# V. Finite-Dimensional Representations, 273-346

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## *Lie Groups Beyond an Introduction*

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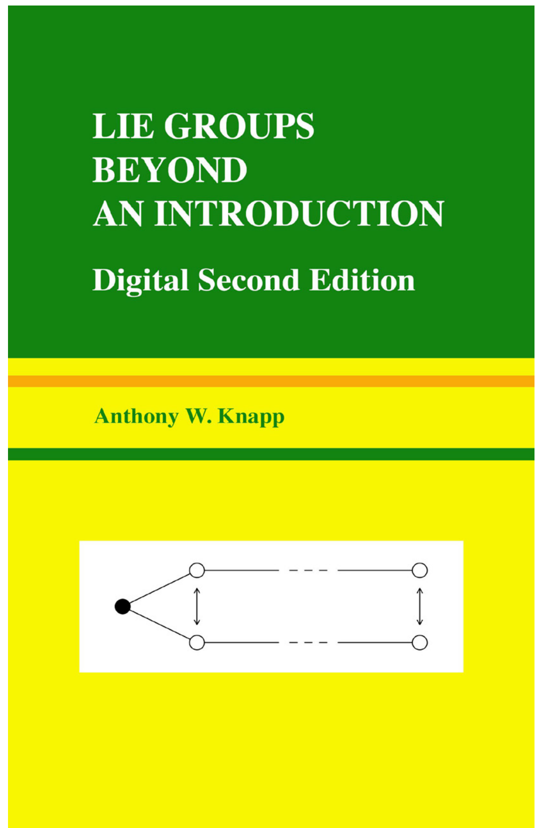


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**Lie Groups Beyond an Introduction, Digital Second Edition**

Pages vii–xviii and 1–812 are the same in the digital and printed second editions. A list of corrections as of June 2023 has been included as pages 813–820 of the digital second edition. The corrections have not been implemented in the text.

Cover: Vogan diagram of $\mathfrak{sl}(2n, \mathbb{R})$. See page 399.

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# CHAPTER V

# Finite-Dimensional Representations

**Abstract.** In any finite-dimensional representation of a complex semisimple Lie algebra $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$ acts completely reducibly, the simultaneous eigenvalues being called "weights." Once a positive system for the roots $\Delta^+(\mathfrak{g}, \mathfrak{h})$ has been fixed, one can speak of highest weights. The Theorem of the Highest Weight says that irreducible finite-dimensional representations are characterized by their highest weights and that the highest weight can be any dominant algebraically integral linear functional on $\mathfrak{h}$. The hard step in the proof is the construction of an irreducible representation corresponding to a given dominant algebraically integral form. This step is carried out by using "Verma modules," which are universal highest weight modules.

All finite-dimensional representations of $\mathfrak{g}$ are completely reducible. Consequently the nature of such a representation can be determined from the representation of $\mathfrak{h}$ in the space of "$\mathfrak{n}$ invariants." The Harish-Chandra Isomorphism identifies the center of the universal enveloping algebra $U(\mathfrak{g})$ with the Weyl-group invariant members of $U(\mathfrak{h})$. The proof uses the complete reducibility of finite-dimensional representations of $\mathfrak{g}$.

The center of $U(\mathfrak{g})$ acts by scalars in any irreducible representation of $\mathfrak{g}$, whether finite dimensional or infinite dimensional. The result is a homomorphism of the center into $\mathbb{C}$ and is known as the "infinitesimal character" of the representation. The Harish-Chandra Isomorphism makes it possible to parametrize all possible homomorphisms of the center into $\mathbb{C}$, thus to parametrize all possible infinitesimal characters. The parametrization is by the quotient of $\mathfrak{h}^*$ by the Weyl group.

The Weyl Character Formula attaches to each irreducible finite-dimensional representation a formal exponential sum corresponding to the character of the representation. The proof uses infinitesimal characters. The formula encodes the multiplicity of each weight, and this multiplicity is made explicit by the Kostant Multiplicity Formula. The formula encodes also the dimension of the representation, which is made explicit by the Weyl Dimension Formula.

Parabolic subalgebras provide a framework for generalizing the Theorem of the Highest Weight so that the Cartan subalgebra is replaced by a larger subalgebra called the "Levi factor" of the parabolic subalgebra.

The theory of finite-dimensional representations of complex semisimple Lie algebras has consequences for compact connected Lie groups. One of these is a formula for the order of the fundamental group. Another is a version of the Theorem of the Highest Weight that takes global properties of the group into account. The Weyl Character Formula becomes more explicit, giving an expression for the character of any irreducible representation when restricted to a maximal torus.

## 1. Weights

For most of this chapter we study finite-dimensional representations of complex semisimple Lie algebras. As introduced in Example 4 of §I.5, these are complex-linear homomorphisms of a complex semisimple Lie algebra into $\mathrm{End}_{\mathbb{C}} V$, where $V$ is a finite-dimensional complex vector space. Historically the motivation for studying such representations comes from two sources—representations of $\mathfrak{sl}(2, \mathbb{C})$ and representations of compact Lie groups. Representations of $\mathfrak{sl}(2, \mathbb{C})$ were studied in §I.9, and the theory of the present chapter may be regarded as generalizing the results of that section to all complex semisimple Lie algebras.

Representations of compact connected Lie groups were studied in Chapter IV. If $G$ is a compact connected Lie group, then a representation of $G$ on a finite-dimensional complex vector space $V$ yields a representation of the Lie algebra $\mathfrak{g}_0$ on $V$ and then a representation of the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ on $V$. The Lie algebra $\mathfrak{g}_0$ is the direct sum of an abelian Lie algebra and a semisimple Lie algebra, and the same thing is true of $\mathfrak{g}$. Through studying the representations of the semisimple part of $\mathfrak{g}$, we shall be able, with only little extra effort, to complete the study of the representations of $G$ at the end of this chapter.

The examples of representations in Chapter IV give us examples for the present chapter, as well as clues for how to proceed. The easy examples, apart from the trivial representation with $\mathfrak{g}$ acting as 0, are the standard representations of $\mathfrak{su}(n)^{\mathbb{C}}$ and $\mathfrak{so}(n)^{\mathbb{C}}$. These are obtained by differentiation of the standard representations of $SU(n)$ and $SO(n)$ and just amount to multiplication of a matrix by a column vector, namely

$$\varphi(X) \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = X \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}.$$

The differentiated versions of the other examples in §IV.1 are more complicated because they involve tensor products. Although tensor products on the group level (4.2) are fairly simple, they become more complicated on the Lie algebra level (4.3) because of the product rule for differentiation. This complication persists for representations in spaces of symmetric or alternating tensors, since such spaces are subspaces of tensor products. Thus the usual representation of $SU(n)$ on $\bigwedge^l \mathbb{C}^n$ is given simply by

$$\Phi(g)(\varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_l}) = g\varepsilon_{j_1} \wedge \cdots \wedge g\varepsilon_{j_l},$$

while the corresponding representation of $\mathfrak{su}(n)^{\mathbb{C}}$ on $\bigwedge^l \mathbb{C}^n$ is given by

$$\varphi(X)(\varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_l}) = \sum_{k=1}^{l} \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_{k-1}} \wedge X\varepsilon_{j_k} \wedge \varepsilon_{j_{k+1}} \wedge \cdots \wedge \varepsilon_{j_l}.$$

The second construction that enters the examples of §IV.1 is contragredient, given on the Lie group level by (4.1) and on the Lie algebra level by (4.4). Corollary A.24b, with $E = \mathbb{C}^n$, shows that the representation in a space $S^n(E^*)$ of polynomials may be regarded as the contragredient of the representation in the space $S^n(E)$ of symmetric tensors.

The clue for how to proceed comes from the representation theory of compact connected Lie groups $G$ in Chapter IV. Let $\mathfrak{g}_0$ be the Lie algebra of $G$, and let $\mathfrak{g}$ be the complexification. If $T$ is a maximal torus in $G$, then the complexified Lie algebra of $T$ is a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Insight into $\mathfrak{g}$ comes from roots relative to $\mathfrak{t}$, which correspond to simultaneous eigenspaces for the action of $T$, according to (4.32). If $\Phi$ is any finite-dimensional representation of $G$ on a complex vector space $V$, then $\Phi$ may be regarded as unitary by Proposition 4.6. Hence $\Phi|_T$ is unitary, and Corollary 4.7 shows that $\Phi|_T$ splits as the direct sum of irreducible representations of $T$. By Corollary 4.9 each of these irreducible representations of $T$ is 1-dimensional. Thus $V$ is the direct sum of simultaneous eigenspaces for the action of $T$, hence also for the action of $\mathfrak{t}$.

At first this kind of decomposition seems unlikely to persist when the compact groups are dropped and we have only a representation of a complex semisimple Lie algebra, since Proposition 2.4 predicts only a generalized weight-space decomposition. But a decomposition into simultaneous eigenspaces is nonetheless valid and is the starting point for our investigation. Before coming to this, let us record that the proofs of Schur's Lemma and its corollary in §IV.2 are valid for representations of Lie algebras.

**Proposition 5.1** (Schur's Lemma). Suppose $\varphi$ and $\varphi'$ are irreducible representations of a Lie algebra $\mathfrak{g}$ on finite-dimensional vector spaces $V$ and $V'$, respectively. If $L : V \to V'$ is a linear map such that $\varphi'(X)L = L\varphi(X)$ for all $X \in \mathfrak{g}$, then $L$ is one-one onto or $L = 0$.

PROOF. We see easily that $\ker L$ and image $L$ are invariant subspaces of $V$ and $V'$, respectively, and then the only possibilities are the ones listed.

**Corollary 5.2.** Suppose $\varphi$ is an irreducible representation of a Lie algebra $\mathfrak{g}$ on a finite-dimensional complex vector space $V$. If $L : V \to V$ is a linear map such that $\varphi(X)L = L\varphi(X)$ for all $X \in \mathfrak{g}$, then $L$ is scalar.

PROOF. Let $\lambda$ be an eigenvalue of $L$. Then $L - \lambda I$ is not one-one onto, but it does commute with $\varphi(X)$ for all $X \in \mathfrak{g}$. By Proposition 5.1, $L - \lambda I = 0$.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$, and let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the set of roots. Following the notation first introduced in Corollary 2.38, let $\mathfrak{h}_0$ be the real form of $\mathfrak{h}$ on which all roots are real valued. Let $B$ be any nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ that is positive definite on $\mathfrak{h}_0$. Relative to $B$, we can define members $H_\alpha$ of $\mathfrak{h}$ for each $\alpha \in \Delta$. Then $\mathfrak{h}_0 = \sum_{\alpha\in\Delta} \mathbb{R}H_\alpha$.

Let $\varphi$ be a representation on the complex vector space $V$. Recall from §II.2 that if $\lambda$ is in $\mathfrak{h}^*$, we define $V_\lambda$ to be the subspace

$$\{v \in V \mid (\varphi(H) - \lambda(H)1)^n v = 0 \text{ for all } H \in \mathfrak{h} \text{ and some } n = n(H, V)\}.$$

If $V_\lambda \neq 0$, then $V_\lambda$ is called a **generalized weight space** and $\lambda$ is a **weight**. Members of $V_\lambda$ are called **generalized weight vectors**. When $V$ is finite dimensional, $V$ is the direct sum of its generalized weight spaces by Proposition 2.4.

The **weight space** corresponding to $\lambda$ is

$$\{v \in V \mid \varphi(H)v = \lambda(H)v \text{ for all } H \in \mathfrak{h}\},$$

i.e., the subspace of $V_\lambda$ for which $n$ can be taken to be 1. Members of the weight space are called **weight vectors**. The examples of weight vectors below continue the discussion of examples in §IV.1.

EXAMPLES FOR $G = SU(n)$. Here $\mathfrak{g} = \mathfrak{su}(n)^{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$. As in Example 1 of §II.1, we define $\mathfrak{h}$ to be the diagonal subalgebra. The roots are all $e_i - e_j$ with $i \neq j$.

1) Let $V$ consist of all polynomials in $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$ homogeneous of degree $N$. Let $H = \operatorname{diag}(it_1, \dots, it_n)$ with $\sum t_j = 0$. Then the Lie algebra representation $\varphi$ has

$$\begin{aligned}
\varphi(H)P(z, \bar{z}) &= \frac{d}{dr} P\left(e^{-rH}\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}, e^{rH}\begin{pmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{pmatrix}\right)_{r=0} \\
&= \frac{d}{dr} P\left(\begin{pmatrix} e^{-irt_1}z_1 \\ \vdots \\ e^{-irt_n}z_n \end{pmatrix}, \begin{pmatrix} e^{irt_1}\bar{z}_1 \\ \vdots \\ e^{irt_n}\bar{z}_n \end{pmatrix}\right)_{r=0} \\
&= \sum_{j=1}^n (-it_j z_j)\frac{\partial P}{\partial z_j}(z, \bar{z}) + \sum_{j=1}^n (it_j \bar{z}_j)\frac{\partial P}{\partial \bar{z}_j}(z, \bar{z}).
\end{aligned}$$

If $P$ is a monomial of the form

$$P(z,\bar z) = z_1^{k_1}\cdots z_n^{k_n}\bar z_1^{l_1}\cdots \bar z_n^{l_n} \qquad \text{with } \sum_{j=1}^n (k_j + l_j) = N,$$

then the above expression simplifies to

$$\varphi(H)P = \Big(\sum_{j=0}^n (l_j - k_j)(it_j)\Big)P.$$

Thus the monomial $P$ is a weight vector of weight $\sum_{j=0}^n (l_j - k_j)e_j$.

2) Let $V = \bigwedge^l \mathbb{C}^n$. Again let $H = \operatorname{diag}(it_1,\dots,it_n)$ with $\sum t_j = 0$. Then the Lie algebra representation $\varphi$ has

$$\begin{aligned}\varphi(H)(\varepsilon_{j_1}\wedge\cdots\wedge\varepsilon_{j_l}) &= \sum_{k=1}^l \varepsilon_{j_1}\wedge\cdots\wedge H\varepsilon_{j_k}\wedge\cdots\wedge\varepsilon_{j_l}\\ &= \sum_{k=1}^l (it_{j_k})(\varepsilon_{j_1}\wedge\cdots\wedge\varepsilon_{j_l}).\end{aligned}$$

Thus $\varepsilon_{j_1}\wedge\cdots\wedge\varepsilon_{j_l}$ is a weight vector of weight $\sum_{k=1}^l e_{j_k}$.

EXAMPLES FOR $G = SO(2n+1)$. Here $\mathfrak{g} = \mathfrak{so}(2n+1)^{\mathbb{C}} = \mathfrak{so}(2n+1,\mathbb{C})$. As in Example 2 of §II.1, we define $\mathfrak{h}$ to be built from the first $n$ diagonal blocks of size 2. The roots are $\pm e_j$ and $\pm e_i \pm e_j$ with $i \neq j$.

1) Let $m = 2n+1$, and let $V$ consist of all complex-valued polynomials on $\mathbb{R}^m$ of degree $\le N$. Let $H_1$ be the member of $\mathfrak{h}$ equal to $\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$ in the first 2-by-2 block and 0 elsewhere. Then the Lie algebra representation $\varphi$ has

(5.3)

$$\varphi(H_1)P\begin{pmatrix}x_1\\ \vdots\\ x_m\end{pmatrix} = \frac{d}{dr}P\begin{pmatrix}x_1\cos r - x_2\sin r\\ x_1\sin r + x_2\cos r\\ x_3\\ \vdots\\ x_m\end{pmatrix}\Bigg|_{r=0} = -x_2\frac{\partial P}{\partial x_1}(x) + x_1\frac{\partial P}{\partial x_2}(x).$$

For $P(x) = (x_1 + ix_2)^k$, $\varphi(H_1)$ thus acts as the scalar $ik$. The other 2-by-2 blocks of $\mathfrak{h}$ annihilate this $P$, and it follows that $(x_1 + ix_2)^k$ is a weight vector of weight $-ke_1$. Similarly $(x_1 - ix_2)^k$ is a weight vector of weight $+ke_1$.

Replacing $P$ in (5.3) by $(x_{2j-1} \pm x_{2j})Q$ and making the obvious adjustments in the computation, we obtain

$$\varphi(H)((x_{2j-1} \pm ix_{2j})Q) = (x_{2j-1} \pm ix_{2j})(\varphi(H) \mp e_j(H))Q \qquad \text{for } H \in \mathfrak{h}.$$

Since $x_{2j-1} + ix_{2j}$ and $x_{2j-1} - ix_{2j}$ together generate $x_{2j-1}$ and $x_{2j}$ and since $\varphi(H)$ acts as 0 on $x_{2n+1}^k$, this equation tells us how to compute $\varphi(H)$ on any monomial, hence on any polynomial.

It is clear that the subspace of polynomials homogeneous of degree $N$ is an invariant subspace under the representation. This invariant subspace is spanned by the weight vectors

$$(x_1 + ix_2)^{k_1}(x_1 - ix_2)^{l_1}(x_3 + ix_4)^{k_2} \cdots (x_{2n-1} - ix_{2n})^{l_n} x_{2n+1}^{k_0},$$

where $\sum_{j=0}^n k_j + \sum_{j=1}^n l_j = N$. Hence the weights of the subspace are all expressions $\sum_{j=1}^n (l_j - k_j)e_j$ with $\sum_{j=0}^n k_j + \sum_{j=1}^n l_j = N$.

2) Let $V = \bigwedge^l \mathbb{C}^{2n+1}$. The element $H_1$ of $\mathfrak{h}$ in the above example acts on $\varepsilon_1 + i\varepsilon_2$ by the scalar $+i$ and on $\varepsilon_1 - i\varepsilon_2$ by the scalar $-i$. Thus $\varepsilon_1 + i\varepsilon_2$ and $\varepsilon_1 - i\varepsilon_2$ are weight vectors in $\mathbb{C}^{2n+1}$ of respective weights $-e_1$ and $+e_1$. Also $\varepsilon_{2n+1}$ has weight 0. Then the product rule for differentiation allows us to compute the weights in $\bigwedge^l \mathbb{C}^{2n+1}$ and find that they are all expressions

$$\pm e_{j_1} \pm \cdots \pm e_{j_r}$$

with

$$j_1 < \cdots < j_r \qquad \text{and} \qquad \begin{cases} r \le l & \text{if } l \le n \\ r \le 2n+1-l & \text{if } l > n. \end{cases}$$

Motivated by Proposition 4.59 for compact Lie groups, we say that a member $\lambda$ of $\mathfrak{h}^*$ is **algebraically integral** if $2\langle \lambda, \alpha \rangle / |\alpha|^2$ is in $\mathbb{Z}$ for each $\alpha \in \Delta$.

**Proposition 5.4.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra, let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the roots, and let $\mathfrak{h}_0 = \sum_{\alpha \in \Delta} \mathbb{R} H_\alpha$. If $\varphi$ is a representation of $\mathfrak{g}$ on the finite-dimensional complex vector space $V$, then

(a) $\varphi(\mathfrak{h})$ acts diagonably on $V$, so that every generalized weight vector is a weight vector and $V$ is the direct sum of all the weight spaces,
(b) every weight is real valued on $\mathfrak{h}_0$ and is algebraically integral,
(c) roots and weights are related by $\varphi(\mathfrak{g}_\alpha)V_\lambda \subseteq V_{\lambda+\alpha}$.

PROOF.

(a, b) If $\alpha$ is a root and $E_\alpha$ and $E_{-\alpha}$ are nonzero root vectors for $\alpha$ and $-\alpha$, then $\{H_\alpha, E_\alpha, E_{-\alpha}\}$ spans a subalgebra $\mathfrak{sl}_\alpha$ of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, with $2|\alpha|^{-2}H_\alpha$ corresponding to $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then the restriction of $\varphi$ to $\mathfrak{sl}_\alpha$ is a finite-dimensional representation of $\mathfrak{sl}_\alpha$, and Corollary 1.72 shows that $\varphi(2|\alpha|^{-2}H_\alpha)$ is diagonable with integer eigenvalues. This proves (a) and the first half of (b). If $\lambda$ is a weight and $v \in V_\lambda$ is nonzero, then we have just seen that $\varphi(2|\alpha|^{-2}H_\alpha)v = 2|\alpha|^{-2}\langle\lambda, \alpha\rangle v$ is an integral multiple of $v$. Hence $2\langle\lambda, \alpha\rangle/|\alpha|^2$ is an integer, and $\lambda$ is algebraically integral.

(c) Let $E_\alpha$ be in $\mathfrak{g}_\alpha$, let $v$ be in $V_\lambda$, and let $H$ be in $\mathfrak{h}$. Then

$$\begin{aligned}\varphi(H)\varphi(E_\alpha)v &= \varphi(E_\alpha)\varphi(H)v + \varphi([H, E_\alpha])v \\ &= \lambda(H)\varphi(E_\alpha)v + \alpha(H)\varphi(E_\alpha)v \\ &= (\lambda + \alpha)(H)\varphi(E_\alpha)v.\end{aligned}$$

Hence $\varphi(E_\alpha)v$ is in $V_{\lambda+\alpha}$.

## 2. Theorem of the Highest Weight

In this section let $\mathfrak{g}$ be a complex semisimple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra, let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the set of roots, and let $W(\Delta)$ be the Weyl group. Let $\mathfrak{h}_0$ be the real form of $\mathfrak{h}$ on which all roots are real valued, and let $B$ be any nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ that is positive definite on $\mathfrak{h}_0$. Introduce an ordering in $\mathfrak{h}_0^*$ in the usual way, and let $\Pi$ be the resulting simple system.

If $\varphi$ is a representation of $\mathfrak{g}$ on a finite-dimensional complex vector space $V$, then the weights of $V$ are in $\mathfrak{h}_0^*$ by Proposition 5.4b. The largest weight in the ordering is called the **highest weight** of $\varphi$.

**Theorem 5.5** (Theorem of the Highest Weight). Apart from equivalence the irreducible finite-dimensional representations $\varphi$ of $\mathfrak{g}$ stand in one-one correspondence with the dominant algebraically integral linear functionals $\lambda$ on $\mathfrak{h}$, the correspondence being that $\lambda$ is the highest weight of $\varphi_\lambda$. The highest weight $\lambda$ of $\varphi_\lambda$ has these additional properties:

(a) $\lambda$ depends only on the simple system $\Pi$ and not on the ordering used to define $\Pi$,

(b) the weight space $V_\lambda$ for $\lambda$ is 1-dimensional,

(c) each root vector $E_\alpha$ for arbitrary $\alpha \in \Delta^+$ annihilates the members of $V_\lambda$, and the members of $V_\lambda$ are the only vectors with this property,

(d) every weight of $\varphi_\lambda$ is of the form $\lambda - \sum_{i=1}^{l} n_i\alpha_i$ with the integers $\geq 0$ and the $\alpha_i$ in $\Pi$,

(e) each weight space $V_\mu$ for $\varphi_\lambda$ has $\dim V_{w\mu} = \dim V_\mu$ for all $w$ in the Weyl group $W(\Delta)$, and each weight $\mu$ has $|\mu| \leq |\lambda|$ with equality only if $\mu$ is in the orbit $W(\Delta)\lambda$.

REMARKS.

1) Because of (e) the weights in the orbit $W(\Delta)\lambda$ are said to be **extreme**. The set of extreme weights does not depend on the choice of $\Pi$.

2) Much of the proof of Theorem 5.5 will be given in this section after some examples. The proof will be completed in §3. The examples continue the notation of the examples in §1.

EXAMPLES.

1) With $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, let $V$ consist of all polynomials in $z_1, \ldots, z_n$, and $\bar{z}_1, \ldots, \bar{z}_n$ homogeneous of total degree $N$. The weights are all expressions $\sum_{j=1}^{n} (l_j - k_j)e_j$ with $\sum_{j=1}^{n} (k_j + l_j) = N$. The highest weight relative to the usual positive system is $Ne_1$. The subspace of holomorphic polynomials is an invariant subspace, and it has highest weight $-Ne_n$. The subspace of antiholomorphic polynomials is another invariant subspace, and it has highest weight $Ne_1$.

2) With $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, let $V = \bigwedge^l \mathbb{C}^n$. The weights are all expressions $\sum_{k=1}^{l} e_{j_k}$. The highest weight relative to the usual positive system is $\sum_{k=1}^{l} e_k$.

3) With $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$, let the representation space consist of all complex-valued polynomials in $x_1, \ldots, x_{2n+1}$ homogeneous of degree $N$. The weights are all expressions $\sum_{j=1}^{n} (l_j - k_j)e_j$ with $k_0 + \sum_{j=1}^{n} (k_j + l_j) = N$. The highest weight relative to the usual positive system is $Ne_1$.

4) With $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$, let $V = \bigwedge^l \mathbb{C}^{2n+1}$. If $l \leq n$, the weights are all expressions $\pm e_{j_1} \pm \cdots \pm e_{j_r}$ with $j_1 < \cdots < j_r$ and $r \leq l$, and the highest weight relative to the usual positive system is $\sum_{k=1}^{l} e_k$.

PROOF OF EXISTENCE OF THE CORRESPONDENCE. Let $\varphi$ be an irreducible finite-dimensional representation of $\mathfrak{g}$ on a space $V$. The representation $\varphi$ has weights by Proposition 2.4, and we let $\lambda$ be the highest. Then $\lambda$ is algebraically integral by Proposition 5.4b.

If $\alpha$ is in $\Delta^+$, then $\lambda+\alpha$ exceeds $\lambda$ and cannot be a weight. Thus $E_\alpha \in \mathfrak{g}_\alpha$ and $v \in V_\lambda$ imply $\varphi(E_\alpha)v = 0$ by Proposition 5.4c. This proves the first part of (c).

Extend $\varphi$ multiplicatively to be defined on all of $U(\mathfrak{g})$ with $\varphi(1) = 1$ by Corollary 3.6. Since $\varphi$ is irreducible, $\varphi(U(\mathfrak{g}))v = V$ for each $v \neq 0$ in $V$. Let $\beta_1, \ldots, \beta_k$ be an enumeration of $\Delta^+$, and let $H_1, \ldots, H_l$ be a basis of $\mathfrak{h}$. By the Poincaré–Birkhoff–Witt Theorem (Theorem 3.8) the monomials

$$E_{-\beta_1}^{q_1} \cdots E_{-\beta_k}^{q_k} H_1^{m_1} \cdots H_l^{m_l} E_{\beta_1}^{p_1} \cdots E_{\beta_k}^{p_k} \tag{5.6}$$

form a basis of $U(\mathfrak{g})$. Let us apply $\varphi$ of each of these monomials to $v$ in $V_\lambda$. The $E_\beta$'s give 0, the $H$'s multiply by constants (by Proposition 5.4a), and the $E_{-\beta}$'s push the weight down (by Proposition 5.4c). Consequently the only members of $V_\lambda$ that can be obtained by applying $\varphi$ of (5.6) to $v$ are the vectors of $\mathbb{C}v$. Thus $V_\lambda$ is 1-dimensional, and (b) is proved.

The effect of $\varphi$ of (5.6) applied to $v$ in $V_\lambda$ is to give a weight vector with weight

$$\lambda - \sum_{j=1}^{k} q_j \beta_j, \tag{5.7}$$

and these weight vectors span $V$. Thus the weights (5.7) are the only weights of $\varphi$, and (d) follows from Proposition 2.49. Also (d) implies (a).

To prove the second half of (c), let $v \notin V_\lambda$ satisfy $\varphi(E_\alpha)v = 0$ for all $\alpha \in \Delta^+$. Subtracting the component in $V_\lambda$, we may assume that $v$ has 0 component in $V_\lambda$. Let $\lambda_0$ be the largest weight such that $v$ has a nonzero component in $V_{\lambda_0}$, and let $v'$ be the component. Then $\varphi(E_\alpha)v' = 0$ for all $\alpha \in \Delta^+$, and $\varphi(\mathfrak{h})v' \subseteq \mathbb{C}v'$. Applying $\varphi$ of (5.6), we see that

$$V = \sum \mathbb{C}\varphi(E_{-\beta_1})^{q_1} \cdots \varphi(E_{-\beta_k})^{q_k} v'.$$

Every weight of vectors on the right side is strictly lower than $\lambda$, and we have a contradiction with the fact that $\lambda$ occurs as a weight.

Next we prove that $\lambda$ is dominant. Let $\alpha$ be in $\Delta^+$, and form $H'_\alpha$, $E'_\alpha$, and $E'_{-\alpha}$ as in (2.26). These vectors span a Lie subalgebra $\mathfrak{sl}_\alpha$ of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, and the isomorphism carries $H'_\alpha$ to $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For $v \neq 0$ in $V_\lambda$, the subspace of $V$ spanned by all

$$\varphi(E'_{-\alpha})^p \varphi(H'_\alpha)^q \varphi(E'_\alpha)^r v$$

is stable under $\mathfrak{sl}_\alpha$, and (c) shows that it is the same as the span of all $\varphi(E'_{-\alpha})^p v$. On these vectors $\varphi(H'_\alpha)$ acts with eigenvalue

$$(\lambda - p\alpha)(H'_\alpha) = \frac{2\langle \lambda, \alpha \rangle}{|\alpha|^2} - 2p,$$

and the largest eigenvalue of $\varphi(H'_\alpha)$ is therefore $2\langle\lambda,\alpha\rangle/|\alpha|^2$. By Corollary 1.72 the largest eigenvalue for $h$ in any finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ is $\geq 0$, and $\lambda$ is therefore dominant.

Finally we prove (e). Fix $\alpha\in\Delta$, and form $\mathfrak{sl}_\alpha$ as above. Proposition 5.4a shows that $V$ is the direct sum of its simultaneous eigenspaces under $\mathfrak{h}$ and hence also under the subspace $\ker\alpha$ of $\mathfrak{h}$. In turn, since $\ker\alpha$ commutes with $\mathfrak{sl}_\alpha$, each of these simultaneous eigenspaces under $\ker\alpha$ is invariant under $\mathfrak{sl}_\alpha$ and is completely reducible by Theorem 1.67.

Thus $V$ is the direct sum of subspaces invariant and irreducible under $\mathfrak{sl}_\alpha\oplus\ker\alpha$. Let $V'$ be one of these irreducible subspaces. Since $\mathfrak{h}\subseteq\mathfrak{sl}_\alpha\oplus\ker\alpha$, $V'$ is the direct sum of its weight spaces: $V'=\bigoplus_\nu(V'\cap V_\nu)$. If $\nu$ and $\nu'$ are two weights occurring in $V'$, then the irreducibility under $\mathfrak{sl}_\alpha\oplus\ker\alpha$ forces $\nu'-\nu=n\alpha$ for some integer $n$.

Fix a weight $\mu$, and consider such a space $V'$. The weights of $V'$ are $\mu+n\alpha$, and these are distinguished from one another by their values on $H'_\alpha$. By Corollary 1.72, $\dim(V'\cap V_\mu)=\dim(V'\cap V_{s_\alpha\mu})$. Summing over $V'$, we obtain $\dim V_\mu=\dim V_{s_\alpha\mu}$. Since the root reflections generate $W(\Delta)$, it follows that $\dim V_\mu=\dim V_{w\mu}$ for all $w\in W(\Delta)$. This proves the first half of (e).

For the second half of (e), Corollary 2.68 and the result just proved show that there is no loss of generality in assuming that $\mu$ is dominant. Under this restriction on $\mu$, let us use (d) to write $\lambda=\mu+\sum_{i=1}^l n_i\alpha_i$ with all $n_i\geq 0$. Then

$$\begin{aligned}|\lambda|^2&=|\mu|^2+\sum_{i=1}^l n_i\langle\mu,\alpha_i\rangle+\Big|\sum_{i=1}^l n_i\alpha_i\Big|^2\\&\geq|\mu|^2+\Big|\sum_{i=1}^l n_i\alpha_i\Big|^2\qquad\text{by dominance of }\mu.\end{aligned}$$

The right side is $\geq|\mu|^2$ with equality only if $\sum_{i=1}^l n_i\alpha_i=0$. In this case $\mu=\lambda$.

PROOF THAT THE CORRESPONDENCE IS ONE-ONE. Let $\varphi$ and $\varphi'$ be irreducible finite dimensional on $V$ and $V'$, respectively, both with highest weight $\lambda$, and regard $\varphi$ and $\varphi'$ as representations of $U(\mathfrak{g})$. Let $v_0$ and $v'_0$ be nonzero highest weight vectors. Form $\varphi\oplus\varphi'$ on $V\oplus V'$. We claim that

$$S=(\varphi\oplus\varphi')(U(\mathfrak{g}))(v_0\oplus v'_0)$$

is an irreducible invariant subspace of $V\oplus V'$.

Certainly $S$ is invariant. Let $T \subseteq S$ be an irreducible invariant subspace, and let $v \oplus v'$ be a nonzero highest weight vector. For $\alpha \in \Delta^+$, we have

$$0 = (\varphi \oplus \varphi')(E_\alpha)(v \oplus v') = \varphi(E_\alpha)v \oplus \varphi'(E_\alpha)v',$$

and thus $\varphi(E_\alpha)v = 0$ and $\varphi'(E_\alpha)v' = 0$. By (c), $v = cv_0$ and $v' = c'v_0'$. Hence $v \oplus v' = cv_0 \oplus c'v_0'$. This vector by assumption is in $\varphi(U(\mathfrak{g}))(v_0 \oplus v_0')$. When we apply $\varphi$ of (5.6) to $v_0 \oplus v_0'$, the $E_\beta$'s give 0, while the $H$'s multiply by constants, namely

$$(\varphi \oplus \varphi')(H)(v_0 \oplus v_0') = \varphi(H)v_0 \oplus \varphi'(H)v_0' = \lambda(H)(v_0 \oplus v_0').$$

Also the $E_{-\beta}$'s push weights down by Proposition 5.4c. We conclude that $c' = c$. Hence $T = S$, and $S$ is irreducible.

The projection of $S$ to $V$ commutes with the representations and is not identically 0. By Schur's Lemma (Proposition 5.1), $\varphi \oplus \varphi'|_S$ is equivalent with $\varphi$. Similarly it is equivalent with $\varphi'$. Hence $\varphi$ and $\varphi'$ are equivalent.

To complete the proof of Theorem 5.5, we need to prove an existence result. The existence result says that for any dominant algebraically integral $\lambda$, there exists an irreducible finite-dimensional representation $\varphi_\lambda$ of $\mathfrak{g}$ with highest weight $\lambda$. We carry out this step in the next section.

## 3. Verma Modules

In this section we complete the proof of the Theorem of the Highest Weight (Theorem 5.5): Under the assumption that $\lambda$ is algebraically integral, we give an algebraic construction of an irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight $\lambda$.

By means of Corollary 3.6, we can identify representations of $\mathfrak{g}$ with unital left $U(\mathfrak{g})$ modules, and henceforth we shall often drop the name of the representation when working in this fashion. The idea is to consider all $U(\mathfrak{g})$ modules, finite dimensional or infinite dimensional, that possess a vector that behaves like a highest weight vector with weight $\lambda$. Among these we shall see that there is one (called a "Verma module") with a universal mapping property. A suitable quotient of the Verma module will give us our irreducible representation, and the main step will be to prove that it is finite dimensional.

We retain the notation of §2, and we write $\Pi = \{\alpha_1, \ldots, \alpha_l\}$. In addition we let

$$(5.8)\qquad \begin{aligned} \mathfrak{n} &= \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha \\ \mathfrak{n}^- &= \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha} \\ \mathfrak{b} &= \mathfrak{h} \oplus \mathfrak{n} \\ \delta &= \tfrac{1}{2} \sum_{\alpha\in\Delta^+} \alpha. \end{aligned}$$

Then $\mathfrak{n}$, $\mathfrak{n}^-$, and $\mathfrak{b}$ are Lie subalgebras of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$ as a direct sum of vector spaces.

Let the complex vector space $V$ be a unital left $U(\mathfrak{g})$ module. We allow $V$ to be infinite dimensional. Because of Corollary 3.6 we have already defined in §1 the notions "weight," "weight space," and "weight vector" for $V$. Departing slightly from the notation of that section, let $V_\mu$ be the weight space for the weight $\mu$. The sum $\sum V_\mu$ is necessarily a direct sum. As in Proposition 5.4c, we have

$$(5.9)\qquad \mathfrak{g}_\alpha(V_\mu) \subseteq V_{\mu+\alpha}$$

if $\alpha$ is in $\Delta$ and $\mu$ is in $\mathfrak{h}^*$. Moreover, (5.9) and the root-space decomposition of $\mathfrak{g}$ show that

$$(5.10)\qquad \mathfrak{g}\Big(\bigoplus_{\mu\in\mathfrak{h}^*} V_\mu\Big) \subseteq \Big(\bigoplus_{\mu\in\mathfrak{h}^*} V_\mu\Big).$$

A **highest weight vector** for $V$ is by definition a weight vector $v \neq 0$ with $\mathfrak{n}(v) = 0$. The set $\mathfrak{n}(v)$ will be 0 as soon as $E_\alpha v = 0$ for the root vectors $E_\alpha$ of simple roots $\alpha$. In fact, we easily see this assertion by expanding any positive $\alpha$ in terms of simple roots as $\sum_i n_i\alpha_i$ and proceeding by induction on the level $\sum_i n_i$.

A **highest weight module** is a $U(\mathfrak{g})$ module generated by a highest weight vector. "Verma modules," to be defined below, will be universal highest weight modules.

**Proposition 5.11.** Let $M$ be a highest weight module for $U(\mathfrak{g})$, and let $v$ be a highest weight vector generating $M$. Suppose $v$ is of weight $\lambda$. Then

(a) $M = U(\mathfrak{n}^-)v$,

(b) $M = \bigoplus_{\mu\in\mathfrak{h}^*} M_\mu$ with each $M_\mu$ finite dimensional and with $\dim M_\lambda = 1$,

(c) every weight of $M$ is of the form $\lambda - \sum_{i=1}^{l} n_i\alpha_i$ with the $\alpha_i$'s in $\Pi$ and with each $n_i$ an integer $\geq 0$.

PROOF.

(a) We have $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. By the Poincaré–Birkhoff–Witt Theorem (Theorem 3.8 and (3.14)), $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})$. On the vector $v$, $U(\mathfrak{n})$ and $U(\mathfrak{h})$ act to give multiples of $v$. Thus $U(\mathfrak{g})v = U(\mathfrak{n}^-)v$. Since $v$ generates $M$, $M = U(\mathfrak{g})v = U(\mathfrak{n}^-)v$.

(b, c) By (5.10), $\bigoplus M_\mu$ is $U(\mathfrak{g})$ stable, and it contains $v$. Since $M = U(\mathfrak{g})v$, $M = \bigoplus M_\mu$. By (a), $M = U(\mathfrak{n}^-)v$, and (5.9) shows that any expression

$$E_{-\beta_1}^{q_1} \cdots E_{-\beta_k}^{q_k} v \qquad \text{with all } \beta_j \in \Delta^+ \tag{5.12}$$

is a weight vector with weight $\mu = \lambda - q_1\beta_1 - \cdots - q_k\beta_k$, from which (c) follows. The number of expressions (5.12) leading to this $\mu$ is finite, and so $\dim M_\mu < \infty$. The number of expressions (5.12) leading to $\lambda$ is 1, from $v$ itself, and so $\dim M_\lambda = 1$.

Before defining Verma modules, we recall some facts about tensor products of associative algebras. (A special case has already been treated in §I.3.) Let $M_1$ and $M_2$ be complex vector spaces, and let $A$ and $B$ be complex associative algebras with identity. Suppose that $M_1$ is a right $B$ module and $M_2$ is a left $B$ module, and suppose that $M_1$ is also a left $A$ module in such a way that $(am_1)b = a(m_1b)$. We define

$$M_1 \otimes_B M_2 = \frac{M_1 \otimes_{\mathbb{C}} M_2}{\text{subspace generated by all } m_1b \otimes m_2 - m_1 \otimes bm_2},$$

and we let $A$ act on the quotient by $a(m_1 \otimes m_2) = (am_1) \otimes m_2$. Then $M_1 \otimes_B M_2$ is a left $A$ module, and it has the following universal mapping property: Whenever $\psi : M_1 \times M_2 \to E$ is a bilinear map into a complex vector space $E$ such that $\psi(m_1b, m_2) = \psi(m_1, bm_2)$, then there exists a unique linear map $\widetilde{\psi} : M_1 \otimes_B M_2 \to E$ such that $\psi(m_1, m_2) = \widetilde{\psi}(m_1 \otimes m_2)$.

Now let $\lambda$ be in $\mathfrak{h}^*$, and make $\mathbb{C}$ into a left $U(\mathfrak{b})$ module $\mathbb{C}_{\lambda-\delta}$ by defining

$$\begin{aligned} Hz &= (\lambda - \delta)(H)z && \text{for } H \in \mathfrak{h},\ z \in \mathbb{C} \\ Xz &= 0 && \text{for } X \in \mathfrak{n}. \end{aligned} \tag{5.13}$$

(Equation (5.13) defines a 1-dimensional representation of $\mathfrak{b}$, and thus $\mathbb{C}_{\lambda-\delta}$ becomes a left $U(\mathfrak{b})$ module.) The algebra $U(\mathfrak{g})$ itself is a right $U(\mathfrak{b})$ module and a left $U(\mathfrak{g})$ module under multiplication, and we define the **Verma module** $V(\lambda)$ to be the left $U(\mathfrak{g})$ module

$$V(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda-\delta}.$$

**Proposition 5.14.** Let $\lambda$ be in $\mathfrak{h}^*$.

(a) $V(\lambda)$ is a highest weight module under $U(\mathfrak{g})$ and is generated by $1 \otimes 1$ (the **canonical generator**), which is of weight $\lambda - \delta$.

(b) The map of $U(\mathfrak{n}^-)$ into $V(\lambda)$ given by $u \mapsto u(1 \otimes 1)$ is one-one onto.

(c) If $M$ is any highest weight module under $U(\mathfrak{g})$ generated by a highest weight vector $v \neq 0$ of weight $\lambda - \delta$, then there exists one and only one $U(\mathfrak{g})$ homomorphism $\widetilde{\psi}$ of $V(\lambda)$ into $M$ such that $\widetilde{\psi}(1 \otimes 1) = v$. The map $\widetilde{\psi}$ is onto. Also $\widetilde{\psi}$ is one-one if and only if $u \neq 0$ in $U(\mathfrak{n}^-)$ implies $u(v) \neq 0$ in $M$.

PROOF.

(a) Clearly $V(\lambda) = U(\mathfrak{g})(1 \otimes 1)$. Also

$$\begin{aligned} H(1 \otimes 1) &= H \otimes 1 = 1 \otimes H(1) = (\lambda - \delta)(H)(1 \otimes 1) && \text{for } H \in \mathfrak{h} \\ X(1 \otimes 1) &= X \otimes 1 = 1 \otimes X(1) = 0 && \text{for } X \in \mathfrak{n}, \end{aligned}$$

and so $1 \otimes 1$ is a highest weight vector of weight $\lambda - \delta$.

(b) By the Poincaré–Birkhoff–Witt Theorem (Theorem 3.8 and (3.14)), we have $U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} U(\mathfrak{b})$, and this isomorphism is clearly an isomorphism of left $U(\mathfrak{n}^-)$ modules. Thus we obtain a chain of canonical left $U(\mathfrak{n}^-)$ module isomorphisms

$$\begin{aligned} V(\lambda) &= U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C} \cong (U(\mathfrak{n}^-) \otimes_{\mathbb{C}} U(\mathfrak{b})) \otimes_{U(\mathfrak{b})} \mathbb{C} \\ &\cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} (U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} \mathbb{C}) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C} \cong U(\mathfrak{n}^-), \end{aligned}$$

and (b) follows.

(c) We consider the bilinear map of $U(\mathfrak{g}) \times \mathbb{C}_{\lambda-\delta}$ into $M$ given by $(u, z) \mapsto u(zv)$. In terms of the action of $U(\mathfrak{b})$ on $\mathbb{C}_{\lambda-\delta}$, we check for $b$ in $\mathfrak{h}$ and then for $b$ in $\mathfrak{n}$ that

$$(u, b(z)) \mapsto u(b(z)v) = zu((b(1))v)$$

and $$(ub, z) \mapsto ub(zv) = zub(v) = zu((b(1))v).$$

By the universal mapping property, there exists one and only one linear map

$$\widetilde{\psi} : U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda-\delta} \to M$$

such that $u(zv) = \widetilde{\psi}(u \otimes z)$ for all $u \in U(\mathfrak{g})$ and $z \in \mathbb{C}$, i.e., such that $u(v) = \widetilde{\psi}(u(1 \otimes 1))$. This condition says that $\widetilde{\psi}$ is a $U(\mathfrak{g})$ homomorphism and that $1 \otimes 1$ maps to $v$. Hence existence and uniqueness follow. Clearly $\widetilde{\psi}$ is onto.

Let $u$ be in $U(\mathfrak{n}^-)$. If $u(v) = 0$ with $u \neq 0$, then $\widetilde{\psi}(u(1 \otimes 1)) = 0$ while $u(1 \otimes 1) \neq 0$, by (b). Hence $\widetilde{\psi}$ is not one-one. Conversely if $\widetilde{\psi}$ is not one-one, then Proposition 5.11a implies that there exists $u \in U(\mathfrak{n}^-)$ with $u \neq 0$ and $\widetilde{\psi}(u \otimes 1) = 0$. Then

$$u(v) = u(\widetilde{\psi}(1 \otimes 1)) = \widetilde{\psi}(u(1 \otimes 1)) = \widetilde{\psi}(u \otimes 1) = 0.$$

This completes the proof.

**Proposition 5.15.** Let $\lambda$ be in $\mathfrak{h}^*$, and let $V(\lambda)_+ = \bigoplus_{\mu \neq \lambda - \delta} V(\lambda)_\mu$. Then every proper $U(\mathfrak{g})$ submodule of $V(\lambda)$ is contained in $V(\lambda)_+$. Consequently the sum $S$ of all proper $U(\mathfrak{g})$ submodules is a proper $U(\mathfrak{g})$ submodule, and $L(\lambda) = V(\lambda)/S$ is an irreducible $U(\mathfrak{g})$ module. Moreover, $L(\lambda)$ is a highest weight module with highest weight $\lambda - \delta$.

PROOF. If $N$ is a $U(\mathfrak{h})$ submodule, then $N = \bigoplus_\mu (N \cap V(\lambda)_\mu)$. Since $V(\lambda)_{\lambda-\delta}$ is 1-dimensional and generates $V(\lambda)$ (by Proposition 5.14a), the $\lambda - \delta$ term must be 0 in the sum for $N$ if $N$ is proper. Thus $N \subseteq V(\lambda)_+$. Hence $S$ is proper, and $L(\lambda) = V(\lambda)/S$ is irreducible. The image of $1 \otimes 1$ in $L(\lambda)$ is not 0, is annihilated by $\mathfrak{n}$, and is acted upon by $\mathfrak{h}$ according to $\lambda - \delta$. Thus $L(\lambda)$ has all the required properties.

**Theorem 5.16.** Suppose that $\lambda \in \mathfrak{h}^*$ is real valued on $\mathfrak{h}_0$ and is dominant and algebraically integral. Then the irreducible highest weight module $L(\lambda + \delta)$ is an irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight $\lambda$.

REMARKS. Theorem 5.16 will complete the proof of the Theorem of the Highest Weight (Theorem 5.5). The proof of Theorem 5.16 will be preceded by two lemmas.

**Lemma 5.17.** In $U(\mathfrak{sl}(2, \mathbb{C}))$, $[e, f^n] = nf^{n-1}(h - (n-1))$.

PROOF. Let

$$\begin{aligned} Lf &= \text{left by } f \text{ in } U(\mathfrak{sl}(2, \mathbb{C})) \\ Rf &= \text{right by } f \\ \text{ad } f &= Lf - Rf. \end{aligned}$$

Then $Rf = Lf - \text{ad } f$, and the terms on the right commute. By the binomial theorem,

$$\begin{aligned}(Rf)^n e &= \sum_{j=0}^{n} \binom{n}{j} (Lf)^{n-j}(-\text{ad } f)^j e \\ &= (Lf)^n e + n(Lf)^{n-1}(-\text{ad } f)e + \frac{n(n-1)}{2}(Lf)^{n-2}(-\text{ad } f)^2 e\end{aligned}$$

since $(\text{ad } f)^3 e = 0$, and this expression is

$$\begin{aligned}&= (Lf)^n e + nf^{n-1}h + \frac{n(n-1)}{2} f^{n-2}(-2f) \\ &= (Lf)^n e + nf^{n-1}(h-(n-1)).\end{aligned}$$

Thus

$$[e, f^n] = (Rf)^n e - (Lf)^n e = nf^{n-1}(h-(n-1)).$$

**Lemma 5.18.** For general complex semisimple $\mathfrak{g}$, let $\lambda$ be in $\mathfrak{h}^*$, let $\alpha$ be a simple root, and suppose that $m = 2\langle\lambda, \alpha\rangle/|\alpha|^2$ is a positive integer. Let $v_{\lambda-\delta}$ be the canonical generator of $V(\lambda)$, and let $M$ be the $U(\mathfrak{g})$ submodule generated by $(E_{-\alpha})^m v_{\lambda-\delta}$, where $E_{-\alpha}$ is a nonzero root vector for the root $-\alpha$. Then $M$ is isomorphic to $V(s_\alpha\lambda)$.

PROOF. The vector $v = (E_{-\alpha})^m v_{\lambda-\delta}$ is not 0 by Proposition 5.14b. Since $s_\alpha\lambda = \lambda - m\alpha$, $v$ is in $V(\lambda)_{\lambda-\delta-m\alpha} = V(\lambda)_{s_\alpha\lambda-\delta}$. Thus the result will follow from Proposition 5.14c if we show that $E_\beta v = 0$ whenever $E_\beta$ is a root vector for a simple root $\beta$. For $\beta \neq \alpha$, $[E_\beta, E_{-\alpha}] = 0$ since $\beta - \alpha$ is not a root (Lemma 2.51). Thus

$$E_\beta v = E_\beta(E_{-\alpha})^m v_{\lambda-\delta} = (E_{-\alpha})^m E_\beta v_{\lambda-\delta} = 0.$$

For $\beta = \alpha$, let us introduce a root vector $E_\alpha$ for $\alpha$ so that $[E_\alpha, E_{-\alpha}] = 2|\alpha|^{-2}H_\alpha$. The isomorphism (2.27) identifies $\text{span}\{H_\alpha, E_\alpha, E_{-\alpha}\}$ with $\mathfrak{sl}(2, \mathbb{C})$, and then Lemma 5.17 gives

$$\begin{aligned}E_\alpha(E_{-\alpha})^m v_{\lambda-\delta} &= [E_\alpha, E_{-\alpha}^m]v_{\lambda-\delta} \\ &= m(E_{-\alpha})^{m-1}(2|\alpha|^{-2}H_\alpha - (m-1))v_{\lambda-\delta} \\ &= m\left(\frac{2\langle\lambda-\delta, \alpha\rangle}{|\alpha|^2} - (m-1)\right)E_{-\alpha}^{m-1}v_{\lambda-\delta} \\ &= 0,\end{aligned}$$

the last step following from Proposition 2.69.

PROOF OF THEOREM 5.16. Let $v_\lambda \neq 0$ be a highest weight vector in $L(\lambda+\delta)$, with weight $\lambda$. We proceed in three steps.

First we show: For every simple root $\alpha$, $E_{-\alpha}^n v_\lambda = 0$ for all $n$ sufficiently large. Here $E_{-\alpha}$ is a nonzero root vector for $-\alpha$. In fact, for $n = \dfrac{2\langle\lambda+\delta,\alpha\rangle}{|\alpha|^2}$ (which is positive by Proposition 2.69), the member $E_{-\alpha}^n(1\otimes 1)$ of $V(\lambda+\delta)$ lies in a proper $U(\mathfrak{g})$ submodule, according to Lemma 5.18, and hence is in the submodule $S$ in Proposition 5.15. Thus $E_{-\alpha}^n v_\lambda = 0$ in $L(\lambda+\delta)$.

Second we show: The set of weights is stable under the Weyl group $W = W(\Delta)$. In fact, let $\alpha$ be a simple root, let $\mathfrak{sl}_\alpha$ be the copy of $\mathfrak{sl}(2,\mathbb{C})$ given by $\mathfrak{sl}_\alpha = \operatorname{span}\{H_\alpha, E_\alpha, E_{-\alpha}\}$, set $v^{(i)} = E_{-\alpha}^i v_\lambda$, and let $n$ be the largest integer such that $v^{(n)} \neq 0$ (existence by the first step above). Then $\mathbb{C}v^{(0)} + \cdots + \mathbb{C}v^{(n)}$ is stable under $\mathfrak{sl}_\alpha$. The sum of all finite-dimensional $U(\mathfrak{sl}_\alpha)$ submodules thus contains $v^{(0)} = v_\lambda$, and we claim it is $\mathfrak{g}$ stable.

In fact, if $T$ is a finite-dimensional $U(\mathfrak{sl}_\alpha)$ submodule, then

$$\mathfrak{g}T = \Big\{ \sum Xt \mid X \in \mathfrak{g} \text{ and } t \in T \Big\}$$

is finite dimensional and for $Y \in \mathfrak{sl}_\alpha$ and $X \in \mathfrak{g}$ we have

$$YXt = XYt + [Y,X]t = Xt' + [Y,X]t \in \mathfrak{g}T.$$

So $\mathfrak{g}T$ is $\mathfrak{sl}_\alpha$ stable, and the claim follows.

Since the sum of all finite-dimensional $U(\mathfrak{sl}_\alpha)$ submodules of $L(\lambda+\delta)$ is $\mathfrak{g}$ stable, the irreducibility of $L(\lambda+\delta)$ implies that this sum is all of $L(\lambda+\delta)$. By Corollary 1.73, $L(\lambda+\delta)$ is the direct sum of finite-dimensional irreducible $U(\mathfrak{sl}_\alpha)$ submodules.

Let $\mu$ be a weight, and let $t \neq 0$ be in $V_\mu$. We have just shown that $t$ lies in a finite direct sum of finite-dimensional irreducible $U(\mathfrak{sl}_\alpha)$ submodules. Let us write $t = \sum_{i\in I} t_i$ with $t_i$ in a $U(\mathfrak{sl}_\alpha)$ submodule $T_i$ and $t_i \neq 0$. Then

$$\sum H_\alpha t_i = H_\alpha t = \mu(H_\alpha)t = \sum \mu(H_\alpha)t_i,$$

and so
$$\frac{2H_\alpha}{|\alpha|^2}\, t_i = \frac{2\langle\mu,\alpha\rangle}{|\alpha|^2}\, t_i \qquad \text{for each } i \in I.$$

If $\langle\mu,\alpha\rangle > 0$, we know that $(E_{-\alpha})^{2\langle\mu,\alpha\rangle/|\alpha|^2} t_i \neq 0$ from Theorem 1.66. Hence $(E_{-\alpha})^{2\langle\mu,\alpha\rangle/|\alpha|^2} t \neq 0$, and we see that

$$\mu - \frac{2\langle\mu,\alpha\rangle}{|\alpha|^2}\,\alpha = s_\alpha\mu$$

is a weight. If $\langle \mu, \alpha \rangle < 0$ instead, we know that $(E_\alpha)^{-2\langle \mu,\alpha\rangle/|\alpha|^2} t_i \neq 0$ from Theorem 1.66. Hence $(E_\alpha)^{-2\langle \mu,\alpha\rangle/|\alpha|^2} t \neq 0$, and so

$$\mu - \frac{2\langle \mu, \alpha \rangle}{|\alpha|^2}\alpha = s_\alpha \mu$$

is a weight. If $\langle \mu, \alpha \rangle = 0$, then $s_\alpha \mu = \mu$. In any case $s_\alpha \mu$ is a weight. So the set of weights is stable under each reflection $s_\alpha$ for $\alpha$ simple, and Proposition 2.62 shows that the set of weights is stable under $W$.

Third we show: The set of weights of $L(\lambda + \delta)$ is finite, and $L(\lambda + \delta)$ is finite dimensional. In fact, Corollary 2.68 shows that any linear functional on $\mathfrak{h}_0$ is $W$ conjugate to a dominant one. Since the second step above says that the set of weights is stable under $W$, the number of weights is at most $|W|$ times the number of dominant weights, which are of the form $\lambda - \sum_{i=1}^{l} n_i \alpha_i$ by Proposition 5.11c. Each such dominant form must satisfy

$$\langle \lambda, \delta \rangle \geq \sum_{i=1}^{l} n_i \langle \alpha_i, \delta \rangle,$$

and Proposition 2.69 shows that $\sum n_i$ is bounded; thus the number of dominant weights is finite. Then $L(\lambda + \delta)$ is finite dimensional by Proposition 5.11b.

## 4. Complete Reducibility

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, and let $U(\mathfrak{g})$ be its universal enveloping algebra. As a consequence of the generalization of Schur's Lemma given in Proposition 5.19 below, the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts by scalars in any irreducible unital left $U(\mathfrak{g})$ module, even an infinite-dimensional one. The resulting homomorphism $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is the first serious algebraic invariant of an irreducible representation of $\mathfrak{g}$ and is called the **infinitesimal character**. This invariant is most useful in situations where $Z(\mathfrak{g})$ can be shown to be large, which will be the case when $\mathfrak{g}$ is semisimple.

**Proposition 5.19** (Dixmier). Let $\mathfrak{g}$ be a complex Lie algebra, and let $V$ be an irreducible unital left $U(\mathfrak{g})$ module. Then the only $U(\mathfrak{g})$ linear maps $L : V \to V$ are the scalars.

PROOF. The space $E = \mathrm{End}_{U(\mathfrak{g})}(V, V)$ is an associative algebra over $\mathbb{C}$, and Schur's Lemma (Proposition 5.1) shows that every nonzero element of $E$ has a two-sided inverse, i.e., $E$ is a division algebra.

If $v \neq 0$ is in $V$, then the irreducibility implies that $V = U(\mathfrak{g})v$. Hence the dimension of $V$ is at most countable. Since every nonzero element of $E$ is invertible, the $\mathbb{C}$ linear map $L \mapsto L(v)$ of $E$ into $V$ is one-one. Therefore the dimension of $E$ over $\mathbb{C}$ is at most countable.

Let $L$ be in $E$. Arguing by contradiction, suppose that $L$ is not a scalar multiple of the identity. Form the field extension $\mathbb{C}(L) \subseteq E$. Since $\mathbb{C}$ is algebraically closed, $L$ is not algebraic over $\mathbb{C}$. Thus $L$ is transcendental over $\mathbb{C}$. In the transcendental extension $\mathbb{C}(X)$, the set of all $(X - c)^{-1}$ for $c \in \mathbb{C}$ is linearly independent, and consequently the dimension of $\mathbb{C}(X)$ is uncountable. Therefore $\mathbb{C}(L)$ has uncountable dimension, and so does $E$, contradiction.

Let us introduce **adjoint representations** on the universal enveloping algebra $U(\mathfrak{g})$ when $\mathfrak{g}$ is a finite-dimensional complex Lie algebra. We define a representation ad of $\mathfrak{g}$ on $U(\mathfrak{g})$ by

$$(\operatorname{ad} X)u = Xu - uX \qquad \text{for } X \in \mathfrak{g} \text{ and } u \in U(\mathfrak{g}).$$

(The representation property follows from the fact that $XY - YX = [X, Y]$ in $U(\mathfrak{g})$.) Lemma 3.9 shows that ad $X$ carries $U_n(\mathfrak{g})$ to itself. Therefore ad provides for all $n$ a consistently defined family of representations of $\mathfrak{g}$ on $U_n(\mathfrak{g})$.

Each $g \in \operatorname{Int}\mathfrak{g}$ gives an automorphism of $\mathfrak{g}$. Composing with the inclusion of $\mathfrak{g}$ into $U(\mathfrak{g})$, we obtain a complex-linear map of $\mathfrak{g}$ into $U(\mathfrak{g})$, and it will be convenient to call this map $\mathrm{Ad}(g)$. This composition has the property that

$$\begin{aligned}\mathrm{Ad}(g)[X, Y] &= [\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y] \\ &= (\mathrm{Ad}(g)X)(\mathrm{Ad}(g)Y) - (\mathrm{Ad}(g)Y)(\mathrm{Ad}(g)X).\end{aligned}$$

By Proposition 3.3 (with $A = U(\mathfrak{g})$), $\mathrm{Ad}(g)$ extends to a homomorphism of $U(\mathfrak{g})$ into itself carrying 1 to 1. Moreover

$$\mathrm{Ad}(g_1)\mathrm{Ad}(g_2) = \mathrm{Ad}(g_1 g_2) \tag{5.20}$$

because of the uniqueness of the extension and the validity of this formula on $U_1(\mathfrak{g})$. Therefore each $\mathrm{Ad}(g)$ is an automorphism of $U(\mathfrak{g})$. Because $\mathrm{Ad}(g)$ leaves $U_1(\mathfrak{g})$ stable, it leaves each $U_n(\mathfrak{g})$ stable. Its smoothness in $g$ on $U_1(\mathfrak{g})$ implies its smoothness in $g$ on $U_n(\mathfrak{g})$. Thus we obtain for all $n$ a consistently defined family Ad of smooth representations of $G$ on $U_n(\mathfrak{g})$.

**Proposition 5.21.** Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra. Then

(a) the differential at 1 of Ad on $U_n(\mathfrak{g})$ is ad, and
(b) on each $U_n(\mathfrak{g})$, $\operatorname{Ad}(\exp X) = e^{\operatorname{ad} X}$ for all $X \in \mathfrak{g}$.

PROOF. For (a) let $u = X_1^{k_1} \cdots X_n^{k_n}$ be a monomial in $U_n(\mathfrak{g})$. For $X$ in $\mathfrak{g}$, we have

$$\operatorname{Ad}(\exp rX)u = (\operatorname{Ad}(\exp rX)X_1)^{k_1} \cdots (\operatorname{Ad}(\exp rX)X_n)^{k_n}$$

since each $\operatorname{Ad}(g)$ for $g \in \operatorname{Int}\mathfrak{g}$ is an automorphism of $U(\mathfrak{g})$. Differentiating both sides with respect to $r$ and applying the product rule for differentiation, we obtain at $r = 0$

$$\begin{aligned} &\frac{d}{dr}\operatorname{Ad}(\exp rX)u\Big|_{r=0} \\ &\quad = \sum_{i=1}^{n}\sum_{j=1}^{k_i} X_1^{k_1}\cdots X_{i-1}^{k_{i-1}}X_i^{j-1}\Big(\frac{d}{dr}\operatorname{Ad}(\exp rX)X_i\Big)_{r=0}X_i^{k_i-j}X_{i+1}^{k_{i+1}}\cdots X_n^{k_n} \\ &\quad = (\operatorname{ad}X)u. \end{aligned}$$

Then (a) follows from Proposition 1.91, and (b) follows from Corollary 1.85.

**Proposition 5.22.** If $\mathfrak{g}$ is a finite-dimensional complex Lie algebra, then the following conditions on an element $u$ of $U(\mathfrak{g})$ are equivalent:

(a) $u$ is in the center $Z(\mathfrak{g})$,
(b) $uX = Xu$ for all $X \in \mathfrak{g}$,
(c) $e^{\operatorname{ad} X}u = u$ for all $X \in \mathfrak{g}$,
(d) $\operatorname{Ad}(g)u = u$ for all $g \in \operatorname{Int}\mathfrak{g}$.

PROOF. Conclusion (a) implies (b) trivially, and (b) implies (a) since $\mathfrak{g}$ generates $U(\mathfrak{g})$. If (b) holds, then $(\operatorname{ad} X)u = 0$, and (c) follows by summing the series for the exponential. Conversely if (c) holds, then we can replace $X$ by $rX$ in (c) and differentiate to obtain (b). Finally (c) follows from (d) by taking $g = \exp X$ and applying Proposition 5.21b, while (d) follows from (c) by (5.20) and Proposition 5.21b.

In the case that $\mathfrak{g}$ is semisimple, we shall construct some explicit elements of $Z(\mathfrak{g})$ and use them to extend to all semisimple $\mathfrak{g}$ the theorem of complete reducibility proved for $\mathfrak{sl}(2, \mathbb{C})$ in Theorem 1.67. To begin with, here is an explicit element of $Z(\mathfrak{g})$ when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.

EXAMPLE. $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Let $Z = \frac{1}{2}h^2 + ef + fe$ with $h, e, f$ as in (1.5). The action of $Z$ in a representation already appeared in Lemma 1.68. We readily check that $Z$ is in $Z(\mathfrak{g})$ by seeing that $Z$ commutes with $h$, $e$, and $f$. The element $Z$ is a multiple of the Casimir element $\Omega$ defined below.

For a general semisimple $\mathfrak{g}$, let $B$ be the Killing form. (To fix the definitions in this section, we shall not allow more general invariant forms in place of $B$.) Let $X_i$ be any basis of $\mathfrak{g}$ over $\mathbb{C}$, and let $\widetilde{X}_i$ be the dual basis relative to $B$, i.e., the basis with

$$B(\widetilde{X}_i, X_j) = \delta_{ij}.$$

The **Casimir element** $\Omega$ is defined by

$$\Omega = \sum_{i,j} B(X_i, X_j)\widetilde{X}_i\widetilde{X}_j. \tag{5.23}$$

**Proposition 5.24.** In a complex semisimple Lie algebra $\mathfrak{g}$, the Casimir element $\Omega$ is defined independently of the basis $X_i$ and is a member of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$.

PROOF. Let a second basis $X_i'$ be given by means of a nonsingular complex matrix $(a_{ij})$ as

$$X_j' = \sum_m a_{mj} X_m. \tag{5.25a}$$

Let $(b_{ij})$ be the inverse of the matrix $(a_{ij})$, and define

$$\widetilde{X}_i' = \sum_l b_{il}\widetilde{X}_l. \tag{5.25b}$$

Then

$$B(\widetilde{X}_i', X_j') = \sum_{l,m} b_{il}a_{mj}B(\widetilde{X}_l, X_m) = \sum_l b_{il}a_{lj} = \delta_{ij}.$$

Thus $\widetilde{X}_i'$ is the dual basis of $X_j'$. The element to consider is

$$\begin{aligned}
\Omega' &= \sum_{i,j} B(X_i', X_j')\widetilde{X}_i'\widetilde{X}_j' \\
&= \sum_{m,m'}\sum_{l,l'}\sum_{i,j} a_{mi}a_{m'j}b_{il}b_{jl'}B(X_m, X_{m'})\widetilde{X}_l\widetilde{X}_{l'} \\
&= \sum_{m,m'}\sum_{l,l'} \delta_{ml}\delta_{m'l'}B(X_m, X_{m'})\widetilde{X}_l\widetilde{X}_{l'} \\
&= \sum_{l,l'} B(X_l, X_{l'})\widetilde{X}_l\widetilde{X}_{l'} \\
&= \Omega.
\end{aligned}$$

This proves that $\Omega$ is independent of the basis.

Let $g$ be in $\operatorname{Int}\mathfrak{g}$, and take the second basis to be $X_i' = gX_i = \operatorname{Ad}(g)X_i$. Because of Proposition 1.119 the invariance of the Killing form gives

$$B(\operatorname{Ad}(g)\widetilde{X}_i, X_j') = B(\widetilde{X}_i, \operatorname{Ad}(g)^{-1}X_j') = B(\widetilde{X}_i, X_j) = \delta_{ij}, \tag{5.26}$$

and we conclude that $\widetilde{X}_i' = \operatorname{Ad}(g)\widetilde{X}_i$. Therefore

$$\begin{aligned}
\operatorname{Ad}(g)\Omega &= \sum_{i,j} B(X_i, X_j)\operatorname{Ad}(g)(\widetilde{X}_i\widetilde{X}_j) \\
&= \sum_{i,j} B(\operatorname{Ad}(g)X_i, \operatorname{Ad}(g)X_j)\widetilde{X}_i'\widetilde{X}_j' && \text{by Proposition 1.119} \\
&= \sum_{i,j} B(X_i', X_j')\widetilde{X}_i'\widetilde{X}_j' \\
&= \sum_{i,j} B(X_i, X_j)\widetilde{X}_i\widetilde{X}_j && \text{by change of basis} \\
&= \Omega.
\end{aligned}$$

By Proposition 5.22, $\Omega$ is in $Z(\mathfrak{g})$.

EXAMPLE. $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. We take as basis the elements $h, e, f$ as in (1.5). The Killing form has already been computed in Example 2 of §I.3, and we find that $\widetilde{h} = \frac{1}{8}h$, $\widetilde{e} = \frac{1}{4}f$, $\widetilde{f} = \frac{1}{4}e$. Then

$$\begin{aligned}
\Omega &= B(h,h)\widetilde{h}^2 + B(e,f)\widetilde{e}\widetilde{f} + B(f,e)\widetilde{f}\widetilde{e} \\
&= 8\widetilde{h}^2 + 4\widetilde{e}\widetilde{f} + 4\widetilde{f}\widetilde{e} \\
&= \tfrac{1}{8}h^2 + \tfrac{1}{4}ef + \tfrac{1}{4}fe, \qquad (5.27)
\end{aligned}$$

which is $\frac{1}{4}$ of the element $Z = \frac{1}{2}h^2 + ef + fe$ whose action in a representation appeared in Lemma 1.68.

Let $\varphi$ be an irreducible finite-dimensional representation of $\mathfrak{g}$ on a space $V$. Schur's Lemma (Proposition 5.1) and Proposition 5.24 imply that $\Omega$ acts as a scalar in $V$. We shall compute this scalar, making use of the Theorem of the Highest Weight (Theorem 5.5). Thus let us introduce a Cartan subalgebra $\mathfrak{h}$, the set $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ of roots, and a positive system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$.

**Proposition 5.28.** In the complex semisimple Lie algebra $\mathfrak{g}$, let $\mathfrak{h}_0$ be the real form of $\mathfrak{h}$ on which all roots are real valued, and let $\{H_i\}_{i=1}^l$ be an orthonormal basis of $\mathfrak{h}_0$ relative to the Killing form $B$ of $\mathfrak{g}$. Choose root vectors $E_\alpha$ so that $B(E_\alpha, E_{-\alpha}) = 1$ for all $\alpha \in \Delta$. Then

(a) $\Omega = \sum_{i=1}^l H_i^2 + \sum_{\alpha\in\Delta} E_\alpha E_{-\alpha}$,
(b) $\Omega$ operates by the scalar $|\lambda|^2 + 2\langle\lambda, \delta\rangle = |\lambda+\delta|^2 - |\delta|^2$ in an irreducible finite-dimensional representation of $\mathfrak{g}$ of highest weight $\lambda$, where $\delta$ is half the sum of the positive roots,
(c) the scalar value by which $\Omega$ operates in an irreducible finite-dimensional representation of $\mathfrak{g}$ is nonzero if the representation is not trivial.

PROOF.

(a) Since $B(\mathfrak{h}, E_\alpha) = 0$ for all $\alpha \in \Delta$, $\widetilde{H}_i = H_i$. Also the normalization $B(E_\alpha, E_{-\alpha}) = 1$ makes $\widetilde{E}_\alpha = E_{-\alpha}$. Then (a) follows immediately from (5.23).

(b) Let $\varphi$ be an irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight $\lambda$, and let $v_\lambda$ be a nonzero vector of weight $\lambda$. Using the relation $[E_\alpha, E_{-\alpha}] = H_\alpha$ from Lemma 2.18a, we rewrite $\Omega$ from (a) as

$$\begin{aligned}\Omega &= \sum_{i=1}^l H_i^2 + \sum_{\alpha\in\Delta^+} E_\alpha E_{-\alpha} + \sum_{\alpha\in\Delta^+} E_{-\alpha}E_\alpha \\ &= \sum_{i=1}^l H_i^2 + \sum_{\alpha\in\Delta^+} H_\alpha + 2\sum_{\alpha\in\Delta^+} E_{-\alpha}E_\alpha \\ &= \sum_{i=1}^l H_i^2 + 2H_\delta + 2\sum_{\alpha\in\Delta^+} E_{-\alpha}E_\alpha.\end{aligned}$$

When we apply $\Omega$ to $v_\lambda$ and use Theorem 5.5c, the last term gives 0. Thus

$$\Omega v_\lambda = \sum_{i=1}^l \lambda(H_i)^2 v_\lambda + 2\lambda(H_\delta)v_\lambda = (|\lambda|^2 + 2\langle\lambda,\delta\rangle)v_\lambda.$$

Schur's Lemma (Proposition 5.1) shows that $\Omega$ acts by a scalar, and hence that scalar must be $|\lambda|^2 + 2\langle\lambda, \delta\rangle$.

(c) We have $\langle\lambda, \delta\rangle = \frac{1}{2}\sum_{\alpha\in\Delta^+}\langle\lambda, \alpha\rangle$. Since $\lambda$ is dominant, this is $\geq 0$ with equality only if $\langle\lambda, \alpha\rangle = 0$ for all $\alpha$, i.e., only if $\lambda = 0$. Thus the scalar in (b) is $\geq |\lambda|^2$ and can be 0 only if $\lambda$ is 0.

**Theorem 5.29.** Let $\varphi$ be a complex-linear representation of the complex semisimple Lie algebra $\mathfrak{g}$ on a finite-dimensional complex vector space $V$. Then $V$ is completely reducible in the sense that there exist invariant subspaces $U_1, \dots, U_r$ of $V$ such that $V = U_1 \oplus \cdots \oplus U_r$ and the restriction of the representation to each $U_i$ is irreducible.

REMARKS. The proof is very similar to the proof of Theorem 1.67. It is enough by induction to show that any invariant subspace $U$ in $V$ has an invariant complement $U'$. For the case that $U$ has codimension 1, we shall prove this result as a lemma. Then we return to the proof of Theorem 5.29.

**Lemma 5.30.** Let $\varphi : \mathfrak{g} \to \operatorname{End} V$ be a finite-dimensional representation, and let $U \subseteq V$ be an invariant subspace of codimension 1. Then there is a 1-dimensional invariant subspace $W$ such that $V = U \oplus W$.

PROOF.
*Case* 1. Suppose $\dim U = 1$. Form the quotient representation $\varphi$ on $V/U$, with $\dim(V/U) = 1$. This quotient representation is irreducible of dimension 1, and Lemma 4.28 shows that it is 0. Consequently

$$\varphi(\mathfrak{g})V \subseteq U \qquad \text{and} \qquad \varphi(\mathfrak{g})U = 0.$$

Hence if $Y = [X_1, X_2]$, we have

$$\begin{aligned} \varphi(Y)V &\subseteq \varphi(X_1)\varphi(X_2)V + \varphi(X_2)\varphi(X_1)V \\ &\subseteq \varphi(X_1)U + \varphi(X_2)U = 0. \end{aligned}$$

Since Corollary 1.55 gives $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, we conclude that $\varphi(\mathfrak{g}) = 0$. Therefore any complementary subspace to $U$ will serve as $W$.

*Case* 2. Suppose that $\varphi(\,\cdot\,)|_U$ is irreducible and $\dim U > 1$. Since $\dim V/U = 1$, the quotient representation is 0 and $\varphi(\mathfrak{g})V \subseteq U$. The formula for $\Omega$ in (5.23) then shows that $\Omega(V) \subseteq U$, and Proposition 5.28c says that $\Omega$ is a nonzero scalar on $U$. Therefore $\dim(\ker \Omega) = 1$ and $U \cap (\ker \Omega) = 0$. Since $\Omega$ commutes with $\varphi(\mathfrak{g})$, $\ker \Omega$ is an invariant subspace. Taking $W = \ker \Omega$, we have $V = U \oplus W$ as required.

*Case* 3. Suppose that $\varphi(\,\cdot\,)|_U$ is not necessarily irreducible and that $\dim U \geq 1$. We induct on $\dim V$. The base case is $\dim V = 2$ and is handled by Case 1. When $\dim V > 2$, let $U_1 \subseteq U$ be an irreducible invariant subspace, and form the quotient representations on

$$U/U_1 \subseteq V/U_1$$

with quotient $V/U$ of dimension 1. By inductive hypothesis we can write

$$V/U_1 = U/U_1 \oplus Y/U_1,$$

where $Y$ is an invariant subspace in $V$ and $\dim Y/U_1 = 1$. Case 1 or Case 2 is applicable to the representation $\varphi(\,\cdot\,)|_Y$ and the irreducible invariant subspace $U_1$. Then $Y = U_1 \oplus W$, where $W$ is a 1-dimensional invariant subspace. Since $W \subseteq Y$ and $Y \cap U \subseteq U_1$, we find that

$$W \cap U = (W \cap Y) \cap U = W \cap (Y \cap U) \subseteq W \cap U_1 = 0.$$

Therefore $V = U \oplus W$ as required.

PROOF OF THEOREM 5.29. Let $\varphi$ be a representation of $\mathfrak{g}$ on $M$, and let $N \neq 0$ be an invariant subspace. Put

$$V = \{\gamma \in \operatorname{End} M \mid \gamma(M) \subseteq N \text{ and } \gamma|_N \text{ is scalar}\}.$$

Linear algebra shows that $V$ is nonzero. Define a linear function $\sigma$ from $\mathfrak{g}$ into End(End $M$) by

$$\sigma(X)\gamma = \varphi(X)\gamma - \gamma\varphi(X) \qquad \text{for } \gamma \in \operatorname{End} M \text{ and } X \in \mathfrak{g}.$$

Checking directly that $\sigma[X, Y]$ and $\sigma(X)\sigma(Y) - \sigma(Y)\sigma(X)$ are equal, we see that $\sigma$ is a representation of $\mathfrak{g}$ on End $M$.

We claim that the subspace $V \subseteq \operatorname{End} M$ is an invariant subspace under $\sigma$. In fact, let $\gamma(M) \subseteq N$ and $\gamma|_N = \lambda 1$. In the right side of the expression

$$\sigma(X)\gamma = \varphi(X)\gamma - \gamma\varphi(X),$$

the first term carries $M$ to $N$ since $\gamma$ carries $M$ to $N$ and $\varphi(X)$ carries $N$ to $N$. The second term carries $M$ into $N$ since $\varphi(X)$ carries $M$ to $M$ and $\gamma$ carries $M$ to $N$. Thus $\sigma(X)\gamma$ carries $M$ into $N$. On $N$, the action of $\sigma(X)\gamma$ is given by

$$\sigma(X)\gamma(n) = \varphi(X)\gamma(n) - \gamma\varphi(X)(n) = \lambda\varphi(X)(n) - \lambda\varphi(X)(n) = 0.$$

Thus $V$ is an invariant subspace.

Actually the above argument shows also that the subspace $U$ of $V$ given by

$$U = \{\gamma \in V \mid \gamma = 0 \text{ on } N\}$$

is an invariant subspace. Clearly $\dim V/U = 1$. By Lemma 5.30, $V = U \oplus W$ for a 1-dimensional invariant subspace $W = \mathbb{C}\gamma$. Here $\gamma$ is a nonzero scalar $\lambda 1$ on $N$. The invariance of $W$ means that $\sigma(X)\gamma = 0$ since 1-dimensional representations are 0 by Lemma 4.28. Therefore $\gamma$ commutes with $\varphi(X)$ for all $X \in \mathfrak{g}$. But then $\ker\gamma$ is a nonzero invariant subspace of $M$. Since $\gamma$ is nonsingular on $N$ (being a nonzero scalar there), we must have $M = N \oplus \ker\gamma$. This completes the proof.

Let us return to the notation introduced before Proposition 5.28, taking $\mathfrak{h}$ to be a Cartan subalgebra, $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ to be the set of roots, and $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$ to be a positive system. Define $\mathfrak{n}$ and $\mathfrak{n}^-$ as in (5.8).

**Corollary 5.31.** Let a finite-dimensional representation of $\mathfrak{g}$ be given on a space $V$, and let $V^{\mathfrak{n}}$ be the subspace of $\mathfrak{n}$ **invariants** given by

$$V^{\mathfrak{n}} = \{v \in V \mid Xv = 0 \text{ for all } X \in \mathfrak{n}\}.$$

Then the subspace $V^{\mathfrak{n}}$ is a $U(\mathfrak{h})$ module, and

(a) $V = V^{\mathfrak{n}} \oplus \mathfrak{n}^- V$ as $U(\mathfrak{h})$ modules,
(b) the natural map $V^{\mathfrak{n}} \to V/(\mathfrak{n}^- V)$ is an isomorphism of $U(\mathfrak{h})$ modules,
(c) the $U(\mathfrak{h})$ module $V^{\mathfrak{n}}$ determines the $U(\mathfrak{g})$ module $V$ up to equivalence; the dimension of $V^{\mathfrak{n}}$ equals the number of irreducible constituents of $V$, and the multiplicity of a weight in $V^{\mathfrak{n}}$ equals the multiplicity in $V$ of the irreducible representation of $\mathfrak{g}$ with that highest weight.

PROOF. To see that $V^{\mathfrak{n}}$ is a $U(\mathfrak{h})$ module, let $H$ be in $\mathfrak{h}$ and $v$ be in $V^{\mathfrak{n}}$. If $X$ is in $\mathfrak{n}$, then $X(Hv) = H(Xv) + [X, H]v = 0 + X'v$ with $X'$ in $\mathfrak{n}$, and it follows that $Hv$ is in $V^{\mathfrak{n}}$. Thus $V^{\mathfrak{n}}$ is a $U(\mathfrak{h})$ module. Similarly $\mathfrak{n}^- V$ is a $U(\mathfrak{h})$ module. Conclusion (b) is immediate from (a), and conclusion (c) is immediate from Theorems 5.29 and 5.5. Thus we are left with proving (a).

By Theorem 5.29, $V$ is a direct sum of irreducible representations, and hence there is no loss of generality for the proof of (a) in assuming that $V$ is irreducible, say of highest weight $\lambda$. With $V$ irreducible, choose nonzero root vectors $E_\alpha$ for every root $\alpha$, and let $H_1, \ldots, H_l$ be a basis of $\mathfrak{h}$. By the Poincaré–Birkhoff–Witt Theorem (Theorem 3.8), $U(\mathfrak{g})$ is spanned by all elements

$$E_{-\beta_1} \cdots E_{-\beta_p} H_{i_1} \cdots H_{i_q} E_{\alpha_1} \cdots E_{\alpha_r},$$

where the $\alpha_i$ and $\beta_j$ are positive roots, not necessarily distinct. Since $V$ is irreducible, $V$ is spanned by all elements

$$E_{-\beta_1} \cdots E_{-\beta_p} H_{i_1} \cdots H_{i_q} E_{\alpha_1} \cdots E_{\alpha_r} v$$

with $v$ in $V_\lambda$. Since $V_\lambda$ is annihilated by $\mathfrak{n}$, such an element is 0 unless $r = 0$. The space $V_\lambda$ is mapped into itself by $\mathfrak{h}$, and we conclude that $V$ is spanned by all elements

$$E_{-\beta_1} \cdots E_{-\beta_p} v$$

with $v$ in $V_\lambda$. If $p > 0$, such an element is in $\mathfrak{n}^- V$ and has weight less than $\lambda$, while if $p = 0$, it is in $V_\lambda$. Consequently

$$V = V_\lambda \oplus \mathfrak{n}^- V.$$

Theorem 5.5c shows that $V^{\mathfrak{n}}$ is just the $\lambda$ weight space of $V$, and (a) follows. This completes the proof of the corollary.

We conclude this section by giving a generalization of Proposition 5.24 that yields many elements in $Z(\mathfrak{g})$ when $\mathfrak{g}$ is semisimple. We shall use this result in the next section.

**Proposition 5.32.** Let $\varphi$ be a finite-dimensional representation of a complex semisimple Lie algebra $\mathfrak{g}$, and let $B$ be the Killing form of $\mathfrak{g}$. If $X_i$ is a basis of $\mathfrak{g}$ over $\mathbb{C}$, let $\widetilde{X}_i$ be the dual basis relative to $B$. Fix an integer $n \geq 1$ and define

$$z = \sum_{i_1,\dots,i_n} \operatorname{Tr}\varphi(X_{i_1}\cdots X_{i_n})\widetilde{X}_{i_1}\cdots\widetilde{X}_{i_n}$$

as a member of $U(\mathfrak{g})$. Then $z$ is independent of the choice of basis $X_i$ and is a member of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$.

PROOF. The proof is modeled on the argument for Proposition 5.24. Let a second basis $X_i'$ be given by (5.25a), with dual basis $\widetilde{X}_i'$ given by (5.25b). The element to consider is

$$\begin{aligned}
z' &= \sum_{i_1,\dots,i_n} \operatorname{Tr}\varphi(X'_{i_1}\cdots X'_{i_n})\widetilde{X}'_{i_1}\cdots\widetilde{X}'_{i_n}\\
&= \sum_{m_1,\dots,m_n}\sum_{l_1,\dots,l_n}\sum_{i_1,\dots,i_n} a_{m_1 i_1}\cdots a_{m_n i_n}\operatorname{Tr}\varphi(X_{m_1}\cdots X_{m_n})\\
&\qquad\qquad\qquad \times b_{i_1 l_1}\cdots b_{i_n l_n}\widetilde{X}_{l_1}\cdots\widetilde{X}_{l_n}\\
&= \sum_{m_1,\dots,m_n}\sum_{l_1,\dots,l_n} \delta_{m_1 l_1}\cdots\delta_{m_n l_n}\operatorname{Tr}\varphi(X_{m_1}\cdots X_{m_n})\widetilde{X}_{l_1}\cdots\widetilde{X}_{l_n}\\
&= \sum_{l_1,\dots,l_n} \operatorname{Tr}\varphi(X_{l_1}\cdots X_{l_n})\widetilde{X}_{l_1}\cdots\widetilde{X}_{l_n}\\
&= z.
\end{aligned}$$

This proves that $z$ is independent of the basis.

The group $G = \operatorname{Int}\mathfrak{g}$ has Lie algebra $(\operatorname{ad}\mathfrak{g})^{\mathbb{R}}$, and its simply connected cover $\widetilde{G}$ is a simply connected analytic group with Lie algebra $\mathfrak{g}^{\mathbb{R}}$. Regarding the representation $\varphi$ of $\mathfrak{g}$ as a representation of $\mathfrak{g}^{\mathbb{R}}$, we can lift it to a representation $\Phi$ of $\widetilde{G}$ since $\widetilde{G}$ is simply connected. Fix $g \in \widetilde{G}$. In the earlier part of the proof let the new basis be $X_i' = \operatorname{Ad}(g)X_i$. Then (5.26) shows that $\widetilde{X}_i' = \operatorname{Ad}(g)\widetilde{X}_i$. Consequently $\operatorname{Ad}(g)z$ is

$$
\begin{aligned}
&= \sum_{i_1,\dots,i_n} \operatorname{Tr}\varphi(X_{i_1}\cdots X_{i_n})\operatorname{Ad}(g)(\widetilde{X}_{i_1}\cdots\widetilde{X}_{i_n})\\
&= \sum_{i_1,\dots,i_n} \operatorname{Tr}(\Phi(g)\varphi(X_{i_1}\cdots X_{i_n})\Phi(g)^{-1})\widetilde{X}_{i_1}'\cdots\widetilde{X}_{i_n}'\\
&= \sum_{i_1,\dots,i_n} \operatorname{Tr}((\Phi(g)\varphi(X_{i_1})\Phi(g)^{-1})\cdots(\Phi(g)\varphi(X_{i_n})\Phi(g)^{-1}))\widetilde{X}_{i_1}'\cdots\widetilde{X}_{i_n}'\\
&= \sum_{i_1,\dots,i_n} \operatorname{Tr}(\varphi(\operatorname{Ad}(g)X_{i_1})\cdots\varphi(\operatorname{Ad}(g)X_{i_n}))\widetilde{X}_{i_1}'\cdots\widetilde{X}_{i_n}'\\
&= \sum_{i_1,\dots,i_n} \operatorname{Tr}(\varphi((\operatorname{Ad}(g)X_{i_1})\cdots(\operatorname{Ad}(g)X_{i_n})))\widetilde{X}_{i_1}'\cdots\widetilde{X}_{i_n}'\\
&= \sum_{i_1,\dots,i_n} \operatorname{Tr}(\varphi(X_{i_1}'\cdots X_{i_n}'))\widetilde{X}_{i_1}'\cdots\widetilde{X}_{i_n}',
\end{aligned}
$$

and this equals $z$, by the result of the earlier part of the proof. By Proposition 5.22, $z$ is in $Z(\mathfrak{g})$.

## 5. Harish-Chandra Isomorphism

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\mathfrak{h}$, $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, $W = W(\Delta)$, and $B$ be as in §2. Define $\mathcal{H} = U(\mathfrak{h})$. Since $\mathfrak{h}$ is abelian, the algebra $\mathcal{H}$ coincides with the symmetric algebra $S(\mathfrak{h})$. By Proposition A.20b every linear transformation of $\mathfrak{h}$ into an associative commutative algebra $A$ with identity extends uniquely to a homomorphism of $\mathcal{H}$ into $A$ sending 1 into 1. Consequently

(i) $W$ acts on $\mathcal{H}$ (since it maps $\mathfrak{h}$ into $\mathfrak{h} \subseteq \mathcal{H}$, with $\lambda^w(H) = \lambda(H^{w^{-1}})$),

(ii) $\mathcal{H}$ may be regarded as the space of polynomial functions on $\mathfrak{h}^*$ (because if $\lambda$ is in $\mathfrak{h}^*$, $\lambda$ is linear from $\mathfrak{h}$ into $\mathbb{C}$ and so extends to a homomorphism of $\mathcal{H}$ into $\mathbb{C}$; we can think of $\lambda$ on a member of $\mathcal{H}$ as the value of the member of $\mathcal{H}$ at the point $\lambda$).

Let $\mathcal{H}^W = U(\mathfrak{h})^W = S(\mathfrak{h})^W$ be the subalgebra of Weyl-group invariants of $\mathcal{H}$. In this section we shall establish the "Harish-Chandra isomorphism" $\gamma : Z(\mathfrak{g}) \to \mathcal{H}^W$, and we shall see an indication of how this isomorphism allows us to work with infinitesimal characters when $\mathfrak{g}$ is semisimple.

The Harish-Chandra mapping is motivated by considering how an element $z \in Z(\mathfrak{g})$ acts in an irreducible finite-dimensional representation with highest weight $\lambda$. The action is by scalars, by Proposition 5.19, and we compute those scalars by testing the action on a nonzero highest-weight vector.

First we use the Poincaré–Birkhoff–Witt Theorem (Theorem 3.8) to introduce a suitable basis of $U(\mathfrak{g})$ for making the computation. Introduce a positive system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$, and define $\mathfrak{n}$, $\mathfrak{n}^-$, $\mathfrak{b}$, and $\delta$ as in (5.8). As in (5.6), enumerate the positive roots as $\beta_1, \dots, \beta_k$, and let $H_1, \dots, H_l$ be a basis of $\mathfrak{h}$ over $\mathbb{C}$. For each root $\alpha \in \Delta$, let $E_\alpha$ be a nonzero root vector. Then the monomials

$$E_{-\beta_1}^{q_1} \cdots E_{-\beta_k}^{q_k} H_1^{m_1} \cdots H_l^{m_l} E_{\beta_1}^{p_1} \cdots E_{\beta_k}^{p_k} \tag{5.33}$$

are a basis of $U(\mathfrak{g})$ over $\mathbb{C}$.

If we expand the central element $z$ in terms of the above basis of $U(\mathfrak{g})$ and consider the effect of the term (5.33), there are two possibilities. One is that some $p_j$ is $> 0$, and then the term acts as 0. The other is that all $p_j$ are 0. In this case, as we shall see in Proposition 5.34b below, all $q_j$ are 0. The $U(\mathfrak{h})$ part acts on a highest weight vector $v_\lambda$ by the scalar

$$\lambda(H_1)^{m_1} \cdots \lambda(H_l)^{m_l},$$

and that is the total effect of the term. Hence we can compute the effect of $z$ if we can extract those terms in the expansion relative to the basis (5.33) such that only the $U(\mathfrak{h})$ part is present. This idea was already used in the proof of Proposition 5.28b.

Thus define

$$\mathcal{P} = \sum_{\alpha \in \Delta^+} U(\mathfrak{g}) E_\alpha \qquad \text{and} \qquad \mathcal{N} = \sum_{\alpha \in \Delta^+} E_{-\alpha} U(\mathfrak{g}).$$

**Proposition 5.34.**

(a) $U(\mathfrak{g}) = \mathcal{H} \oplus (\mathcal{P} + \mathcal{N})$,

(b) Any member of $Z(\mathfrak{g})$ has its $\mathcal{P} + \mathcal{N}$ component in $\mathcal{P}$.

PROOF.

(a) The fact that $U(\mathfrak{g}) = \mathcal{H} + (\mathcal{P} + \mathcal{N})$ follows by the Poincaré–Birkhoff–Witt Theorem (Theorem 3.8) from the fact that the elements (5.33) span $U(\mathfrak{g})$. Fix the basis of elements (5.33). For any nonzero element of $U(\mathfrak{g})E_\alpha$ with $\alpha \in \Delta^+$, write out the $U(\mathfrak{g})$ factor in terms of the basis (5.33), and consider a single term of the product, say

$$cE_{-\beta_1}^{q_1} \cdots E_{-\beta_k}^{q_k} H_1^{m_1} \cdots H_l^{m_l} E_{\beta_1}^{p_1} \cdots E_{\beta_k}^{p_k} E_\alpha. \tag{5.35}$$

The factor $E_{\beta_1}^{p_1} \cdots E_{\beta_k}^{p_k} E_\alpha$ is in $U(\mathfrak{n})$ and has no constant term. By the Poincaré–Birkhoff–Witt Theorem, we can rewrite it as a linear combination of terms $E_{\beta_1}^{r_1} \cdots E_{\beta_k}^{r_k}$ with $r_1 + \cdots + r_k > 0$. Putting

$$cE_{-\beta_1}^{q_1} \cdots E_{-\beta_k}^{q_k} H_1^{m_1} \cdots H_l^{m_l}$$

in place on the left of each term, we see that (5.35) is a linear combination of terms (5.33) with $p_1 + \cdots + p_k > 0$. Similarly any member of $\mathcal{N}$ is a linear combination of terms (5.33) with $q_1 + \cdots + q_k > 0$. Thus any member of $\mathcal{P} + \mathcal{N}$ is a linear combination of terms (5.33) with $p_1 + \cdots + p_k > 0$ or $q_1 + \cdots + q_k > 0$. Any member of $\mathcal{H}$ has $p_1 + \cdots + p_k = 0$ and $q_1 + \cdots + q_k = 0$ in every term of its expansion, and thus (a) follows.

(b) In terms of the representation ad on $U(\mathfrak{g})$ given in Proposition 5.21, the monomials (5.33) are a basis of $U(\mathfrak{g})$ of weight vectors for ad $\mathfrak{h}$, the weight of (5.33) being

$$-q_1\beta_1 - \cdots - q_k\beta_k + p_1\beta_1 + \cdots + p_k\beta_k. \tag{5.36}$$

Any member $z$ of $Z(\mathfrak{g})$ satisfies $(\operatorname{ad} H)z = Hz - zH = 0$ for $H \in \mathfrak{h}$ and thus is of weight 0. Hence its expansion in terms of the basis (5.33) involves only terms of weight 0. In the proof of (a) we saw that any member of $\mathcal{P} + \mathcal{N}$ has each term with $p_1 + \cdots + p_k > 0$ or $q_1 + \cdots + q_k > 0$. Since the $p$'s and $q$'s are constrained by the condition that (5.36) equal 0, each term must have both $p_1 + \cdots + p_k > 0$ and $q_1 + \cdots + q_k > 0$. Hence each term is in $\mathcal{P}$.

Let $\gamma'_{\mathfrak{n}}$ be the projection of $Z(\mathfrak{g})$ into the $\mathcal{H}$ term in Proposition 5.34a. Applying the basis elements (5.33) to a highest weight vector of a finite-dimensional representation, we see that

(5.37) $\lambda(\gamma'_{\mathfrak{n}}(z))$ is the scalar by which $z$ acts in an irreducible finite-dimensional representation of highest weight $\lambda$.

Despite the tidiness of this result, Harish-Chandra found that a slight adjustment of $\gamma'_{\mathfrak{n}}$ leads to an even more symmetric formula. Define a linear map $\tau_{\mathfrak{n}} : \mathfrak{h} \to \mathcal{H}$ by

$$\tau_{\mathfrak{n}}(H) = H - \delta(H)1, \tag{5.38}$$

and extend $\tau_{\mathfrak{n}}$ to an algebra automorphism of $\mathcal{H}$ by the universal mapping property for symmetric algebras. The **Harish-Chandra map** $\gamma$ is defined by

$$\gamma = \tau_{\mathfrak{n}} \circ \gamma'_{\mathfrak{n}} \tag{5.39}$$

as a mapping of $Z(\mathfrak{g})$ into $\mathcal{H}$.

Any element $\lambda \in \mathfrak{h}^*$ defines an algebra homomorphism $\lambda : \mathcal{H} \to \mathbb{C}$ with $\lambda(1) = 1$, because the universal mapping property of symmetric algebras allows us to extend $\lambda : \mathfrak{h} \to \mathbb{C}$ to $\mathcal{H}$. In terms of this extension, the maps $\gamma$ and $\gamma'_{\mathfrak{n}}$ are related by

$$\lambda(\gamma(z)) = (\lambda - \delta)(\gamma'_{\mathfrak{n}}(z)) \qquad \text{for } z \in Z(\mathfrak{g}),\ \lambda \in \mathfrak{h}^*. \tag{5.40a}$$

If instead we think of $\mathcal{H}$ as the space of polynomial functions on $\mathfrak{h}^*$, this formula may be rewritten as

$$\gamma(z)(\lambda) = \gamma'_{\mathfrak{n}}(z)(\lambda - \delta) \qquad \text{for } z \in Z(\mathfrak{g}),\ \lambda \in \mathfrak{h}^*. \tag{5.40b}$$

We define

$$\chi_\lambda(z) = \lambda(\gamma(z)) \qquad \text{for } z \in Z(\mathfrak{g}), \tag{5.41}$$

so that $\chi_\lambda$ is a map of $Z(\mathfrak{g})$ into $\mathbb{C}$. This map has the following interpretation.

**Proposition 5.42.** For $\lambda \in \mathfrak{h}^*$ and $z \in Z(\mathfrak{g})$, $\chi_\lambda(z)$ is the scalar by which $z$ operates on the Verma module $V(\lambda)$.

REMARK. In this notation we can restate (5.37) as follows:

(5.43) $\chi_{\lambda+\delta}(z)$ is the scalar by which $z$ acts in an irreducible finite-dimensional representation of highest weight $\lambda$.

PROOF. Write $z = \gamma'_{\mathfrak{n}}(z) + p$ with $p \in \mathcal{P}$. If $v_{\lambda-\delta}$ denotes the canonical generator of $V(\lambda)$, then

$$\begin{aligned} z v_{\lambda-\delta} &= \gamma'_{\mathfrak{n}}(z) v_{\lambda-\delta} + p v_{\lambda-\delta} \\ &= (\lambda - \delta)(\gamma'_{\eta}(z)) v_{\lambda-\delta} \\ &= \lambda(\gamma(z)) v_{\lambda-\delta} && \text{by (5.40)} \\ &= \chi_\lambda(z) v_{\lambda-\delta} && \text{by (5.41).} \end{aligned}$$

For $u \in U(\mathfrak{g})$, we therefore have $zuv_{\lambda-\delta} = uzv_{\lambda-\delta} = \chi_\lambda(z)uv_{\lambda-\delta}$. Since $V(\lambda) = U(\mathfrak{g})v_{\lambda-\delta}$, the result follows.

**Theorem 5.44** (Harish-Chandra). The mapping $\gamma$ in (5.40) is an algebra isomorphism of $Z(\mathfrak{g})$ onto the algebra $\mathcal{H}^W$ of Weyl-group invariants in $\mathcal{H}$, and it does not depend on the choice of the positive system $\Delta^+$.

EXAMPLE. $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Let $Z = \frac{1}{2}h^2 + ef + fe$ with $h, e, f$ as in (1.5). We noted in the first example in §4 that $Z$ is in $Z(\mathfrak{sl}(2, \mathbb{C}))$. Let us agree that $e$ corresponds to the positive root $\alpha$. Then $ef = fe + [e, f] = fe + h$ implies

$$Z = \tfrac{1}{2}h^2 + ef + fe = (\tfrac{1}{2}h^2 + h) + 2fe \in \mathcal{H} \oplus \mathcal{P}.$$

Hence

$$\gamma'_{\mathfrak{n}}(Z) = \tfrac{1}{2}h^2 + h.$$

Now $\delta(h) = \frac{1}{2}\alpha\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1$, and so

$$\tau_{\mathfrak{n}}(h) = h - 1.$$

Thus

$$\gamma(Z) = \tfrac{1}{2}(h-1)^2 + (h-1) = \tfrac{1}{2}h^2 - \tfrac{1}{2}.$$

The nontrivial element of the 2-element Weyl group acts on $\mathcal{H}$ by sending $h$ to $-h$, and thus we have a verification that $\gamma(Z)$ is invariant under the Weyl group. Moreover it is now clear that $\mathcal{H}^W = \mathbb{C}[h^2]$ and that $\gamma(\mathbb{C}[Z]) = \mathbb{C}[h^2]$. Theorem 5.44 therefore implies that $Z(\mathfrak{sl}(2, \mathbb{C})) = \mathbb{C}[Z]$.

The proof of Theorem 5.44 will occupy the remainder of this section and will take five steps.

PROOF THAT image$(\gamma) \subseteq \mathcal{H}^W$.

Since members of $\mathcal{H}$ are determined by the effect of all $\lambda \in \mathfrak{h}^*$ on them, we need to prove that

$$\lambda(w(\gamma(z))) = \lambda(\gamma(z))$$

for all $\lambda \in \mathfrak{h}^*$ and $w \in W$. In other words, we need to see that every $w \in W$ has

$$(w^{-1}\lambda)(\gamma(z)) = \lambda(\gamma(z)), \tag{5.45}$$

and it is enough to handle $w$ equal to a reflection in a simple root by Proposition 2.62. Moreover each side for fixed $z$ is a polynomial in $\lambda$, and thus it is enough to prove (5.45) for $\lambda$ dominant integral.

Form the Verma module $V(\lambda)$. We know from Proposition 5.42 that $z$ acts in $V(\lambda)$ by the scalar $\lambda(\gamma(z))$. Also $z$ acts in $V(s_\alpha\lambda)$ by the scalar $(s_\alpha\lambda)(\gamma(z))$. Since $2\langle\lambda, \alpha\rangle/|\alpha|^2$ is an integer $\geq 0$, Lemma 5.18 says that $V(s_\alpha\lambda)$ is isomorphic to a (clearly nonzero) $U(\mathfrak{g})$ submodule of $V(\lambda)$. Thus the two scalars must match, and (5.45) is proved.

PROOF THAT $\gamma$ DOES NOT DEPEND ON THE CHOICE OF $\Delta^+$.

Let $\lambda$ be algebraically integral and dominant for $\Delta^+$, let $V$ be a finite-dimensional irreducible representation of $\mathfrak{g}$ with highest weight $\lambda$ (Theorem 5.5), and let $\chi$ be the infinitesimal character of $V$. Temporarily, let us drop the subscript $\mathfrak{n}$ from $\gamma'$. By Theorem 2.63 any other positive system of roots is related in $\Delta^+$ by a member of $W(\Delta)$. Thus let $w$ be in $W(\Delta)$, and let $\widetilde{\gamma}'$ and $\widetilde{\gamma}$ be defined relative to $\Delta^{+\sim} = w\Delta^+$. We are to prove that $\gamma = \widetilde{\gamma}$. The highest weight of $V$ relative to $w\Delta^+$ is $w\lambda$. If $z$ is in $Z(\mathfrak{g})$, then (5.37) gives

$$\lambda(\gamma'(z)) = \chi(z) = w\lambda(\widetilde{\gamma}'(z)). \tag{5.46}$$

Since $\gamma(z)$ is invariant under $W(\Delta)$,

$$\begin{aligned}(w\lambda + w\delta)(\gamma(z)) &= (\lambda + \delta)(\gamma(z)) = \lambda(\gamma'(z)) \\ &= w\lambda(\widetilde{\gamma}'(z)) = (w\lambda + w\delta)(\widetilde{\gamma}(z)),\end{aligned}$$

the next-to-last step following from (5.46). Since $\gamma(z)$ and $\widetilde{\gamma}(z)$ are polynomial functions equal at the lattice points of an octant, they are equal everywhere.

PROOF THAT $\gamma$ IS MULTIPLICATIVE.

Since $\tau_{\mathfrak{n}}$ is an algebra isomorphism, we need to show that

$$\gamma'_{\mathfrak{n}}(z_1 z_2) = \gamma'_{\mathfrak{n}}(z_1)\gamma'_{\mathfrak{n}}(z_2). \tag{5.47}$$

We have

$$z_1 z_2 - \gamma'_{\mathfrak{n}}(z_1)\gamma'_{\mathfrak{n}}(z_2) = z_1(z_2 - \gamma'_{\mathfrak{n}}(z_2)) + \gamma'_{\mathfrak{n}}(z_2)(z_1 - \gamma'_{\mathfrak{n}}(z_1)),$$

which is in $\mathcal{P}$, and therefore (5.47) follows.

PROOF THAT $\gamma$ IS ONE-ONE.

If $\gamma(z) = 0$, then $\gamma'_{\mathfrak{n}}(z) = 0$, and (5.37) shows that $z$ acts as 0 in every irreducible finite-dimensional representation of $\mathfrak{g}$. By Theorem 5.29, $z$ acts as 0 in every finite-dimensional representation of $\mathfrak{g}$.

In the representation ad of $\mathfrak{g}$ on $U_n(\mathfrak{g})$, $U_{n-1}(\mathfrak{g})$ is an invariant subspace. Thus we obtain a representation ad of $\mathfrak{g}$ on $U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ for each $n$. It is enough to show that if $u \in U(\mathfrak{g})$ acts as 0 in each of these representations, then $u = 0$. Specifically let us expand $u$ in terms of the basis

$$E_{-\beta_1}^{q_1} \cdots E_{-\beta_k}^{q_k} H_1^{m_1} \cdots H_l^{m_l} E_{\beta_1}^{p_1} \cdots E_{\beta_k}^{p_k} \tag{5.48}$$

of $U(\mathfrak{g})$. We show that if $\operatorname{ad} u$ is 0 on all elements

$$H_\delta^m E_{\beta_1}^{r_1} \cdots E_{\beta^k}^{r_k} \mod U^{m+\sum r_j - 1}(\mathfrak{g}), \tag{5.49}$$

then $u = 0$. (Here as usual, $\delta$ is half the sum of the positive roots.)

In (5.48) let $m' = \sum_{j=1}^k (p_j + q_j)$. The effect of a monomial term of $u$ on (5.49) will be to produce a sum of monomials, all of whose $\mathcal{H}$ factors have total degree $\geq m - m'$. There will be one monomial whose $\mathcal{H}$ factors have total degree $= m - m'$, and we shall be able to identify that monomial and its coefficient exactly.

Let us verify this assertion. If $X$ is in $\mathfrak{g}$, the action of $\operatorname{ad} X$ on a monomial $X_1 \cdots X_n$ is

(5.50)
$$\begin{aligned}(\operatorname{ad} X)(X_1 \cdots X_n) &= X X_1 \cdots X_n - X_1 \cdots X_n X \\ &= [X, X_1] X_2 \cdots X_n + X_1 [X, X_2] X_3 \cdots X_n + \cdots + X_1 \cdots X_{n-1}[X, X_n].\end{aligned}$$

If $X_1, \ldots, X_n$ are root vectors or members of $\mathfrak{h}$ and if $X$ has the same property, then so does each $[X, X_j]$. Moreover, Lemma 3.9 allows us to commute a bracket into its correct position in (5.49), modulo lower-order terms.

Consider the effect of $\operatorname{ad} E_{\pm\alpha}$ when applied to an expression (5.49). The result is a sum of terms as in (5.50). When $\operatorname{ad} E_{\pm\alpha}$ acts on the $\mathcal{H}$ part, the degree of the $\mathcal{H}$ part of the resulting term goes down by 1, whereas if $\operatorname{ad} E_{\pm\alpha}$ acts on a root vector, the degree of the $\mathcal{H}$ part of the resulting term goes up by 1 or stays the same. When some $\operatorname{ad} H_j$ acts on an expression of the form (5.49), the degree of the $\mathcal{H}$ part of each term stays the same.

Thus when ad of (5.48) acts on (5.49), every term of the result has $\mathcal{H}$ part of degree $\geq m - m'$, and degree $= m - m'$ arises only when all $\operatorname{ad} E_{\pm\alpha}$'s act on one of the factors $H_\delta$. To compute exactly the term at the end with $\mathcal{H}$ part of degree $= m - m'$, let us follow this process step by step. When we apply $\operatorname{ad} E_{\beta_k}$ to (5.49), we get a contribution of $\langle -\beta_k, \delta \rangle$ from each factor of $H_\delta$ in (5.49), plus irrelevant terms. Thus $\operatorname{ad} E_{\beta_k}$ of (5.49) gives

$$m \langle -\beta_n, \delta \rangle H_\delta^{m-1} E_{\beta_1}^{r_1} \cdots E_{\beta_k}^{r_k+1} + \text{irrelevant terms.}$$

By the time we have applied all of $\operatorname{ad}(E_{\beta_1}^{p_1} \cdots E_{\beta_k}^{p_k})$ to (5.49), the result is

(5.51)
$$\frac{m!}{(m - \sum p_j)!} \Big( \prod_{j=1}^k \langle -\beta_j, \delta \rangle^{p_j} \Big) H_\delta^{m - \sum p_j} E_{\beta_1}^{p_1 + r_1} \cdots E_{\beta_k}^{p_k + r_k} + \text{irrelevant terms.}$$

Next we apply $\operatorname{ad} H_l$ to (5.51). The main term is multiplied by the constant $\sum_{j=1}^{k}(p_j+r_j)\beta_j(H_l)$. Repeating this kind of computation for the other factors from $\operatorname{ad}(\mathcal{H})$, we see that $\operatorname{ad}(H_1^{m_1}\cdots H_l^{m_l})$ of (5.51) is

$$\frac{m!}{m-\sum p_j}\Big(\prod_{j=1}^{k}\langle-\beta_j,\delta\rangle^{p_j}\Big)\prod_{i=1}^{l}\Big(\sum_{j=1}^{k}(p_j+r_j)\beta_j(H_i)\Big)^{m_i} \tag{5.52}$$
$$\times H_\delta^{m-\sum p_j}E_{\beta_1}^{p_1+r_1}\cdots E_{\beta_k}^{p_k+r_k}+\text{irrelevant terms.}$$

Finally we apply $\operatorname{ad} E_{-\beta_k}$ to (5.52). The main term gets multiplied by $(m-\sum p_j)\langle\beta_k,\delta\rangle$, another factor of $H_\delta$ gets dropped, and a factor of $E_{-\beta_k}$ appears. Repeating this kind of computation for the other factors $\operatorname{ad} E_{-\beta_j}$, we see that $\operatorname{ad}(E_{-\beta_1}^{q_1}\cdots E_{-\beta_k}^{q_k})$ of (5.52) is

$$\frac{m!}{(m-m')!}\Big(\prod_{j=1}^{k}(-1)^{p_j}\langle\beta_j,\delta\rangle^{p_j+q_j}\Big)\prod_{i=1}^{l}\Big(\sum_{j=1}^{k}(p_j+r_j)\beta_j(H_i)\Big)^{m_i}$$
$$\times E_{-\beta_1}^{q_1}\cdots E_{-\beta_k}^{q_k}H_\delta^{m-m'}E_{\beta_1}^{p_1+r_1}\cdots E_{\beta_k}^{p_k+r_k}+\text{irrelevant terms.} \tag{5.53}$$

This completes our exact computation of the main term of ad of (5.48) on (5.49).

We regard $m$ and the $r_j$'s fixed for the present. Among the terms of $u$, we consider the effect of ad of only those with $m'$ as large as possible. From these, the powers of the root vectors in (5.53) allow us to reconstruct the $p_j$'s and $q_j$'s. The question is whether the different terms of $u$ for which $m'$ is maximal and the $p_j$'s and $q_j$'s take on given values can have their main contributions to (5.53) add to 0. Thus we ask whether a finite sum

$$\sum_{m_1,\dots,m_l}c_{m_1,\dots,m_l}\prod_{i=1}^{l}\Big(\sum_{j=1}^{k}(p_j+r_j)\beta_j(H_i)\Big)^{m_i}$$

can be 0 for all choices of integers $r_j\geq 0$.

Assume it is 0 for all such choices. Then

$$\sum_{m_1,\dots,m_l}c_{m_1,\dots,m_l}\prod_{i=1}^{l}\Big(\sum_{j=1}^{k}z_j\beta_j(H_i)\Big)^{m_i}=0$$

for all complex $z_1,\dots,z_k$. Hence

$$\sum_{m_1,\dots,m_l}c_{m_1,\dots,m_l}\prod_{i=1}^{l}(\mu(H_i))^{m_i}=0$$

for all $\mu \in \mathfrak{h}^*$, and we obtain

$$\mu\Big(\sum_{m_1,\dots,m_l} c_{m_1,\dots,m_l} H_1^{m_1}\cdots H_l^{m_l}\Big) = 0$$

for all $\mu \in \mathfrak{h}^*$. Therefore

$$\sum_{m_1,\dots,m_l} c_{m_1,\dots,m_l} H_1^{m_1}\cdots H_l^{m_l} = 0,$$

and it follows that all the terms under consideration in $u$ were 0. Thus $\gamma$ is one-one.

PROOF THAT $\gamma$ IS ONTO.

To prove that $\gamma$ is onto $\mathcal{H}^W$, we need a supply of members of $Z(\mathfrak{g})$. Proposition 5.32 will fulfill this need. Let $\mathcal{H}_n$ and $\mathcal{H}_n^W$ be the subspaces of $\mathcal{H}$ and $\mathcal{H}^W$ of elements homogeneous of degree $n$. It is clear from the Poincaré–Birkhoff–Witt Theorem that

$$\gamma(Z(\mathfrak{g}) \cap U_n(\mathfrak{g})) \subseteq \bigoplus_{d=0}^{n} \mathcal{H}_d^W. \tag{5.54}$$

Let $\lambda$ be any dominant algebraically integral member of $\mathfrak{h}^*$, and let $\varphi_\lambda$ be the irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight $\lambda$. Let $\Lambda(\lambda)$ be the weights of $\varphi_\lambda$, repeated as often as their multiplicities. In Proposition 5.32 let $X_i$ be the ordered basis dual to one consisting of a basis $H_1, \dots, H_l$ of $\mathfrak{h}$ followed by the root vectors $E_\alpha$. The proposition says that the following element $z$ is in $Z(\mathfrak{g})$:

$$\begin{aligned} z &= \sum_{i_1,\dots,i_n} \operatorname{Tr} \varphi_\lambda(\widetilde{X}_{i_1}\cdots \widetilde{X}_{i_n}) X_{i_1}\cdots X_{i_n} \\ &= \sum_{\substack{i_1,\dots,i_n,\\ \text{all } \le l}} \operatorname{Tr} \varphi_\lambda(\widetilde{H}_{i_1}\cdots \widetilde{H}_{i_n}) H_{i_1}\cdots H_{i_n} + \sum_{\substack{j_1,\dots,j_n,\\ \text{at least one } > l}} \operatorname{Tr} \varphi_\lambda(\widetilde{X}_{j_1}\cdots \widetilde{X}_{j_n}) X_{j_1}\cdots X_{j_n}. \end{aligned}$$

In the second sum on the right side of the equality, some factor of $X_{j_1}\cdots X_{j_n}$ is a root vector. Commuting the factors into their positions to match terms with the basis vectors (5.33) of $U(\mathfrak{g})$, we see that

$$X_{j_1}\cdots X_{j_n} \equiv u \mod U_{n-1}(\mathfrak{g}) \qquad \text{with } u \in \mathcal{P} + \mathcal{N},$$

i.e.,

$$X_{j_1}\cdots X_{j_n} \equiv 0 \mod \Big(\bigoplus_{d=0}^{n-1} \mathcal{H}_d \oplus (\mathcal{P} + \mathcal{N})\Big).$$

Application of $\gamma'_{\mathfrak{n}}$ to $z$ therefore gives

$$\gamma'_{\mathfrak{n}}(z) \equiv \sum_{\substack{i_1,\dots,i_n,\\ \text{all } \le l}} \operatorname{Tr} \varphi_\lambda(\widetilde{H}_{i_1} \cdots \widetilde{H}_{i_n}) H_{i_1} \cdots H_{i_n} \mod \Big(\bigoplus_{d=0}^{n-1} \mathcal{H}_d\Big).$$

The automorphism $\tau_{\mathfrak{n}}$ of $\mathcal{H}$ affects elements only modulo lower-order terms, and thus

$$\begin{aligned} \gamma(z) &\equiv \sum_{\substack{i_1,\dots,i_n,\\ \text{all } \le l}} \operatorname{Tr} \varphi_\lambda(\widetilde{H}_{i_1} \cdots \widetilde{H}_{i_n}) H_{i_1} \cdots H_{i_n} \mod \Big(\bigoplus_{d=0}^{n-1} \mathcal{H}_d\Big) \\ &= \sum_{\mu \in \Lambda(\lambda)} \sum_{\substack{i_1,\dots,i_n,\\ \text{all } \le l}} \mu(\widetilde{H}_{i_1}) \cdots \mu(\widetilde{H}_{i_n}) H_{i_1} \cdots H_{i_n} \mod \Big(\bigoplus_{d=0}^{n-1} \mathcal{H}_d\Big). \end{aligned}$$

Now

$$\sum_i \mu(\widetilde{H}_i) H_i = H_\mu \tag{5.55}$$

since

$$\Big\langle \sum_i \mu(\widetilde{H}_i) H_i,\ \widetilde{H}_j \Big\rangle = \mu(\widetilde{H}_j) = \langle H_\mu, \widetilde{H}_j \rangle \qquad \text{for all } j.$$

Thus

$$\gamma(z) \equiv \sum_{\mu \in \Lambda(\lambda)} (H_\mu)^n \mod \Big(\bigoplus_{d=0}^{n-1} \mathcal{H}_d\Big).$$

The set of weights of $\varphi_\lambda$, together with their multiplicities, is invariant under $W$ by Theorem 5.5e. Hence $\sum_{\mu \in \Lambda(\lambda)} (H_\mu)^n$ is in $\mathcal{H}^W$, and we can write

$$\gamma(z) \equiv \sum_{\mu \in \Lambda(\lambda)} (H_\mu)^n \mod \Big(\bigoplus_{d=0}^{n-1} \mathcal{H}_d^W\Big). \tag{5.56}$$

To prove that $\gamma$ is onto $\mathcal{H}^W$, we show that the image of $\gamma$ contains $\bigoplus_{d=0}^m \mathcal{H}_d^W$ for every $m$. For $m = 0$, we have $\gamma(1) = 1$, and there is nothing further to prove. Assuming the result for $m = n - 1$, we see from (5.56) that we can choose $z_1 \in Z(\mathfrak{g})$ with

$$\gamma(z - z_1) = \sum_{\mu \in \Lambda(\lambda)} (H_\mu)^n. \tag{5.57}$$

To complete the induction, we shall show that

$$\text{the elements } \sum_{\mu \in \Lambda(\lambda)} (H_\mu)^n \text{ span } \mathcal{H}_n^W. \tag{5.58}$$

Let $\Lambda_D(\lambda)$ be the set of dominant weights of $\varphi_\lambda$, repeated according to their multiplicities. Since again the set of weights, together with their multiplicities, is invariant under $W$, we can rewrite the right side of (5.58) as

$$= \sum_{\mu \in \Lambda_D(\lambda)} c_\mu \sum_{w \in W} (H_{w\mu})^n, \tag{5.59}$$

where $c_\mu^{-1}$ is the order of the stabilizer of $\mu$ in $W$. We know that $\varphi_\lambda$ contains the weight $\lambda$ with multiplicity 1. Equation (5.57) shows that the elements (5.59) are in the image of $\gamma$ in $\mathcal{H}_n^W$. To complete the induction, it is thus enough to show that

$$\text{the elements (5.59) span } \mathcal{H}_n^W. \tag{5.60}$$

We do so by showing that

(5.61a) the span of all elements (5.59) includes all elements $\sum_{w \in W} (H_{w\nu})^n$ for $\nu$ dominant and algebraically integral,

(5.61b) $\mathcal{H}_n^W$ is spanned by all elements $\sum_{w \in W} (H_{w\nu})^n$ for $\nu$ dominant and algebraically integral.

To prove (5.61a), note that the set of dominant algebraically integral $\nu$ in a compact set is finite because the set of integral points forms a lattice in the real linear span of the roots. Hence it is permissible to induct on $|\nu|$. The trivial case for the induction is $|\nu| = 0$. Suppose inductively that (5.61a) has been proved for all dominant algebraically integral $\nu$ with $|\nu| < |\lambda|$. If $\mu$ is any dominant weight of $\varphi_\lambda$ other than $\lambda$, then $|\mu| < |\lambda|$ by Theorem 5.5e. Thus the expression (5.59) involving $\lambda$ is the sum of $c_\lambda \sum_{w \in W} (H_{w\lambda})^n$ and a linear combination of terms for which (5.61a) is assumed by induction already to be proved. Since $c_\lambda \neq 0$, (5.61a) holds for $\sum_{w\in W}(H_{w\lambda})^n$. This completes the induction and the proof of (5.61a).

To prove (5.61b), it is enough (by summing over $w \in W$) to prove that

(5.61c) $\mathcal{H}_n$ is spanned by all elements $(H_\nu)^n$ for $\nu$ dominant and algebraically integral,

and we do so by induction on $n$. The trivial case of the induction is $n = 0$.

For $1 \leq i \leq \dim \mathfrak{h}$, we can choose dominant algebraically integral forms $\lambda_i$ such that $\{\lambda_i\}$ is a $\mathbb{C}$ basis for $\mathfrak{h}^*$. Since the $\lambda_i$'s span $\mathfrak{h}^*$, the $H_{\lambda_i}$ span $\mathfrak{h}$. Consequently the $n^{\text{th}}$ degree monomials in the $H_{\lambda_i}$ span $\mathcal{H}_n$.

Assuming (5.61c) inductively for $n-1$, we now prove it for $n$. Let $\nu_1, \dots, \nu_n$ be dominant and algebraically integral. It is enough to show that the monomial $H_{\nu_1}\cdots H_{\nu_n}$ is a linear combination of elements $(H_\nu)^n$ with $\nu$ dominant and algebraically integral. By the induction hypothesis,

$$(H_{\nu_1}\cdots H_{\nu_{n-1}})H_{\nu_n} = \sum_\nu c_\nu H_\nu^{n-1} H_{\nu_n},$$

and it is enough to show that $H_\nu^{n-1}H_{\nu'}$ is a linear combination of terms $(H_{\nu+r\nu'})^n$ with $r \geq 0$ in $\mathbb{Z}$. By the invertibility of a Vandermonde matrix, choose constants $c_1, \dots, c_n$ with

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ & & \vdots & & \\ 1 & 2^{n-1} & 3^{n-1} & \cdots & n^{n-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \sum_{j=1}^n c_j (H_{\nu+j\nu'})^n &= \sum_{j=1}^n c_j (H_\nu + jH_{\nu'})^n \\ &= \sum_{k=0}^n \binom{n}{k} H_\nu^{n-k} H_{\nu'}^k \sum_{j=1}^n c_j j^k \\ &= nH_\nu^{n-1}H_{\nu'}. \end{aligned}$$

Thus $H_\nu^{n-1}H_{\nu'}$ has the required expansion, and the induction is complete. This proves (5.61c), and consequently $\gamma$ is onto $\mathcal{H}^W$. This completes the proof of Theorem 5.44.

For $\mathfrak{g}$ complex semisimple we say that a unital left $U(\mathfrak{g})$ module $V$ "has an infinitesimal character" if $Z(\mathfrak{g})$ acts by scalars in $V$. In this case the **infinitesimal character** of $V$ is the homomorphism $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ with $\chi(z)$ equal to the scalar by which $z$ acts. Proposition 5.19 says that every irreducible unital left $U(\mathfrak{g})$ module has an infinitesimal character.

The Harish-Chandra isomorphism allows us to determine explicitly all possible infinitesimal characters. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. If $\lambda$ is in $\mathfrak{h}^*$, then $\lambda$ is meaningful on the element $\gamma(z)$ of $\mathcal{H}$. Earlier we defined in (5.41) a homomorphism $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$ by $\chi_\lambda(z) = \lambda(\gamma(z))$.

**Theorem 5.62.** If $\mathfrak{g}$ is a reductive Lie algebra and $\mathfrak{h}$ is a Cartan subalgebra, then every homomorphism of $Z(\mathfrak{g})$ into $\mathbb{C}$ sending 1 into 1 is of the form $\chi_\lambda$ for some $\lambda \in \mathfrak{h}^*$. If $\lambda'$ and $\lambda$ are in $\mathfrak{h}^*$, then $\chi_{\lambda'} = \chi_\lambda$ if and only if $\lambda'$ and $\lambda$ are in the same orbit under the Weyl group $W = W(\mathfrak{g}, \mathfrak{h})$.

PROOF. Let $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ be a homomorphism with $\chi(1) = 1$. By Theorem 5.44, $\gamma$ carries $Z(\mathfrak{g})$ onto $\mathcal{H}^W$, and therefore $\gamma(\ker \chi)$ is an ideal in $\mathcal{H}^W$. Let us check that the corresponding ideal $I = \mathcal{H}\gamma(\ker \chi)$ in $\mathcal{H}$ is proper. Assuming the contrary, suppose $u_1, \dots, u_n$ in $\mathcal{H}$ and $H_1, \dots, H_n$ in $\gamma(\ker \chi)$ are such that $\sum_i u_i H_i = 1$. Application of $w \in W$ gives $\sum_i (wu_i) H_i = 1$. Summing on $w$, we obtain

$$\sum_i \Big( \sum_{w \in W} wu_i \Big) H_i = |W|.$$

Since $\sum_{w \in W} wu_i$ is in $\mathcal{H}^W$, we can apply $\chi \circ \gamma^{-1}$ to both sides. Since $\chi(1) = 1$, the result is

$$\sum_i \chi\Big(\gamma^{-1}\Big(\sum_{w \in W} wu_i\Big)\Big) \chi(\gamma^{-1}(H_i)) = |W|.$$

But the left side is 0 since $\chi(\gamma^{-1}(H_i)) = 0$ for all $i$, and we have a contradiction. We conclude that the ideal $I$ is proper.

By Zorn's Lemma, extend $I$ to a maximal ideal $\widetilde{I}$ of $\mathcal{H}$. The Hilbert Nullstellensatz tells us that there is some $\lambda \in \mathfrak{h}^*$ with

$$\widetilde{I} = \{H \in \mathcal{H} \mid \lambda(H) = 0\}.$$

Since $\gamma(\ker \chi) \subseteq I \subseteq \widetilde{I}$, we have $\chi_\lambda(z) = \lambda(\gamma(z)) = 0$ for all $z \in \ker \chi$. In other words, $\chi(z) = \chi_\lambda(z)$ for $z \in \ker \chi$ and for $z = 1$. These $z$'s span $\mathcal{H}^W$, and hence $\chi = \chi_\lambda$.

If $\lambda'$ and $\lambda$ are in the same orbit under $W$, say $\lambda' = w\lambda$, then the identity $w(\gamma(z)) = \gamma(z)$ for $w \in W$ forces

$$\chi_{\lambda'}(z) = \lambda'(\gamma(z)) = \lambda'(w(\gamma(z))) = w^{-1}\lambda'(\gamma(z)) = \lambda(\gamma(z)) = \chi_\lambda(z).$$

Finally suppose $\lambda'$ and $\lambda$ are not in the same orbit under $W$. Choose a polynomial $p$ on $\mathfrak{h}^*$ that is 1 on $W\lambda$ and 0 on $W\lambda'$. The polynomial $p$ on $\mathfrak{h}^*$ is nothing more than an element $H$ of $\mathcal{H}$ with

$$w\lambda(H) = 1 \quad \text{and} \quad w\lambda'(H) = 0 \quad \text{for all } w \in W. \tag{5.63}$$

The element $\widetilde{H}$ of $\mathcal{H}$ with $\widetilde{H} = |W|^{-1}\sum_{w\in W} wH$ is in $\mathcal{H}^W$ and satisfies the same properties (5.63) as $H$. By Theorem 5.44 we can choose $z \in Z(\mathfrak{g})$ with $\gamma(z) = \widetilde{H}$. Then $\chi_\lambda(z) = \lambda(\gamma(z)) = \lambda(\widetilde{H}) = 1$ while $\chi_{\lambda'}(z) = 0$. Hence $\chi_{\lambda'} \neq \chi_\lambda$.

Now suppose that $V$ is a $U(\mathfrak{g})$ module with infinitesimal character $\chi$. By Theorem 5.62, $\chi = \chi_\lambda$ for some $\lambda \in \mathfrak{h}^*$. We often abuse notation and say that $V$ has **infinitesimal character** $\lambda$. The element $\lambda$ is determined up to the operation of the Weyl group, again by Theorem 5.62.

EXAMPLES.

1) Let $V$ be a finite-dimensional irreducible $U(\mathfrak{g})$ module with highest weight $\lambda$. By (5.43), $V$ has infinitesimal character $\lambda + \delta$.

2) If $\lambda$ is in $\mathfrak{h}^*$, then the Verma module $V(\lambda)$ has infinitesimal character $\lambda$ by Proposition 5.42.

3) When $B$ is the Killing form and $\Omega$ is the Casimir element, Proposition 5.28b shows that $\lambda(\gamma'_{\mathfrak{n}}(\Omega)) = |\lambda - \delta|^2 - |\delta|^2$ if $\lambda$ is dominant and algebraically integral. The same proof shows that this formula remains valid as long as $\lambda$ is in the real linear span of the roots. Combining this result with the definition (5.41), we obtain

$$\chi_\lambda(\Omega) = |\lambda|^2 - |\delta|^2 \tag{5.64}$$

for $\lambda$ in the real linear span of the roots.

## 6. Weyl Character Formula

We saw in §IV.2 that the character of a finite-dimensional representation of a compact group determines the representation up to equivalence. Thus characters provide an effective tool for working with representations in a canonical fashion. In this section we shall deal with characters in a formal way, working in the context of complex semisimple Lie algebras, deferring until §8 the interpretation in terms of compact connected Lie groups.

To understand where the formalism comes from, it is helpful to think of the group $SL(2,\mathbb{C})$ and its compact subgroup $SU(2)$. The group $SU(2)$ is simply connected, being homeomorphic to the 3-sphere, and it follows from Proposition 1.143 that $SL(2,\mathbb{C})$ is simply connected also. A finite-dimensional representation of $SU(2)$ is automatically smooth. Thus it leads via differentiation to a representation of $\mathfrak{su}(2)$, then via complexification to a representation of $\mathfrak{sl}(2,\mathbb{C})$, and then via passage to the simply connected group to a holomorphic representation of $SL(2,\mathbb{C})$. We can recover the original representation of $SU(2)$ by restriction, and we can begin this cycle at any stage, continuing all the way around. This construction is an instance of "Weyl's unitary trick," which we shall study later.

Let us see the effect of this construction as we follow the character of an irreducible representation $\Phi$ with differential $\varphi$. Let $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The diagonal subalgebra $\mathfrak{h} = \{zh \mid z \in \mathbb{C}\}$ is a Cartan subalgebra of $\mathfrak{sl}(2,\mathbb{C})$, and the roots are 2 and $-2$ on $h$. We take the root that is 2 on $h$ (and has $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ as root vector) to be positive, and we call it $\alpha$. The weights of $\varphi$ are determined by the eigenvalues of $\varphi(h)$. According to Theorem 1.65, the eigenvalues are of the form $n, n-2, \dots, -n$. Hence if we define $\lambda \in \mathfrak{h}^*$ by $\lambda(zh) = zn$, then the weights are

$$\lambda,\ \lambda-\alpha,\ \lambda-2\alpha,\ \dots, -\lambda.$$

Thus the matrix of $\varphi(zh)$ relative to a basis of weight vectors is

$$\varphi(zh) = \operatorname{diag}(\lambda(zh),\ (\lambda-\alpha)(zh),\ (\lambda-2\alpha)(zh),\ \dots, -\lambda(zh)).$$

Exponentiating this formula in order to pass to the group $SL(2,\mathbb{C})$, we obtain

$$\Phi(\exp zh) = \operatorname{diag}(e^{\lambda(zh)},\ e^{(\lambda-\alpha)(zh)},\ e^{(\lambda-2\alpha)(zh)},\ \dots, e^{-\lambda(zh)}).$$

This formula makes sense within $SU(2)$ if $z$ is purely imaginary. In any event if $\chi_\Phi$ denotes the character of $\Phi$ (i.e., the trace of $\Phi$ of a group element), then we obtain

$$\begin{aligned}\chi_\Phi(\exp zh) &= e^{\lambda(zh)} + e^{(\lambda-\alpha)(zh)} + e^{(\lambda-2\alpha)(zh)} + \cdots + e^{-\lambda(zh)} \\ &= \frac{e^{(\lambda+\delta)(zh)} - e^{-(\lambda+\delta)(zh)}}{e^{\delta(zh)} - e^{-\delta(zh)}},\end{aligned}$$

where $\delta = \frac{1}{2}\alpha$ takes the value 1 on $h$. We can drop the group element from the notation if we introduce formal exponentials. Then we can write

$$\chi_\Phi = e^\lambda + e^{\lambda-\alpha} + e^{\lambda-2\alpha} + \cdots + e^{-\lambda} = \frac{e^{\lambda+\delta} - e^{-(\lambda+\delta)}}{e^\delta - e^{-\delta}}.$$

In this section we shall derive a similar expression involving formal exponentials for the character of an irreducible representation of a complex semisimple Lie algebra with a given highest weight. This result is the "Weyl Character Formula." We shall interpret the result in terms of compact connected Lie groups in §8.

The first step is to develop the formalism of exponentials. We fix a complex semisimple Lie algebra $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$, the set $\Delta$ of roots, the Weyl group $W$, and a simple system $\Pi = \{\alpha_1, \ldots, \alpha_l\}$. Let $\Delta^+$ be the set of positive roots, and let $\delta$ be half the sum of the positive roots.

Following customary set-theory notation, let $\mathbb{Z}^{\mathfrak{h}^*}$ be the additive group of all functions from $\mathfrak{h}^*$ to $\mathbb{Z}$. If $f$ is in $\mathbb{Z}^{\mathfrak{h}^*}$, then the **support** of $f$ is the set of $\lambda \in \mathfrak{h}^*$ where $f(\lambda) \neq 0$. For $\lambda \in \mathfrak{h}^*$, define $e^\lambda$ to be the member of $\mathbb{Z}^{\mathfrak{h}^*}$ that is 1 at $\lambda$ and 0 elsewhere.

Within $\mathbb{Z}^{\mathfrak{h}^*}$, let $\mathbb{Z}[\mathfrak{h}^*]$ be the subgroup of elements of finite support. For such elements we can write $f = \sum_{\lambda \in \mathfrak{h}^*} f(\lambda) e^\lambda$ since the sum is really a finite sum. However, it will be convenient to allow this notation also for $f$ in the larger group $\mathbb{Z}^{\mathfrak{h}^*}$, since the notation is unambiguous in this larger context.

Let $Q^+$ be the set of all members of $\mathfrak{h}^*$ given as $\sum_{i=1}^l n_i \alpha_i$ with all the $n_i$ equal to integers $\geq 0$. The **Kostant partition function** $\mathcal{P}$ is the function from $Q^+$ to the nonnegative integers that tells the number of ways, apart from order, that a member of $Q^+$ can be written as the sum of positive roots. By convention, $\mathcal{P}(0) = 1$.

Let $\mathbb{Z}\langle\mathfrak{h}^*\rangle$ be the set of all $f \in \mathbb{Z}^{\mathfrak{h}^*}$ whose support is contained in the union of a finite number of sets $\nu_i - Q^+$ with each $\nu_i$ in $\mathfrak{h}^*$. This is an abelian group, and we have

$$\mathbb{Z}[\mathfrak{h}^*] \subseteq \mathbb{Z}\langle\mathfrak{h}^*\rangle \subseteq \mathbb{Z}^{\mathfrak{h}^*}.$$

Within $\mathbb{Z}\langle\mathfrak{h}^*\rangle$, we introduce the multiplication

$$\Big(\sum_{\lambda\in\mathfrak{h}^*} c_\lambda e^\lambda\Big)\Big(\sum_{\mu\in\mathfrak{h}^*} \widetilde{c}_\mu e^\mu\Big) = \sum_{\nu\in\mathfrak{h}^*}\Big(\sum_{\lambda+\mu=\nu} c_\lambda \widetilde{c}_\mu\Big)e^\nu. \tag{5.65}$$

To see that (5.65) makes sense, we have to check that the interior sum on the right side is finite. Because we are working within $\mathbb{Z}\langle\mathfrak{h}^*\rangle$, we can write $\lambda = \lambda_0 - q_\lambda^+$ with $q_\lambda^+ \in Q^+$ and with only finitely many possibilities for $\lambda_0$, and we can similarly write $\mu = \mu_0 - q_\mu^+$. Then

$$(\lambda_0 - q_\lambda^+) + (\mu_0 - q_\mu^+) = \nu$$

and hence
$$q_\lambda^+ + q_\mu^+ = \nu - \lambda_0 - \mu_0.$$

Finiteness follows since there are only finitely many possibilities for $\lambda_0$ and $\mu_0$ and since $\mathcal{P}(\nu - \lambda_0 - \mu_0) < \infty$ for each.

Under the definition of multiplication in (5.65), $\mathbb{Z}\langle\mathfrak{h}^*\rangle$ is a commutative ring with identity $e^0$. Since $e^\lambda e^\mu = e^{\lambda+\mu}$, the natural multiplication in $\mathbb{Z}[\mathfrak{h}^*]$ is consistent with the multiplication in $\mathbb{Z}\langle\mathfrak{h}^*\rangle$.

The Weyl group $W$ acts on $\mathbb{Z}^{\mathfrak{h}^*}$. The definition is $wf(\mu) = f(w^{-1}\mu)$ for $f \in \mathbb{Z}^{\mathfrak{h}^*}$, $\mu \in \mathfrak{h}^*$, and $w \in W$. Then $w(e^\lambda) = e^{w\lambda}$. Each $w \in W$ leaves $\mathbb{Z}[\mathfrak{h}^*]$ stable, but in general $w$ does not leave $\mathbb{Z}\langle\mathfrak{h}^*\rangle$ stable.

We shall make use of the sign function on $W$. Let $\varepsilon(w) = \det w$ for $w \in W$. This is always $\pm1$. Any root reflection $s_\alpha$ has $\varepsilon(s_\alpha) = -1$. Thus if $w$ is written as the product of $k$ root reflections, then $\varepsilon(w) = (-1)^k$. By Proposition 2.70,

$$\varepsilon(w) = (-1)^{l(w)}, \tag{5.66}$$

where $l(w)$ is the length of $w$ as defined in §II.6.

When $\varphi$ is a representation of $\mathfrak{g}$ on $V$, we shall sometimes abuse notation and refer to $V$ as the representation. If $V$ is a representation, we say that $V$ **has a character** if $V$ is the direct sum of its weight spaces under $\mathfrak{h}$, i.e., $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$, and if $\dim V_\mu < \infty$ for $\mu \in \mathfrak{h}^*$. In this case the **character** is

$$\operatorname{char}(V) = \sum_{\mu\in\mathfrak{h}^*} (\dim V_\mu) e^\mu.$$

EXAMPLE. Let $V(\lambda)$ be a Verma module, and let $v_{\lambda-\delta}$ be the canonical generator. Let $\mathfrak{n}^-$ be the sum of the root spaces in $\mathfrak{g}$ for the negative roots. By Proposition 5.14b the map of $U(\mathfrak{n}^-)$ into $V(\lambda)$ given by $u \mapsto uv_{\lambda-\delta}$ is one-one onto. Also the action of $U(\mathfrak{h})$ on $V(\lambda)$ matches the action of $U(\mathfrak{h})$ on $U(\mathfrak{n}^-) \otimes \mathbb{C}v_{\lambda-\delta}$. Thus

$$\dim V(\lambda)_\mu = \dim U(\mathfrak{n}^-)_{\mu-\lambda+\delta}.$$

Let $E_{-\beta_1}, \ldots, E_{-\beta_k}$ be a basis of $\mathfrak{n}^-$ consisting of root vectors. The Poincaré–Birkhoff–Witt Theorem (Theorem 3.8) shows that monomials in this basis form a basis of $U(\mathfrak{n}^-)$, and it follows that $\dim U(\mathfrak{n}^-)_{-\nu} = \mathcal{P}(\nu)$. Therefore

$$\dim V(\lambda)_\mu = \mathcal{P}(\lambda - \delta - \mu),$$

and $V(\lambda)$ has a character. The character is given by

$$\text{char}(V(\lambda)) = \sum_{\mu \in \mathfrak{h}^*} \mathcal{P}(\lambda - \delta - \mu) e^\mu = e^{\lambda-\delta} \sum_{\gamma \in Q^+} \mathcal{P}(\gamma) e^{-\gamma}. \tag{5.67}$$

Let us establish some properties of characters. Let $V$ be a representation of $\mathfrak{g}$ with a character, and suppose that $V'$ is a subrepresentation. Then the representations $V'$ and $V/V'$ have characters, and

$$\text{char}(V) = \text{char}(V') + \text{char}(V/V'). \tag{5.68}$$

In fact, we just extend a basis of weight vectors for $V'$ to a basis of weight vectors of $V$. Then it is apparent that

$$\dim V_\mu = \dim V'_\mu + \dim(V/V')_\mu,$$

and (5.68) follows.

The relationship among $V$, $V'$, and $V/V'$ is summarized by saying that

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V/V' \longrightarrow 0$$

is an **exact sequence**. This means that the kernel of each map going out equals the image of each map going in.

In these terms, we can generalize (5.68) as follows. Whenever

$$0 \longrightarrow V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} V_3 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_{n-1}} V_n \longrightarrow 0$$

is an exact sequence of representations of $\mathfrak{g}$ with characters, then

$$\sum_{j=1}^{n} (-1)^j \text{char}(V_j) = 0. \tag{5.69}$$

To prove (5.69), we note that the following are exact sequences; in each case "inc" denotes an inclusion:

$$0 \longrightarrow \text{image}(\varphi_1) \xrightarrow{\text{inc}} V_2 \xrightarrow{\varphi_2} \text{image}(\varphi_2) \longrightarrow 0,$$

$$0 \longrightarrow \text{image}(\varphi_2) \xrightarrow{\text{inc}} V_3 \xrightarrow{\varphi_3} \text{image}(\varphi_3) \longrightarrow 0,$$

$$\vdots$$

$$0 \longrightarrow \text{image}(\varphi_{n-2}) \xrightarrow{\text{inc}} V_{n-1} \xrightarrow{\varphi_{n-1}} \text{image}(\varphi_{n-1}) \longrightarrow 0.$$

For $2 \le j \le n-1$, (5.68) gives

$$-\text{char}(\text{image}(\varphi_{j-1})) + \text{char}(V_j) - \text{char}(\text{image}(\varphi_j)) = 0.$$

Multiplying by $(-1)^j$ and summing, we obtain

$$\begin{aligned} 0 = -\text{char}(\text{image}(\varphi_1)) + \text{char}(V_2) - \text{char}(V_3) \\ + \cdots + (-1)^{n-1}\text{char}(V_{n-1}) + (-1)^n\text{char}(\text{image}(\varphi_{n-1})). \end{aligned}$$

Since $V_1 \cong \text{image}(\varphi_1)$ and $V_n \cong \text{image}(\varphi_{n-1})$, (5.69) follows.

Suppose that $V_1$ and $V_2$ are representations of $\mathfrak{g}$ having characters that are in $\mathbb{Z}\langle\mathfrak{h}^*\rangle$. Then $V_1 \otimes V_2$, which is a representation under the definition (4.3), has a character, and

$$(V_1 \otimes V_2) = (\text{char}(V_1))(\text{char}(V_2)). \tag{5.70}$$

In fact, the tensor product of weight vectors is a weight vector, and we can form a basis of $V_1 \otimes V_2$ from such tensor-product vectors. Hence (5.70) is an immediate consequence of (5.65).

The **Weyl denominator** is the member of $\mathbb{Z}[\mathfrak{h}^*]$ given by

$$d = e^\delta \prod_{\alpha\in\Delta^+} (1 - e^{-\alpha}). \tag{5.71}$$

Define

$$K = \sum_{\gamma\in Q^+} \mathcal{P}(\gamma)e^{-\gamma}.$$

This is a member of $\mathbb{Z}\langle\mathfrak{h}^*\rangle$.

**Lemma 5.72.** In the ring $\mathbb{Z}\langle\mathfrak{h}^*\rangle$, $Ke^{-\delta}d = 1$. Hence $d^{-1}$ exists in $\mathbb{Z}\langle\mathfrak{h}^*\rangle$.

PROOF. From the definition in (5.71), we have

$$e^{-\delta}d = \prod_{\alpha\in\Delta^+}(1 - e^{-\alpha}). \tag{5.73}$$

Meanwhile

$$\prod_{\alpha\in\Delta^+}(1 + e^{-\alpha} + e^{-2\alpha} + \cdots) = \sum_{\gamma\in Q^+}\mathcal{P}(\gamma)e^{-\gamma} = K. \tag{5.74}$$

Since $(1 - e^{-\alpha})(1 + e^{-\alpha} + e^{-2\alpha} + \cdots) = 1$ for $\alpha$ positive, the lemma follows by multiplying (5.74) by (5.73).

**Theorem 5.75** (Weyl Character Formula). Let $V$ be an irreducible finite-dimensional representation of the complex semisimple Lie algebra $\mathfrak{g}$ with highest weight $\lambda$. Then

$$\operatorname{char}(V) = d^{-1}\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\delta)}.$$

REMARKS. We shall prove this theorem below after giving three lemmas. But first we deduce an alternative formulation of the theorem.

**Corollary 5.76** (Weyl Denominator Formula).

$$e^{\delta}\prod_{\alpha\in\Delta^+}(1 - e^{-\alpha}) = \sum_{w\in W}\varepsilon(w)e^{w\delta}.$$

PROOF. Take $\lambda = 0$ in Theorem 5.75. Then $V$ is the 1-dimensional trivial representation, and $\operatorname{char}(V) = e^0 = 1$.

**Theorem 5.77** (Weyl Character Formula, alternative formulation). Let $V$ be an irreducible finite-dimensional representation of the complex semisimple Lie algebra $\mathfrak{g}$ with highest weight $\lambda$. Then

$$\Big(\sum_{w\in W}\varepsilon(w)e^{w\delta}\Big)\operatorname{char}(V) = \sum_{w\in W}\varepsilon(w)e^{w(\lambda+\delta)}.$$

PROOF. This follows by substituting the result of Corollary 5.76 into the formula of Theorem 5.75.

**Lemma 5.78.** If $\lambda$ in $\mathfrak{h}^*$ is dominant, then no $w \neq 1$ in $W$ fixes $\lambda + \delta$.

PROOF. If $w \neq 1$ fixes $\lambda + \delta$, then Chevalley's Lemma in the form of Corollary 2.73 shows that some root $\alpha$ has $\langle \lambda + \delta, \alpha \rangle = 0$. We may assume that $\alpha$ is positive. But then $\langle \lambda, \alpha \rangle \geq 0$ by dominance and $\langle \delta, \alpha \rangle > 0$ by Proposition 2.69, and we have a contradiction.

**Lemma 5.79.** The Verma module $V(\lambda)$ has a character belonging to $\mathbb{Z}\langle \mathfrak{h}^* \rangle$, and $\operatorname{char}(V(\lambda)) = d^{-1}e^{\lambda}$.

PROOF. Formula (5.67) shows that

$$\operatorname{char}(V(\lambda)) = e^{\lambda-\delta} \sum_{\gamma \in Q^+} \mathcal{P}(\gamma) e^{-\gamma} = K e^{-\delta} e^{\lambda},$$

and thus the result follows by substituting from Lemma 5.72.

**Lemma 5.80.** Let $\lambda_0$ be in $\mathfrak{h}^*$, and suppose that $M$ is a representation of $\mathfrak{g}$ such that

(i) $M$ has infinitesimal character $\lambda_0$ and
(ii) $M$ has a character belonging to $\mathbb{Z}\langle \mathfrak{h}^* \rangle$.

Let

$$D_M = \{\lambda \in W\lambda_0 \mid (\lambda - \delta + Q^+) \cap \operatorname{support}(\operatorname{char}(M)) \neq \emptyset\}.$$

Then $\operatorname{char}(M)$ is a finite $\mathbb{Z}$ linear combination of $\operatorname{char}(V(\lambda))$ for $\lambda$ in $D_M$.

REMARK. $D_M$ is a finite set, being a subset of an orbit of the finite group $W$.

PROOF. We may assume that $M \neq 0$, and we proceed by induction on $|D_M|$. First assume that $|D_M| = 0$. Since $M$ has a character belonging to $\mathbb{Z}\langle \mathfrak{h}^* \rangle$, we can find $\mu$ in $\mathfrak{h}^*$ such that $\mu - \delta$ is a weight of $M$ but $\mu - \delta + q^+$ is not a weight of $M$ for any $q^+ \neq 0$ in $Q^+$. Set $m = \dim M_{\mu-\delta}$. Since the root vectors for positive roots evidently annihilate $M_{\mu-\delta}$, the universal mapping property for Verma modules (Proposition 5.14c) shows that we can find a $U(\mathfrak{g})$ homomorphism $\varphi : V(\mu)^m \to M$ such that $(V(\mu)^m)_{\mu-\delta}$ maps one-one onto $M_{\mu-\delta}$. The infinitesimal character $\lambda_0$ of $M$ must match the infinitesimal character of $V(\mu)$, which is $\mu$ by Proposition 5.42. By Theorem 5.62, $\mu$ is in $W\lambda_0$. Then $\mu$ is in $D_M$, and $|D_M| = 0$ is impossible. This completes the base case of the induction.

Now assume the result of the lemma for modules $N$ satisfying (i) and (ii) such that $D_N$ has fewer than $|D_M|$ members. Construct $\mu$, $m$, and $\varphi$ as above. Let $L$ be the kernel of $\varphi$, and put $N = M/\text{image}\,\varphi$. Then

$$0 \longrightarrow L \longrightarrow V(\mu)^m \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$$

is an exact sequence of representations. By (5.68), $\text{char}(L)$ and $\text{char}(N)$ exist. Thus (5.69) gives

$$\text{char}(M) = -\text{char}(L) + m\,\text{char}(V(\mu)) + \text{char}(N).$$

Moreover $L$ and $N$ satisfy (i) and (ii). The induction will be complete if we show that $|D_L| < |D_M|$ and $|D_N| < |D_M|$.

In the case of $N$, we clearly have $D_N \subseteq D_M$. Since $\psi$ is onto, the equality $M_{\mu-\delta} = \text{image}\,\varphi$ implies that $N_{\mu-\delta} = 0$. Thus $\mu$ is not in $D_N$, and $|D_N| < |D_M|$.

In the case of $L$, if $\lambda$ is in $D_L$, then $\lambda - \delta + Q^+$ has nonempty intersection with $\text{support}(\text{char}(L))$ and hence with $\text{support}(\text{char}(V(\mu)))$. Then $\mu - \delta$ is in $\lambda - \delta + Q^+$, and hence $\mu - \delta$ is a member of the intersection $(\lambda-\delta+Q^+)\cap\text{support}(\text{char}(M))$. That is, $\lambda$ is in $D_M$. Therefore $D_L \subseteq D_M$. But $\mu$ is not in $D_L$, and hence $|D_L| < |D_M|$. This completes the proof.

PROOF OF THEOREM 5.75. By (5.43), $V$ has infinitesimal character $\lambda+\delta$. Lemma 5.80 applies to $V$ with $\lambda_0$ replaced by $\lambda+\delta$, and Lemma 5.79 allows us to conclude that

$$\text{char}(V) = d^{-1} \sum_{w\in W} c_w e^{w(\lambda+\delta)}$$

for some unknown integers $c_w$. We rewrite this formula as

$$d\,\text{char}(V) = \sum_{w\in W} c_w e^{w(\lambda+\delta)}. \tag{5.81}$$

Let us say that a member $f$ of $\mathbb{Z}[\mathfrak{h}^*]$ is **even** (under $W$) if $wf = f$ for all $w$ in $W$. It is **odd** if $wf = \varepsilon(w)f$ for all $w$ in $W$. Theorem 5.5e shows that $\text{char}(V)$ is even. Let us see that $d$ is odd. In fact, we can write $d$ as

$$d = \prod_{\alpha\in\Delta^+} (e^{\alpha/2} - e^{-\alpha/2}). \tag{5.82}$$

If we replace each $\alpha$ by $w\alpha$, we get the same factors on the right side of (5.82) except for minus signs, and the number of minus signs is the number of positive roots $\alpha$ such that $w\alpha$ is negative. By (5.66) this product of minus signs is just $\varepsilon(w)$.

Consequently the left side of (5.81) is odd under $W$, and application of $w_0$ to both sides of (5.81) gives

$$\begin{aligned}\sum_{w\in W} c_w \varepsilon(w_0) e^{w(\lambda+\delta)} &= \varepsilon(w_0) d\ \mathrm{char}(V) = w_0(d\ \mathrm{char}(V)) \\ &= \sum_{w\in W} c_w e^{w_0 w(\lambda+\delta)} = \sum_{w\in W} c_{w_0^{-1}w} e^{w(\lambda+\delta)}.\end{aligned}$$

By Lemma 5.78 the two sides of this formula are equal term by term. Thus we have $c_{w_0^{-1}w} = c_w \varepsilon(w_0)$ for $w$ in $W$. Taking $w = 1$ gives $c_{w_0^{-1}} = c_1 \varepsilon(w_0) = c_1 \varepsilon(w_0^{-1})$, and hence $c_{w_0} = c_1 \varepsilon(w_0)$. Therefore

$$d\ \mathrm{char}(V) = c_1 \sum_{w\in W} \varepsilon(w) e^{w(\lambda+\delta)}.$$

Expanding the left side and taking Theorem 5.5b into account, we see that the coefficient of $e^{\lambda+\delta}$ on the left side is 1. Thus another application of Lemma 5.78 gives $c_1 = 1$.

**Corollary 5.83** (Kostant Multiplicity Formula). Let $V$ be an irreducible finite-dimensional representation of the complex semisimple Lie algebra $\mathfrak{g}$ with highest weight $\lambda$. If $\mu$ is in $\mathfrak{h}^*$, then the multiplicity of $\mu$ as a weight of $V$ is

$$\sum_{w\in W} \varepsilon(w) \mathcal{P}(w(\lambda+\delta) - (\mu+\delta)).$$

REMARK. By convention in this formula, $\mathcal{P}(\nu) = 0$ if $\nu$ is not in $Q^+$.

PROOF. Lemma 5.72 and Theorem 5.75 combine to give

$$\begin{aligned}\mathrm{char}(V) &= d^{-1}(d\ \mathrm{char}(V)) \\ &= (Ke^{-\delta})(d\ \mathrm{char}(V)) \\ &= \Big(\sum_{\gamma\in Q^+} \mathcal{P}(\gamma) e^{-\delta-\gamma}\Big)\Big(\sum_{w\in W} \varepsilon(w) e^{w(\lambda+\delta)}\Big).\end{aligned}$$

Hence the required multiplicity is

$$\sum_{\substack{\gamma\in Q^+,\ w\in W \\ -\delta-\gamma+w(\lambda+\delta)=\mu}} \mathcal{P}(\gamma)\varepsilon(w) = \sum_{w\in W} \varepsilon(w) \mathcal{P}(w(\lambda+\delta) - \mu - \delta).$$

**Theorem 5.84** (Weyl Dimension Formula). Let $V$ be an irreducible finite-dimensional representation of the complex semisimple Lie algebra $\mathfrak{g}$ with highest weight $\lambda$. Then

$$\dim V = \frac{\prod_{\alpha\in\Delta^+}\langle\lambda+\delta,\alpha\rangle}{\prod_{\alpha\in\Delta^+}\langle\delta,\alpha\rangle}.$$

PROOF. For $H \in \mathfrak{h}^*$, we introduce the ring homomorphism called "evaluation at $H$," which is written $\epsilon_H : \mathbb{Z}[\mathfrak{h}^*] \to \mathbb{C}$ and is given by

$$f = \sum f(\lambda)e^{\lambda} \mapsto \sum f(\lambda)e^{\lambda(H)}.$$

Then $\dim V = \epsilon_0(\mathrm{char}(V))$. The idea is thus to apply $\epsilon_0$ to the Weyl Character Formula as given in Theorem 5.75 or Theorem 5.77. But a direct application will give $0/0$ for the value of $\epsilon_0(\mathrm{char}(V))$, and we have to proceed more carefully. In effect, we shall use a version of l'Hôpital's Rule.

For $f \in \mathbb{Z}[\mathfrak{h}^*]$ and $\varphi \in \mathfrak{h}^*$, we define

$$\partial_\varphi f(H) = \frac{d}{dr}\, f(H + rH_\varphi)|_{r=0}.$$

Then

$$\partial_\varphi e^{\lambda(H)} = \frac{d}{dr}\, e^{\lambda(H+rH_\varphi)}|_{r=0} = \langle\lambda,\varphi\rangle e^{\lambda(H)}. \tag{5.85}$$

Consider any derivative $\partial_{\varphi_1}\cdots\partial_{\varphi_n}$ of order less than the number of positive roots, and apply it to the Weyl denominator (5.71), evaluating at $H$. We are then considering

$$\partial_{\varphi_1}\cdots\partial_{\varphi_n}\Big(e^{-\delta(H)}\prod_{\alpha\in\Delta^+}(e^{\alpha(H)}-1)\Big).$$

Each $\partial_{\varphi_j}$ operates by the product rule and differentiates one factor, leaving the others alone. Thus each term in the derivative has an undifferentiated $e^{\alpha(H)} - 1$ and will give 0 when evaluated at $H = 0$.

We apply $\prod_{\alpha\in\Delta^+}\partial_\alpha$ to both sides of the identity given by the Weyl Character Formula

$$d\,\mathrm{char}(V) = \sum_{w\in W}\varepsilon(w)e^{w(\lambda+\delta)}.$$

Then we evaluate at $H = 0$. The result on the left side comes from the Leibniz rule and involves many terms, but all of them give 0 (according to the previous paragraph) except the one that comes from applying all the derivatives to $d$ and evaluating the other factor at $H = 0$. Thus we obtain

$$\Big(\Big(\prod_{\alpha\in\Delta^+}\partial_\alpha\Big)d(H)\Big)(0)\dim V = \Big(\Big(\prod_{\alpha\in\Delta^+}\partial_\alpha\Big)\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\delta)(H)}\Big)(0).$$

By Corollary 5.76 we can rewrite this formula as

$$\Big(\Big(\prod_{\alpha\in\Delta^+}\partial_\alpha\Big)\sum_{w\in W}\varepsilon(w)e^{(w\delta)(H)}\Big)(0)\dim V = \Big(\Big(\prod_{\alpha\in\Delta^+}\partial_\alpha\Big)\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\delta)(H)}\Big)(0). \tag{5.86}$$

We calculate

$$\begin{aligned}
\Big(\prod_{\alpha\in\Delta^+}\partial_\alpha\Big)&\Big(\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\delta)(H)}\Big)\\
&= \sum_{w\in W}\varepsilon(w)\prod_{\alpha\in\Delta^+}\langle w(\lambda+\delta),\alpha\rangle e^{w(\lambda+\delta)(H)} && \text{by (5.85)}\\
&= \sum_{w\in W}\varepsilon(w^{-1})\prod_{\alpha\in\Delta^+}\langle \lambda+\delta, w^{-1}\alpha\rangle e^{w(\lambda+\delta)(H)}\\
&= \sum_{w\in W}\prod_{\alpha\in\Delta^+}\langle \lambda+\delta,\alpha\rangle e^{w(\lambda+\delta)(H)} && \text{by (5.66)}\\
&= \Big(\prod_{\alpha\in\Delta^+}\langle\lambda+\delta,\alpha\rangle\Big)\sum_{w\in W}e^{w(\lambda+\delta)(H)}. && (5.87)
\end{aligned}$$

When $\lambda = 0$, (5.87) has a nonzero limit as $H$ tends to 0 by Proposition 2.69. Therefore we can evaluate $\dim V$ from (5.86) by taking the quotient with $H$ in place and then letting $H$ tend to 0. By (5.87) the result is the formula of the theorem.

The Weyl Dimension Formula provides a convenient tool for deciding irreducibility. Let $\varphi$ be a finite-dimensional representation of $\mathfrak{g}$, and suppose that $\lambda$ is the highest weight of $\varphi$. Theorem 5.29 shows that $\varphi$ is completely reducible, and one of the irreducible summands must have $\lambda$ as highest weight. Call this summand $\varphi_\lambda$. Theorem 5.84 allows us to compute $\dim\varphi_\lambda$. Then it follows that $\varphi$ is irreducible if and only if $\dim\varphi$ matches the value of $\dim\varphi_\lambda$ given by Theorem 5.84.

EXAMPLE. With $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, let $\varphi$ be the representation on the space consisting of all holomorphic polynomials in $z_1, \dots, z_n$ homogeneous of degree $N$. We shall prove that this representation is irreducible. From the first example in §2, we know that this representation has highest weight $-Ne_n$. Its dimension is $\binom{N+n-1}{N}$, the number of ways of labeling $n-1$ of $N+n-1$ objects as dividers and the others as monomials $z_j$. To check that $\varphi$ is irreducible, it is enough to see from the Weyl Dimension Formula that the irreducible representation $\varphi_{-Ne_n}$ with highest weight $\lambda = -Ne_n$ has dimension $\binom{N+n-1}{N}$. Easy calculation gives

$$\delta = \tfrac{1}{2}(n-1)e_1 + \tfrac{1}{2}(n-3)e_2 + \cdots + \tfrac{1}{2}(1-n)e_n.$$

A quotient $\dfrac{\langle \lambda + \delta, \alpha \rangle}{\langle \delta, \alpha \rangle}$ will be 1 unless $\langle \lambda, \alpha \rangle \neq 0$. Therefore

$$\dim \varphi_{-Ne_n} = \prod_{j=1}^{n-1} \frac{\langle -Ne_n + \delta, e_j - e_n \rangle}{\langle \delta, e_j - e_n \rangle} = \prod_{j=1}^{n-1} \frac{N+n-j}{n-j} = \binom{N+n-1}{N},$$

as required.

## 7. Parabolic Subalgebras

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\mathfrak{h}$, $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, and $B$ be as in §2. A **Borel subalgebra** of $\mathfrak{g}$ is a subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ for some positive system $\Delta^+$ within $\Delta$. Any subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ containing a Borel subalgebra is called a **parabolic subalgebra** of $\mathfrak{g}$. Our goal in this section is to classify parabolic subalgebras and to relate them to finite-dimensional representations of $\mathfrak{g}$.

We regard $\mathfrak{h}$ and $\mathfrak{n}$ as fixed in our discussion, and we study only parabolic subalgebras $\mathfrak{q}$ that contain $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Let $\Pi$ be the simple system determining $\Delta^+$ and $\mathfrak{n}$, and define $\mathfrak{n}^-$ as in (5.8). Since $\mathfrak{q} \supseteq \mathfrak{h}$ and since the root spaces are 1-dimensional, $\mathfrak{q}$ is necessarily of the form

$$\mathfrak{q} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha, \tag{5.88}$$

where $\Gamma$ is a subset of $\Delta(\mathfrak{g}, \mathfrak{h})$ containing $\Delta^+(\mathfrak{g}, \mathfrak{h})$. The extreme cases are $\mathfrak{q} = \mathfrak{b}$ (with $\Gamma = \Delta^+(\mathfrak{g}, \mathfrak{h})$) and $\mathfrak{q} = \mathfrak{g}$ (with $\Gamma = \Delta(\mathfrak{g}, \mathfrak{h})$).

To obtain further examples of parabolic subalgebras, we fix a subset $\Pi'$ of the set $\Pi$ of simple roots and let

$$\Gamma = \Delta^+(\mathfrak{g}, \mathfrak{h}) \cup \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \alpha \in \operatorname{span}(\Pi')\}. \tag{5.89}$$

Then (5.88) is a parabolic subalgebra containing the given Borel subalgebra $\mathfrak{b}$. (Closure under brackets follows from the fact that if $\alpha$ and $\beta$ are in $\Gamma$ and if $\alpha+\beta$ is a root, then $\alpha+\beta$ is in $\Gamma$; this fact is an immediate consequence of Proposition 2.49.) All examples are of this form, according to Proposition 5.90 below. With $\Gamma$ as in (5.88), define $-\Gamma$ to be the set of negatives of the members of $\Gamma$.

**Proposition 5.90.** The parabolic subalgebras $\mathfrak{q}$ containing $\mathfrak{b}$ are parametrized by the set of subsets of simple roots; the one corresponding to a subset $\Pi'$ is of the form (5.88) with $\Gamma$ as in (5.89).

PROOF. If $\mathfrak{q}$ is given, we define $\Gamma(\mathfrak{q})$ to be the $\Gamma$ in (5.88), and we define $\Pi'(\mathfrak{q})$ to be the set of simple roots in the linear span of $\Gamma(\mathfrak{q}) \cap -\Gamma(\mathfrak{q})$. Then $\mathfrak{q} \mapsto \Pi'(\mathfrak{q})$ is a map from parabolic subalgebras $\mathfrak{q}$ containing $\mathfrak{b}$ to subsets of simple roots. In the reverse direction, if $\Pi'$ is given, we define $\Gamma(\Pi')$ to be the $\Gamma$ in (5.89), and then $\mathfrak{q}(\Pi')$ is defined by means of (5.88). We have seen that $\mathfrak{q}(\Pi')$ is a subalgebra, and thus $\Pi' \mapsto \mathfrak{q}(\Pi')$ is a map from subsets of simple roots to parabolic subalgebras containing $\mathfrak{b}$.

To complete the proof we have to show that these two maps are inverse to one another. To see that $\Pi'(\mathfrak{q}(\Pi')) = \Pi'$, we observe that

$$\{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \alpha \in \operatorname{span}(\Pi')\}$$

is closed under negatives. Therefore (5.89) gives

$$\begin{aligned} \Gamma(\Pi') \cap -\Gamma(\Pi') &= (\Delta^+(\mathfrak{g}, \mathfrak{h}) \cup \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \alpha \in \operatorname{span}(\Pi')\}) \\ &\qquad \cap (-\Delta^+(\mathfrak{g}, \mathfrak{h}) \cup \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \alpha \in \operatorname{span}(\Pi')\}) \\ &= (\Delta^+(\mathfrak{g}, \mathfrak{h}) \cap -\Delta^+(\mathfrak{g}, \mathfrak{h})) \\ &\qquad \cup \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \alpha \in \operatorname{span}(\Pi')\} \\ &= \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \alpha \in \operatorname{span}(\Pi')\}. \end{aligned}$$

The simple roots in the span of the right side are the members of $\Pi'$, by the independence in Proposition 2.49, and it follows that $\Pi'(\mathfrak{q}(\Pi')) = \Pi'$.

To see that $\mathfrak{q}(\Pi'(\mathfrak{q})) = \mathfrak{q}$, we are to show that $\Gamma(\Pi'(\mathfrak{q})) = \Gamma(\mathfrak{q})$. Since $\Delta^+(\mathfrak{g}, \mathfrak{h}) \subseteq \Gamma(\mathfrak{q})$, the inclusion $\Gamma(\Pi'(\mathfrak{q})) \subseteq \Gamma(\mathfrak{q})$ will follow if we show that

$$\{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \alpha \in \operatorname{span}(\Pi'(\mathfrak{q}))\} \subseteq \Gamma(\mathfrak{q}). \tag{5.91}$$

Since $\Gamma(\mathfrak{q}) = \Delta^+(\mathfrak{g}, \mathfrak{h}) \cup (\Gamma(\mathfrak{q}) \cap -\Gamma(\mathfrak{q}))$, the inclusion $\Gamma(\Pi'(\mathfrak{q})) \supseteq \Gamma(\mathfrak{q})$ will follow if we show that

$$\Gamma(\mathfrak{q}) \cap -\Gamma(\mathfrak{q}) \subseteq \Gamma(\Pi'(\mathfrak{q})). \tag{5.92}$$

Let us first prove (5.91). The positive members of the left side of (5.91) are elements of the right side since $\mathfrak{b} \subseteq \mathfrak{q}$. Any negative root in the left side is a negative-integer combination of members of $\Pi'(\mathfrak{q})$ by Proposition 2.49. Let $-\alpha$ be such a root, and expand $\alpha$ in terms of the simple roots $\Pi = \{\alpha_i\}_{i=1}^l$ as $\alpha = \sum_i n_i\alpha_i$. We prove by induction on the level $\sum n_i$ that a nonzero root vector $E_{-\alpha}$ for $-\alpha$ is in $\mathfrak{q}$. When the level is 1, this assertion is just the definition of $\Pi'(\mathfrak{q})$. When the level of $\alpha$ is $> 1$, we can choose positive roots $\beta$ and $\gamma$ with $\alpha = \beta + \gamma$. Then $\beta$ and $\gamma$ are positive integer combinations of members of $\Pi'(\mathfrak{q})$. By inductive hypothesis, $-\beta$ and $-\gamma$ are in $\Gamma(\mathfrak{q})$. Hence the corresponding root vectors $E_{-\beta}$ and $E_{-\gamma}$ are in $\mathfrak{q}$. By Corollary 2.35, $[E_{-\beta}, E_{-\gamma}]$ is a nonzero root vector for $-\alpha$. Since $\mathfrak{q}$ is a subalgebra, $-\alpha$ must be in $\Gamma(\mathfrak{q})$. This proves (5.91).

Finally let us prove (5.92). Let $-\alpha$ be a negative root in $\Gamma(\mathfrak{q})$, and expand $\alpha$ in terms of simple roots as $\alpha = \sum_i n_i\alpha_i$. The assertion is that each $\alpha_i$ for which $n_i > 0$ is in $\Pi'(\mathfrak{q})$, i.e., has $-\alpha_i \in \Gamma(\mathfrak{q})$. We prove this assertion by induction on the level $\sum n_i$, the case of level 1 being trivial. If the level of $\alpha$ is $> 1$, then $\alpha = \beta + \gamma$ with $\beta$ and $\gamma$ in $\Delta^+(\mathfrak{g}, \mathfrak{h})$. The root vectors $E_{-\alpha}$ and $E_\beta$ are in $\mathfrak{q}$, and hence so is their bracket, which is a nonzero multiple of $E_{-\gamma}$ by Corollary 2.35. Similarly $E_{-\alpha}$ and $E_\gamma$ are in $\mathfrak{q}$, and hence so is $E_{-\beta}$. Thus $-\gamma$ and $-\beta$ are in $\Gamma(\mathfrak{q})$. By induction the constituent simple roots of $\beta$ and $\gamma$ are in $\Pi'(\mathfrak{q})$, and thus the same thing is true of $\alpha$. This proves (5.92) and completes the proof of the proposition.

Now define

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma \cap -\Gamma} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{u} = \bigoplus_{\substack{\alpha \in \Gamma, \\ \alpha \notin -\Gamma}} \mathfrak{g}_\alpha, \tag{5.93a}$$

so that

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}. \tag{5.93b}$$

**Corollary 5.94.** Relative to a parabolic subalgebra $\mathfrak{q}$ containing $\mathfrak{b}$,

(a) $\mathfrak{l}$ and $\mathfrak{u}$ are subalgebras of $\mathfrak{q}$, and $\mathfrak{u}$ is an ideal in $\mathfrak{q}$,
(b) $\mathfrak{u}$ is nilpotent,

(c) $\mathfrak{l}$ is reductive with center $\mathfrak{h}'' = \bigcap_{\alpha\in\Gamma\cap-\Gamma} \ker\alpha \subseteq \mathfrak{h}$ and with semisimple part $\mathfrak{l}_{ss}$ having root-space decomposition

$$\mathfrak{l}_{ss} = \mathfrak{h}' \oplus \bigoplus_{\alpha\in\Gamma\cap-\Gamma} \mathfrak{g}_\alpha,$$

where $\mathfrak{h}' = \sum_{\alpha\in\Gamma\cap-\Gamma} \mathbb{C}H_\alpha$.

PROOF. By Proposition 5.90 let $\mathfrak{q}$ be built from $\Pi'$ by means of (5.89) and (5.88). Then (a) is clear. In (b), we have $\mathfrak{u} \subseteq \mathfrak{n}$, and hence $\mathfrak{u}$ is nilpotent.

Let us prove (c). Let $\mathfrak{h}_0$ be the real form of $\mathfrak{h}$ on which all roots are real valued. Then $\mathfrak{h}_0' = \mathfrak{h}_0 \cap \mathfrak{h}'$ and $\mathfrak{h}_0'' = \mathfrak{h}_0 \cap \mathfrak{h}''$ are real forms of $\mathfrak{h}'$ and $\mathfrak{h}''$, respectively. The form $B$ for $\mathfrak{g}$ has $B|_{\mathfrak{h}_0\times\mathfrak{h}_0}$ positive definite, and it is clear that $\mathfrak{h}_0'$ and $\mathfrak{h}_0''$ are orthogonal complements of each other. Therefore $\mathfrak{h}_0 = \mathfrak{h}_0' \oplus \mathfrak{h}_0''$ and $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$. Thus with $\mathfrak{l}_{ss}$ defined as in the statement of (c), $\mathfrak{l} = \mathfrak{h}'' \oplus \mathfrak{l}_{ss}$. Moreover it is clear that $\mathfrak{h}''$ and $\mathfrak{l}_{ss}$ are ideals in $\mathfrak{l}$ and that $\mathfrak{h}''$ is contained in the center. To complete the proof, it is enough to show that $\mathfrak{l}_{ss}$ is semisimple.

Thus let $B'$ be the Killing form of $\mathfrak{l}_{ss}$. Relative to $B'$, $\mathfrak{h}'$ is orthogonal to each $\mathfrak{g}_\alpha$ in $\mathfrak{l}$, and each $\mathfrak{g}_\alpha$ in $\mathfrak{l}$ is orthogonal to all $\mathfrak{g}_\beta$ in $\mathfrak{l}$ except $\mathfrak{g}_{-\alpha}$. For $\alpha \in \Gamma \cap -\Gamma$, choose root vectors $E_\alpha$ and $E_{-\alpha}$ with $B(E_\alpha, E_{-\alpha}) = 1$, so that $[E_\alpha, E_{-\alpha}] = H_\alpha$. We shall show that $B'(E_\alpha, E_{-\alpha}) > 0$ and that $B'$ is positive definite on $\mathfrak{h}_0' \times \mathfrak{h}_0'$. Then it follows that $B'$ is nondegenerate, and $\mathfrak{l}_{ss}$ is semisimple by Cartan's Criterion for Semisimplicity (Theorem 1.45).

In considering $B'(E_\alpha, E_{-\alpha})$, we observe from Corollary 2.37 that $\operatorname{ad} E_\alpha \operatorname{ad} E_{-\alpha}$ acts with eigenvalue $\geq 0$ on any $\mathfrak{g}_\beta$. On $H \in \mathfrak{h}$, it gives $\alpha(H)H_\alpha$, which is a positive multiple of $H_\alpha$ if $H = H_\alpha$ and is 0 if $H$ is in $\ker\alpha$. Thus $\operatorname{ad} E_\alpha \operatorname{ad} E_{-\alpha}$ has trace $> 0$ on $\mathfrak{h}$ and trace $\geq 0$ on each $\mathfrak{g}_\beta$. Consequently $B'(E_\alpha, E_{-\alpha}) > 0$.

If $H$ is in $\mathfrak{h}_0'$, then $B'(H, H) = \sum_{\alpha\in\Gamma\cap-\Gamma} \alpha(H)^2$, and each term is $\geq 0$. To get 0, we must have $\alpha(H) = 0$ for all $\alpha \in \Gamma \cap -\Gamma$. This condition forces $H$ to be in $\mathfrak{h}''$. Since $\mathfrak{h}' \cap \mathfrak{h}'' = 0$, we find that $H = 0$. Consequently $B'$ is positive definite on $\mathfrak{h}_0' \times \mathfrak{h}_0'$, as asserted.

In the decomposition (5.93) of $\mathfrak{q}$, $\mathfrak{l}$ is called the **Levi factor** and $\mathfrak{u}$ is called the **nilpotent radical**. The nilpotent radical can be characterized solely in terms of $\mathfrak{q}$ as the radical of the symmetric bilinear form $B|_{\mathfrak{q}\times\mathfrak{q}}$, where $B$ is the invariant form for $\mathfrak{g}$. But the Levi factor $\mathfrak{l}$ depends on $\mathfrak{h}$ as well as $\mathfrak{q}$.

Define

$$\mathfrak{u}^- = \bigoplus_{\substack{\alpha\in\Gamma,\\ \alpha\notin-\Gamma}} \mathfrak{g}_{-\alpha}. \tag{5.95a}$$

and

$$\mathfrak{q}^- = \mathfrak{l} \oplus \mathfrak{u}^-, \tag{5.95b}$$

(The subalgebra $\mathfrak{q}^-$ is a parabolic subalgebra containing the Borel subalgebra $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$.) Then we have the important identities

$$\mathfrak{l} = \mathfrak{q} \cap \mathfrak{q}^- \tag{5.96}$$

and

$$\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}. \tag{5.97}$$

Now we shall examine parabolic subalgebras in terms of centralizers and eigenvalues. We begin with some notation. In the background will be our Cartan subalgebra $\mathfrak{h}$ and the Borel subalgebra $\mathfrak{b}$. We suppose that $V$ is a finite-dimensional completely reducible representation of $\mathfrak{h}$, and we denote by $\Delta(V)$ the set of weights of $\mathfrak{h}$ in $V$. Some examples are

$$\begin{aligned}
\Delta(\mathfrak{g}) &= \Delta(\mathfrak{g}, \mathfrak{h}) \cup \{0\}\\
\Delta(\mathfrak{n}) &= \Delta^+(\mathfrak{g}, \mathfrak{h})\\
\Delta(\mathfrak{q}) &= \Gamma \cup \{0\}\\
\Delta(\mathfrak{l}) &= (\Gamma \cap -\Gamma) \cup \{0\}\\
\Delta(\mathfrak{u}) &= \{\alpha \in \Gamma \mid -\alpha \notin \Gamma\}.
\end{aligned}$$

For each weight $\omega \in \Delta(V)$, let $m_\omega$ be the multiplicity of $\omega$. We define

$$\delta(V) = \tfrac{1}{2} \sum_{\omega\in\Delta(V)} m_\omega \omega, \tag{5.98}$$

half the sum of the weights with multiplicities counted. An example is that $\delta(\mathfrak{n}) = \delta$, with $\delta$ defined as in §II.6 and again in (5.8). The following result generalizes Proposition 2.69.

**Proposition 5.99.** Let $V$ be a finite-dimensional representation of $\mathfrak{g}$, and let $\Lambda$ be a subset of $\Delta(V)$. Suppose that $\alpha$ is a root such that $\lambda \in \Lambda$ and $\alpha + \lambda \in \Delta(V)$ together imply $\alpha + \lambda \in \Lambda$. Then $\Big\langle \sum_{\lambda\in\Lambda} m_\lambda \lambda,\ \alpha \Big\rangle \geq 0$. Strict inequality holds when the representation is the adjoint representation of $\mathfrak{g}$ on $V = \mathfrak{g}$ and $\alpha$ is in $\Lambda$ and $-\alpha$ is not in $\Lambda$.

PROOF. Theorem 5.29 shows that $V$ is completely reducible. If $E_\alpha$ and $E_{-\alpha}$ denote nonzero root vectors for $\alpha$ and $-\alpha$, $V$ is therefore completely reducible under $\mathfrak{h} + \operatorname{span}\{H_\alpha, E_\alpha, E_{-\alpha}\}$. Let $\lambda$ be in $\Lambda$, and suppose that $\langle \lambda, \alpha\rangle < 0$. Then the theory for $\mathfrak{sl}(2, \mathbb{C})$ shows that $\lambda, \lambda + \alpha, \dots, s_\alpha\lambda$ are in $\Delta(V)$, and the hypothesis forces all of these weights to be in $\Lambda$. In particular $s_\alpha\lambda$ is in $\Lambda$. Theorem 5.5e says that $m_\lambda = m_{s_\alpha\lambda}$. Therefore

$$\sum_{\lambda\in\Lambda} m_\lambda\lambda = \sum_{\substack{\lambda\in\Lambda,\\ \langle\lambda,\alpha\rangle<0}} m_\lambda(\lambda + s_\alpha\lambda) + \sum_{\substack{\lambda\in\Lambda,\\ \langle\lambda,\alpha\rangle=0}} m_\lambda\lambda + \sum_{\substack{\lambda\in\Lambda,\ s_\alpha\lambda\notin\Lambda,\\ \langle\lambda,\alpha\rangle>0}} m_\lambda\lambda.$$

The inner product of $\alpha$ with the first two sums on the right is 0, and the inner product of $\alpha$ with the third sum is term-by-term positive. This proves the first assertion. In the case of the adjoint representation, if $\alpha \in \Lambda$ and $-\alpha \notin \Lambda$, then $\alpha$ occurs in the third sum and gives a positive inner product. This proves the second assertion.

**Corollary 5.100.** Let $\mathfrak{q}$ be a parabolic subalgebra containing $\mathfrak{b}$. If $\alpha$ is in $\Delta^+(\mathfrak{g}, \mathfrak{h})$, then

$$\langle \delta(\mathfrak{u}), \alpha\rangle \quad \text{is} \quad \begin{cases} = 0 & \text{if } \alpha \in \Delta(\mathfrak{l}, \mathfrak{h}) \\ > 0 & \text{if } \alpha \in \Delta(\mathfrak{u}). \end{cases}$$

PROOF. In Proposition 5.99 let $V = \mathfrak{g}$ and $\Lambda = \Delta(\mathfrak{u})$. If $\alpha$ is in $\Delta(\mathfrak{l}, \mathfrak{h})$, the proposition applies to $\alpha$ and $-\alpha$ and gives $\langle\delta(\mathfrak{u}), \alpha\rangle = 0$. If $\alpha$ is in $\Delta(\mathfrak{u})$, then $-\alpha$ is not in $\Lambda$ and the proposition gives $\langle\delta(\mathfrak{u}), \alpha\rangle > 0$.

**Corollary 5.101.** Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra containing $\mathfrak{b}$. Then the element $H = H_{\delta(\mathfrak{u})}$ of $\mathfrak{h}$ has the property that all roots are real valued on $H$ and

$$\begin{aligned} \mathfrak{u} &= \text{sum of eigenspaces of ad } H \text{ for positive eigenvalues} \\ \mathfrak{l} &= Z_{\mathfrak{g}}(H) = \text{eigenspace of ad } H \text{ for eigenvalue } 0 \\ \mathfrak{u}^- &= \text{sum of eigenspaces of ad } H \text{ for negative eigenvalues.} \end{aligned}$$

PROOF. This is immediate from Corollary 5.100.

We are ready to examine the role of parabolic subalgebras in finite-dimensional representations. The idea is to obtain a generalization of the Theorem of the Highest Weight (Theorem 5.5) in which $\mathfrak{h}$ and $\mathfrak{n}$ get replaced by $\mathfrak{l}$ and $\mathfrak{u}$.

The Levi factor $\mathfrak{l}$ of a parabolic subalgebra $\mathfrak{q}$ containing $\mathfrak{b}$ is reductive by Corollary 5.94c, but it is usually not semisimple. In the representations that we shall study, $\mathfrak{h}$ will act completely reducibly, and hence the subalgebra $\mathfrak{h}''$ in that corollary will act completely reducibly. Each simultaneous eigenspace of $\mathfrak{h}''$ will give a representation of $\mathfrak{l}_{ss}$, which will be completely reducible by Theorem 5.29. We summarize these remarks as follows.

**Proposition 5.102.** Let $\mathfrak{q}$ be a parabolic subalgebra containing $\mathfrak{b}$. In any finite-dimensional representation of $\mathfrak{l}$ for which $\mathfrak{h}$ acts completely reducibly, $\mathfrak{l}$ acts completely reducibly. This happens in particular when the action of a representation of $\mathfrak{g}$ is restricted to $\mathfrak{l}$.

Each irreducible constituent from Proposition 5.102 consists of a scalar action by $\mathfrak{h}''$ and an irreducible representation of $\mathfrak{l}_{ss}$, and the Theorem of the Highest Weight (Theorem 5.5) is applicable for the latter. Reassembling matters, we see that we can treat $\mathfrak{h}$ as a Cartan subalgebra of $\mathfrak{l}$ and treat $\Gamma \cap -\Gamma$ as the root system $\Delta(\mathfrak{l}, \mathfrak{h})$. The Theorem of the Highest Weight may then be reinterpreted as valid for $\mathfrak{l}$. Even though $\mathfrak{l}$ is merely reductive, we shall work with $\mathfrak{l}$ in this fashion without further special comment.

Let a finite-dimensional representation of $\mathfrak{g}$ be given on a space $V$, and fix a parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ containing $\mathfrak{b}$. The key tool for our investigation will be the subspace of $\mathfrak{u}$ **invariants** given by

$$V^{\mathfrak{u}} = \{v \in V \mid Xv = 0 \text{ for all } X \in \mathfrak{u}\}.$$

This subspace carries a representation of $\mathfrak{l}$ since $H \in \mathfrak{l}$, $v \in V^{\mathfrak{u}}$, and $X \in \mathfrak{u}$ imply

$$X(Hv) = H(Xv) + [X, H]v = 0 + 0 = 0$$

by Corollary 5.94a. By Corollary 5.31c the representation of $\mathfrak{l}$ on $V^{\mathfrak{u}}$ is determined up to equivalence by the representation of $\mathfrak{h}$ on the space of $\mathfrak{l} \cap \mathfrak{n}$ invariants. But

$$(V^{\mathfrak{u}})^{\mathfrak{l}\cap\mathfrak{n}} = V^{\mathfrak{u}\oplus(\mathfrak{l}\cap\mathfrak{n})} = V^{\mathfrak{n}}, \tag{5.103}$$

and the right side is given by the Theorem of the Highest Weight for $\mathfrak{g}$. This fact allows us to treat the representation of $\mathfrak{l}$ on $V^{\mathfrak{u}}$ as a generalization of the highest weight of the representation of $\mathfrak{g}$ on $V$.

**Theorem 5.104.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra, let $\Delta^+(\mathfrak{g}, \mathfrak{h})$ be a positive system for the set of roots, and define $\mathfrak{n}$ by (5.8). Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra containing the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$.

(a) If an irreducible finite-dimensional representation of $\mathfrak{g}$ is given on $V$, then the corresponding representation of $\mathfrak{l}$ on $V^{\mathfrak{u}}$ is irreducible. The highest weight of this representation of $\mathfrak{l}$ matches the highest weight of $V$ and is therefore algebraically integral and dominant for $\Delta^+(\mathfrak{g}, \mathfrak{h})$.

(b) If irreducible finite-dimensional representations of $\mathfrak{g}$ are given on $V_1$ and $V_2$ such that the associated irreducible representations of $\mathfrak{l}$ on $V_1^{\mathfrak{u}}$ and $V_2^{\mathfrak{u}}$ are equivalent, then $V_1$ and $V_2$ are equivalent.

(c) If an irreducible finite-dimensional representation of $\mathfrak{l}$ on $M$ is given whose highest weight is algebraically integral and dominant for $\Delta^+(\mathfrak{g}, \mathfrak{h})$, then there exists an irreducible finite-dimensional representation of $\mathfrak{g}$ on a space $V$ such that $V^{\mathfrak{u}} \cong M$ as representations of $\mathfrak{l}$.

PROOF.

(a) By (5.103), $(V^{\mathfrak{u}})^{\mathfrak{l}\cap\mathfrak{n}} = V^{\mathfrak{n}}$. Parts (b) and (c) of Theorem 5.5 for $\mathfrak{g}$ say that $V^{\mathfrak{n}}$ is 1-dimensional. Hence the space of $\mathfrak{l} \cap \mathfrak{n}$ invariants for $V^{\mathfrak{u}}$ is 1-dimensional. Since $V^{\mathfrak{u}}$ is completely reducible under $\mathfrak{l}$ by Proposition 5.102, Theorem 5.5c for $\mathfrak{l}$ shows that $V^{\mathfrak{u}}$ is irreducible under $\mathfrak{l}$. If $\lambda$ is the highest weight of $V$ under $\mathfrak{g}$, then $\lambda$ is the highest weight of $V^{\mathfrak{u}}$ under $\mathfrak{l}$ since $V_\lambda = V^{\mathfrak{n}} \subseteq V^{\mathfrak{u}}$. Then $\lambda$ is algebraically integral and dominant for $\Delta^+(\mathfrak{g}, \mathfrak{h})$ by Theorem 5.5 for $\mathfrak{g}$.

(b) If $V_1^{\mathfrak{u}}$ and $V_2^{\mathfrak{u}}$ are equivalent under $\mathfrak{l}$, then $(V_1^{\mathfrak{u}})^{\mathfrak{l}\cap\mathfrak{n}}$ and $(V_2^{\mathfrak{u}})^{\mathfrak{l}\cap\mathfrak{n}}$ are equivalent under $\mathfrak{h}$. By (5.103), $V_1^{\mathfrak{n}}$ and $V_2^{\mathfrak{n}}$ are equivalent under $\mathfrak{h}$. By uniqueness in Theorem 5.5, $V_1$ and $V_2$ are equivalent under $\mathfrak{g}$.

(c) Let $M$ have highest weight $\lambda$, which is assumed algebraically integral and dominant for $\Delta^+(\mathfrak{g}, \mathfrak{h})$. By Theorem 5.5 we can form an irreducible finite-dimensional representation of $\mathfrak{g}$ on a space $V$ with highest weight $\lambda$. Then $V^{\mathfrak{u}}$ has highest weight $\lambda$ by (a), and $V^{\mathfrak{u}} \cong M$ as representations of $\mathfrak{l}$ by uniqueness in Theorem 5.5 for $\mathfrak{l}$.

**Proposition 5.105.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra containing $\mathfrak{b}$. If $V$ is any finite-dimensional $U(\mathfrak{g})$ module, then

(a) $V = V^{\mathfrak{u}} \oplus \mathfrak{u}^- V$,

(b) the natural map $V^{\mathfrak{u}} \to V/(\mathfrak{u}^- V)$ is an isomorphism of $U(\mathfrak{l})$ modules,

(c) the $U(\mathfrak{l})$ module $V^{\mathfrak{u}}$ determines the $U(\mathfrak{g})$ module $V$ up to equivalence; the number of irreducible constituents of $V^{\mathfrak{u}}$ equals the number of irreducible constituents of $V$, and the multiplicity of an irreducible $U(\mathfrak{l})$ module in $V^{\mathfrak{u}}$ equals the multiplicity in $V$ of the irreducible $U(\mathfrak{g})$ module with that same highest weight.

PROOF. We have seen that $V^{\mathfrak{u}}$ is a $U(\mathfrak{l})$ module, and similarly $\mathfrak{u}^- V$ is a $U(\mathfrak{l})$ module. Conclusion (b) is immediate from (a), and conclusion (c) is immediate from Theorems 5.29 and 5.104. Thus we are left with proving (a).

By Theorem 5.29, $V$ is a direct sum of irreducible representations, and there is no loss of generality in assuming that $V$ is irreducible, say of highest weight $\lambda$.

With $V$ irreducible, we argue as in the proof of Corollary 5.31, using a Poincaré–Birkhoff–Witt basis of $U(\mathfrak{g})$ built from root vectors in $\mathfrak{u}^-$, root vectors in $\mathfrak{l}$ together with members of $\mathfrak{h}$, and root vectors in $\mathfrak{u}$. We may do so because of (5.97). Each such root vector is an eigenvector under ad $H_{\delta(\mathfrak{u})}$, and the eigenvalues are negative, zero, and positive in the three cases by Corollary 5.101. Using this eigenvalue as a substitute for "weight" in the proof of Corollary 5.31, we see that

$$V = U(\mathfrak{l})V_\lambda \oplus \mathfrak{u}^- V.$$

But $\mathfrak{l}$ acts irreducibly on $V^{\mathfrak{u}}$ by Theorem 5.104a, and $V_\lambda = V^{\mathfrak{n}} \subseteq V^{\mathfrak{u}}$. Hence $U(\mathfrak{l})V_\lambda = V^{\mathfrak{u}}$, and (a) is proved. This completes the proof of the proposition.

## 8. Application to Compact Lie Groups

As was mentioned in §1, one of the lines of motivation for studying finite-dimensional representations of complex semisimple Lie algebras is the representation theory of compact connected Lie groups. We now return to that theory in order to interpret the results of this chapter in that context.

Throughout this section we let $G$ be a compact connected Lie group with Lie algebra $\mathfrak{g}_0$ and complexified Lie algebra $\mathfrak{g}$, and we let $T$ be a maximal torus with Lie algebra $\mathfrak{t}_0$ and complexified Lie algebra $\mathfrak{t}$. The Lie algebra $\mathfrak{g}$ is reductive (Corollary 4.25), and we saw in §IV.4 how to interpret $\mathfrak{t}$ as a Cartan subalgebra and how the theory of roots extended from the semisimple case to this reductive case. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ be the set of roots, and let $W = W(\Delta)$ be the Weyl group.

Recall that a member $\lambda$ of $\mathfrak{t}^*$ is **analytically integral** if it is the differential of a multiplicative character $\xi_\lambda$ of $T$, i.e., if $\xi_\lambda(\exp H) = e^{\lambda(H)}$ for all $H \in \mathfrak{t}_0$. If $\lambda$ is analytically integral, then $\lambda$ takes purely imaginary values on $\mathfrak{t}_0$ by Proposition 4.58. Every root is analytically integral by Proposition 4.58. Every analytically integral member of $\mathfrak{t}^*$ is algebraically integral by Proposition 4.59.

**Lemma 5.106.** If $\Phi$ is a finite-dimensional representation of the compact connected Lie group $G$ and if $\lambda$ is a weight of the differential of $\Phi$, then $\lambda$ is analytically integral.

PROOF. We observed in §1 that $\Phi|_T$ is the direct sum of 1-dimensional invariant subspaces with $\Phi|_T$ acting in each by a multiplicative character $\xi_{\lambda_j}$. Then the weights are the various $\lambda_j$'s. Since each weight is the differential of a multiplicative character of $T$, each weight is analytically integral.

**Theorem 5.107.** Let $G$ be a simply connected compact semisimple Lie group, let $T$ be a maximal torus, and let $\mathfrak{t}$ be the complexified Lie algebra of $T$. Then every algebraically integral member of $\mathfrak{t}^*$ is analytically integral.

PROOF. Let $\lambda \in \mathfrak{t}^*$ be algebraically integral. Then $\lambda$ is real valued on $i\mathfrak{t}_0$, and the real span of the roots is $(i\mathfrak{t}_0)^*$ by semisimplicity of $\mathfrak{g}$. Hence $\lambda$ is in the real span of the roots. By Proposition 2.67 we can introduce a positive system $\Delta^+(\mathfrak{g}, \mathfrak{t})$ such that $\lambda$ is dominant. By the Theorem of the Highest Weight (Theorem 5.5), there exists an irreducible finite-dimensional representation $\varphi$ of $\mathfrak{g}$ with highest weight $\lambda$. Since $G$ is simply connected, there exists an irreducible finite-dimensional representation $\Phi$ of $G$ with differential $\varphi|_{\mathfrak{g}_0}$. By Lemma 5.106, $\lambda$ is analytically integral.

**Corollary 5.108.** If $G$ is a compact semisimple Lie group, then the order of the fundamental group of $G$ equals the index of the group of analytically integral forms for $G$ in the group of algebraically integral forms.

PROOF. Let $\widetilde{G}$ be a simply connected covering group of $G$. By Weyl's Theorem (Theorem 4.69), $\widetilde{G}$ is compact. Theorem 5.107 shows that the analytically integral forms for $\widetilde{G}$ coincide with the algebraically integral forms. Then it follows from Proposition 4.67 that the index of the group of analytically integral forms for $G$ in the group of algebraically integral forms equals the order of the kernel of the covering homomorphism $\widetilde{G} \to G$. Since $\widetilde{G}$ is simply connected, this kernel is isomorphic to the fundamental group of $G$.

EXAMPLE. Let $G = SO(2n+1)$ with $n \geq 1$ or $G = SO(2n)$ with $n \geq 2$. The analytically integral forms in standard notation are all expressions $\sum_{j=1}^{n} c_j e_j$ with all $c_j$ in $\mathbb{Z}$. The algebraically integral forms are all expressions $\sum_{j=1}^{n} c_j e_j$ with all $c_j$ in $\mathbb{Z}$ or all $c_j$ in $\mathbb{Z} + \frac{1}{2}$. Corollary 5.108 therefore implies that the fundamental group of $G$ has order 2. This conclusion sharpens Proposition 1.136.

**Corollary 5.109.** If $G$ is a simply connected compact semisimple Lie group, then the order of the center $Z_G$ of $G$ equals the determinant of the Cartan matrix.

PROOF. Let $G'$ be the adjoint group of $G$ so that $Z_G$ is the kernel of the covering map $G \to G'$. The analytically integral forms for $G$ coincide with the algebraically integral forms by Theorem 5.107, and the analytically integral forms for $G'$ coincide with the $\mathbb{Z}$ combinations of roots by Proposition 4.68. Thus the corollary follows by combining Propositions 4.64 and 4.67.

Now we give results that do not assume that $G$ is semisimple. Since $\mathfrak{g}_0$ is reductive, we can write $\mathfrak{g}_0 = Z_{\mathfrak{g}_0} \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$ with $[\mathfrak{g}_0, \mathfrak{g}_0]$ semisimple. Put $\mathfrak{t}_0' = \mathfrak{t}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0]$. The root-space decomposition of $\mathfrak{g}$ is then

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g},\mathfrak{t})} \mathfrak{g}_\alpha = Z_{\mathfrak{g}} \oplus \Big( \mathfrak{t}' \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g},\mathfrak{t})} \mathfrak{g}_\alpha \Big).$$

By Proposition 4.24 the compactness of $G$ implies that there is an invariant inner product on the Lie algebra $\mathfrak{g}_0$, and we let $B$ be its negative. (This form was used in Chapter IV, beginning in §5.) If we were assuming that $\mathfrak{g}_0$ is semisimple, then $B$ could be taken to be the Killing form, according to Corollary 4.26. We extend $B$ to be complex bilinear on $\mathfrak{g} \times \mathfrak{g}$. The restriction of $B$ to $i\mathfrak{t}_0 \times i\mathfrak{t}_0$ is an inner product, which transfers to give an inner product on $(i\mathfrak{t}_0)^*$. Analytically integral forms are always in $(i\mathfrak{t}_0)^*$. If a positive system $\Delta^+(\mathfrak{g}, \mathfrak{t})$ is given for the roots, then the condition of dominance for the form depends only on the restriction of the form to $i\mathfrak{t}_0'$.

**Theorem 5.110** (Theorem of the Highest Weight). Let $G$ be a compact connected Lie group with complexified Lie algebra $\mathfrak{g}$, let $T$ be a maximal torus with complexified Lie algebra $\mathfrak{t}$, and let $\Delta^+(\mathfrak{g}, \mathfrak{t})$ be a positive system for the roots. Apart from equivalence the irreducible finite-dimensional representations $\Phi$ of $G$ stand in one-one correspondence with the dominant analytically integral linear functionals $\lambda$ on $\mathfrak{t}$, the correspondence being that $\lambda$ is the highest weight of $\Phi$.

REMARK. The highest weight has the additional properties given in Theorem 5.5.

PROOF. Let notation be as above. If $\Phi$ is given, then the highest weight $\lambda$ of $\Phi$ is analytically integral by Lemma 5.106. To see dominance, let $\varphi$ be the differential of $\Phi$. Extend $\varphi$ complex linearly from $\mathfrak{g}_0$ to $\mathfrak{g}$, and restrict to $[\mathfrak{g}, \mathfrak{g}]$. The highest weight of $\varphi$ on $[\mathfrak{g}, \mathfrak{g}]$ is the restriction of $\lambda$ to $\mathfrak{t}'$, and this must be dominant by Theorem 5.5. Therefore $\lambda$ is dominant.

By Theorem 4.29, $G$ is a commuting product $G = (Z_G)_0 G_{ss}$ with $G_{ss}$ compact semisimple. Suppose that $\Phi$ and $\Phi'$ are irreducible representations of $G$, both with highest weight $\lambda$. By Schur's Lemma (Corollary 4.9), $\Phi|_{(Z_G)_0}$ and $\Phi'|_{(Z_G)_0}$ are scalar, and the scalar is determined by the restriction of $\lambda$ to the Lie algebra $Z_{\mathfrak{g}_0}$ of $(Z_G)_0$. Hence $\Phi|_{(Z_G)_0} = \Phi'|_{(Z_G)_0}$. On $G_{ss}$, the differentials $\varphi$ and $\varphi'$ give irreducible representations of $[\mathfrak{g}, \mathfrak{g}]$ with the same highest weight $\lambda|_{\mathfrak{t}'}$, and these are equivalent by Theorem 5.5. Then it follows that $\varphi$ and $\varphi'$ are equivalent as representations of $\mathfrak{g}$, and $\Phi$ and $\Phi'$ are equivalent as representations of $G$.

Finally if an analytically integral dominant $\lambda$ is given, we shall produce a representation $\Phi$ of $G$ with highest weight $\lambda$. The form $\lambda$ is algebraically integral by Proposition 4.59. We construct an irreducible representation $\varphi$ of $\mathfrak{g}$ with highest weight $\lambda$: This comes in two parts, with $\varphi|_{[\mathfrak{g},\mathfrak{g}]}$ equal to the representation in Theorem 5.5 corresponding to $\lambda|_{\mathfrak{t}'}$ and with $\varphi|_{Z_{\mathfrak{g}}}$ given by scalar operators equal to $\lambda|_{Z_{\mathfrak{g}}}$.

Let $\widetilde{G}$ be the universal covering group of $G$. Since $\widetilde{G}$ is simply connected, there exists an irreducible representation $\widetilde{\Phi}$ of $\widetilde{G}$ with differential $\varphi|_{\mathfrak{g}_0}$, hence with highest weight $\lambda$. To complete the proof, we need to show that $\widetilde{\Phi}$ descends to a representation $\Phi$ of $G$.

Since $G = (Z_G)_0 G_{ss}$, $\widetilde{G}$ is of the form $\mathbb{R}^n \times \widetilde{G}_{ss}$, where $\widetilde{G}_{ss}$ is the universal covering group of $G_{ss}$. Let $Z$ be the discrete subgroup of the center $Z_{\widetilde{G}}$ of $\widetilde{G}$ such that $G \cong \widetilde{G}/Z$. By Weyl's Theorem (Theorem 4.69), $\widetilde{G}_{ss}$ is compact. Thus Corollary 4.47 shows that the center of $\widetilde{G}_{ss}$ is contained in every maximal torus of $\widetilde{G}_{ss}$. Since $Z_{\widetilde{G}} \subseteq \mathbb{R}^n \times Z_{\widetilde{G}_{ss}}$, it follows that $Z_{\widetilde{G}} \subseteq \exp \mathfrak{t}_0$. Now $\lambda$ is analytically integral for $G$, and consequently the corresponding multiplicative character $\xi_\lambda$ on $\exp \mathfrak{t}_0 \subseteq \widetilde{G}$ is trivial on $Z$. By Schur's Lemma, $\widetilde{\Phi}$ is scalar on $Z_{\widetilde{G}}$, and its scalar values must agree with those of $\xi_\lambda$ since $\lambda$ is a weight. Thus $\widetilde{\Phi}$ is trivial on $Z$, and $\widetilde{\Phi}$ descends to a representation $\Phi$ of $G$, as required.

Next we take up characters. Let $\Phi$ be an irreducible finite-dimensional representation of the compact connected Lie group $G$ with highest weight $\lambda$, let $V$ be the underlying vector space, and let $\varphi$ be the differential, regarded as a representation of $\mathfrak{g}$. The Weyl Character Formula, as stated in Theorem 5.75, gives a kind of generating function for the weights of an irreducible Lie algebra representation in the semisimple case. Hence it is applicable to the semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$, the Cartan subalgebra $\mathfrak{t}'$, the representation $\varphi|_{[\mathfrak{g},\mathfrak{g}]}$, and the highest weight $\lambda|_{\mathfrak{t}'}$. By Schur's Lemma, $\Phi|_{(Z_G)_0}$ is scalar, necessarily with differential $\varphi|_{Z_\mathfrak{g}} = \lambda|_{Z_\mathfrak{g}}$. Thus we can extend the Weyl Character Formula as stated in Theorem 5.75 to be meaningful for our reductive $\mathfrak{g}$ by extending all weights from $\mathfrak{t}'$ to $\mathfrak{t}$ with $\lambda|_{Z_\mathfrak{g}}$ as their values on $Z_\mathfrak{g}$. The formula looks the same:

$$\Big(e^\delta \prod_{\alpha\in\Delta^+}(1-e^{-\alpha})\Big)\operatorname{char}(V) = \sum_{w\in W}\varepsilon(w)e^{w(\lambda+\delta)}. \tag{5.111}$$

We can apply the evaluation homomorphism $\epsilon_H$ to both sides for any $H \in \mathfrak{t}$, but we want to end up with an expression for $\operatorname{char}(V)$ as a function on the maximal torus $T$. This is a question of analytic integrality. The expressions $\operatorname{char}(V)$ and $\prod(1-e^{-\alpha})$ give well defined functions on $T$ since each weight and root is analytically integral. But $e^\delta$ need not give a well defined function on $T$ since $\delta$ need not be analytically integral. (It is not analytically integral for $SO(3)$, for example.) Matters are resolved by the following lemma.

**Lemma 5.112.** For each $w \in W$, $\delta - w\delta$ is analytically integral. In fact, $\delta - w\delta$ is the sum of all positive roots $\beta$ such that $w^{-1}\beta$ is negative.

PROOF. We write

$$\delta = \tfrac12 \sum\{\beta \mid \beta > 0,\ w^{-1}\beta > 0\} + \tfrac12\sum\{\beta \mid \beta > 0,\ w^{-1}\beta < 0\}$$

and

$$\begin{aligned} w\delta &= \tfrac12 w\sum\{\alpha \mid \alpha > 0,\ w\alpha > 0\} + \tfrac12 w\sum\{\alpha \mid \alpha > 0,\ w\alpha < 0\} \\ &= \tfrac12 \sum\{w\alpha \mid \alpha > 0,\ w\alpha > 0\} + \tfrac12 \sum\{w\alpha \mid \alpha > 0,\ w\alpha < 0\} \\ &= \tfrac12 \sum\{\beta \mid w^{-1}\beta > 0,\ \beta > 0\} + \tfrac12 \sum\{\eta \mid w^{-1}\eta > 0,\ \eta < 0\} \\ &\qquad\qquad \text{under } \beta = w\alpha \text{ and } \eta = w\alpha \\ &= \tfrac12 \sum\{\beta \mid w^{-1}\beta > 0,\ \beta > 0\} - \tfrac12 \sum\{\beta \mid w^{-1}\beta < 0,\ \beta > 0\} \\ &\qquad\qquad \text{under } \beta = -\eta. \end{aligned}$$

Subtracting, we obtain

$$\delta - w\delta = \sum\{\beta \mid \beta > 0,\ w^{-1}\beta < 0\}$$

as required.

**Theorem 5.113** (Weyl Character Formula). Let $G$ be a compact connected Lie group, let $T$ be a maximal torus, let $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t})$ be a positive system for the roots, and let $\lambda \in \mathfrak{t}^*$ be analytically integral and dominant. Then the character $\chi_{\Phi_\lambda}$ of the irreducible finite-dimensional representation $\Phi_\lambda$ of $G$ with highest weight $\lambda$ is given by

$$\chi_{\Phi_\lambda}(t) = \frac{\sum_{w \in W} \varepsilon(w) \xi_{w(\lambda+\delta)-\delta}(t)}{\prod_{\alpha \in \Delta^+} (1 - \xi_{-\alpha}(t))}$$

at every $t \in T$ where no $\xi_\alpha$ takes the value 1 on $t$. If $G$ is simply connected, then this formula can be rewritten as

$$\chi_{\Phi_\lambda}(t) = \frac{\sum_{w \in W} \varepsilon(w) \xi_{w(\lambda+\delta)}(t)}{\xi_\delta(t) \prod_{\alpha \in \Delta^+} (1 - \xi_{-\alpha}(t))} = \frac{\sum_{w \in W} \varepsilon(w) \xi_{w(\lambda+\delta)}(t)}{\sum_{w \in W} \varepsilon(w) \xi_{w\delta}(t)}.$$

REMARK. Theorem 4.36 says that every member of $G$ is conjugate to a member of $T$. Since characters are constant on conjugacy classes, the above formulas determine the characters everywhere on $G$.

PROOF. Theorem 5.110 shows that $\Phi_\lambda$ exists when $\lambda$ is analytically integral and dominant. We apply Theorem 5.75 in the form of (5.111). When we divide (5.111) by $e^\delta$, Lemma 5.112 says that all the exponentials yield well defined functions on $T$. The first formula follows. If $G$ is simply connected, then $G$ is semisimple as a consequence of Proposition 1.122. The linear functional $\delta$ is algebraically integral by Proposition 2.69, hence analytically integral by Theorem 5.107. Thus we can regroup the formula as indicated. The version of the formula with an alternating sum in the denominator uses Theorem 5.77 in place of Theorem 5.75.

Finally we discuss how parabolic subalgebras play a role in the representation theory of compact Lie groups. With $G$ and $T$ given, fix a positive system $\Delta^+(\mathfrak{g}, \mathfrak{t})$ for the roots, define $\mathfrak{n}$ as in (5.8), and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Corollary 5.101 shows that $\mathfrak{l} = Z_{\mathfrak{g}}(H_{\delta(\mathfrak{u})})$, and we can equally well write $\mathfrak{l} = Z_{\mathfrak{g}}(iH_{\delta(\mathfrak{u})})$. Since $iH_{\delta(\mathfrak{u})}$ is in $\mathfrak{t}_0 \subseteq \mathfrak{g}_0$, $\mathfrak{l}$ is the complexification of the subalgebra

$$\mathfrak{l}_0 = Z_{\mathfrak{g}_0}(iH_{\delta(\mathfrak{u})})$$

of $\mathfrak{g}_0$. Define

$$L = Z_G(iH_{\delta(\mathfrak{u})}).$$

This is a compact subgroup of $G$ containing $T$. Since the closure of $\exp i\mathbb{R}H_{\delta(\mathfrak{u})}$ is a torus in $G$, $L$ is the centralizer of a torus in $G$ and is connected by Corollary 4.51. Thus we have an inclusion of compact connected Lie groups $T \subseteq L \subseteq G$, and $T$ is a maximal torus in both $L$ and $G$. Hence analytic integrality is the same for $L$ as for $G$. Combining Theorems 5.104 and 5.110, we obtain the following result.

**Theorem 5.114.** Let $G$ be a compact connected Lie group with maximal torus $T$, let $\mathfrak{g}_0$ and $\mathfrak{t}_0$ be the Lie algebras, and let $\mathfrak{g}$ and $\mathfrak{t}$ be the complexifications. Let $\Delta^+(\mathfrak{g}, \mathfrak{t})$ be a positive system for the roots, and define $\mathfrak{n}$ by (5.8). Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra containing $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, let $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0$, and let $L$ be the analytic subgroup of $G$ with Lie algebra $\mathfrak{l}_0$.

(a) The subgroup $L$ is compact connected, and $T$ is a maximal torus in it.

(b) If an irreducible finite-dimensional representation of $G$ is given on $V$, then the corresponding representation of $L$ on $V^{\mathfrak{u}}$ is irreducible. The highest weight of this representation of $L$ matches the highest weight of $V$ and is therefore analytically integral and dominant for $\Delta^+(\mathfrak{g}, \mathfrak{h})$.

(c) If irreducible finite-dimensional representations of $G$ are given on $V_1$ and $V_2$ such that the associated irreducible representations of $L$ on $V_1^{\mathfrak{u}}$ and $V_2^{\mathfrak{u}}$ are equivalent, then $V_1$ and $V_2$ are equivalent.

(d) If an irreducible finite-dimensional representation of $L$ on $M$ is given whose highest weight is analytically integral and dominant for $\Delta^+(\mathfrak{g}, \mathfrak{h})$, then there exists an irreducible finite-dimensional representation of $G$ on a space $V$ such that $V^{\mathfrak{u}} \cong M$ as representations of $L$.

## 9. Problems

1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\varphi$ be a finite-dimensional representation of $\mathfrak{g}$ on the space $V$. The contragredient $\varphi^c$ is defined in (4.4).
   (a) Show that the weights of $\varphi^c$ are the negatives of the weights of $\varphi$.
   (b) Let $w_0$ be the element of the Weyl group produced in Problem 18 of Chapter II such that $w_0\Delta^+ = -\Delta^+$. If $\varphi$ is irreducible with highest weight $\lambda$, prove that $\varphi^c$ is irreducible with highest weight $-w_0\lambda$.
2. As in Problems 9–14 of Chapter IV, let $V_N$ be the space of polynomials in $x_1, \ldots, x_n$ homogeneous of degree $N$, and let $H_N$ be the subspace of harmonic polynomials. The compact group $G = SO(n)$ acts on $V_N$, and hence so does the complexified Lie algebra $\mathfrak{so}(n, \mathbb{C})$. The subspace $H_N$ is an invariant subspace. In the parts of this problem, it is appropriate to handle separately the cases of $n$ odd and $n$ even.
   (a) The weights of $V_N$ are identified in §1. Check that $Ne_1$ is the highest weight, and conclude that $Ne_1$ is the highest weight of $H_N$.

(b) Calculate the dimension of the irreducible representation of $\mathfrak{so}(n, \mathbb{C})$ with highest weight $Ne_1$, compare with the result of Problem 14 of Chapter IV, and conclude that $\mathfrak{so}(n, \mathbb{C})$ acts irreducibly on $H_N$.

3. As in Problems 15–17 of Chapter IV, let $V_N$ be the space of polynomials in $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$ homogeneous of degree $N$, and let $V_{p,q}$ be the subspace of polynomials with $p$ $z$-type factors and $q$ $\bar{z}$-type factors. The compact group $G = SU(n)$ acts on $V_N$, and hence so does the complexified Lie algebra $\mathfrak{sl}(n, \mathbb{C})$. The subspace $H_{p,q}$ of harmonic polynomials in $V_{p,q}$ is an invariant subspace.
   (a) The weights of $V_{p,q}$ are identified in §1. Check that $qe_1 - pe_n$ is the highest weight, and conclude that $qe_1 - pe_n$ is the highest weight of $H_{p,q}$.
   (b) Calculate the dimension of the irreducible representation of $\mathfrak{sl}(n, \mathbb{C})$ with highest weight $qe_1 - pe_n$, compare with the result of Problem 17 of Chapter IV, and conclude that $\mathfrak{sl}(n, \mathbb{C})$ acts irreducibly on $H_{p,q}$.

4. For $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$, show that the space $\mathcal{H}^W$ of Weyl-group invariants contains a nonzero element homogeneous of degree 3.

5. Give an interpretation of the Weyl Denominator Formula for $\mathfrak{sl}(n, \mathbb{C})$ in terms of the evaluation of Vandermonde determinants.

6. Prove that the Kostant partition function $\mathcal{P}$ satisfies the recursion formula

$$\mathcal{P}(\lambda) = -\sum_{\substack{w \in W, \\ w \neq 1}} \varepsilon(w)\mathcal{P}(\lambda - (\delta - w\delta))$$

   for $\lambda \neq 0$ in $Q^+$. Here $\mathcal{P}(\nu)$ is understood to be 0 if $\nu$ is not in $Q^+$.

Problems 7–10 address irreducibility of certain representations in spaces of alternating tensors.

7. Show that the representation of $\mathfrak{sl}(n, \mathbb{C})$ on $\bigwedge^l \mathbb{C}^n$ is irreducible by showing that the dimension of the irreducible representation with highest weight $\sum_{k=1}^l e_k$ is $\binom{n}{l}$.

8. Show that the representation of $\mathfrak{so}(2n+1, \mathbb{C})$ on $\bigwedge^l \mathbb{C}^{2n+1}$ is irreducible for $l \leq n$ by showing that the dimension of the irreducible representation with highest weight $\sum_{k=1}^l e_k$ is $\binom{2n+1}{l}$.

9. Show that the representation of $\mathfrak{so}(2n, \mathbb{C})$ on $\bigwedge^l \mathbb{C}^{2n}$ is irreducible for $l < n$ by showing that the dimension of the irreducible representation with highest weight $\sum_{k=1}^l e_k$ is $\binom{2n}{l}$.

10. Show that the representation of $\mathfrak{so}(2n, \mathbb{C})$ on $\bigwedge^n \mathbb{C}^{2n}$ is reducible, being the sum of two irreducible representations with respective highest weights $\big(\sum_{k=1}^{n-1} e_k\big) \pm e_n$.

Problems 11–13 concern Verma modules.

11. Prove for arbitrary $\lambda$ and $\mu$ in $\mathfrak{h}^*$ that every nonzero $U(\mathfrak{g})$ linear map of $V(\mu)$ into $V(\lambda)$ is one-one.

12. Prove for arbitrary $\lambda$ and $\mu$ in $\mathfrak{h}^*$ that if $V(\mu)$ is isomorphic to a $U(\mathfrak{g})$ submodule of $V(\lambda)$, then $\mu$ is in $\lambda - Q^+$ and is in the orbit of $\lambda$ under the Weyl group.

13. Let $\lambda$ be in $\mathfrak{h}^*$, and let $M$ be an irreducible quotient of a $U(\mathfrak{g})$ submodule of $V(\lambda)$. Prove that $M$ is isomorphic to the $U(\mathfrak{g})$ module $L(\mu)$ of Proposition 5.15 for some $\mu$ in $\lambda - Q^+$ such that $\mu$ is in the orbit of $\lambda$ under the Weyl group.

Problems 14–15 concern tensor products of irreducible representations. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let notation be as in §2.

14. Let $\varphi_\lambda$ and $\varphi_{\lambda'}$ be irreducible representations of $\mathfrak{g}$ with highest weights $\lambda$ and $\lambda'$, respectively. Prove that the weights of $\varphi_\lambda \otimes \varphi_{\lambda'}$ are all sums $\mu + \mu'$, where $\mu$ is a weight of $\varphi_\lambda$ and $\mu'$ is a weight of $\varphi_{\lambda'}$. How is the multiplicity of $\mu + \mu'$ related to multiplicities in $\varphi_\lambda$ and $\varphi_{\lambda'}$?

15. Let $v_\lambda$ and $v_{\lambda'}$ be highest weight vectors in $\varphi_\lambda$ and $\varphi_{\lambda'}$, respectively. Prove that $v_\lambda \otimes v_{\lambda'}$ is a highest weight vector in $\varphi_\lambda \otimes \varphi_{\lambda'}$. Conclude that $\varphi_{\lambda+\lambda'}$ occurs exactly once in $\varphi_\lambda \otimes \varphi_{\lambda'}$. (This occurrence is sometimes called the **Cartan composition** of $\varphi_\lambda$ and $\varphi_{\lambda'}$.)

Problems 16–18 begin a construction of “spin representations.” Let $u_1, \dots, u_n$ be the standard orthonormal basis of $\mathbb{R}^n$. The **Clifford algebra** $\mathrm{Cliff}(\mathbb{R}^n)$ is an associative algebra over $\mathbb{R}$ of dimension $2^n$ with a basis parametrized by subsets of $\{1, \dots, n\}$ and given by

$$\{u_{i_1} u_{i_2} \cdots u_{i_k} \mid i_1 < i_2 < \cdots < i_k\}.$$

The generators multiply by the rules

$$u_i^2 = -1, \qquad u_i u_j = -u_j u_i \ \text{ if } i \neq j.$$

16. Verify that the Clifford algebra is associative.

17. The Clifford algebra, like any associative algebra, becomes a Lie algebra under the bracket operation $[x, y] = xy - yx$. Put

$$\mathfrak{q} = \sum_{i \neq j} \mathbb{R} u_i u_j.$$

Verify that $\mathfrak{q}$ is a Lie subalgebra of $\mathrm{Cliff}(\mathbb{R}^n)$ isomorphic to $\mathfrak{so}(n)$, the isomorphism being $\varphi : \mathfrak{so}(n) \to \mathfrak{q}$ with

$$\varphi(E_{ji} - E_{ij}) = \tfrac{1}{2} u_i u_j.$$

18. With $\varphi$ as in Problem 17, verify that

$$[\varphi(x), u_j] = x u_j \qquad \text{for all } x \in \mathfrak{so}(n).$$

Here the left side is a bracket in $\mathrm{Cliff}(\mathbb{R}^n)$, and the right side is the product of the matrix $x$ by the column vector $u_j$, the product being reinterpreted as a member of $\mathrm{Cliff}(\mathbb{R}^n)$.

Problems 19–27 continue the construction of spin representations. We form the complexification $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{R}^n)$ and denote left multiplication by $c$, putting $c(x)y = xy$. Then $c$ is a representation of the associative algebra $\mathrm{Cliff}^{\mathbb{C}}((\mathbb{R}^n)$ on itself, hence also of the Lie algebra $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{R}^n)$ on itself, hence also of the Lie subalgebra $\mathfrak{q}^{\mathbb{C}} \cong \mathfrak{so}(n, \mathbb{C})$ on $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{R}^n)$. Let $n = 2m + 1$ or $n = 2m$, according as $n$ is odd or even. For $1 \le j \le m$, let

$$z_j = u_{2j-1} + i u_{2j} \qquad \text{and} \qquad \bar{z}_j = u_{2j-1} - i u_{2j}.$$

For each subset $S$ of $\{1, \dots, m\}$, define

$$z_S = \Big(\prod_{j \in S} z_j\Big)\Big(\prod_{j=1}^{m} \bar{z}_j\Big),$$

with each product arranged so that the indices are in increasing order. If $n$ is odd, define also

$$z'_S = \Big(\prod_{j \in S} z_j\Big)\Big(\prod_{j=1}^{m} \bar{z}_j\Big) u_{2m+1}.$$

19. Check that

$$z_j^2 = \bar{z}_j^2 = 0 \qquad \text{and} \qquad \bar{z}_j z_j \bar{z}_j = -4 z_j,$$

and deduce that

$$c(z_j) z_S = \begin{cases} \pm z_{S \cup \{j\}} & \text{if } j \notin S \\ 0 & \text{if } j \in S \end{cases}$$

$$c(\bar{z}_j) z_S = \begin{cases} 0 & \text{if } j \notin S \\ \pm 4 z_{S - \{j\}} & \text{if } j \in S. \end{cases}$$

20. When $n$ is odd, check that $c(z_j) z'_S$ and $c(\bar{z}_j) z'_S$ are given by formulas similar to those in Problem 19, and compute also $c(u_{2m+1}) z_S$ and $c(u_{2m+1}) z'_S$, up to sign.

21. For $n$ even let

$$\mathcal{S} = \sum_{S \subseteq \{1,\dots,m\}} \mathbb{C} z_S,$$

of dimension $2^m$. For $n$ odd let

$$\mathcal{S} = \sum_{S \subseteq \{1,\dots,m\}} \mathbb{C} z_S + \sum_{T \subseteq \{1,\dots,m\}} \mathbb{C} z'_T,$$

of dimension $2^{m+1}$. Prove that $c(\mathrm{Cliff}^{\mathbb{C}}(\mathbb{R}^n))$ carries $\mathcal{S}$ to itself, hence that $c(\mathfrak{q}^{\mathbb{C}})$ carries $\mathcal{S}$ to itself.

22. For $n$ even, write $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, where $\mathcal{S}^+$ refers to sets $S$ with an even number of elements and where $\mathcal{S}^-$ corresponds to sets $S$ with an odd number of elements. Prove that $\mathcal{S}^+$ and $\mathcal{S}^-$ are invariant subspaces under $c(\mathfrak{q}^{\mathbb{C}})$, of dimension $2^{m-1}$. (The representations $\mathcal{S}^+$ and $\mathcal{S}^-$ are the **spin representations** of $\mathfrak{so}(2m, \mathbb{C})$.)

23. For $n$ odd, write $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, where $\mathcal{S}^+$ corresponds to sets $S$ with an even number of elements and sets $T$ with an odd number of elements and where $\mathcal{S}^-$ corresponds to sets $S$ with an odd number of elements and sets $T$ with an even number of elements. Prove that $\mathcal{S}^+$ and $\mathcal{S}^-$ are invariant subspaces under $c(\mathfrak{q}^{\mathbb{C}})$, of dimension $2^m$, and that they are equivalent under right multiplication by $u_{2m+1}$. (The **spin representation** of $\mathfrak{so}(2m+1, \mathbb{C})$ is either of the equivalent representations $\mathcal{S}^+$ and $\mathcal{S}^-$.)

24. Let $\mathfrak{t}_0$ be the maximal abelian subspace of $\mathfrak{so}(n)$ in §IV.5. In terms of the isomorphism $\varphi$ in Problem 17, check that the corresponding maximal abelian subspace of $\mathfrak{q}$ is $\varphi(\mathfrak{t}_0) = \sum \mathbb{R} u_{2j} u_{2j-1}$. In the notation of §II.1, check also that $\frac{1}{2} i u_{2j} u_{2j-1}$ is $\varphi$ of the element of $\mathfrak{t}$ on which $e_j$ is 1 and $e_i$ is 0 for $i \neq j$.

25. In the notation of the previous problem, prove that

$$c(\varphi(h)) z_S = \frac{1}{2} \Big( \sum_{j \notin S} e_j - \sum_{j \in S} e_j \Big)(h) z_S$$

for $h \in \mathfrak{t}$. Prove also that a similar formula holds for the action on $z'_S$ when $n$ is odd.

26. Suppose that $n$ is even.
    (a) Conclude from Problem 25 that the weights of $\mathcal{S}^+$ are all expressions $\frac{1}{2}(\pm e_1 \pm \cdots \pm e_m)$ with an even number of minus signs, while the weights of $\mathcal{S}^-$ are all expressions $\frac{1}{2}(\pm e_1 \pm \cdots \pm e_m)$ with an odd number of minus signs.
    (b) Compute the dimensions of the irreducible representations with highest weights $\frac{1}{2}(e_1 + \cdots + e_{m-1} + e_m)$ and $\frac{1}{2}(e_1 + \cdots + e_{m-1} - e_m)$, and conclude that $\mathfrak{so}(2m, \mathbb{C})$ acts irreducibly on $\mathcal{S}^+$ and $\mathcal{S}^-$.

27. Suppose that $n$ is odd.
   (a) Conclude from Problem 25 that the weights of $\mathcal{S}^+$ are all expressions $\frac{1}{2}(\pm e_1 \pm \cdots \pm e_m)$ and that the weights of $\mathcal{S}^-$ are the same.
   (b) Compute the dimension of the irreducible representation with highest weight $\frac{1}{2}(e_1 + \cdots + e_m)$, and conclude that $\mathfrak{so}(2m+1, \mathbb{C})$ acts irreducibly on $\mathcal{S}^+$ and $\mathcal{S}^-$.

Problems 28–33 concern fundamental representations. Let $\alpha_1, \ldots, \alpha_l$ be the simple roots, and define $\varpi_1, \ldots, \varpi_l$ by $2\langle \varpi_i, \alpha_j \rangle / |\alpha_j|^2 = \delta_{ij}$. The dominant algebraically integral linear functionals are then all expressions $\sum_i n_i \varpi_i$ with all $n_i$ integers $\geq 0$. We call $\varpi_i$ the **fundamental weight** attached to the simple root $\alpha_i$, and the corresponding irreducible representation is called the **fundamental representation** attached to that simple root.

28. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$.
   (a) Verify that the fundamental weights are $\sum_{k=1}^{l} e_k$ for $1 \leq l \leq n-1$.
   (b) Using Problem 7, verify that the fundamental representations are the usual alternating-tensor representations.

29. Let $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$. Let $\alpha_i = e_i - e_{i+1}$ for $i < n$, and let $\alpha_i = e_n$.
   (a) Verify that the fundamental weights are $\varpi_l = \sum_{k=1}^{l} e_k$ for $1 \leq l \leq n-1$ and $\varpi_n = \frac{1}{2}\sum_{k=1}^{n} e_k$.
   (b) Using Problem 8, verify that the fundamental representations attached to simple roots other than the last one are alternating-tensor representations.
   (c) Using Problem 27, verify that the fundamental representation attached to the last simple root is the spin representation.

30. Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$. Let $\alpha_i = e_i - e_{i+1}$ for $i < n-1$, and let $\alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = e_{n-1} + e_n$.
   (a) Verify that the fundamental weights are $\varpi_l = \sum_{k=1}^{l} e_k$ for $1 \leq l \leq n-2$, $\varpi_{n-1} = \frac{1}{2}\sum_{k=1}^{n} e_k$, and $\varpi_n = \frac{1}{2}\big(\sum_{k=1}^{n-1} e_k - e_n\big)$.
   (b) Using Problem 9, verify that the fundamental representations attached to simple roots other than the last two are alternating-tensor representations.
   (c) Using Problem 26, verify that the fundamental representations attached to the last two simple roots are the spin representations.

31. Let $\lambda$ and $\lambda'$ be dominant algebraically integral, and suppose that $\lambda - \lambda'$ is dominant and nonzero. Prove that the dimension of an irreducible representation with highest weight $\lambda$ is greater than the dimension of an irreducible representation with highest weight $\lambda'$.

32. Given $\mathfrak{g}$, prove for each integer $N$ that there are only finitely many irreducible representations of $\mathfrak{g}$, up to equivalence, of dimension $\leq N$.

33. Let $\mathfrak{g}$ be a complex simple Lie algebra of type $G_2$.
    (a) Using Problem 42 in Chapter II, construct a 7-dimensional nonzero representation of $\mathfrak{g}$.
    (b) Let $\alpha_1$ be the long simple root, and let $\alpha_2$ be the short simple root. Verify that $\varpi_1 = 2\alpha_1 + 3\alpha_2$ and that $\varpi_2 = \alpha_1 + 2\alpha_2$.
    (c) Verify that the dimensions of the fundamental representations of $\mathfrak{g}$ are 7 and 14. Which one has dimension 7?
    (d) Using Problem 31, conclude that the representation constructed in (a) is irreducible.

Problems 34–35 concern Borel subalgebras $\mathfrak{b}$ of a complex semisimple Lie algebra $\mathfrak{g}$.

34. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the system of roots, let $\Delta^+$ be a system of positive roots, let $\mathfrak{n} = \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ be the sum of the root spaces corresponding to $\Delta^+$, and let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ be the corresponding Borel subalgebra of $\mathfrak{g}$. If $H \in \mathfrak{h}$ has $\alpha(H) \neq 0$ for all $\alpha \in \Delta^+$ and if $X$ is in $\mathfrak{n}$, prove that the centralizer $Z_\mathfrak{b}(H + X)$ is a Cartan subalgebra of $\mathfrak{g}$.

35. Within the complex semisimple Lie algebra $\mathfrak{g}$, let $(\mathfrak{b}, \mathfrak{h}, \{X_\alpha\})$ be a triple consisting of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that lies in $\mathfrak{b}$, and a system of nonzero root vectors for the simple roots in the positive system of roots defining $\mathfrak{b}$. Let $(\mathfrak{b}', \mathfrak{h}', \{X_{\alpha'}\})$ be a second such triple. Suppose that there is a compact Lie algebra $\mathfrak{u}_0$ that is a real form of $\mathfrak{g}$ and has the property that $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{u}_0$ is a maximal abelian subalgebra of $\mathfrak{u}_0$. Prove that there exists an element $g \in \operatorname{Int}\mathfrak{g}$ such that $\operatorname{Ad}(g)\mathfrak{b} = \mathfrak{b}'$, $\operatorname{Ad}(g)\mathfrak{h} = \mathfrak{h}'$, and $\operatorname{Ad}(g)\{X_\alpha\} = \{X_{\alpha'}\}$.