrak{h}_0$. Hence T is a torus. If α is a real root in $\Delta(\mathfrak{g}, \mathfrak{h})$, then the same argument as for (7.54) shows that

$$\gamma_\alpha = \exp 2\pi i |\alpha|^{-2} H_\alpha \tag{7.109}$$

is an element of K with $\gamma_\alpha^2 = 1$. As α varies, the elements γ_α commute. Define $F(T)$ to be the subgroup of K generated by all the elements γ_α for α real. Theorem 7.55 identifies $F(T)$ in the special case that $\mathfrak{h}_0$ is maximally noncompact; the theorem says that $F(T) = F$ in this case.

Proposition 7.110. Let G be semisimple with a complexification $G^{\mathbb{C}}$, and let $\mathfrak{h}_0$ be a θ stable Cartan subalgebra. Then the corresponding Cartan subgroup is $H = ATF(T)$.

PROOF. By Proposition 7.25, $Z_G(\mathfrak{t}_0)$ is a reductive Lie group, and then it satisfies $Z_G(\mathfrak{t}_0) = Z_K(\mathfrak{t}_0)\exp(\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0}(\mathfrak{t}_0))$. By Corollary 4.51, $Z_K(\mathfrak{t}_0)$ is connected. Therefore $Z_G(\mathfrak{t}_0)$ is connected.

Consequently $Z_G(\mathfrak{t}_0)$ is the analytic subgroup corresponding to

$$Z_{\mathfrak{g}_0}(\mathfrak{t}_0) = \mathfrak{g}_0 \cap \big(\mathfrak{h} + \sum_{\alpha \text{ real}} \mathfrak{g}_\alpha\big) = \mathfrak{h}_0 + \big(\sum_{\alpha \text{ real}} \mathbb{R}H_\alpha + \sum_{\alpha \text{ real}} (\mathfrak{g}_\alpha \cap \mathfrak{g}_0)\big).$$

The grouped term on the right is a split semisimple Lie algebra $\mathfrak{s}_0$. Let S be the corresponding analytic subgroup, so that $Z_G(\mathfrak{t}_0) = (\exp \mathfrak{h}_0)S = ATS$. Since the subspace $\mathfrak{a}_0' = \sum_{\alpha \text{ real}} \mathbb{R}H_\alpha$ of $\mathfrak{s}$ is a maximal abelian subspace of $\mathfrak{s}_0 \cap \mathfrak{p}_0$, Theorem 7.55 shows that the corresponding F group is just $F(T)$. By Theorem 7.53c, $Z_S(\mathfrak{a}_0') = (\exp \mathfrak{a}_0')F(T)$. Then

$$Z_G(\mathfrak{h}_0) = Z_{ATS}(\mathfrak{a}_0) = AT\,Z_S(\mathfrak{a}_0) = AT\,Z_S(\mathfrak{a}_0') = AT\,F(T).$$

Corollary 7.111. Let G be semisimple with a complexification $G^{\mathbb{C}}$, and let $Q = MAN$ be the Langlands decomposition of a cuspidal parabolic subgroup. Let $\mathfrak{t}_0$ be a θ stable compact Cartan subalgebra of $\mathfrak{m}_0$, and let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be the corresponding θ stable Cartan subalgebra of $\mathfrak{g}_0$. Define T and $F(T)$ from $\mathfrak{t}_0$. Then

(a) $Z_M(\mathfrak{t}_0) = TF(T)$,
(b) $Z_{M_0} = Z_M \cap T$,
(c) $Z_M = (Z_M \cap T)F(T) = Z_{M_0}F(T)$,
(d) $M_0 Z_M = M_0 F(T)$.

REMARK. When Q is a minimal parabolic subgroup, the subgroup M_0Z_M is all of M. But for general Q, M_0Z_M need not exhaust M. For some purposes in representation theory, M_0Z_M plays an intermediate role in passing from representations of M_0 to representations of M.

PROOF.

(a) Proposition 7.110 gives $Z_M(\mathfrak{t}_0) = {}^0Z_G(\mathfrak{t}_0 \oplus \mathfrak{a}_0) = {}^0(ATF(T)) = TF(T)$.

(b) Certainly $Z_M \cap T \subseteq Z_{M_0}$. In the reverse direction, Z_{M_0} is contained in $K \cap M_0$, hence is contained in the center of $K \cap M_0$. The center of a compact connected Lie group is contained in every maximal torus (Corollary 4.47), and thus $Z_{M_0} \subseteq T$. To complete the proof of (b), we show that $Z_{M_0} \subseteq Z_M$. The sum of $\mathfrak{a}_0$ and a maximally noncompact Cartan subalgebra of $\mathfrak{m}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, and the corresponding Cartan subgroup of G is abelian by Proposition 7.110. The intersection of this Cartan subgroup with M is a maximal noncompact Cartan subgroup of M and is abelian. By Corollary 7.91, $Z_{M_0} \subseteq Z_M$.

(c) The subgroup $F(T)$ is contained in Z_M since it is in $K \cap \exp i\mathfrak{a}_0$.

Therefore $Z_M = Z_M \cap Z_M(\mathfrak{t}_0) = Z_M \cap (TF(T)) = (Z_M \cap T)F(T)$, which proves the first equality of (c). The second equality follows from (b).

(d) By (c), $M_0 Z_M = M_0 Z_{M_0} F(T) = M_0 F(T)$.

9. Harish-Chandra Decomposition

For $G = SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \,\middle|\, |\alpha|^2 - |\beta|^2 = 1 \right\}$, the subgroup K can be taken to be $K = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\}$, and G/K may be identified with the disc $\{|z| < 1\}$ by $gK \leftrightarrow \beta/\bar\alpha$. If $g' = \begin{pmatrix} \alpha' & \beta' \\ \bar\beta' & \bar\alpha' \end{pmatrix}$ is given, then the equality $g'g = \begin{pmatrix} \alpha'\alpha + \beta'\bar\beta & \alpha'\beta + \beta'\bar\alpha \\ \bar\beta'\alpha + \bar\alpha'\bar\beta & \bar\beta'\beta + \bar\alpha'\bar\alpha \end{pmatrix}$ implies that

$$g'(gK) \leftrightarrow \frac{\alpha'\beta + \beta'\bar\alpha}{\bar\beta'\beta + \bar\alpha'\bar\alpha} = \frac{\alpha'(\beta/\bar\alpha) + \beta'}{\bar\beta'(\beta/\bar\alpha) + \bar\alpha'}.$$

In other words, under this identification, g' acts by the associated linear fractional transformation $z \mapsto \dfrac{\alpha' z + \beta'}{\bar\beta' z + \bar\alpha'}$. The transformations by which G acts on G/K are thus holomorphic once we have imposed a suitable complex-manifold structure on G/K.

If G is a semisimple Lie group, then we say that G/K is **Hermitian** if G/K admits a complex-manifold structure such that G acts by holomorphic transformations. In this section we shall classify the semisimple groups G for which G/K is Hermitian. Since the center of G is contained in K (Theorem 6.31e), we could assume, if we wanted, that G is an adjoint group. At any rate there is no loss of generality in assuming that G is linear and hence has a complexification. We begin with a more complicated example.

EXAMPLE. Let $n \ge m$, let $M_{nm}(\mathbb{C})$ be the complex vector space of all n-by-m complex matrices, and let 1_m be the m-by-m identity matrix. Define

$$\Omega = \{Z \in M_{nm}(\mathbb{C}) \mid 1_m - Z^* Z \text{ is positive definite}\}.$$

We shall identify Ω with a quotient G/K, taking $G = SU(n,m)$ and

$$\begin{aligned} K &= S(U(n) \times U(m)) \\ &= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \,\middle|\, A \in U(n),\ D \in U(m),\ \det A \det D = 1 \right\}. \end{aligned}$$

The group action of G on Ω will be by

$$g(Z) = (AZ + B)(CZ + D)^{-1} \qquad \text{if } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{7.112}$$

To see that (7.112) defines an action of G on Ω, we shall verify that $(CZ + D)^{-1}$ is defined in (7.112) and that $g(Z)$ is in Ω if Z is in Ω. To do so, we write

$$\begin{aligned} (AZ + B)^*(AZ + B) &- (CZ + D)^*(CZ + D) \\ &= \begin{pmatrix} Z^* & 1_m \end{pmatrix} g^* \begin{pmatrix} 1_n & 0 \\ 0 & -1_m \end{pmatrix} g \begin{pmatrix} Z \\ 1_m \end{pmatrix} \\ &= \begin{pmatrix} Z^* & 1_m \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ 0 & -1_m \end{pmatrix} \begin{pmatrix} Z \\ 1_m \end{pmatrix} \quad \text{since } g \text{ is in } SU(n, m) \\ &= Z^*Z - 1_m. \end{aligned} \tag{7.113}$$

With Z in Ω, suppose $(CZ + D)v = 0$. Unless $v = 0$, we see from (7.113) that

$$0 \le v^*(AZ + B)^*(AZ + B)v = v^*(Z^*Z - 1_m)v < 0,$$

a contradiction. Hence $(CZ + D)^{-1}$ exists, and then (7.113) gives

$$g(Z)^*g(Z) - 1_m = (CZ + D)^{*-1}(Z^*Z - 1_m)(CZ + D)^*.$$

The right side is negative definite, and hence $g(Z)$ is in Ω.

The isotropy subgroup at $Z = 0$ is the subgroup with $B = 0$, and this subgroup reduces to K. Let us see that G acts transitively on Ω. Let $Z \in M_{nm}(\mathbb{C})$ be given. The claim is that Z decomposes as

$$Z = udv \qquad \text{with } u \in U(n),\ v \in U(m), \tag{7.114}$$

and d of the form $d = \begin{pmatrix} d_0 \\ 0 \end{pmatrix}$, where $d_0 = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$ with all $\lambda_j \ge 0$ and where 0 is of size $(n - m)$-by-m. To prove (7.114), we extend Z to a square matrix $\begin{pmatrix} Z & 0 \end{pmatrix}$ of size n-by-n and let the polar decomposition of $\begin{pmatrix} Z & 0 \end{pmatrix}$ be $\begin{pmatrix} Z & 0 \end{pmatrix} = u_1 p$ with $u_1 \in U(n)$ and p positive semidefinite. Since $\begin{pmatrix} Z & 0 \end{pmatrix}$ is 0 in the last $n - m$ columns, u_1 gives 0 when applied to the last $n - m$ columns of p. The matrix u_1 is nonsingular, and thus the last $n - m$ columns of p are 0. Since p is Hermitian, $p = \begin{pmatrix} p' & 0 \\ 0 & 0 \end{pmatrix}$ with p'

positive semidefinite of size m-by-m. By the finite-dimensional Spectral Theorem, write $p' = u_2 d_0 u_2^{-1}$ with $u_2 \in U(m)$ and $d_0 = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$. Then (7.114) holds with $u = u_1 \begin{pmatrix} u_2 & 0 \\ 0 & 1_{n-m} \end{pmatrix}$, $d = \begin{pmatrix} d_0 \\ 0 \end{pmatrix}$, and $v = u_2^{-1}$.

With Z as in (7.114), the matrix $Z^*Z = v^*d^*dv$ has the same eigenvalues as d^*d, which has eigenvalues $\lambda_1^2, \dots, \lambda_m^2$. Thus Z is in Ω if and only if $0 \le \lambda_j < 1$ for $1 \le j \le m$. In the formula (7.114) there is no loss of generality in assuming that $(\det u)(\det v)^{-1} = 1$, so that $\begin{pmatrix} u & 0 \\ 0 & v^{-1} \end{pmatrix}$ is in K. Let a be the member of $SU(n, m)$ that is $\begin{pmatrix} \cosh t_j & \sinh t_j \\ \sinh t_j & \cosh t_j \end{pmatrix}$ in the j^{th} and $(n+j)^{\text{th}}$ rows and columns for $1 \le j \le m$ and is otherwise the identity. Then $a(0) = d$, and $\begin{pmatrix} u & 0 \\ 0 & v^{-1} \end{pmatrix}(d) = udv = Z$. Hence $g = \begin{pmatrix} u & 0 \\ 0 & v^{-1} \end{pmatrix} a$ maps 0 to Z, and the action of G on Ω is transitive.

Throughout this section we let G be a semisimple Lie group with a complexification $G^{\mathbb{C}}$. We continue with the usual notation for G as a reductive Lie group. Let $\mathfrak{c}_0$ be the center of $\mathfrak{k}_0$. We shall see that a necessary and sufficient condition for G/K to be Hermitian is that $Z_{\mathfrak{g}_0}(\mathfrak{c}_0) = \mathfrak{k}_0$. In this case we shall exhibit G/K as holomorphically equivalent to a bounded domain in $\mathbb{C}^n$ for a suitable n. The explicit realization of G/K as a bounded domain is achieved through the "Harish-Chandra decomposition" of a certain open dense subset of $G^{\mathbb{C}}$.

First we shall prove that if G/K is Hermitian, then $Z_{\mathfrak{g}_0}(\mathfrak{c}_0) = \mathfrak{k}_0$. Before stating a precise theorem of this kind, we recall the "multiplication-by-i" mapping introduced in connection with holomorphic mappings in §I.12. If M is a complex manifold of dimension n, we can associate to M an almost-complex structure consisting of a multiplication-by-i mapping $J_p \in \operatorname{End}(T_p(M))$ for each p. For each p, we have $J_p^2 = -1$. If $\Phi : M \to N$ is a smooth mapping between complex manifolds, then Φ is holomorphic if and only if the Cauchy–Riemann equations hold. If $\{J_p\}$ and $\{J'_q\}$ are the respective almost-complex structures for M and N, these equations may be written as

$$J'_{\Phi(p)} \circ d\Phi_p = d\Phi_p \circ J_p \tag{7.115}$$

for all p.

Now let us consider the case that $M = N = G/K$ and p is the identity coset. If G/K is Hermitian, then each left translation L_k by $k \in K$ (defined by $L_k(k') = kk'$) is holomorphic and fixes the identity coset. If J denotes the multiplication-by-i mapping at the identity coset, then (7.115) gives

$$J \circ dL_k = dL_k \circ J.$$

We may identify the tangent space at the identity coset with $\mathfrak{p}_0$, and then $dL_k = \mathrm{Ad}(k)|_{\mathfrak{p}_0}$. Differentiating, we obtain

$$J \circ (\mathrm{ad}\, X)|_{\mathfrak{p}_0} = (\mathrm{ad}\, X)|_{\mathfrak{p}_0} \circ J \qquad \text{for all } X \in \mathfrak{k}_0. \tag{7.116}$$

Theorem 7.117. If G/K is Hermitian, then the multiplication-by-i mapping $J : \mathfrak{p}_0 \to \mathfrak{p}_0$ at the identity coset is of the form $J = (\mathrm{ad}\, X_0)|_{\mathfrak{p}_0}$ for some $X_0 \in \mathfrak{k}_0$. This element X_0 is in $\mathfrak{c}_0$ and satisfies $Z_{\mathfrak{g}_0}(X_0) = \mathfrak{k}_0$. Hence $Z_{\mathfrak{g}_0}(\mathfrak{c}_0) = \mathfrak{k}_0$.

PROOF. Since $J^2 = -1$ on $\mathfrak{p}_0$, the complexification $\mathfrak{p}$ is the direct sum of its $+i$ and $-i$ eigenspaces $\mathfrak{p}^+$ and $\mathfrak{p}^-$. The main step is to prove that

$$[X, Y] = 0 \qquad \text{if } X \in \mathfrak{p}^+ \text{ and } Y \in \mathfrak{p}^+. \tag{7.118}$$

Let B be the bilinear form on $\mathfrak{g}_0$ and $\mathfrak{g}$ that is part of the data of a reductive group, and define a bilinear form C on $\mathfrak{p}$ by

$$C(X, Y) = B(X, Y) + B(JX, JY).$$

Since B is positive definite on $\mathfrak{p}_0$, so is C. Hence C is nondegenerate on $\mathfrak{p}$. Let us prove that

$$C([[X, Y], Z], T) = C([[Z, T], X], Y) \tag{7.119}$$

for X, Y, Z, T in $\mathfrak{p}$. When X, Y, Z are in $\mathfrak{p}$, the bracket $[Y, Z]$ is in $\mathfrak{k}$, and therefore (7.116) implies that

$$J[X, [Y, Z]] = [JX, [Y, Z]]. \tag{7.120}$$

Using the Jacobi identity and (7.120) repeatedly, together with the invariance of B, we compute

$$\begin{aligned} B(J[[X, Y], Z], JT) &= B(J[X, [Y, Z]], JT) - B(J[Y, [X, Z]], JT) \\ &= B([JX, [Y, Z]], JT) - B([JY, [X, Z]], JT) \\ &= -B([JT, [Y, Z]], JX) + B([JT, [X, Z]], JY) \\ &= -B(J[T, [Y, Z]], JX) + B(J[T, [X, Z]], JY). \end{aligned} \tag{7.121}$$

Using the result (7.121) with Z and T interchanged, we obtain

$$\begin{aligned} B(J[[X,Y],Z],JT) &= B([[X,Y],JZ],JT) \\ &= -B([[X,Y],JT],JZ) \\ &= -B(J[[X,Y],T],JZ) \\ &= B(J[Z,[Y,T]],JX) - B(J[Z,[X,T]],JY). \end{aligned} \tag{7.122}$$

The sum of (7.121) and (7.122) is

$$\begin{aligned} 2B(J[[X,Y],Z],JT) &= -B(J[T,[Y,Z]],JX) + B(J[T,[X,Z]],JY) \\ &\quad + B(J[Z,[Y,T]],JX) - B(J[Z,[X,T]],JY) \\ &= B(J[Y,[Z,T]],JX) - B(J[X,[Z,T]],JY) \\ &= B([JY,[Z,T]],JX) - B([JX,[Z,T]],JY) \\ &= 2B([Z,T],[JX,JY]) \\ &= 2B([[Z,T],JX],JY) \\ &= 2B(J[[Z,T],X],JY). \end{aligned} \tag{7.123}$$

The calculation that leads to (7.123) remains valid if J is dropped throughout. If we add the results with J present and with J absent, we obtain (7.119). To prove (7.118), suppose that X and Y are in $\mathfrak{p}^+$, so that $JX = iX$ and $JY = iY$. Then

$$\begin{aligned} C([[Z,T],X],Y) &= C(J[[Z,T],X],JY) \\ &= C([[Z,T],JX],JY) \\ &= -C([[Z,T],X],Y) \end{aligned}$$

says $C([[Z,T],X],Y) = 0$. By (7.119), $C([[X,Y],Z],T) = 0$. Since T is arbitrary and C is nondegenerate,

$$[[X,Y],Z] = 0 \qquad \text{for all } Z \in \mathfrak{p}. \tag{7.124}$$

If bar denotes conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$, then $B(W,\overline{W}) < 0$ for all $W \neq 0$ in $\mathfrak{k}$. For $W = [X,Y]$, we have

$$B([X,Y],\overline{[X,Y]}) = B([X,Y],[\overline{X},\overline{Y}]) = B([[X,Y],\overline{X}],\overline{Y}),$$

and the right side is 0 by (7.124). Therefore $[X,Y] = 0$, and (7.118) is proved.

Let us extend J to a linear map $\widetilde{J}$ defined on $\mathfrak{g}$, putting $\widetilde{J} = 0$ on $\mathfrak{k}$. We shall deduce from (7.118) that $\widetilde{J}$ is a derivation of $\mathfrak{g}_0$, i.e., that

$$\widetilde{J}[X, Y] = [\widetilde{J}X, Y] + [X, \widetilde{J}Y] \qquad \text{for } X, Y \in \mathfrak{g}_0. \tag{7.125}$$

If X and Y are in $\mathfrak{k}_0$, all terms are 0, and (7.125) is automatic. If X is in $\mathfrak{k}_0$ and Y is in $\mathfrak{p}_0$, then $[\widetilde{J}X, Y] = 0$ since $\widetilde{J}X = 0$, and (7.125) reduces to (7.116). Thus suppose X and Y are in $\mathfrak{p}_0$. The element $X - iJX$ is in $\mathfrak{p}^+$ since

$$J(X - iJX) = JX - iJ^2X = JX + iX = i(X - iJX),$$

and similarly $Y - iJY$ is in $\mathfrak{p}^+$. By (7.118),

$$0 = [X - iJX, Y - iJY] = ([X, Y] - [JX, JY]) - i([JX, Y] + [X, JY]).$$

The real and imaginary parts must each be 0. Since the imaginary part is 0, the right side of (7.125) is 0. The left side of (7.125) is 0 since $\widetilde{J}$ is 0 on $\mathfrak{k}_0$. Hence $\widetilde{J}$ is a derivation of $\mathfrak{g}_0$.

By Proposition 1.121, $\widetilde{J} = \operatorname{ad} X_0$ for some $X_0 \in \mathfrak{g}_0$. Let $Y \in \mathfrak{p}_0$ be given. Since $J^2 = -1$ on $\mathfrak{p}_0$, the element $Y' = -JY$ of $\mathfrak{p}_0$ has $JY' = Y$. Then

$$B(X_0, Y) = B(X_0, JY') = B(X_0, [X_0, Y']) = B([X_0, X_0], Y') = 0.$$

Hence X_0 is orthogonal to $\mathfrak{p}_0$, and X_0 must be in $\mathfrak{k}_0$. Since $\widetilde{J} = \operatorname{ad} X_0$ is 0 on $\mathfrak{k}_0$, X_0 is in $\mathfrak{c}_0$.

If Y is in $Z_{\mathfrak{g}_0}(X_0)$, then the $\mathfrak{k}_0$ component of Y already commutes with X_0 since X_0 is in $\mathfrak{c}_0$. Thus we may assume that Y is in $\mathfrak{p}_0$. But then $[X_0, Y] = JY$. Since J is nonsingular on $\mathfrak{p}_0$, $0 = [X_0, Y]$ implies $Y = 0$. We conclude that $Z_{\mathfrak{g}_0}(X_0) = \mathfrak{k}_0$. Finally we have

$$\mathfrak{k}_0 \subseteq Z_{\mathfrak{g}_0}(\mathfrak{c}_0) \subseteq Z_{\mathfrak{g}_0}(X_0) = \mathfrak{k}_0,$$

and equality must hold throughout. Therefore $Z_{\mathfrak{g}_0}(\mathfrak{c}_0) = \mathfrak{k}_0$.

For the converse we assume that $Z_{\mathfrak{g}_0}(\mathfrak{c}_0) = \mathfrak{k}_0$, and we shall exhibit a complex structure on G/K such that G operates by holomorphic transformations. Fix a maximal abelian subspace $\mathfrak{t}_0$ of $\mathfrak{k}_0$. Then $\mathfrak{c}_0 \subseteq \mathfrak{t}_0$, so that $Z_{\mathfrak{g}_0}(\mathfrak{t}_0) \subseteq Z_{\mathfrak{g}_0}(\mathfrak{c}_0) = \mathfrak{k}_0$. Consequently $\mathfrak{t}_0$ is a compact Cartan subalgebra of $\mathfrak{g}_0$. The corresponding Cartan subgroup T is connected by Proposition 7.90b, hence is a torus.

Every root in $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ is imaginary, hence compact or noncompact in the sense of §VI.7. If Δ_K and Δ_n denote the sets of compact and noncompact roots, then we have

$$\mathfrak{k} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta_K} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{p} = \bigoplus_{\alpha \in \Delta_n} \mathfrak{g}_\alpha, \tag{7.126}$$

just as in (6.103).

Lemma 7.127. A root α is compact if and only if α vanishes on the center $\mathfrak{c}$ of $\mathfrak{k}$.

PROOF. If α is in Δ, then $\alpha(\mathfrak{c}) = 0$ if and only if $[\mathfrak{c}, \mathfrak{g}_\alpha] = 0$, if and only if $\mathfrak{g}_\alpha \subseteq Z_{\mathfrak{g}}(\mathfrak{c})$, if and only if $\mathfrak{g}_\alpha \subseteq \mathfrak{k}$, if and only if α is compact.

By a **good ordering** for $i\mathfrak{t}_0$, we mean a system of positivity in which every noncompact positive root is larger than every compact root. A good ordering always exists; we can, for instance, use a lexicographic ordering that takes $i\mathfrak{c}_0$ before its orthogonal complement in $i\mathfrak{t}_0$. Fixing a good ordering, let Δ^+, Δ_K^+, and Δ_n^+ be the sets of positive roots in Δ, Δ_K, and Δ_n. Define

$$\mathfrak{p}^+ = \bigoplus_{\alpha \in \Delta_n^+} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{p}^- = \bigoplus_{\alpha \in \Delta_n^+} \mathfrak{g}_{-\alpha},$$

so that $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$.

In the example of $SU(n, m)$ earlier in this section, we have

$$i\mathfrak{c}_0 = \mathbb{R}\,\mathrm{diag}(\tfrac{1}{n}, \dots, \tfrac{1}{n}, -\tfrac{1}{m}, \dots, -\tfrac{1}{m})$$

with n entries $\frac{1}{n}$ and m entries $-\frac{1}{m}$, and we may take $\mathfrak{t}_0$ to be the diagonal subalgebra. If roots $e_i - e_j$ that are positive on

$$\mathrm{diag}(\tfrac{1}{n}, \dots, \tfrac{1}{n}, -\tfrac{1}{m}, \dots, -\tfrac{1}{m})$$

are declared to be positive, then $\mathfrak{p}^+$ has the block form $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ and $\mathfrak{p}^-$ has the block form $\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$.

Lemma 7.128. The subspaces $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are abelian subspaces of $\mathfrak{p}$, and $[\mathfrak{k}, \mathfrak{p}^+] \subseteq \mathfrak{p}^+$ and $[\mathfrak{k}, \mathfrak{p}^-] \subseteq \mathfrak{p}^-$.

PROOF. Let α, β, and $\alpha+\beta$ be in Δ with α compact and β noncompact. Then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, and β and $\alpha+\beta$ are both positive or both negative because the ordering is good. Summing on α and β, we see that $[\mathfrak{k}, \mathfrak{p}^+] \subseteq \mathfrak{p}^+$ and $[\mathfrak{k}, \mathfrak{p}^-] \subseteq \mathfrak{p}^-$.

If α and β are in Δ_n^+, then $\alpha+\beta$ cannot be a root since it would have to be a compact root larger than the noncompact positive root α. Summing on α and β, we obtain $[\mathfrak{p}^+, \mathfrak{p}^+] = 0$. Similarly $[\mathfrak{p}^-, \mathfrak{p}^-] = 0$.

Let $\mathfrak{b}$ be the Lie subalgebra

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$$

of $\mathfrak{g}$, and let P^+, $K^{\mathbb{C}}$, P^-, and B be the analytic subgroups of $G^{\mathbb{C}}$ with Lie algebras $\mathfrak{p}^+$, $\mathfrak{k}$, $\mathfrak{p}^-$, and $\mathfrak{b}$. Since $G^{\mathbb{C}}$ is complex and $\mathfrak{p}^+, \mathfrak{k}, \mathfrak{p}^-, \mathfrak{b}$ are closed under multiplication by i, all the groups P^+, $K^{\mathbb{C}}$, P^-, B are complex subgroups.

Theorem 7.129 (Harish-Chandra decomposition). Let G be semisimple with a complexification $G^{\mathbb{C}}$, and suppose that the center $\mathfrak{c}_0$ of $\mathfrak{k}_0$ has $Z_{\mathfrak{g}_0}(\mathfrak{c}_0) = \mathfrak{k}_0$. Then multiplication from $P^+ \times K^{\mathbb{C}} \times P^-$ into $G^{\mathbb{C}}$ is one-one, holomorphic, and regular (with image open in $G^{\mathbb{C}}$), GB is open in $G^{\mathbb{C}}$, and there exists a bounded open subset $\Omega \subseteq P^+$ such that

$$GB = GK^{\mathbb{C}}P^- = \Omega K^{\mathbb{C}} P^-.$$

Moreover, G/K is Hermitian. In fact, the map of G into Ω given by $g \mapsto$ (P^+ component of g) exhibits G/K and Ω as diffeomorphic, and G acts holomorphically on Ω by $g(\omega) =$ (P^+ component of $g\omega$).

REMARKS.

1) We shall see in the proof that the complex group P^+ is holomorphically isomorphic with some $\mathbb{C}^n$, and the theorem asserts that Ω is a bounded open subset when regarded as in $\mathbb{C}^n$ in this fashion.

2) When $G = SU(n,m)$, $G^{\mathbb{C}}$ may be taken as $SL(n+m, \mathbb{C})$. The decomposition of an open subset of $G^{\mathbb{C}}$ as $P^+ \times K^{\mathbb{C}} \times P^-$ is

(7.130)
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix},$$

valid whenever D is nonsingular. Whatever Ω is in the theorem, if $\omega = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}$ is in Ω and $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is in G, then $g\omega = \begin{pmatrix} A & AZ+B \\ C & CZ+D \end{pmatrix}$; hence (7.130) shows that the P^+ component of $g\omega$ is

$$\begin{pmatrix} 1 & (AZ+B)(CZ+D)^{-1} \\ 0 & 1 \end{pmatrix}.$$

So the action is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\left(\begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & (AZ+B)(CZ+D)^{-1} \\ 0 & 1 \end{pmatrix}. \tag{7.131}$$

We know from the example earlier in this section that the image of $Z = 0$ under $Z \mapsto (AZ+B)(CZ+D)^{-1}$ for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $SU(n,m)$ is all Z with $1_m - Z^*Z$ positive definite. Therefore Ω consists of all $\begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}$ such that $1_m - Z^*Z$ is positive definite, and the action (7.131) corresponds to the action by linear fractional transformations in the example.

3) The proof will reduce matters to two lemmas, which we shall consider separately.

PROOF. Define

$$\mathfrak{n} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}, \qquad \mathfrak{b}_K = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta_K^+} \mathfrak{g}_{-\alpha},$$

$$N,\ N^-,\ B_K = \text{corresponding analytic subgroups of } G^{\mathbb{C}}.$$

Let $H_{\mathbb{R}}$ and H be the analytic subgroups of $G^{\mathbb{C}}$ with Lie algebras $i\mathfrak{t}_0$ and $\mathfrak{t}$, so that $H = TH_{\mathbb{R}}$ as a direct product. By (7.100) a Cartan subgroup of a complex semisimple Lie group is connected, and therefore H is a Cartan subgroup. The involution $\theta \circ \text{bar}$, where bar is the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$, is a Cartan involution of $\mathfrak{g}$, and $i\mathfrak{t}_0$ is a maximal abelian subspace of the -1 eigenspace. The $+1$ eigenspace is $\mathfrak{k}_0 \oplus i\mathfrak{p}_0$, and the corresponding analytic subgroup of $G^{\mathbb{C}}$ we call U. Then

$$Z_U(i\mathfrak{t}_0) = Z_U(\mathfrak{t}) = U \cap Z_{G^{\mathbb{C}}}(\mathfrak{t}) = U \cap H = T.$$

So the $M_{\mathfrak{p}}$ group is just T. By Proposition 7.82 the M of every parabolic subgroup of $G^{\mathbb{C}}$ is connected.

The restricted roots of $\mathfrak{g}^{\mathbb{R}}$ relative to $i\,\mathfrak{t}_0$ are evidently the restrictions from $\mathfrak{t}$ to $i\,\mathfrak{t}_0$ of the roots. Therefore $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}^-$ is a minimal parabolic subalgebra of $\mathfrak{g}^{\mathbb{R}}$. Since parabolic subgroups of $G^{\mathbb{C}}$ are closed (by Proposition 7.83b) and connected, B is closed.

The subspace $\mathfrak{k} \oplus \mathfrak{p}^-$ is a Lie subalgebra of $\mathfrak{g}^{\mathbb{R}}$ containing $\mathfrak{b}$ and hence is a parabolic subalgebra. Then Proposition 7.83 shows that $K^{\mathbb{C}}$ and P^- are closed, $K^{\mathbb{C}}P^-$ is closed, and multiplication $K^{\mathbb{C}} \times P^-$ is a diffeomorphism onto. Similarly P^+ is closed.

Moreover the Lie algebra $\mathfrak{k} \oplus \mathfrak{p}^-$ of $K^{\mathbb{C}}P^-$ is complex, and hence $K^{\mathbb{C}}P^-$ is a complex manifold. Then multiplication $K^{\mathbb{C}} \times P^-$ is evidently holomorphic and has been observed to be one-one and regular. Since $\mathfrak{p}^+ \oplus (\mathfrak{k} \oplus \mathfrak{p}^-) = \mathfrak{g}$, Lemma 6.44 shows that the holomorphic multiplication map $P^+ \times (K^{\mathbb{C}}P^-) \to G^{\mathbb{C}}$ is everywhere regular. It is one-one by Proposition 7.83e. Hence $P^+ \times K^{\mathbb{C}} \times P^- \to G$ is one-one, holomorphic, and regular.

Next we shall show that GB is open in $G^{\mathbb{C}}$. First let us observe that

$$\mathfrak{g}_0 \cap (i\,\mathfrak{t}_0 \oplus \mathfrak{n}^-) = 0. \tag{7.132}$$

In fact, since roots are imaginary on $\mathfrak{t}_0$, we have $\overline{\mathfrak{g}_\alpha} = \mathfrak{g}_{-\alpha}$. Thus if h is in $i\,\mathfrak{t}_0$ and $X_{-\alpha}$ is in $\mathfrak{n}^-$, then

$$\overline{h + \sum_{\alpha\in\Delta^+} X_{-\alpha}} = -h + \sum_{\alpha\in\Delta^+} \overline{X_{-\alpha}} \in -h + \mathfrak{n},$$

and (7.132) follows since members of $\mathfrak{g}_0$ equal their own conjugates. The real dimension of $i\,\mathfrak{t}_0 \oplus \mathfrak{n}^-$ is half the real dimension of $\mathfrak{t} \oplus \mathfrak{n} \oplus \mathfrak{n}^- = \mathfrak{g}$, and hence

$$\dim_{\mathbb{R}}(\mathfrak{g}_0 \oplus (i\,\mathfrak{t}_0 \oplus \mathfrak{n}^-)) = \dim_{\mathbb{R}} \mathfrak{g}. \tag{7.133}$$

Combining (7.132) and (7.133), we see that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus (i\,\mathfrak{t}_0 \oplus \mathfrak{n}^-). \tag{7.134}$$

The subgroup $H_{\mathbb{R}}N^-$ of $G^{\mathbb{C}}$ is closed by Proposition 7.83, and hence $H_{\mathbb{R}}N^-$ is an analytic subgroup, necessarily with Lie algebra $i\,\mathfrak{t}_0 \oplus \mathfrak{n}^-$. By Lemma 6.44 it follows from (7.134) that multiplication $G \times H_{\mathbb{R}}N^- \to G^{\mathbb{C}}$ is everywhere regular. The dimension relation (7.133) therefore implies that $GH_{\mathbb{R}}N^-$ is open in $G^{\mathbb{C}}$. Since $B = TH_{\mathbb{R}}N^-$ and $T \subseteq G$, GB equals $GH_{\mathbb{R}}N^-$ and is open in $G^{\mathbb{C}}$.

The subgroups P^+ and P^- are the N groups of parabolic subalgebras, and their Lie algebras are abelian by Lemma 7.128. Hence P^+ and P^- are Euclidean groups. Then $\exp : \mathfrak{p}^+ \to P^+$ is biholomorphic, and P^+ is biholomorphic with $\mathbb{C}^n$ for some n. Similarly P^- is biholomorphic with $\mathbb{C}^n$.

The subgroup $K^{\mathbb{C}}$ is a reductive group, being connected and having bar as a Cartan involution for its Lie algebra. It is the product of the identity component of its center by a complex semisimple Lie group, and our above considerations show that its parabolic subgroups are connected. Then B_K is a parabolic subgroup, and

$$K^{\mathbb{C}} = K B_K \tag{7.135}$$

by Proposition 7.83f.

Let A denote a specific $A_{\mathfrak{p}}$ component for the Iwasawa decomposition of G, to be specified in Lemma 7.143 below. We shall show in Lemma 7.145 that this A satisfies

$$A \subseteq P^+ K^{\mathbb{C}} P^- \tag{7.136a}$$

and

(7.136b) P^+ components of members of A are bounded.

Theorem 7.39 shows that $G = KAK$. Since $\mathfrak{b} \subseteq \mathfrak{k} \oplus \mathfrak{p}^-$, we have $B \subseteq K^{\mathbb{C}} P^-$. Since Lemma 7.128 shows that $K^{\mathbb{C}}$ normalizes P^+ and P^-, (7.136a) gives

$$\begin{aligned} GB &\subseteq GK^{\mathbb{C}}P^- \subseteq KAKK^{\mathbb{C}}P^- \\ &\subseteq KP^+K^{\mathbb{C}}P^-K^{\mathbb{C}}P^- = P^+K^{\mathbb{C}}P^-. \end{aligned} \tag{7.137}$$

By (7.135) we have

$$GK^{\mathbb{C}}P^- = GKB_KP^- \subseteq GB_KP^- \subseteq GB. \tag{7.138}$$

Inclusions (7.137) and (7.138) together imply that

$$GB = GK^{\mathbb{C}}P^- \subseteq P^+K^{\mathbb{C}}P^-.$$

Since GB is open,

$$GB = GK^{\mathbb{C}}P^- = \Omega K^{\mathbb{C}}P^- \tag{7.139}$$

for some open set Ω in P^+.

Let us write $p^+(\,\cdot\,)$ for the P^+ component. For $gb \in GB$, we have $p^+(gb) = p^+(g)$, and thus p^+ restricts to a smooth map carrying G onto Ω. From (7.139) it follows that the map $G \times \Omega \to \Omega$ given by

$$(g, \omega) \mapsto p^+(g\omega) \tag{7.140}$$

is well defined. For fixed g, this is holomorphic since left translation by g is holomorphic on $G^{\mathbb{C}}$ and since p^+ is holomorphic from $P^+K^{\mathbb{C}}P^-$ to P^+. To see that (7.140) is a group action, we use that $K^{\mathbb{C}}P^-$ is a subgroup. Let g_1 and g_2 be given, and write $g_2\omega = p^+(g_2\omega)k_2p_2^-$ and $g_1g_2\omega = p^+(g_1g_2\omega)k^{\mathbb{C}}p^-$. Then

$$g_1p^+(g_2\omega) = g_1g_2\omega(k_2p_2^-)^{-1} = p^+(g_1g_2\omega)(k^{\mathbb{C}}p^-)(k_2p_2^-)^{-1}.$$

Since $(k^{\mathbb{C}}p^-)(k_2p_2^-)^{-1}$ is in $K^{\mathbb{C}}P^-$, $p^+(g_1p^+(g_2\omega)) = p^+(g_1g_2\omega)$. Therefore (7.140) is a group action. The action is evidently smooth, and we have seen that it is transitive.

If g is in G and k is in K, we can regard 1 as in Ω and write

$$p^+(gk) = p^+(gk1) = p^+(gp^+(k1)) = p^+(g1)$$

since $k1$ is in $K \subseteq K^{\mathbb{C}}$ and has P^+ component 1. Therefore $p^+ : G \to \Omega$ descends to a smooth map of G/K onto Ω. Let us see that it is one-one. If $p^+(g_1) = p^+(g_2)$, then $g_1 = g_2k^{\mathbb{C}}p^-$ since $K^{\mathbb{C}}P^-$ is a group, and hence $g_2^{-1}g_1 = k^{\mathbb{C}}p^-$. Thus the map $G/K \to \Omega$ will be one-one if we show that

$$G \cap K^{\mathbb{C}}P^- = K. \tag{7.141}$$

To prove (7.141), we note that $\supseteq$ is clear. Then we argue in the same way as for (7.132) that

$$\mathfrak{g}_0 \cap (\mathfrak{k} \oplus \mathfrak{p}^-) = \mathfrak{k}_0. \tag{7.142}$$

Since G and $K^{\mathbb{C}}P^-$ are closed in $G^{\mathbb{C}}$, their intersection is a closed subgroup of G with Lie algebra $\mathfrak{k}_0$. Let $g = k \exp X$ be the global Cartan decomposition of an element g of $G \cap K^{\mathbb{C}}P^-$. Then $\operatorname{Ad}(g)\mathfrak{k}_0 = \mathfrak{k}_0$, and Lemma 7.22 implies that $(\operatorname{ad} X)\mathfrak{k}_0 \subseteq \mathfrak{k}_0$. Since $\operatorname{ad} X$ is skew symmetric relative to B, $(\operatorname{ad} X)\mathfrak{p}_0 \subseteq \mathfrak{p}_0$. But $X \in \mathfrak{p}_0$ implies that $(\operatorname{ad} X)\mathfrak{k}_0 \subseteq \mathfrak{p}_0$ and $(\operatorname{ad} X)\mathfrak{p}_0 \subseteq \mathfrak{k}_0$. Hence $\operatorname{ad} X = 0$ and $X = 0$. This proves (7.141).

To see that $G/K \to \Omega$ is everywhere regular, it is enough, since (7.140) is a smooth group action, to show that the differential of $p^+ : G \to \Omega$ at the identity is one-one on $\mathfrak{p}_0$. But dp^+ complexifies to the projection of $\mathfrak{g} = \mathfrak{p}^+ \oplus \mathfrak{k} \oplus \mathfrak{p}^-$ on $\mathfrak{p}^+$, and (7.142) shows that the kernel of this projection meets $\mathfrak{p}_0$ only in 0. Therefore the map $G/K \to \Omega$ is a diffeomorphism.

To see that Ω is bounded, we need to see that $p^+(g)$ remains bounded as g varies in G. If $g \in G$ is given, write $g = k_1 a k_2$ according to $G = KAK$. Then $p^+(g) = p^+(k_1 a) = k_1 p^+(a) k_1^{-1}$ by (7.139) and Lemma 7.128. Therefore it is enough to prove that $\|\log p^+(a)\|$ remains bounded, and this is just (7.136b). Thus the theorem reduces to proving (7.136), which we do in Lemmas 7.143 and 7.145 below.

Lemma 7.143. Inductively define $\gamma_1, \dots, \gamma_s$ in Δ_n^+ as follows: γ_1 is the largest member of Δ_n^+, and γ_j is the largest member of Δ_n^+ orthogonal to $\gamma_1, \dots, \gamma_{j-1}$. For $1 \le j \le s$, let E_{γ_j} be a nonzero root vector for γ_j. Then the roots $\gamma_1, \dots, \gamma_s$ are strongly orthogonal, and

$$\mathfrak{a}_0 = \bigoplus_{j=1}^{s} \mathbb{R}(E_{\gamma_j} + \overline{E_{\gamma_j}})$$

is a maximal abelian subspace of $\mathfrak{p}_0$.

PROOF. We make repeated use of the fact that if E_β is in $\mathfrak{g}_\beta$, then $\overline{E_\beta}$ is in $\mathfrak{g}_{-\beta}$. Since $[\mathfrak{p}^+, \mathfrak{p}^+] = 0$ by Lemma 7.128, $\gamma_j + \gamma_i$ is never a root, and the γ_j's are strongly orthogonal. Then it follows that $\mathfrak{a}_0$ is abelian.

To see that $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{p}_0$, let X be a member of $\mathfrak{p}_0$ commuting with $\mathfrak{a}_0$. By (7.126) we can write $X = \sum_{\beta \in \Delta_n} X_\beta$ with $X_\beta \in \mathfrak{g}_\beta$. Without loss of generality, we may assume that X is orthogonal to $\mathfrak{a}_0$, and then we are to prove that $X = 0$. Assuming that $X \ne 0$, let β_0 be the largest member of Δ_n such that $X_{\beta_0} \ne 0$. Since $X = \overline{X}$, $X_{-\beta_0} \ne 0$ also; thus β_0 is positive. Choose j as small as possible so that β_0 is not orthogonal to γ_j.

First suppose that $\beta_0 \ne \gamma_j$. Since $[\mathfrak{p}^+, \mathfrak{p}^+] = 0$, $\beta_0 + \gamma_j$ is not a root. Therefore $\beta_0 - \gamma_j$ is a root. The root β_0 is orthogonal to $\gamma_1, \dots, \gamma_{j-1}$, and γ_j is the largest noncompact root orthogonal to $\gamma_1, \dots, \gamma_{j-1}$. Thus $\beta_0 < \gamma_j$, and $\beta_0 - \gamma_j$ is negative. We have

$$0 = [X, E_{\gamma_j} + \overline{E_{\gamma_j}}] = \sum_{\beta \in \Delta_n} ([X_\beta, E_{\gamma_j}] + [X_\beta, \overline{E_{\gamma_j}}]), \tag{7.144}$$

and $[X_{\beta_0}, \overline{E_{\gamma_j}}]$ is not 0, by Corollary 2.35. Thus there is a compensating term $[X_\beta, E_{\gamma_j}]$, i.e., there exists $\beta \in \Delta_n$ with $\beta + \gamma_j = \beta_0 - \gamma_j$ and with $X_\beta \ne 0$. Since $X = \overline{X}$, $X_{-\beta} \ne 0$. By maximality of β_0, $\beta_0 > -\beta$. Since

$\gamma_j - \beta_0$ is positive, $\gamma_j > \beta_0 > -\beta$. Therefore $\beta + \gamma_j$ is positive. But $\beta + \gamma_j = \beta_0 - \gamma_j$, and the right side is negative, contradiction.

Next suppose that $\beta_0 = \gamma_j$. Then $[X_{\gamma_j}, \overline{E_{\gamma_j}}] \neq 0$, and (7.144) gives

$$[X_{-\gamma_j}, E_{\gamma_j}] + [X_{\gamma_j}, \overline{E_{\gamma_j}}] = 0.$$

Define scalars c^+ and c^- by $X_{\gamma_j} = c^+ E_{\gamma_j}$ and $X_{-\gamma_j} = c^- \overline{E_{\gamma_j}}$. Substituting, we obtain

$$-c^-[E_{\gamma_j}, \overline{E_{\gamma_j}}] + c^+[E_{\gamma_j}, \overline{E_{\gamma_j}}] = 0,$$

and therefore $c^+ = c^-$. Consequently $X_{\gamma_j} + X_{-\gamma_j} = c^+(E_{\gamma_j} + \overline{E_{\gamma_j}})$ makes a contribution to X that is nonorthogonal to $E_{\gamma_j} + \overline{E_{\gamma_j}}$. Since the other terms of X are orthogonal to $E_{\gamma_j} + \overline{E_{\gamma_j}}$, we have a contradiction. We conclude that $X = 0$ and hence that $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{p}_0$.

Lemma 7.145. With notation as in Lemma 7.143 and with the E_{γ_j}'s normalized so that $[E_{\gamma_j}, \overline{E_{\gamma_j}}] = 2|\gamma_j|^{-2} H_{\gamma_j}$, let $Z = \sum_{j=1}^s t_j(E_{\gamma_j} + \overline{E_{\gamma_j}})$ be in $\mathfrak{a}_0$. Then

$$\exp Z = \exp X_0 \exp H_0 \exp Y_0 \tag{7.146}$$

with

$$X_0 = \sum (\tanh t_j) E_{\gamma_j} \in \mathfrak{p}^+, \qquad Y_0 = \sum (\tanh t_j) \overline{E_{\gamma_j}} \in \mathfrak{p}^-,$$
$$H_0 = -\sum (\log \cosh t_j)[E_{\gamma_j}, \overline{E_{\gamma_j}}] \in i\mathfrak{t}_0 \subseteq \mathfrak{k}.$$

Moreover the P^+ components $\exp X_0$ of $\exp Z$ remain bounded as Z varies through $\mathfrak{a}_0$.

REMARK. The given normalization is the one used with Cayley transforms in §VI.7 and in particular is permissible.

PROOF. For the special case that $G = SU(1,1) \subseteq SL(2,\mathbb{C})$, (7.146) is just the identity

$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} = \begin{pmatrix} 1 & \tanh t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\cosh t)^{-1} & 0 \\ 0 & \cosh t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tanh t & 1 \end{pmatrix}.$$

Here we are using $E_\gamma = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\overline{E_\gamma} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

We can embed the special case in the general case for each γ_j, $1 \leq j \leq s$, since the inclusion

$$\mathfrak{sl}(2, \mathbb{C}) = \mathbb{C}H_{\gamma_j} + \mathbb{C}E_{\gamma_j} + \mathbb{C}\overline{E_{\gamma_j}} \subseteq \mathfrak{g}$$

induces a homomorphism $SL(2, \mathbb{C}) \to G^{\mathbb{C}}$, $SL(2, \mathbb{C})$ being simply connected. This embedding handles each of the s terms of Z separately. Since the γ_j's are strongly orthogonal, the contributions to X_0, Y_0, and H_0 for γ_i commute with those for γ_j when $i \neq j$, and (7.146) follows for general Z.

Finally in the expression for X_0, the coefficients of each E_{γ_j} lie between -1 and $+1$ for all Z. Hence $\exp X_0$ remains bounded in P^+.

This completes the proof of Theorem 7.129. Let us see what it means in examples. First suppose that $\mathfrak{g}_0$ is simple. For $\mathfrak{c}_0$ to be nonzero, $\mathfrak{g}_0$ must certainly be noncompact. Consider the Vogan diagram of $\mathfrak{g}_0$ in a good ordering. Lemma 7.128 rules out having the sum of two positive noncompact roots be a root. Since the sum of any connected set of simple roots in a Dynkin diagram is a root, it follows that there cannot be two or more noncompact simple roots in the Vogan diagram. Hence there is just one noncompact simple root, and the Vogan diagram is one of those considered in §VI.10. Since there is just one noncompact simple root and that root cannot occur twice in any positive root, every positive noncompact root has the same restriction to $\mathfrak{c}_0$. In particular, $\dim \mathfrak{c}_0 = 1$.

To see the possibilities, we can refer to the classification in §VI.10 and see that $\mathfrak{c}_0 \neq 0$ for the following cases and only these up to isomorphism:

(7.147)

$\mathfrak{g}_0$	$\mathfrak{k}_0$
$\mathfrak{su}(p, q)$	$\mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus \mathbb{R}$
$\mathfrak{so}(2, n)$	$\mathfrak{so}(n) \oplus \mathbb{R}$
$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{su}(n) \oplus \mathbb{R}$
$\mathfrak{so}^*(2n)$	$\mathfrak{su}(n) \oplus \mathbb{R}$
E III	$\mathfrak{so}(10) \oplus \mathbb{R}$
E VII	$\mathfrak{e}_6 \oplus \mathbb{R}$

Conversely each of these cases corresponds to a group G satisfying the condition $Z_{\mathfrak{g}_0}(\mathfrak{c}_0) = \mathfrak{k}_0$, and hence G/K is Hermitian in each case.

If $\mathfrak{g}_0$ is merely semisimple, then the condition $Z_{\mathfrak{g}_0}(\mathfrak{c}_0) = \mathfrak{k}_0$ forces the center of the component of $\mathfrak{k}_0$ in each noncompact simple component of $\mathfrak{g}_0$ to be nonzero. The corresponding G/K is then the product of spaces obtained in the preceding paragraph.

10. Problems

1. Prove that the orthogonal group $O(2n)$ does not satisfy property (v) of a reductive Lie group.

2. Let $\widetilde{SL}(2,\mathbb{R})$ be the universal covering group of $SL(2,\mathbb{R})$, and let φ be the covering homomorphism. Let $\widetilde{K}$ be the subgroup of $\widetilde{SL}(2,\mathbb{R})$ fixed by the global Cartan involution Θ. Parametrize $\widetilde{K} \cong \mathbb{R}$ so that $\ker\varphi = \mathbb{Z}$. Define $\widetilde{G} = \widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$, and extend Θ to $\widetilde{G}$ so as to be 1 in the second factor. Within the subgroup $\mathbb{R} \times \mathbb{R}$ where Θ is 1, let D be the discrete subgroup generated by $(0,1)$ and $(1,\sqrt{2})$, so that D is central in $\widetilde{G}$. Define $G = \widetilde{G}/D$.
   (a) Prove that G is a connected reductive Lie group with ${}^0G = G$.
   (b) Prove that G_{ss} has infinite center and is not closed in G.

3. In $G = SL(n,\mathbb{R})$, take $M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p}$ to be the upper-triangular subgroup.
   (a) Follow the prescription of Proposition 7.76 to see that the proposition leads to all possible full block upper-triangular subgroups of $SL(n,\mathbb{R})$.
   (b) Give a direct proof for $SL(n,\mathbb{R})$ that the only closed subgroups containing $M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p}$ are the full block upper-triangular subgroups.
   (c) Give a direct proof for $SL(n,\mathbb{R})$ that no two distinct full block upper-triangular subgroups are conjugate within $SL(n,\mathbb{R})$.

4. In the notation for $G = SL(4,\mathbb{R})$ as in §VI.4, form the parabolic subgroup MAN containing the upper-triangular group and corresponding to the subset $\{f_3 - f_4\}$ of simple restricted roots.
   (a) Prove that the $\mathfrak{a}_0$ roots are $\pm(f_1 - f_2)$, $\pm(f_1 - \frac{1}{2}(f_3 + f_4))$, and $\pm(f_2 - \frac{1}{2}(f_3 + f_4))$.
   (b) Prove that the $\mathfrak{a}_0$ roots do not all have the same length and do not form a root system.

5. Show that a maximal proper parabolic subgroup MAN of $SL(3,\mathbb{R})$ is cuspidal and that $M \neq M_0 Z_M$.

6. For G equal to split G_2, show that there is a cuspidal maximal proper parabolic subgroup MAN such that the set of $\mathfrak{a}_0$ roots is of the form $\{\pm\eta, \pm 2\eta, \pm 3\eta\}$.

7. The group $G = Sp(2,\mathbb{R})$ has at most four nonconjugate Cartan subalgebras, according to §VI.7, and a representative of each conjugacy class is given in that section.
   (a) For each of the four, construct the MA of an associated cuspidal parabolic subgroup as in Proposition 7.87.
   (b) Use the result of (a) to show that the two Cartan subalgebras of noncompact dimension one are not conjugate.

8. Let G be $SO(n,2)_0$.
   (a) Show that $G^\mathbb{C} \cong SO(n+2,\mathbb{C})$.

(b) Show that $Z_{\mathfrak{g}_0}(\mathfrak{c}_0) = \mathfrak{k}_0$.

(c) The isomorphism in (a) identifies the root system of $SO(n, 2)$ as of type $B_{(n+1)/2}$ if n is odd and of type $D_{(n+2)/2}$ if n is even. Identify which roots are compact and which are noncompact.

(d) Decide on some particular good ordering in the sense of §9, and identify the positive roots.

Problems 9–12 concern a reductive Lie group G. Notation is as in §2.

9. Let $\mathfrak{a}_0$ be maximal abelian in $\mathfrak{p}_0$. The natural inclusion $N_K(\mathfrak{a}_0) \subseteq N_G(\mathfrak{a}_0)$ induces a homomorphism $N_K(\mathfrak{a}_0)/Z_K(\mathfrak{a}_0) \to N_G(\mathfrak{a}_0)/Z_G(\mathfrak{a}_0)$. Prove that this homomorphism is an isomorphism.

10. Let $\mathfrak{t}_0 \oplus \mathfrak{a}_0$ be a maximally noncompact θ stable Cartan subalgebra of $\mathfrak{g}_0$. Prove that every element of $N_K(\mathfrak{a}_0)$ decomposes as a product zn, where n is in $N_K(\mathfrak{t}_0 \oplus \mathfrak{a}_0)$ and z is in $Z_K(\mathfrak{a}_0)$.

11. Let H be a Cartan subgroup of G, and let s_α be a root reflection in $W(\mathfrak{g}, \mathfrak{h})$.

(a) Prove that s_α is in $W(G, H)$ if α is real or α is compact imaginary.

(b) Prove that if H is compact and G is connected, then s_α is not in $W(G, H)$ when α is noncompact imaginary.

(c) Give an example of a reductive Lie group G with a compact Cartan subgroup H such that s_α is in $W(G, H)$ for some noncompact imaginary root α.

12. Let $H = TA$ be the global Cartan decomposition of a Θ stable Cartan subgroup of G. Let $W(G, A) = N_G(\mathfrak{a}_0)/Z_G(\mathfrak{a}_0)$, and let $M = {}^0Z_G(\mathfrak{a}_0)$. Let $W_1(G, H)$ be the subgroup of $W(G, H)$ of elements normalizing $i\mathfrak{t}_0$ and $\mathfrak{a}_0$ separately.

(a) Show that restriction to $\mathfrak{a}_0$ defines a homomorphism of $W_1(G, H)$ into $W(G, A)$.

(b) Prove that the homomorphism in (a) is onto.

(c) Prove that the kernel of the homomorphism in (a) may be identified with $W(M, T)$.

Problems 13–21 concern a reductive Lie group G that is semisimple. Notation is as in §2.

13. Let $\mathfrak{t}_0 \oplus \mathfrak{a}_0$ be a maximally noncompact θ stable Cartan subalgebra of $\mathfrak{g}_0$, impose an ordering on the roots that takes $\mathfrak{a}_0$ before $i\mathfrak{t}_0$, let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{t} \oplus \mathfrak{a}$ and built from that ordering, and let bar denote the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Prove that the smallest Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ and $\bar{\mathfrak{b}}$ is the complexification of a minimal parabolic subalgebra of $\mathfrak{g}_0$.

14. Prove that $N_{\mathfrak{g}_0}(\mathfrak{k}_0) = \mathfrak{k}_0$.

15. Let G have a complexification $G^{\mathbb{C}}$. Prove that the normalizer of $\mathfrak{g}_0$ in $G^{\mathbb{C}}$ is a reductive Lie group.

16. Let G have a complexification $G^{\mathbb{C}}$, let $U \subseteq G^{\mathbb{C}}$ be the analytic subgroup with Lie algebra $\mathfrak{k}_0 \oplus i\mathfrak{p}_0$, and let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be the decomposition into $+1$ and -1 eigenspaces of a θ stable Cartan subalgebra of $\mathfrak{g}_0$. Prove that $\exp i\mathfrak{a}_0$ is closed in U.

17. Give an example of a semisimple G with complexification $G^{\mathbb{C}}$ such that $K \cap \exp i\mathfrak{a}_0$ strictly contains $K_{\text{split}} \cap \exp i\mathfrak{a}_0$. Here $\mathfrak{a}_0$ is assumed maximal abelian in $\mathfrak{p}_0$.

18. Suppose that G has a complexification $G^{\mathbb{C}}$ and that $\operatorname{rank} G = \operatorname{rank} K$. Prove that $Z_{G^{\mathbb{C}}} = Z_G$.

19. Suppose that $\operatorname{rank} G = \operatorname{rank} K$. Prove that any two complexifications of G are holomorphically isomorphic.

20. Show that the conclusions of Problems 18 and 19 are false for $G = SL(3, \mathbb{R})$.

21. Suppose that G/K is Hermitian and that $\mathfrak{g}_0$ is simple. Show that there are only two ways to impose a G invariant complex structure on G/K.

Problems 22–24 compare the integer span of the roots with the integer span of the compact roots. It is assumed that G is a reductive Lie group with $\operatorname{rank} G = \operatorname{rank} K$.

22. Fix a positive system Δ^+. Attach to each simple noncompact root the integer 1 and to each simple compact root the integer 0; extend additively to the group generated by the roots, obtaining a function $\gamma \mapsto n(\gamma)$. Arguing as in Lemma 6.98, prove that $n(\gamma)$ is odd when γ is a positive noncompact root and is even when γ is a positive compact root.

23. Making use of the function $\gamma \mapsto (-1)^{n(\gamma)}$, prove that a noncompact root can never be an integer combination of compact roots.

24. Suppose that G is semisimple, that $\mathfrak{g}_0$ is simple, and that G/K is not Hermitian. Prove that the lattice generated by the compact roots has index 2 in the lattice generated by all the roots.

Problems 25–29 give further properties of semisimple groups with $\operatorname{rank} G = \operatorname{rank} K$. Let $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$, and form roots, compact and noncompact.

25. K acts on $\mathfrak{p}$ via the adjoint representation. Identify the weights as the noncompact roots, showing in particular that 0 is not a weight.

26. Show that the subalgebras of $\mathfrak{g}$ containing $\mathfrak{k}$ are of the form $\mathfrak{k} \oplus \bigoplus_{\alpha \in E} \mathfrak{g}_\alpha$ for some subset E of noncompact roots.

27. Suppose that $\mathfrak{k} \oplus \bigoplus_{\alpha \in E} \mathfrak{g}_\alpha$ is a subalgebra of $\mathfrak{g}$. Prove that

$$\mathfrak{k} \oplus \sum_{\alpha \in E} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \qquad \text{and} \qquad \mathfrak{k} \oplus \bigoplus_{\alpha \in (E \cap (-E))} \mathfrak{g}_\alpha$$

are subalgebras of $\mathfrak{g}$ that are the complexifications of subalgebras of $\mathfrak{g}_0$.

28. Suppose that $\mathfrak{g}_0$ is simple. Prove that the adjoint representation of K on $\mathfrak{p}$ splits into at most two irreducible pieces.

29. Suppose that $\mathfrak{g}_0$ is simple, and suppose that the adjoint representation of K on $\mathfrak{p}$ is reducible (necessarily into two pieces, according to Problem 28). Show that the center $\mathfrak{c}_0$ of $\mathfrak{k}_0$ is nonzero, that $Z_{\mathfrak{g}_0}(\mathfrak{c}_0) = \mathfrak{k}_0$, and that the irreducible pieces are $\mathfrak{p}^+$ and $\mathfrak{p}^-$.

Problems 30–33 concern the group $G = SU(n, n) \cap Sp(n, \mathbb{C})$. In the notation of §9, let Ω be the set of all $Z \in M_{nn}(\mathbb{C})$ such that $1_n - Z^*Z$ is positive definite and $Z = Z^t$.

30. Using Problem 15b from Chapter VI, prove that $G \cong Sp(n, \mathbb{R})$.

31. With the members of G written in block form, show that (7.112) defines an action of G on Ω by holomorphic transformations.

32. Identify the isotropy subgroup of G at 0 with

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \middle| \, A \in U(n) \right\}.$$

33. The diagonal subalgebra of $\mathfrak{g}_0$ is a compact Cartan subalgebra. Exhibit a good ordering such that $\mathfrak{p}^+$ consists of block strictly upper-triangular matrices.

Problems 34–36 concern the group $G = SO^*(2n)$. In the notation of §9, let Ω be the set of all $Z \in M_{nn}(\mathbb{C})$ such that $1_n - Z^*Z$ is positive definite and $Z = -Z^t$.

34. With the members of G written in block form, show that (7.112) defines an action of G on Ω by holomorphic transformations.

35. Identify the isotropy subgroup of G at 0 with

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \middle| \, A \in U(n) \right\}.$$

36. The diagonal subalgebra of $\mathfrak{g}_0$ is a compact Cartan subalgebra. Exhibit a good ordering such that $\mathfrak{p}^+$ consists of block strictly upper-triangular matrices.

Problems 37–41 concern the restricted roots in cases when G is semisimple and G/K is Hermitian.

37. In the example of §9 with $G = SU(n, m)$,

(a) show that the roots γ_j produced in Lemma 7.143 are $\gamma_1 = e_1 - e_{n+m}$, $\gamma_2 = e_2 - e_{n+m-1}, \dots, \gamma_m = e_m - e_{m+1}$.

(b) show that the restricted roots (apart from Cayley transform) always include all $\pm\gamma_j$ and all $\frac{1}{2}(\pm\gamma_i \pm \gamma_j)$. Show that there are no other restricted roots if $m = n$ and that $\pm\frac{1}{2}\gamma_i$ are the only other restricted roots if $m < n$.

38. In the example of Problems 30–33 with $G = SU(n,n) \cap Sp(n,\mathbb{C})$, a group that is shown in Problem 30 to be isomorphic to $Sp(n,\mathbb{R})$,

(a) show that the roots γ_j produced in Lemma 7.143 are $\gamma_1 = 2e_1, \dots, \gamma_n = 2e_n$.

(b) show that the restricted roots (apart from Cayley transform) are all $\pm\gamma_j$ and all $\frac{1}{2}(\pm\gamma_i \pm \gamma_j)$.

39. In the example of Problem 6 of Chapter VI and Problems 34–36 above with $G = SO^*(2n)$,

(a) show that the roots γ_j produced in Lemma 7.143 are $\gamma_1 = e_1 + e_n$, $\gamma_2 = e_2 + e_{n-1}, \dots, \gamma_{[n/2]} = e_{[n/2]} + e_{n-[n/2]+1}$.

(b) find the restricted roots apart from Cayley transform.

40. For general G with G/K Hermitian, suppose that α, β, and γ are roots with α compact and with β and γ positive noncompact in a good ordering. Prove that $\alpha + \beta$ and $\alpha + \beta + \gamma$ cannot both be roots.

41. Let the expansion of a root in terms of Lemma 7.143 be $\gamma = \sum_{i=1}^{s} c_i\gamma_i + \gamma'$ with γ' orthogonal to $\gamma_1, \dots, \gamma_s$.

(a) Prove for each i that $2c_i$ is an integer with $|2c_i| \le 3$.

(b) Rule out $c_i = -\frac{3}{2}$ by using Problem 40 and the γ_i string containing γ, and rule out $c_i = +\frac{3}{2}$ by applying this conclusion to $-\gamma$.

(c) Rule out $c_i = \pm 1$ for some $j \ne i$ by a similar argument.

(d) Show that $c_i \ne 0$ for at most two indices i by a similar argument.

(e) Deduce that each restricted root, apart from Cayley transform, is of one of the forms $\pm\gamma_i$, $\frac{1}{2}(\pm\gamma_i \pm \gamma_j)$, or $\pm\frac{1}{2}\gamma_i$.

(f) If $\mathfrak{g}_0$ is simple, conclude that the restricted root system is of type $(BC)_s$ or C_s.

Problems 42–44 yield a realization of G/K, in the Hermitian case, as a particularly nice unbounded open subset Ω' of P^+. Let notation be as in §9.

42. In the special case that $G = SU(1,1)$, let u be the Cayley transform matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$, let $G' = SL(2,\mathbb{R})$, and let

$$\Omega' = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \,\middle|\, \operatorname{Im} z > 0 \right\}.$$

It is easily verified that $uGu^{-1} = G'$. Prove that $uGB = G'uB = \Omega' K^{\mathbb{C}} P^-$ and that G' acts on Ω' by the usual action of $SL(2, \mathbb{R})$ on the upper half plane.

43. In the general case as in §9, let $\gamma_1, \ldots, \gamma_s$ be constructed as in Lemma 7.143. For each j, construct an element u_j in $G^{\mathbb{C}}$ that behaves for the 3-dimensional group corresponding to γ_j like the element u of Problem 42. Put $u = \prod_{j=1}^{s} u_j$.
    (a) Exhibit u as in $P^+ K^{\mathbb{C}} P^-$.
    (b) Let $\mathfrak{a}_0$ be the maximal abelian subspace of $\mathfrak{p}_0$ constructed in Lemma 7.143, and let $A_{\mathfrak{p}} = \exp \mathfrak{a}_0$. Show that $uA_{\mathfrak{p}}u^{-1} \subseteq K^{\mathbb{C}}$.
    (c) Show for a particular ordering on $\mathfrak{a}_0^*$ that $uN_{\mathfrak{p}}u^{-1} \subseteq P^+ K^{\mathbb{C}}$ if $N_{\mathfrak{p}}$ is built from the positive restricted roots.
    (d) Writing $G = N_{\mathfrak{p}} A_{\mathfrak{p}} K$ by the Iwasawa decomposition, prove that $uGB \subseteq P^+ K^{\mathbb{C}} P^-$.

44. Let $G' = uGu^{-1}$. Prove that $G'uB = \Omega' K^{\mathbb{C}} P^-$ for some open subset Ω' of P^+. Prove also that the resulting action of G' on Ω' is holomorphic and transitive, and identify Ω' with G/K.

Problems 45–51 give further information about quasisplit Lie algebras and inner forms, which were introduced in Problems 28–35 of Chapter VI. Fix a complex semisimple Lie algebra $\mathfrak{g}$, and let N be the order of the automorphism group of the Dynkin diagram of $\mathfrak{g}$. If $\mathfrak{g}$ is simple, then N is 1, 2, or 6, but other values of N are possible for general complex semisimple $\mathfrak{g}$.

45. For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C})$ with $n > 2$, show that $\mathfrak{sl}(n, \mathbb{R}) \oplus \mathfrak{su}(n)$ and $\mathfrak{su}(n) \oplus \mathfrak{sl}(n, \mathbb{R})$ are isomorphic real forms of $\mathfrak{g}$ but are not inner forms of one another.

46. Prove the following:
    (a) The number of inner classes of real forms of $\mathfrak{g}$ is $\leq N$.
    (b) The number of isomorphism classes of quasisplit real forms of $\mathfrak{g}$ is $\leq N$.
    (c) If the number of isomorphism classes of quasisplit real forms equals N, then the number of inner classes of real forms of $\mathfrak{g}$ equals N and any two isomorphic real forms of $\mathfrak{g}$ are inner forms of one another.

47. Under the assumption that $N = 1$, deduce the following from Problem 46:
    (a) Any two real forms of $\mathfrak{g}$ are inner forms of one another.
    (b) The Lie algebra $\mathfrak{g}$ has no real form that is quasisplit but not split.

48. Prove that $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})/\mathrm{Int}(\mathfrak{g}^{\mathbb{R}})$ has order $2N$ if $\mathfrak{g}$ is simple.

49. Under the assumption that $N = 2$, deduce from Problems 46 and 48 that any two isomorphic real forms of $\mathfrak{g}$ are inner forms of one another.

50. By referring to the tables in Appendix C, observe that there are 2 nonisomorphic quasisplit real forms of each of the complex simple Lie algebras of types

A_n for $n > 1$, D_n for $n > 4$, and E_6. Conclude that there are two inner classes of real forms in each case and that any two isomorphic real forms are inner forms of one another.

51. This problem uses **triality**, which, for current purposes, refers to members of $\operatorname{Aut}\mathfrak{g}/\operatorname{Int}\mathfrak{g}$ of order 3 when $\mathfrak{g}$ is a complex Lie algebra of type D_4. The objective is to show that $\mathfrak{g} = \mathfrak{so}(8, \mathbb{C})$ contains at least two distinct real forms $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ that are isomorphic to $\mathfrak{so}(5, 3)$ but that are not inner forms of one another. Let $\mathfrak{g}_0$ be a Lie algebra isomorphic to $\mathfrak{so}(5, 3)$, let θ be a Cartan involution, and introduce a maximally noncompact Cartan subalgebra given in standard notation by $\mathfrak{h}_0 = \mathfrak{a}_0 \oplus \mathfrak{t}_0$. Choose an ordering that takes $\mathfrak{a}_0$ before $i\mathfrak{t}_0$. In the usual notation for a Dynkin diagram of type D_4, the simple roots $e_1 - e_2$ and $e_2 - e_3$ are real, and $e_3 - e_4$ and $e_3 + e_4$ are complex. Introduce an automorphism τ of $\mathfrak{so}(8, \mathbb{C})$ that corresponds to a counterclockwise rotation τ of the D_4 diagram through 1/3 of a revolution. Put $\mathfrak{g}_0' = \tau(\mathfrak{g}_0)$. For a suitable normalization of root vectors used in defining τ, show that the conjugations σ and σ' of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ satisfy $\sigma'\sigma = \tau^{-1}$, and conclude that $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ are not inner forms of one another.

Problems 52–57 give further information about groups of real rank one beyond that in §6. Let G be an analytic group whose Lie algebra $\mathfrak{g}$ is simple of real rank one, let θ be a Cartan involution of $\mathfrak{g}$, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition, let $\mathfrak{a}$ be a (1-dimensional) maximal abelian subspace of $\mathfrak{p}$, let $\mathfrak{g} = \mathfrak{g}_{-2\beta} \oplus \mathfrak{g}_{-\beta} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_{\beta} \oplus \mathfrak{g}_{2\beta}$ be the restricted-root space decomposition, and let m_β and $m_{2\beta}$ be the dimensions of $\mathfrak{g}_\beta$ and $\mathfrak{g}_{2\beta}$. Select a maximal abelian subspace $\mathfrak{t}$ of $\mathfrak{m}$, so that the restricted roots are the restrictions to $\mathfrak{a}$ of the roots relative to the Cartan subalgebra $\mathfrak{a} \oplus \mathfrak{t}$. Let $\mathfrak{g}_1 = \mathfrak{g}_{-2\beta} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_{2\beta}$ and $\mathfrak{k}_1 = \mathfrak{g}_1 \cap \mathfrak{k}$. Finally let K, A, G_1, and K_1 be the analytic subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{g}_1$, and $\mathfrak{k}_1$, and let M be the centralizer of A in K.

52. If α is a root, write $\alpha_R + \alpha_I$ with α_R the restriction to α and α_I the restriction to $\mathfrak{t}$. The complex conjugate root is $\bar{\alpha} = \alpha_R - \alpha_I$. Suppose α is complex.
    (a) Prove that $2\langle\alpha, \bar{\alpha}\rangle/|\alpha|^2$ is 0 or -1.
    (b) Prove that $2\langle\alpha, \bar{\alpha}\rangle/|\alpha|^2 = 0$ implies $|\alpha|^2 = \frac{1}{2}|2\alpha_R|^2$ and that $2\langle\alpha, \bar{\alpha}\rangle/|\alpha|^2 = -1$ implies $|\alpha|^2 = |2\alpha_R|^2$.

53. Prove that if m_β and $m_{2\beta}$ are both nonzero, then 2β is a root when extended to be 0 on $\mathfrak{t}$. Conclude that m_β is even and $m_{2\beta}$ is odd.

54. Prove that if $m_{2\beta} \neq 0$ and α is a complex root with $2\langle\alpha, \bar{\alpha}\rangle/|\alpha|^2 = 0$, then α_R is $\pm 2\beta$.

55. Prove that if m_β and $m_{2\beta}$ are both nonzero, then $\mathfrak{g}$ has a Cartan subalgebra that lies in $\mathfrak{k}$. Prove that this Cartan subalgebra may be assumed to be of the form $\mathfrak{t} \oplus \mathbb{R}(X + \theta X)$ with $X \in \mathfrak{g}_{2\beta}$, so that it lies in $\mathfrak{k}_1$.

56. Suppose that $m_{2\beta} \neq 0$ and that $\mathfrak{g}$ has a Cartan subalgebra lying in $\mathfrak{k}$. Prove the following:
   (a) 2β is a root when extended to be 0 on $\mathfrak{t}$.
   (b) If there are roots of two different lengths, then every noncompact root is short.

57. Suppose that G has a complexification $G^{\mathbb{C}}$, that $m_{2\beta} \neq 0$, and that $\mathfrak{g}$ has a Cartan subalgebra lying in $\mathfrak{k}_1$. Problem 10 of Chapter VI produces an element g_θ of G such that $\mathrm{Ad}(g_\theta) = \theta$, and (7.54) produces a certain element $\gamma_{2\beta}$ in M. Prove the following:
   (a) $\mathrm{Ad}(\gamma_{2\beta}) = -1$ on $\mathfrak{g}_\beta$ and $\mathfrak{g}_{-\beta}$.
   (b) $\gamma_{2\beta}$ is in the center of M, the center of K_1, and the center of G_1, but it is not in the center of K if $m_\beta \neq 0$.
   (c) g_θ is in the center of K_1 and the center of K, but it is not in M and is not in the center of G.