II. Complex Semisimple Lie Algebras, 123-212

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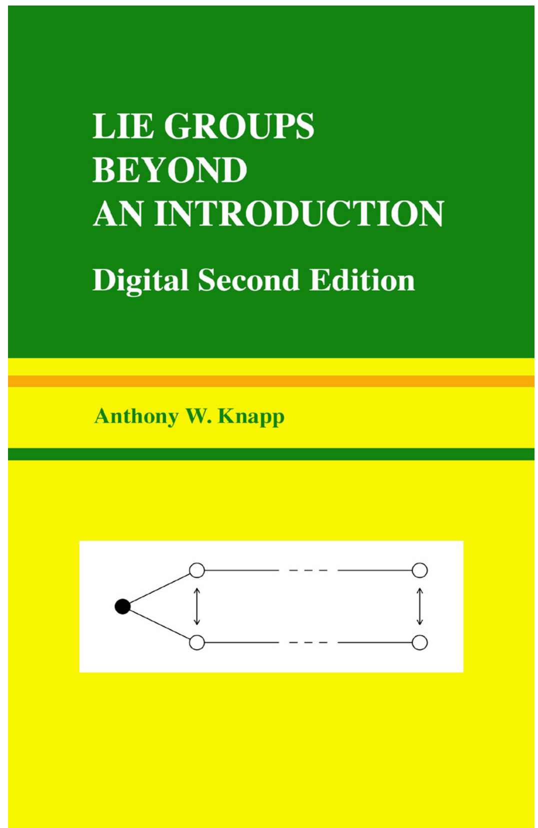

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Pages vii–xviii and 1–812 are the same in the digital and printed second editions. A list of corrections as of June 2023 has been included as pages 813–820 of the digital second edition. The corrections have not been implemented in the text.

Cover: Vogan diagram of $\mathfrak{sl}(2n, \mathbb{R})$. See page 399.

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CHAPTER II

Complex Semisimple Lie Algebras

Abstract. The theme of this chapter is an investigation of complex semisimple Lie algebras by a two-step process, first by passing from such a Lie algebra to a reduced abstract root system via a choice of Cartan subalgebra and then by passing from the root system to an abstract Cartan matrix and an abstract Dynkin diagram via a choice of an ordering.

The chapter begins by making explicit a certain amount of this structure for four infinite classes of classical complex semisimple Lie algebras. Then for a general finite-dimensional complex Lie algebra, it is proved that Cartan subalgebras exist and are unique up to conjugacy.

When the given Lie algebra is semisimple, the Cartan subalgebra is abelian. The adjoint action of the Cartan subalgebra on the given semisimple Lie algebra leads to a root-space decomposition of the given Lie algebra, and the set of roots forms a reduced abstract root system.

If a suitable ordering is imposed on the underlying vector space of an abstract root system, one can define simple roots as those positive roots that are not sums of positive roots. The simple roots form a particularly nice basis of the underlying vector space, and a Cartan matrix and Dynkin diagram may be defined in terms of them. The definitions of abstract Cartan matrix and abstract Dynkin diagram are arranged so as to include the matrix and diagram obtained from a root system.

Use of the Weyl group shows that the Cartan matrix and Dynkin diagram obtained from a root system by imposing an ordering are in fact independent of the ordering. Moreover, nonisomorphic reduced abstract root systems have distinct Cartan matrices. It is possible to classify the abstract Cartan matrices and then to see by a case-by-case argument that every abstract Cartan matrix arises from a reduced abstract root system. Consequently the correspondence between reduced abstract root systems and abstract Cartan matrices is one-one onto, up to isomorphism.

The correspondence between complex semisimple Lie algebras and reduced abstract root systems lies deeper. Apart from isomorphism, the correspondence does not depend upon the choice of Cartan subalgebra, as a consequence of the conjugacy of Cartan subalgebras proved earlier in the chapter. To examine the correspondence more closely, one first finds generators and relations for any complex semisimple Lie algebra. The Isomorphism Theorem then explains how much freedom there is in lifting an isomorphism between root systems to an isomorphism between complex semisimple Lie algebras. Finally the Existence Theorem says that every reduced abstract root system arises from some complex semisimple Lie algebra. Consequently the correspondence between complex semisimple Lie algebras and reduced abstract root systems is one-one onto, up to isomorphism.

1. Classical Root-space Decompositions

Recall from §I.8 that the complex Lie algebras $\mathfrak{sl}(n,\mathbb{C})$ for $n \geq 2$, $\mathfrak{so}(n,\mathbb{C})$ for $n \geq 3$, and $\mathfrak{sp}(n,\mathbb{C})$ for $n \geq 1$ are all semisimple. As we shall see in this section, each of these Lie algebras has an abelian subalgebra $\mathfrak{h}$ such that an analysis of $\operatorname{ad}\mathfrak{h}$ leads to a rather complete understanding of the bracket law in the full Lie algebra. We shall give the analysis of $\operatorname{ad}\mathfrak{h}$ in each example and then, to illustrate the power of the formulas we have, identify which of these Lie algebras are simple over $\mathbb{C}$.

EXAMPLE 1. The complex Lie algebra is $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$. Let

$$\begin{aligned} \mathfrak{h}_0 &= \text{real diagonal matrices in } \mathfrak{g} \\ \mathfrak{h} &= \text{all diagonal matrices in } \mathfrak{g}. \end{aligned}$$

Then $\mathfrak{h} = \mathfrak{h}_0 \oplus i\mathfrak{h}_0 = (\mathfrak{h}_0)^{\mathbb{C}}$. Define a matrix E_{ij} to be 1 in the $(i,j)^{\text{th}}$ place and 0 elsewhere, and define a member e_j of the dual space $\mathfrak{h}^*$ by

$$e_j \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{pmatrix} = h_j.$$

For each $H \in \mathfrak{h}$, $\operatorname{ad} H$ is diagonalized by the basis of $\mathfrak{g}$ consisting of members of $\mathfrak{h}$ and the E_{ij} for $i \neq j$. We have

$$(\operatorname{ad} H)E_{ij} = [H, E_{ij}] = (e_i(H) - e_j(H))E_{ij}.$$

In other words, E_{ij} is a simultaneous eigenvector for all $\operatorname{ad} H$, with eigenvalue $e_i(H) - e_j(H)$. In its dependence on H, the eigenvalue is linear. Thus the eigenvalue is a linear functional on $\mathfrak{h}$, namely $e_i - e_j$. The $(e_i - e_j)$'s, for $i \neq j$, are called **roots**. The set of roots is denoted Δ. We have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}E_{ij},$$

which we can rewrite as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{e_i - e_j}, \tag{2.1}$$

where

$$\mathfrak{g}_{e_i - e_j} = \{X \in \mathfrak{g} \mid (\operatorname{ad} H)X = (e_i - e_j)(H)X \text{ for all } H \in \mathfrak{h}\}.$$

The decomposition (2.1) is called a **root-space decomposition**. The set Δ of roots spans $\mathfrak{h}^*$ over $\mathbb{C}$.

The bracket relations are easy, relative to (2.1). If α and β are roots, we can compute $[E_{ij}, E_{i'j'}]$ and see that

$$(2.2) \qquad [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \begin{cases} = \mathfrak{g}_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ = 0 & \text{if } \alpha+\beta \text{ is not a root or } 0 \\ \subseteq \mathfrak{h} & \text{if } \alpha+\beta = 0. \end{cases}$$

In the last case the exact formula is

$$[E_{ij}, E_{ji}] = E_{ii} - E_{jj} \in \mathfrak{h}.$$

All the roots are real on $\mathfrak{h}_0$ and thus, by restriction, can be considered as members of $\mathfrak{h}_0^*$. The next step is to introduce a notion of positivity within $\mathfrak{h}_0^*$ such that

(i) for any nonzero $\varphi \in \mathfrak{h}_0^*$, exactly one of φ and $-\varphi$ is positive,
(ii) the sum of positive elements is positive, and any positive multiple of a positive element is positive.

The way in which such a notion of positivity is introduced is not important, and we shall just choose one at this stage.

To do so, we observe a canonical form for members of $\mathfrak{h}_0^*$. The linear functionals $e_1, \ldots, e_n$ span $\mathfrak{h}_0^*$, and their sum is 0. Any member of $\mathfrak{h}_0^*$ can therefore be written nonuniquely as $\sum_j c_j e_j$, and $(\sum_i c_i)(e_1 + \cdots + e_n) = 0$. Therefore our given linear functional equals

$$\sum_{j=1}^{n} \Big(c_j - \frac{1}{n}\sum_{i=1}^{n} c_i\Big) e_j.$$

In this latter representation the sum of the coefficients is 0. Thus any member of $\mathfrak{h}_0^*$ can be realized as $\sum_j a_j e_j$ with $\sum_j a_j = 0$. No such nonzero expression can vanish on $E_{ii} - E_{nn}$ for all i with $1 \le i < n$, and thus the realization as $\sum_j a_j e_j$ with $\sum_j a_j = 0$ is unique.

If $\varphi = \sum_j a_j e_j$ is given as a member of $\mathfrak{h}_0^*$ with $\sum_j a_j = 0$, we say that a nonzero φ is **positive** (written $\varphi > 0$) if the first nonzero coefficient a_j is > 0. It is clear that this notion of positivity satisfies properties (i) and (ii) above.

We say that $\varphi > \psi$ if $\varphi - \psi$ is positive. The result is a simple ordering on $\mathfrak{h}_0^*$ that is preserved under addition and under multiplication by positive scalars.

For the roots the effect is that

$$\begin{aligned} e_1 - e_n > e_1 - e_{n-1} > \cdots > e_1 - e_2 \\ > e_2 - e_n > e_2 - e_{n-1} > \cdots > e_2 - e_3 \\ > \cdots > e_{n-2} - e_n > e_{n-2} - e_{n-1} > e_{n-1} - e_n > 0, \end{aligned}$$

and afterward we have the negatives. The positive roots are the $e_i - e_j$ with $i < j$.

Now let us prove that $\mathfrak{g}$ is simple over $\mathbb{C}$ for $n \geq 2$. Let $\mathfrak{a} \subseteq \mathfrak{g}$ be an ideal, and first suppose $\mathfrak{a} \subseteq \mathfrak{h}$. Let $H \neq 0$ be in $\mathfrak{a}$. Since the roots span $\mathfrak{h}^*$, we can find a root α with $\alpha(H) \neq 0$. If X is in $\mathfrak{g}_\alpha$ and $X \neq 0$, then

$$\alpha(H)X = [H, X] \in [\mathfrak{a}, \mathfrak{g}] \subseteq \mathfrak{a} \subseteq \mathfrak{h},$$

and so X is in $\mathfrak{h}$, contradiction. Hence $\mathfrak{a} \subseteq \mathfrak{h}$ implies $\mathfrak{a} = 0$.

Next, suppose $\mathfrak{a}$ is not contained in $\mathfrak{h}$. Let $X = H + \sum X_\alpha$ be in $\mathfrak{a}$ with each X_α in $\mathfrak{g}_\alpha$ and with some $X_\alpha \neq 0$. For the moment assume that there is some root $\alpha < 0$ with $X_\alpha \neq 0$, and let β be the smallest such α. Say $X_\beta = cE_{ij}$ with $i > j$ and $c \neq 0$. Form

$$[E_{1i}, [X, E_{jn}]]. \tag{2.3}$$

The claim is that (2.3) is a nonzero multiple of E_{1n}. In fact, we cannot have $i = 1$ since $j < i$. If $i < n$, then $[E_{ij}, E_{jn}] = aE_{in}$ with $a \neq 0$, and also $[E_{1i}, E_{in}] = bE_{1n}$ with $b \neq 0$. Thus (2.3) has a nonzero component in $\mathfrak{g}_{e_1 - e_n}$ in the decomposition (2.1). The other components of (2.3) must correspond to larger roots than $e_1 - e_n$ if they are nonzero, but $e_1 - e_n$ is the largest root. Hence the claim follows if $i < n$. If $i = n$, then (2.3) is

$$= [E_{1n}, [cE_{nj} + \cdots, E_{jn}]] = c[E_{1n}, E_{nn} - E_{jj}] + \cdots = cE_{1n}.$$

Thus the claim follows if $i = n$.

In any case we conclude that E_{1n} is in $\mathfrak{a}$. For $i \neq j$, the formula

$$E_{kl} = c'[E_{k1}, [E_{1n}, E_{nl}]] \qquad \text{with } c' \neq 0$$

(with obvious changes if $k = 1$ or $l = n$) shows that E_{kl} is in $\mathfrak{a}$, and

$$[E_{kl}, E_{lk}] = E_{kk} - E_{ll}$$

shows that a spanning set of $\mathfrak{h}$ is in $\mathfrak{a}$. Hence $\mathfrak{a} = \mathfrak{g}$.

Thus an ideal $\mathfrak{a}$ that is not in $\mathfrak{h}$ has to be all of $\mathfrak{g}$ if there is some $\alpha < 0$ with $X_\alpha \neq 0$ above. Similarly if there is some $\alpha > 0$ with $X_\alpha \neq 0$, let β be the largest such α, say $\alpha = e_i - e_j$ with $i < j$. Form $[E_{ni}, [X, E_{j1}]]$ and argue with E_{n1} in the same way to get $\mathfrak{a} = \mathfrak{g}$. Thus $\mathfrak{g}$ is simple over $\mathbb{C}$. This completes the first example.

We can abstract these properties. The complex Lie algebra $\mathfrak{g}$ will be simple whenever we can arrange that

1) $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$ such that $\mathfrak{g}$ has a simultaneous eigenspace decomposition relative to $\operatorname{ad}\mathfrak{h}$ and

(a) the 0 eigenspace is $\mathfrak{h}$,
(b) the other eigenspaces are 1-dimensional,
(c) with the set Δ of roots defined as before, (2.2) holds,
(d) the roots are all real on some real form $\mathfrak{h}_0$ of $\mathfrak{h}$.

2) the roots span $\mathfrak{h}^*$. If α is a root, so is $-\alpha$.

3) $\sum_{\alpha\in\Delta}[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathfrak{h}$.

4) each root $\beta < 0$ relative to an ordering of $\mathfrak{h}_0^*$ defined from a notion of positivity satisfying (i) and (ii) above has the following property: There exists a sequence of roots $\alpha_1, \dots, \alpha_k$ such that each partial sum from the left of $\beta + \alpha_1 + \cdots + \alpha_k$ is a root or 0 and the full sum is the largest root. If a partial sum $\beta + \cdots + \alpha_j$ is 0, then the member $[E_{\alpha_j}, E_{-\alpha_j}]$ of $\mathfrak{h}$ is such that $\alpha_{j+1}([E_{\alpha_j}, E_{-\alpha_j}]) \neq 0$.

We shall see that the other complex Lie algebras from §I.8, namely $\mathfrak{so}(n,\mathbb{C})$ and $\mathfrak{sp}(n,\mathbb{C})$, have the same kind of structure, provided n is restricted suitably.

EXAMPLE 2. The complex Lie algebra is $\mathfrak{g} = \mathfrak{so}(2n+1,\mathbb{C})$. Here a similar analysis by means of $\operatorname{ad}\mathfrak{h}$ for an abelian subalgebra $\mathfrak{h}$ is possible, and we shall say what the constructs are that lead to the conclusion that $\mathfrak{g}$ is simple for $n \geq 1$. We define

$$\mathfrak{h} = \{H \in \mathfrak{so}(2n+1,\mathbb{C}) \mid H = \text{matrix below}\}$$

$$H = \begin{pmatrix} \begin{pmatrix} 0 & ih_1 \\ -ih_1 & 0 \end{pmatrix} & & & & \\ & \begin{pmatrix} 0 & ih_2 \\ -ih_2 & 0 \end{pmatrix} & & & \\ & & \ddots & & \\ & & & \begin{pmatrix} 0 & ih_n \\ -ih_n & 0 \end{pmatrix} & \\ & & & & 0 \end{pmatrix}$$

$$e_j(\text{above } H) = h_j, \qquad 1 \leq j \leq n$$
$$\mathfrak{h}_0 = \{H \in \mathfrak{h} \mid \text{entries are purely imaginary}\}$$
$$\Delta = \{\pm e_i \pm e_j \text{ with } i \neq j\} \cup \{\pm e_k\}.$$

The members of $\mathfrak{h}_0^*$ are the linear functionals $\sum_j a_j e_j$ with all a_j real, and every root is of this form. A member $\varphi = \sum_j a_j e_j$ of $\mathfrak{h}_0^*$ is defined to be positive if $\varphi \neq 0$ and if the first nonzero a_j is positive. In the resulting ordering the largest root is $e_1 + e_2$. The root-space decomposition is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \qquad \text{with } \mathfrak{g}_\alpha = \mathbb{C}E_\alpha$$

and with E_α as defined below. To define E_α, first let $i < j$ and let $\alpha = \pm e_i \pm e_j$. Then E_α is 0 except in the sixteen entries corresponding to the i^{th} and j^{th} pairs of indices, where it is

$$E_\alpha = \begin{array}{cc} \begin{array}{cc} \;\; i \;\; & \;\; j \;\; \end{array} & \\ \begin{pmatrix} 0 & X_\alpha \\ -X_\alpha^t & 0 \end{pmatrix} & \begin{array}{c} i \\ j \end{array} \end{array}$$

with

$$X_{e_i - e_j} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \qquad X_{e_i + e_j} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$

$$X_{-e_i + e_j} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \qquad X_{-e_i - e_j} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

To define E_α for $\alpha = \pm e_k$, write

$$E_\alpha = \begin{array}{c} \begin{array}{cc} \text{pair} & \text{entry} \\ k & 2n+1 \end{array} \\ \begin{pmatrix} 0 & X_\alpha \\ -X_\alpha^t & 0 \end{pmatrix} \end{array}$$

with 0's elsewhere and with

$$X_{e_k} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \text{and} \qquad X_{-e_k} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

EXAMPLE 3. The complex Lie algebra is $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. Again an analysis by means of $\operatorname{ad} \mathfrak{h}$ for an abelian subalgebra $\mathfrak{h}$ is possible, and we shall say what the constructs are that lead to the conclusion that $\mathfrak{g}$ is simple for $n \geq 1$. We define

$$\mathfrak{h} = \left\{ H = \begin{pmatrix} h_1 & & & & & \\ & \ddots & & & & \\ & & h_n & & & \\ & & & -h_1 & & \\ & & & & \ddots & \\ & & & & & -h_n \end{pmatrix} \right\}$$

$$e_j(\text{above } H) = h_j, \qquad 1 \leq j \leq n$$
$$\mathfrak{h}_0 = \{H \in \mathfrak{h} \mid \text{entries are real}\}$$
$$\Delta = \{\pm e_i \pm e_j \text{ with } i \neq j\} \cup \{\pm 2e_k\}$$
$$\begin{aligned} E_{e_i - e_j} &= E_{i,j} - E_{j+n,i+n}, \qquad & E_{2e_k} &= E_{k,k+n}, \\ E_{e_i + e_j} &= E_{i,j+n} + E_{j,i+n}, \qquad & E_{-2e_k} &= E_{k+n,k}, \\ E_{-e_i - e_j} &= E_{i+n,j} + E_{j+n,i}. \end{aligned}$$

EXAMPLE 4. The complex Lie algebra is $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$. The analysis is similar to that for $\mathfrak{so}(2n+1, \mathbb{C})$. The Lie algebra $\mathfrak{so}(2n, \mathbb{C})$ is simple over $\mathbb{C}$ for $n \geq 3$, the constructs for this example being

$\mathfrak{h}$ as with $\mathfrak{so}(2n+1, \mathbb{C})$ but with the last row and column deleted
$e_j(H) = h_j, \quad 1 \leq j \leq n,$ as with $\mathfrak{so}(2n+1, \mathbb{C})$
$\mathfrak{h}_0 = \{H \in \mathfrak{h} \mid \text{entries are purely imaginary}\}$
$\Delta = \{\pm e_i \pm e_j \text{ with } i \neq j\}$
E_α as for $\mathfrak{so}(2n+1, \mathbb{C})$ when $\alpha = \pm e_i \pm e_j$.

When $n = 2$, condition (4) in the list of abstracted properties fails. In fact, take $\beta = -e_1 + e_2$. The only choice for α_1 is $e_1 - e_2$, and then $\beta + \alpha_1 = 0$. We have to choose $\alpha_2 = e_1 + e_2$, and $\alpha_2([E_{\alpha_1}, E_{-\alpha_1}]) = 0$. In §5 we shall see that $\mathfrak{so}(4, \mathbb{C})$ is actually not simple.

2. Existence of Cartan Subalgebras

The idea is to approach a general complex semisimple Lie algebra $\mathfrak{g}$ by imposing on it the same kind of structure as in §1. We try to construct an $\mathfrak{h}$, a set of roots, a real form $\mathfrak{h}_0$ on which the roots are real, and an ordering on $\mathfrak{h}_0^*$. Properties (1) through (3) in §1 turn out actually to be equivalent with $\mathfrak{g}$ semisimple. In the presence of the first three properties, property (4) will be equivalent with $\mathfrak{g}$ simple. But we shall obtain better formulations of property (4) later, and that property should be disregarded, at least for the time being.

The hypothesis of semisimplicity of $\mathfrak{g}$ enters the construction only by forcing special features of $\mathfrak{h}$ and the roots. Accordingly we work with a general finite-dimensional complex Lie algebra $\mathfrak{g}$ until near the end of this section.

Let $\mathfrak{h}$ be a finite-dimensional Lie algebra over $\mathbb{C}$. Recall from §I.5 that a representation π of $\mathfrak{h}$ on a complex vector space V is a complex-linear Lie algebra homomorphism of $\mathfrak{h}$ into $\operatorname{End}_{\mathbb{C}}(V)$. For such π and V, whenever α is in the dual $\mathfrak{h}^*$, we let V_α be defined as

$$\{v \in V \mid (\pi(H) - \alpha(H)1)^n v = 0 \text{ for all } H \in \mathfrak{h} \text{ and some } n = n(H, v)\}.$$

If $V_\alpha \neq 0$, V_α is called a **generalized weight space** and α is a **weight**. Members of V_α are called **generalized weight vectors**.

For now, we shall be interested only in the case that V is finite dimensional. In this case $\pi(H) - \alpha(H)1$ has 0 as its only generalized eigenvalue on V_α and is nilpotent on this space, as a consequence of the theory of Jordan normal form. Therefore $n(H, v)$ can be taken to be $\dim V$.

Proposition 2.4. Suppose that $\mathfrak{h}$ is a nilpotent Lie algebra over $\mathbb{C}$ and that π is a representation of $\mathfrak{h}$ on a finite-dimensional complex vector space V. Then there are finitely many generalized weights, each generalized weight space is stable under $\pi(\mathfrak{h})$, and V is the direct sum of all the generalized weight spaces.

REMARKS.

1) The direct-sum decomposition of V as the sum of the generalized weight spaces is called a **weight-space decomposition** of V.

2) The weights need not be linearly independent. For example, they are dependent in our root-space decompositions in the previous section.

3) Since $\mathfrak{h}$ is nilpotent, it is solvable, and Lie's Theorem (Corollary 1.29) applies to it. In a suitable basis of V, $\pi(\mathfrak{h})$ is therefore simultaneously triangular. The generalized weights will be the distinct diagonal entries, as functions on $\mathfrak{h}$. To get the direct sum decomposition, however, is subtler; we need to make more serious use of the fact that $\mathfrak{h}$ is nilpotent.

PROOF. First we check that V_α is invariant under $\pi(\mathfrak{h})$. Fix $H \in \mathfrak{h}$ and let

$$V_{\alpha,H} = \{v \in V \mid (\pi(H) - \alpha(H)1)^n v = 0 \text{ for some } n = n(v)\},$$

so that $V_\alpha = \cap_{H\in\mathfrak{h}} V_{\alpha,H}$. It is enough to prove that $V_{\alpha,H}$ is invariant under $\pi(\mathfrak{h})$ if $H \neq 0$. Since $\mathfrak{h}$ is nilpotent, ad H is nilpotent. Let

$$\mathfrak{h}_{(m)} = \{Y \in \mathfrak{h} \mid (\operatorname{ad} H)^m Y = 0\},$$

so that $\mathfrak{h} = \cup_{m=0}^d \mathfrak{h}_{(m)}$ with $d = \dim \mathfrak{h}$. We prove that $\pi(Y)V_{\alpha,H} \subseteq V_{\alpha,H}$ for $Y \in \mathfrak{h}_{(m)}$ by induction on m.

For $m = 0$, we have $\mathfrak{h}_{(0)} = 0$ since $(\operatorname{ad} H)^0 = 1$. So $\pi(Y) = \pi(0) = 0$, and $\pi(Y)V_{\alpha,H} \subseteq V_{\alpha,H}$ trivially.

We now address general m under the assumption that our assertion is true for all $Z \in \mathfrak{h}_{(m-1)}$. Let Y be in $\mathfrak{h}_{(m)}$. Then $[H, Y]$ is in $\mathfrak{h}_{(m-1)}$, and we have

$$\begin{aligned}(\pi(H) - \alpha(H)1)\pi(Y) &= \pi([H, Y]) + \pi(Y)\pi(H) - \alpha(H)\pi(Y)\\ &= \pi(Y)(\pi(H) - \alpha(H)1) + \pi([H, Y])\end{aligned}$$

and

$$\begin{aligned}&(\pi(H) - \alpha(H)1)^2\pi(Y)\\ &\quad= (\pi(H)-\alpha(H)1)\pi(Y)(\pi(H)-\alpha(H)1) + (\pi(H)-\alpha(H)1)\pi([H, Y])\\ &\quad= \pi(Y)(\pi(H) - \alpha(H)1)^2 + \pi([H, Y])(\pi(H) - \alpha(H)1)\\ &\qquad + (\pi(H) - \alpha(H)1)\pi([H, Y]).\end{aligned}$$

Iterating, we obtain

$$\begin{aligned}&(\pi(H) - \alpha(H)1)^l\pi(Y)\\ &\qquad= \pi(Y)(\pi(H) - \alpha(H)1)^l\\ &\qquad\quad + \sum_{s=0}^{l-1}(\pi(H) - \alpha(H)1)^{l-1-s}\pi([H, Y])(\pi(H) - \alpha(H)1)^s.\end{aligned}$$

For $v \in V_{\alpha,H}$, we have $(\pi(H) - \alpha(H)1)^N v = 0$ if $N \geq \dim V$. Take $l = 2N$. When the above expression is applied to v, the only terms in the sum on the right side that can survive are those with $s < N$. For these we have $l - 1 - s \geq N$. Then $(\pi(H) - \alpha(H)1)^s v$ is in $V_{\alpha,H}$, $\pi([H, Y])$ leaves $V_{\alpha,H}$ stable since $[H, Y]$ is in $\mathfrak{h}_{(m-1)}$, and

$$(\pi(H) - \alpha(H)1)^{l-1-s}\pi([H, Y])(\pi(H) - \alpha(H)1)^s v = 0.$$

Hence $(\pi(H) - \alpha(H)1)^l\pi(Y)v = 0$, and $V_{\alpha,H}$ is stable under $\pi(Y)$. This completes the induction and the proof that V_α is invariant under $\pi(\mathfrak{h})$.

Now we can obtain the decomposition $V = \bigoplus_\alpha V_\alpha$. Let $H_1, \ldots, H_r$ be a basis for $\mathfrak{h}$. The Jordan decomposition of $\pi(H_1)$ gives us a generalized eigenspace decomposition that we can write as

$$V = \bigoplus_\lambda V_{\lambda, H_1}.$$

Here we can regard the complex number λ as running over all distinct values of $\alpha(H_1)$ for α arbitrary in $\mathfrak{h}^*$. Thus we can rewrite the Jordan decomposition as

$$V = \bigoplus_{\substack{\text{values of} \\ \alpha(H_1)}} V_{\alpha(H_1), H_1}.$$

For fixed $\alpha \in \mathfrak{h}^*$, $V_{\alpha(H_1),H_1}$ is nothing more than the space V_{α,H_1} defined at the start of the proof. From what we have already shown, the space $V_{\alpha(H_1),H_1} = V_{\alpha,H_1}$ is stable under $\pi(\mathfrak{h})$. Thus we can decompose it under $\pi(H_2)$ as

$$V = \bigoplus_{\alpha(H_1)} \bigoplus_{\alpha(H_2)} (V_{\alpha(H_1),H_1} \cap V_{\alpha(H_2),H_2}),$$

and we can iterate to obtain

$$V = \bigoplus_{\alpha(H_1),\ldots,\alpha(H_r)} \Big(\bigcap_{j=1}^{r} V_{\alpha(H_j),H_j} \Big)$$

with each of the spaces invariant under $\pi(\mathfrak{h})$. By Lie's Theorem (Corollary 1.29), we can regard all $\pi(H_i)$ as acting simultaneously by triangular matrices on $\bigcap_{j=1}^r V_{\alpha(H_j),H_j}$, evidently with all diagonal entries $\alpha(H_i)$. Then $\pi(\sum c_i H_i)$ must act as a triangular matrix with all diagonal entries $\sum c_i \alpha(H_i)$. Thus if we define a linear functional α by $\alpha(\sum c_i H_i) = \sum c_i \alpha(H_i)$, we see that $\bigcap_{j=1}^r V_{\alpha(H_j),H_j}$ is exactly V_α. Thus $V = \bigoplus_\alpha V_\alpha$, and in particular there are only finitely many weights.

Proposition 2.5. If $\mathfrak{g}$ is any finite-dimensional Lie algebra over $\mathbb{C}$ and if $\mathfrak{h}$ is a nilpotent Lie subalgebra, then the generalized weight spaces of $\mathfrak{g}$ relative to $\operatorname{ad}_{\mathfrak{g}} \mathfrak{h}$ satisfy

(a) $\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is defined as

$$\{X \in \mathfrak{g} \mid (\operatorname{ad} H - \alpha(H)1)^n X = 0 \text{ for all } H \in \mathfrak{h} \text{ and some } n = n(H, X)\},$$

(b) $\mathfrak{h} \subseteq \mathfrak{g}_0$,

(c) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ (with $\mathfrak{g}_{\alpha+\beta}$ understood to be 0 if $\alpha + \beta$ is not a generalized weight).

PROOF.
(a) This is by Proposition 2.4.
(b) Since $\mathfrak{h}$ is nilpotent, $\operatorname{ad}\mathfrak{h}$ is nilpotent on $\mathfrak{h}$. Thus $\mathfrak{h} \subseteq \mathfrak{g}_0$.
(c) Let $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, and $H \in \mathfrak{h}$. Then

$$\begin{aligned}(\operatorname{ad} H - (\alpha(H) + \beta(H))1)[X, Y]\\ &= [H, [X, Y]] - \alpha(H)[X, Y] - \beta(H)[X, Y]\\ &= [(\operatorname{ad} H - \alpha(H)1)X, Y] + [X, (\operatorname{ad} H - \beta(H)1)Y],\end{aligned}$$

and we can readily set up an induction to see that

$$\begin{aligned}(\operatorname{ad} H - (\alpha(H) + \beta(H))1)^n[X, Y]\\ &= \sum_{k=0}^{n} \binom{n}{k} [(\operatorname{ad} H - \alpha(H)1)^k X, (\operatorname{ad} H - \beta(H)1)^{n-k} Y].\end{aligned}$$

If $n \geq 2 \dim \mathfrak{g}$, either k or $n-k$ is $\geq \dim \mathfrak{g}$, and hence every term on the right side is 0.

Corollary 2.6. $\mathfrak{g}_0$ is a subalgebra.

PROOF. This follows from Proposition 2.5c.

To match the behavior of our examples in the previous section, we make the following definition. A nilpotent Lie subalgebra $\mathfrak{h}$ of a finite-dimensional complex Lie algebra $\mathfrak{g}$ is a **Cartan subalgebra** if $\mathfrak{h} = \mathfrak{g}_0$. The inclusion $\mathfrak{h} \subseteq \mathfrak{g}_0$ is always guaranteed by Proposition 2.5b.

Proposition 2.7. A nilpotent Lie subalgebra $\mathfrak{h}$ of a finite-dimensional complex Lie algebra $\mathfrak{g}$ is a Cartan subalgebra if and only if $\mathfrak{h}$ equals the normalizer $N_{\mathfrak{g}}(\mathfrak{h}) = \{X \in \mathfrak{g} \mid [X, \mathfrak{h}] \subseteq \mathfrak{h}\}$.

PROOF. We always have

$$\mathfrak{h} \subseteq N_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{g}_0. \tag{2.8}$$

The first of these inclusions holds because $\mathfrak{h}$ is a Lie subalgebra. The second holds because $(\operatorname{ad} H)^n X = (\operatorname{ad} H)^{n-1}[H, X]$ and $\operatorname{ad} H$ is nilpotent on $\mathfrak{h}$.

Now assume that $\mathfrak{h}$ is a Cartan subalgebra. Then $\mathfrak{g}_0 = \mathfrak{h}$ by definition. By (2.8), $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_0$. Conversely assume that $\mathfrak{h}$ is not a Cartan subalgebra, i.e., that $\mathfrak{g}_0 \neq \mathfrak{h}$. Form $\operatorname{ad}\mathfrak{h} : \mathfrak{g}_0/\mathfrak{h} \to \mathfrak{g}_0/\mathfrak{h}$ as a Lie algebra of transformations of the nonzero vector space $\mathfrak{g}_0/\mathfrak{h}$. Since $\mathfrak{h}$ is solvable, this Lie algebra of transformations is solvable. By Lie's Theorem (Theorem 1.25) there exists an $X+\mathfrak{h}$ in $\mathfrak{g}_0/\mathfrak{h}$ with $X \notin \mathfrak{h}$ that is a simultaneous eigenvector for $\operatorname{ad}\mathfrak{h}$, and we know that its simultaneous eigenvalue has to be 0. This means that $(\operatorname{ad} H)(X+\mathfrak{h}) \subseteq \mathfrak{h}$, i.e., $[H, X]$ is in $\mathfrak{h}$. Hence X is not in $\mathfrak{h}$ but X is in $N_{\mathfrak{g}}(\mathfrak{h})$. Thus $\mathfrak{h} \neq N_{\mathfrak{g}}(\mathfrak{h})$.

Theorem 2.9. Any finite-dimensional complex Lie algebra $\mathfrak{g}$ has a Cartan subalgebra.

Before coming to the proof, we introduce "regular" elements of $\mathfrak{g}$. In $\mathfrak{sl}(n, \mathbb{C})$ the regular elements will be the matrices with distinct eigenvalues. Let us consider matters more generally.

If π is a representation of $\mathfrak{g}$ on a finite-dimensional vector space V, we can regard each $X \in \mathfrak{g}$ as generating a 1-dimensional abelian subalgebra, and we can then form $V_{0,X}$, the generalized eigenspace for eigenvalue 0 under $\pi(X)$. Let

$$l_{\mathfrak{g}}(V) = \min_{X \in \mathfrak{g}} \dim V_{0,X}$$

$$R_{\mathfrak{g}}(V) = \{X \in \mathfrak{g} \mid \dim V_{0,X} = l_{\mathfrak{g}}(V)\}.$$

To understand $l_{\mathfrak{g}}(V)$ and $R_{\mathfrak{g}}(V)$ better, form the characteristic polynomial

$$\det(\lambda 1 - \pi(X)) = \lambda^n + \sum_{j=0}^{n-1} d_j(X)\lambda^j.$$

In any basis of $\mathfrak{g}$, the $d_j(X)$ are polynomial functions on $\mathfrak{g}$, as we see by expanding $\det(\lambda 1 - \sum \mu_i \pi(X_i))$. For given X, if j is the smallest value for which $d_j(X) \neq 0$, then $j = \dim V_{0,X}$, since the degree of the last term in the characteristic polynomial is the multiplicity of 0 as a generalized eigenvalue of $\pi(X)$. Thus $l_{\mathfrak{g}}(V)$ is the minimum j such that $d_j(X) \not\equiv 0$, and

$$R_{\mathfrak{g}}(V) = \{X \in \mathfrak{g} \mid d_{l_{\mathfrak{g}}(V)}(X) \neq 0\}.$$

Let us apply these considerations to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$. The elements of $R_{\mathfrak{g}}(\mathfrak{g})$, relative to the adjoint representation, are the **regular elements** of $\mathfrak{g}$. For any X in $\mathfrak{g}$, $\mathfrak{g}_{0,X}$ is a Lie subalgebra of $\mathfrak{g}$ by the corollary of Proposition 2.5, with $\mathfrak{h} = \mathbb{C}X$.

Theorem 2.9′. If X is a regular element of the finite-dimensional complex Lie algebra $\mathfrak{g}$, then the Lie algebra $\mathfrak{g}_{0,X}$ is a Cartan subalgebra of $\mathfrak{g}$.

PROOF. First we show that $\mathfrak{g}_{0,X}$ is nilpotent. Assuming the contrary, we construct two sets:

(i) the set of $Z \in \mathfrak{g}_{0,X}$ such that $((\operatorname{ad} Z)|_{\mathfrak{g}_{0,X}})^{\dim \mathfrak{g}_{0,X}} \neq 0$, which is nonempty by Engel's Theorem (Corollary 1.38) and is open,

(ii) the set of $W \in \mathfrak{g}_{0,X}$ such that $\operatorname{ad} W|_{\mathfrak{g}/\mathfrak{g}_{0,X}}$ is nonsingular, which is nonempty since X is in it (regularity is not used here) and is the set where some polynomial is nonvanishing, hence is dense (because if a polynomial vanishes on a nonempty open set, it vanishes identically).

These two sets must have nonempty intersection, and so we can find Z in $\mathfrak{g}_{0,X}$ such that

$$((\operatorname{ad} Z)|_{\mathfrak{g}_{0,X}})^{\dim \mathfrak{g}_{0,X}} \neq 0 \qquad \text{and} \qquad \operatorname{ad} Z|_{\mathfrak{g}/\mathfrak{g}_{0,X}} \text{ is nonsingular.}$$

Then the generalized multiplicity of the eigenvalue 0 for $\operatorname{ad} Z$ is less than $\dim \mathfrak{g}_{0,X}$, and hence $\dim \mathfrak{g}_{0,Z} < \dim \mathfrak{g}_{0,X}$, in contradiction with the regularity of X. We conclude that $\mathfrak{g}_{0,X}$ is nilpotent.

Since $\mathfrak{g}_{0,X}$ is nilpotent, we can use $\mathfrak{g}_{0,X}$ to decompose $\mathfrak{g}$ as in Proposition 2.4. Let $\mathfrak{g}_0$ be the 0 generalized weight space. Then we have

$$\mathfrak{g}_{0,X} \subseteq \mathfrak{g}_0 = \bigcap_{Y \in \mathfrak{g}_{0,X}} \mathfrak{g}_{0,Y} \subseteq \mathfrak{g}_{0,X}.$$

So $\mathfrak{g}_{0,X} = \mathfrak{g}_0$, and $\mathfrak{g}_{0,X}$ is a Cartan subalgebra.

In this book we shall be interested in Cartan subalgebras $\mathfrak{h}$ only when $\mathfrak{g}$ is semisimple. In this case $\mathfrak{h}$ has special properties, as follows.

Proposition 2.10. If $\mathfrak{g}$ is a complex semisimple Lie algebra and $\mathfrak{h}$ is a Cartan subalgebra, then $\mathfrak{h}$ is abelian.

PROOF. Since $\mathfrak{h}$ is nilpotent and therefore solvable, $\operatorname{ad} \mathfrak{h}$ is solvable as a Lie algebra of transformations of $\mathfrak{g}$. By Lie's Theorem (Corollary 1.29) it is simultaneously triangular in some basis. For any three triangular matrices A, B, C, we have $\operatorname{Tr}(ABC) = \operatorname{Tr}(BAC)$. Therefore

$$\operatorname{Tr}(\operatorname{ad}[H_1, H_2] \operatorname{ad} H) = 0 \qquad \text{for } H_1,\ H_2,\ H \in \mathfrak{h}. \tag{2.11}$$

Next let α be any nonzero generalized weight, let X be in $\mathfrak{g}_\alpha$, and let H be in $\mathfrak{h}$. By Proposition 2.5c, $\operatorname{ad} H \operatorname{ad} X$ carries $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\alpha+\beta}$. Thus Proposition 2.5a shows that

$$\operatorname{Tr}(\operatorname{ad} H \operatorname{ad} X) = 0. \tag{2.12}$$

Specializing (2.12) to $H = [H_1, H_2]$ and using (2.11) and Proposition 2.5a, we see that the Killing form B of $\mathfrak{g}$ satisfies

$$B([H_1, H_2], X) = 0 \qquad \text{for all } X \in \mathfrak{g}.$$

By Cartan's Criterion for Semisimplicity (Theorem 1.45), B is nondegenerate. Therefore $[H_1, H_2] = 0$, and $\mathfrak{h}$ is abelian.

Proposition 2.13. In a complex semisimple Lie algebra $\mathfrak{g}$, a Lie subalgebra is a Cartan subalgebra if it is maximal among the abelian subalgebras $\mathfrak{h}$ such that $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{h}$ is simultaneously diagonable.

REMARKS.

1) It is immediate from this corollary that the subalgebras $\mathfrak{h}$ in the examples of §1 are Cartan subalgebras.

2) Proposition 2.13 implies the existence of Cartan subalgebras, but only in the semisimple case. A uniqueness theorem, Theorem 2.15 below, will say that any two Cartan subalgebras are conjugate, and hence every Cartan subalgebra in the semisimple case must satisfy the properties in the proposition.

3) The properties in the proposition can also be seen directly without using the uniqueness theorem. Proposition 2.10 shows that any Cartan subalgebra $\mathfrak{h}$ in the semisimple case is abelian, and it is maximal abelian since $\mathfrak{h} = \mathfrak{g}_0$. Corollary 2.23 will show for a Cartan subalgebra $\mathfrak{h}$ in the semisimple case that $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{h}$ is simultaneously diagonable.

PROOF. Let $\mathfrak{h}$ be maximal among the abelian subalgebras such that $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{h}$ is simultaneously diagonable. Since $\mathfrak{h}$ is abelian and hence nilpotent, Proposition 2.4 shows that $\mathfrak{g}$ has a weight-space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\beta \neq 0} \mathfrak{g}_\beta$ under $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{h}$. Since $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{h}$ is simultaneously diagonable, $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{r}$ with $[\mathfrak{h}, \mathfrak{r}] = 0$. In view of Proposition 2.7, we are to prove that $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$. Here $\mathfrak{h} \subseteq N_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{g}_0$ by (2.8), and it is enough to show that $\mathfrak{r} = 0$. Arguing by contradiction, suppose that $X \neq 0$ is in $\mathfrak{r}$. Then $\mathfrak{h} \oplus \mathbb{C}X$ is an abelian subalgebra properly containing $\mathfrak{h}$, and the hypothesis of maximality says that $\mathrm{ad}\,X$ must not be diagonable. We apply Proposition 2.4 again, this time using $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h} \oplus \mathbb{C}X)$ and obtaining $\mathfrak{g} = \bigoplus_\beta \bigoplus_{\beta'|_{\mathfrak{h}} = \beta} \mathfrak{g}_{\beta'}$. By Theorem 1.48 we can write $\mathrm{ad}\,X = s + n$ with s diagonable, n nilpotent, $sn = ns$, and $s = p(\mathrm{ad}\,X)$ for some polynomial p without constant term. Since $\mathrm{ad}\,X$ carries each $\mathfrak{g}_{\beta'}$ to itself, so does s. The transformation s must then act by the scalar $\beta'(X)$ on $\mathfrak{g}_{\beta'}$. Since $[\mathfrak{g}_{\beta'}, \mathfrak{g}_{\gamma'}] \subseteq \mathfrak{g}_{\beta'+\gamma'}$ by Proposition 2.5c, it follows for $Y \in \mathfrak{g}_{\beta'}$ and $Z \in \mathfrak{g}_{\gamma'}$

that $s[Y, Z] = (\beta'(X) + \gamma'(X))[Y, Z] = [s(Y), Z] + [Y, s(Z)]$. In other words, s is a derivation of $\mathfrak{g}$. By Proposition 1.121, $s = \operatorname{ad} S$ for some S in $\mathfrak{g}$. Since $s = p(\operatorname{ad} X)$ and $[\mathfrak{h}, X] = 0$, we find that $[\mathfrak{h}, S] = 0$. By the hypothesis of maximality, S is in $\mathfrak{h}$. From $\operatorname{ad} X = \operatorname{ad} S + n$, we conclude that $n = \operatorname{ad} N$ for some N in $\mathfrak{h} \oplus \mathbb{C}X$. In other words we could have assumed that $\operatorname{ad} X$ is nilpotent from the outset. Since $\operatorname{ad} X$ is nilpotent on $\mathfrak{g}$ and since $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{r}$ is a subalgebra (Corollary 2.6), $\operatorname{ad} X$ is nilpotent on $\mathfrak{g}_0$. Thus every member of $\operatorname{ad}(\mathfrak{h} \oplus \mathbb{C}X)$ is nilpotent on $\mathfrak{g}_0$. But X is arbitrary in $\mathfrak{r}$, and thus every member of $\operatorname{ad} \mathfrak{g}_0$ is nilpotent on $\mathfrak{g}_0$. By Engel's Theorem (Corollary 1.38), $\mathfrak{g}_0$ is a nilpotent Lie algebra. Consequently we can use $\operatorname{ad}_{\mathfrak{g}} \mathfrak{g}_0$ to decompose $\mathfrak{g}$ according to Proposition 2.4, and the 0 weight space can be no bigger than it was when we used $\operatorname{ad}_{\mathfrak{g}} \mathfrak{h}$ at the start. Thus the 0 weight space has to be $\mathfrak{g}_0$, and $\mathfrak{g}_0$ is a Cartan subalgebra. If we write the decomposition according to $\operatorname{ad}_{\mathfrak{g}} \mathfrak{g}_0$ as $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha$, then we have $B(X, X_0) = \sum (\dim \mathfrak{g}_\alpha)\alpha(X)\alpha(X_0)$ when X is the element above and X_0 is in $\mathfrak{g}_0$. This sum is 0 since the nilpotence of $\operatorname{ad} X$ makes $\alpha(X) = 0$ for all α. As in (2.12), $B(X, X_\alpha) = 0$ for $X_\alpha \in \mathfrak{g}_\alpha$ with $\alpha \neq 0$. Thus $B(X, \mathfrak{g}) = 0$. Since B is nondegenerate, it follows that $X = 0$, and we have arrived at a contradiction.

3. Uniqueness of Cartan Subalgebras

We turn to the question of uniqueness of Cartan subalgebras. We begin with a lemma about polynomial mappings.

Lemma 2.14. Let $P : \mathbb{C}^m \to \mathbb{C}^n$ be a holomorphic polynomial function not identically 0. Then the set of vectors z in $\mathbb{C}^m$ for which $P(z)$ is not the 0 vector is connected in $\mathbb{C}^m$.

PROOF. Suppose that z_0 and w_0 in $\mathbb{C}^m$ have $P(z_0) \neq 0$ and $P(w_0) \neq 0$. As a function of $z \in \mathbb{C}$, $P(z_0 + z(w_0 - z_0))$ is a vector-valued holomorphic polynomial nonvanishing at $z = 0$ and $z = 1$. The subset of $z \in \mathbb{C}$ where it vanishes is finite, and the complement in $\mathbb{C}$ is connected. Thus z_0 and w_0 lie in a connected set in $\mathbb{C}^m$ where P is nonvanishing. Taking the union of these connected sets with z_0 fixed and w_0 varying, we see that the set where $P(w_0) \neq 0$ is connected.

Theorem 2.15. If $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are Cartan subalgebras of a finite-dimensional complex Lie algebra $\mathfrak{g}$, then there exists $a \in \operatorname{Int} \mathfrak{g}$ with $a(\mathfrak{h}_1) = \mathfrak{h}_2$.

REMARKS.

1) In particular any two Cartan subalgebras are conjugate by an automorphism of $\mathfrak{g}$. As was explained after the introduction of $\operatorname{Int}\mathfrak{g}$ in §I.11, $\operatorname{Int}\mathfrak{g} = \operatorname{Int}\mathfrak{g}^{\mathbb{R}}$ is a universal version of $\operatorname{Ad}(G)$ for analytic groups G with Lie algebra $\mathfrak{g}^{\mathbb{R}}$. Thus if G is some analytic group with Lie algebra $\mathfrak{g}^{\mathbb{R}}$, the theorem asserts that the conjugacy can be achieved by some automorphism $\operatorname{Ad}(g)$ with $g \in G$.

2) By the theorem all Cartan subalgebras of $\mathfrak{g}$ have the same dimension. The common value of this dimension is called the **rank** of $\mathfrak{g}$.

PROOF. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Under the definitions in §2,

$$R_{\mathfrak{h}}(\mathfrak{g}) = \{Y \in \mathfrak{h} \mid \dim \mathfrak{g}_{0,Y} \text{ is a minimum for elements of } \mathfrak{h}\}.$$

We shall show that

(a) two alternative formulas for $R_{\mathfrak{h}}(\mathfrak{g})$ are

$$\begin{aligned} R_{\mathfrak{h}}(\mathfrak{g}) &= \{Y \in \mathfrak{h} \mid \alpha(Y) \neq 0 \text{ for all generalized weights } \alpha \neq 0\} \\ &= \{Y \in \mathfrak{h} \mid \mathfrak{g}_{0,Y} = \mathfrak{h}\}, \end{aligned}$$

(b) $Y \in R_{\mathfrak{h}}(\mathfrak{g})$ implies $\operatorname{ad} Y$ is nonsingular on $\bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha$,

(c) the image of the map

$$\sigma : \operatorname{Int}\mathfrak{g} \times R_{\mathfrak{h}}(\mathfrak{g}) \to \mathfrak{g}$$

given by $\sigma(a, Y) = a(Y)$ is open in $\mathfrak{g}$ and is contained in $R_{\mathfrak{g}}(\mathfrak{g})$,

(d) if $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are Cartan subalgebras that are not conjugate by $\operatorname{Int}\mathfrak{g}$, then the corresponding images of the maps in (c) are disjoint,

(e) every member of $R_{\mathfrak{g}}(\mathfrak{g})$ is in the image of the map in (c) for some Cartan subalgebra $\mathfrak{h}$,

(f) $R_{\mathfrak{g}}(\mathfrak{g})$ is connected.

These six statements prove the theorem. In fact, (c) through (e) exhibit $R_{\mathfrak{g}}(\mathfrak{g})$ as a nontrivial disjoint union of open sets if we have nonconjugacy. But (f) says that such a nontrivial disjoint union is impossible. Thus let us prove the six statements.

(a) Since $\mathfrak{h}$ is a Cartan subalgebra, $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha$. If Y is in $\mathfrak{h}$, then $\mathfrak{g}_{0,Y} = \{X \in \mathfrak{g} \mid (\operatorname{ad} Y)^n X = 0\}$, where $n = \dim \mathfrak{g}$. Thus elements X in $\mathfrak{g}_{0,Y}$ are characterized by being in the generalized eigenspace for $\operatorname{ad} Y$ with eigenvalue 0. So $\mathfrak{g}_{0,Y} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0, \alpha(Y)=0} \mathfrak{g}_\alpha$. Since finitely many hyperplanes in $\mathfrak{h}$ cannot have union $\mathfrak{h}$ ($\mathbb{C}$ being an infinite field), we can find Y with $\alpha(Y) \neq 0$ for all $\alpha \neq 0$. Then we see that $\mathfrak{g}_{0,Y}$ is smallest when it is $\mathfrak{h}$, and (a) follows.

(b) The linear map $\operatorname{ad} Y$ acts on $\mathfrak{g}_\alpha$ with generalized eigenvalue $\alpha(Y) \neq 0$, by (a). Hence $\operatorname{ad} Y$ is nonsingular on each $\mathfrak{g}_\alpha$.

(c) Since $\operatorname{Int} \mathfrak{g}$ is a group, it is enough to show that $Y \in R_{\mathfrak{h}}(\mathfrak{g})$ implies that $(\operatorname{Int} \mathfrak{g})(R_{\mathfrak{h}}(\mathfrak{g}))$ contains a neighborhood of Y in $\mathfrak{g}$. Form the differential $d\sigma$ at the point $(1, Y)$. Since $R_{\mathfrak{h}}(\mathfrak{g})$ is open in $\mathfrak{h}$, the tangent space at Y may be regarded as $\mathfrak{h}$ (with $c_H(t) = Y + tH$ being a curve with derivative $H \in \mathfrak{h}$). Similarly the tangent space at the point $\sigma(1, Y)$ of $\mathfrak{g}$ may be identified with $\mathfrak{g}$. Finally the tangent space at the point 1 of $\operatorname{Int} \mathfrak{g}$ is the Lie algebra $\operatorname{ad} \mathfrak{g}$. Hence $d\sigma$ is a map

$$d\sigma : \operatorname{ad} \mathfrak{g} \times \mathfrak{h} \to \mathfrak{g}.$$

Now

$$\begin{aligned} d\sigma(\operatorname{ad} X, 0) &= \frac{d}{dt} \sigma(e^{t \operatorname{ad} X}, Y)|_{t=0} \\ &= \frac{d}{dt}(e^{t \operatorname{ad} X})Y|_{t=0} = (\operatorname{ad} X)Y = [X, Y] \end{aligned}$$

and

$$d\sigma(0, H) = \frac{d}{dt} \sigma(1, Y + tH)|_{t=0} = \frac{d}{dt}(Y + tH)|_{t=0} = H.$$

Thus $\operatorname{image}(d\sigma) = [Y, \mathfrak{g}] + \mathfrak{h}$. By (b), $d\sigma$ is onto $\mathfrak{g}$. Hence the image of σ includes a neighborhood of $\sigma(1, Y)$ in $\mathfrak{g}$. Therefore $\operatorname{image}(\sigma)$ is open. But $R_{\mathfrak{g}}(\mathfrak{g})$ is dense. So $\operatorname{image}(\sigma)$ contains a member X of $R_{\mathfrak{g}}(\mathfrak{g})$. Then $a(Y) = X$ for some $a \in \operatorname{Int} \mathfrak{g}$ and $Y \in \mathfrak{h}$. From $a(Y) = X$ we easily check that $a(\mathfrak{g}_{0,Y}) = \mathfrak{g}_{0,X}$. Hence $\dim \mathfrak{g}_{0,Y} = \dim \mathfrak{g}_{0,X}$. Since $\dim \mathfrak{g}_{0,Y} = l_{\mathfrak{h}}(\mathfrak{g})$ and $\dim \mathfrak{g}_{0,X} = l_{\mathfrak{g}}(\mathfrak{g})$, we obtain $l_{\mathfrak{h}}(\mathfrak{g}) = l_{\mathfrak{g}}(\mathfrak{g})$. Thus $R_{\mathfrak{h}}(\mathfrak{g}) \subseteq R_{\mathfrak{g}}(\mathfrak{g})$. Now $R_{\mathfrak{g}}(\mathfrak{g})$ is stable under $\operatorname{Aut}_{\mathbb{C}} \mathfrak{g}$, and so $\operatorname{image}(\sigma) \subseteq R_{\mathfrak{g}}(\mathfrak{g})$.

(d) Let $a_1(Y_1) = a_2(Y_2)$ with $Y_1 \in R_{\mathfrak{h}_1}(\mathfrak{g})$ and $Y_2 \in R_{\mathfrak{h}_2}(\mathfrak{g})$. Then $a = a_2^{-1}a_1$ has $a(Y_1) = Y_2$. As in the previous step, we obtain $a(\mathfrak{g}_{0,Y_1}) = \mathfrak{g}_{0,Y_2}$. By (a), $\mathfrak{g}_{0,Y_1} = \mathfrak{h}_1$ and $\mathfrak{g}_{0,Y_2} = \mathfrak{h}_2$. Hence $a(\mathfrak{h}_1) = \mathfrak{h}_2$.

(e) If X is in $R_{\mathfrak{g}}(\mathfrak{g})$, let $\mathfrak{h} = \mathfrak{g}_{0,X}$. This is a Cartan subalgebra, by Theorem 2.9′, and (a) says that X is in $R_{\mathfrak{h}}(\mathfrak{g})$ for this $\mathfrak{h}$. Then $\sigma(1, X) = X$ shows that X is in the image of the σ defined relative to this $\mathfrak{h}$.

(f) We have seen that $R_{\mathfrak{g}}(\mathfrak{g})$ is the complement of the set where a nonzero polynomial vanishes. By Lemma 2.14 this set is connected.

4. Roots

Throughout this section, $\mathfrak{g}$ denotes a complex semisimple Lie algebra, B is its Killing form, and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. We saw in Proposition 2.10 that $\mathfrak{h}$ is abelian. The nonzero generalized weights of $\operatorname{ad}\mathfrak{h}$ on $\mathfrak{g}$ are called the **roots** of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We denote the set of roots by Δ or $\Delta(\mathfrak{g},\mathfrak{h})$. Then we can rewrite the weight-space decomposition of Proposition 2.5a as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha. \tag{2.16}$$

This decomposition is called the **root-space decomposition** of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Members of $\mathfrak{g}_\alpha$ are called **root vectors** for the root α.

Proposition 2.17.

(a) If α and β are in $\Delta\cup\{0\}$ and $\alpha+\beta\neq 0$, then $B(\mathfrak{g}_\alpha,\mathfrak{g}_\beta)=0$.

(b) If α is in $\Delta\cup\{0\}$, then B is nonsingular on $\mathfrak{g}_\alpha\times\mathfrak{g}_{-\alpha}$.

(c) If α is in Δ, then so is $-\alpha$.

(d) $B|_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate; consequently to each root α corresponds H_α in $\mathfrak{h}$ with $\alpha(H)=B(H,H_\alpha)$ for all $H\in\mathfrak{h}$.

(e) Δ spans $\mathfrak{h}^*$.

PROOF.

(a) By Proposition 2.5c, $\operatorname{ad}\mathfrak{g}_\alpha \operatorname{ad}\mathfrak{g}_\beta$ carries $\mathfrak{g}_\lambda$ into $\mathfrak{g}_{\lambda+\alpha+\beta}$ and consequently, when written as a matrix in terms of a basis of $\mathfrak{g}$ compatible with (2.16), has zero in every diagonal entry. Therefore its trace is 0.

(b) Since B is nondegenerate (Theorem 1.45), $B(X,\mathfrak{g})\neq 0$ for each $X\in\mathfrak{g}_\alpha$. Since (a) shows that $B(X,\mathfrak{g}_\beta)=0$ for every β other than $-\alpha$, we must have $B(X,\mathfrak{g}_{-\alpha})\neq 0$.

(c, d) These are immediate from (b).

(e) Suppose $H\in\mathfrak{h}$ has $\alpha(H)=0$ for all $\alpha\in\Delta$. By (2.16), $\operatorname{ad}H$ is nilpotent. Since $\mathfrak{h}$ is abelian, $\operatorname{ad}H\operatorname{ad}H'$ is nilpotent for all $H'\in\mathfrak{h}$. Therefore $B(H,\mathfrak{h})=0$. By (d), $H=0$. Consequently Δ spans $\mathfrak{h}^*$.

For each root α, choose and fix, by Lie's Theorem (Theorem 1.25) applied to the action of $\mathfrak{h}$ on $\mathfrak{g}_\alpha$, a vector $E_\alpha\neq 0$ in $\mathfrak{g}_\alpha$ with $[H,E_\alpha]=\alpha(H)E_\alpha$ for all $H\in\mathfrak{h}$.

Lemma 2.18.

(a) If α is a root and X is in $\mathfrak{g}_{-\alpha}$, then $[E_\alpha, X] = B(E_\alpha, X)H_\alpha$.
(b) If α and β are in Δ, then $\beta(H_\alpha)$ is a rational multiple of $\alpha(H_\alpha)$.
(c) If α is in Δ, then $\alpha(H_\alpha) \neq 0$.

PROOF.

(a) Since $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_0$ by Proposition 2.5c, $[E_\alpha, X]$ is in $\mathfrak{h}$. For H in $\mathfrak{h}$, we have

$$\begin{aligned} B([E_\alpha, X], H) &= -B(X, [E_\alpha, H]) = B(X, [H, E_\alpha]) \\ &= \alpha(H)B(X, E_\alpha) = B(H_\alpha, H)B(E_\alpha, X) \\ &= B(B(E_\alpha, X)H_\alpha, H). \end{aligned}$$

Then the conclusion follows from Proposition 2.17d.

(b) By Proposition 2.17b, we can choose $X_{-\alpha}$ in $\mathfrak{g}_{-\alpha}$ such that $B(E_\alpha, X_{-\alpha}) = 1$. Then (a) shows that

$$[E_\alpha, X_{-\alpha}] = H_\alpha. \tag{2.19}$$

With β fixed in Δ, let $\mathfrak{g}' = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{\beta+n\alpha}$. This subspace is invariant under $\operatorname{ad} H_\alpha$, and we shall compute the trace of $\operatorname{ad} H_\alpha$ on this subspace in two ways. Noting that $\operatorname{ad} H_\alpha$ acts on $\mathfrak{g}_{\beta+n\alpha}$ with the single generalized eigenvalue $(\beta + n\alpha)(H_\alpha)$ and adding the contribution to the trace over all values of n, we obtain

$$\sum_{n\in\mathbb{Z}} (\beta(H_\alpha) + n\alpha(H_\alpha)) \dim \mathfrak{g}_{\beta+n\alpha} \tag{2.20}$$

as the trace. On the other hand, Proposition 2.5c shows that $\mathfrak{g}'$ is invariant under $\operatorname{ad} E_\alpha$ and $\operatorname{ad} X_{-\alpha}$. By (2.19) the trace is

$$= \operatorname{Tr} \operatorname{ad} H_\alpha = \operatorname{Tr}(\operatorname{ad} E_\alpha \operatorname{ad} X_{-\alpha} - \operatorname{ad} X_{-\alpha} \operatorname{ad} E_\alpha) = 0.$$

Thus (2.20) equals 0, and the conclusion follows.

(c) Suppose $\alpha(H_\alpha) = 0$. By (b), $\beta(H_\alpha) = 0$ for all $\beta \in \Delta$. By Proposition 2.17e every member of $\mathfrak{h}^*$ vanishes on H_α. Thus $H_\alpha = 0$. But this conclusion contradicts Proposition 2.17d, since α is assumed to be nonzero.

Proposition 2.21. If α is in Δ, then $\dim \mathfrak{g}_\alpha = 1$. Also $n\alpha$ is not in Δ for any integer $n \geq 2$.

REMARK. Thus we no longer need to use the cumbersome condition $(\operatorname{ad} H - \alpha(H)1)^k X = 0$ for $X \in \mathfrak{g}_\alpha$ but can work with $k = 1$. Briefly

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid (\operatorname{ad} H)X = \alpha(H)X\}. \tag{2.22}$$

PROOF. As in the proof of Lemma 2.18b, we can choose $X_{-\alpha}$ in $\mathfrak{g}_{-\alpha}$ with $B(E_\alpha, X_{-\alpha}) = 1$ and obtain the bracket relation (2.19). Put $\mathfrak{g}'' = \mathbb{C}E_\alpha \oplus \mathbb{C}H_\alpha \oplus \bigoplus_{n<0} \mathfrak{g}_{n\alpha}$. This subspace is invariant under $\operatorname{ad} H_\alpha$ and $\operatorname{ad} E_\alpha$, by Proposition 2.5c, and it is invariant under $\operatorname{ad} X_{-\alpha}$ by Proposition 2.5c and Lemma 2.18a. By (2.19), $\operatorname{ad} H_\alpha$ has trace 0 in its action on $\mathfrak{g}''$. But $\operatorname{ad} H_\alpha$ acts on each summand with a single generalized eigenvalue, and thus the trace is

$$= \alpha(H_\alpha) + 0 + \sum_{n<0} n\alpha(H_\alpha) \dim \mathfrak{g}_{n\alpha} = 0.$$

Using Lemma 2.18c, we see that

$$\sum_{n=1}^{\infty} n \dim \mathfrak{g}_{-n\alpha} = 1.$$

Consequently $\dim \mathfrak{g}_{-\alpha} = 1$ and $\dim \mathfrak{g}_{-n\alpha} = 0$ for $n \geq 2$. Proposition 2.17c shows that we may replace α by $-\alpha$ everywhere in the above argument, and then we obtain the conclusion of the proposition.

Corollary 2.23. The action of $\operatorname{ad} \mathfrak{h}$ on $\mathfrak{g}$ is simultaneously diagonable.

REMARK. This corollary completes the promised converse to Proposition 2.13.

PROOF. This follows by combining (2.16), Proposition 2.10, and Proposition 2.21.

Corollary 2.24. On $\mathfrak{h} \times \mathfrak{h}$, the Killing form is given by

$$B(H, H') = \sum_{\alpha \in \Delta} \alpha(H)\alpha(H').$$

REMARK. This formula is a special property of the Killing form. By contrast the previous results of this section remain valid if B is replaced by any nondegenerate symmetric invariant bilinear form. We shall examine the role of special properties of B further when we come to Corollary 2.38.

PROOF. Let $\{H_i\}$ be a basis of $\mathfrak{h}$. By Proposition 2.21 and Corollary 2.23, $\{H_i\} \cup \{E_\alpha\}$ is a basis of $\mathfrak{g}$, and each $\operatorname{ad} H$ acts diagonally. Then $\operatorname{ad} H \operatorname{ad} H'$ acts diagonally, and the respective eigenvalues are 0 and $\{\alpha(H)\alpha(H')\}$. Hence

$$B(H, H') = \operatorname{Tr}(\operatorname{ad} H \operatorname{ad} H') = \sum_{\alpha \in \Delta} \alpha(H)\alpha(H').$$

Corollary 2.25. The pair of vectors $\{E_\alpha, E_{-\alpha}\}$ selected before Lemma 2.18 may be normalized so that $B(E_\alpha, E_{-\alpha}) = 1$.

PROOF. By Proposition 2.17b, $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are nonsingularly paired. Since Proposition 2.21 shows each of these spaces to be 1-dimensional, the result follows.

The above results may be interpreted as saying that $\mathfrak{g}$ is built out of copies of $\mathfrak{sl}(2, \mathbb{C})$ in a certain way. To see this, let E_α and $E_{-\alpha}$ be normalized as in Corollary 2.25. Then Lemma 2.18a gives us the bracket relations

$$\begin{aligned} [H_\alpha, E_\alpha] &= \alpha(H_\alpha)E_\alpha \\ [H_\alpha, E_{-\alpha}] &= -\alpha(H_\alpha)E_{-\alpha} \\ [E_\alpha, E_{-\alpha}] &= H_\alpha. \end{aligned}$$

We normalize these vectors suitably, for instance by

$$H'_\alpha = \frac{2}{\alpha(H_\alpha)} H_\alpha, \quad E'_\alpha = \frac{2}{\alpha(H_\alpha)} E_\alpha, \quad E'_{-\alpha} = E_{-\alpha}. \tag{2.26}$$

Then

$$\begin{aligned} [H'_\alpha, E'_\alpha] &= 2E'_\alpha \\ [H'_\alpha, E'_{-\alpha}] &= -2E'_{-\alpha} \\ [E'_\alpha, E'_{-\alpha}] &= H'_\alpha. \end{aligned}$$

As in (1.5) let us define elements of $\mathfrak{sl}(2, \mathbb{C})$ by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These satisfy

$$\begin{aligned} [h, e] &= 2e \\ [h, f] &= -2f \\ [e, f] &= h. \end{aligned}$$

Consequently

$$H'_\alpha \mapsto h, \quad E'_\alpha \mapsto e, \quad E'_{-\alpha} \mapsto f \tag{2.27}$$

extends linearly to an isomorphism of $\operatorname{span}\{H_\alpha, E_\alpha, E_{-\alpha}\}$ onto $\mathfrak{sl}(2,\mathbb{C})$. Thus $\mathfrak{g}$ is spanned by embedded copies of $\mathfrak{sl}(2,\mathbb{C})$. The detailed structure of $\mathfrak{g}$ comes by understanding how these copies of $\mathfrak{sl}(2,\mathbb{C})$ fit together. To investigate this question, we study the action of such an $\mathfrak{sl}(2,\mathbb{C})$ subalgebra on all of $\mathfrak{g}$, i.e., we study a complex-linear representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathfrak{g}$. We already know some invariant subspaces for this representation, and we study these one at a time.

Thus the representation to study is the one in the proof of Lemma 2.18b, with the version of $\mathfrak{sl}(2,\mathbb{C})$ built from a root α acting on the vector space $\mathfrak{g}' = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{\beta+n\alpha}$ by ad. Correspondingly we make the following definition of **root string**. Let α be in Δ, and let β be in $\Delta \cup \{0\}$. The α **string containing** β is the set of all members of $\Delta \cup \{0\}$ of the form $\beta + n\alpha$ for $n \in \mathbb{Z}$. Two examples of root strings appear in Figure 2.1.

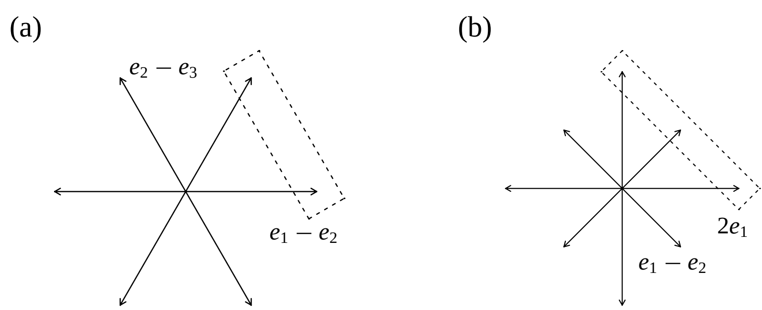


FIGURE 2.1. Root strings: (a) $e_2 - e_3$ string containing $e_1 - e_2$ for $\mathfrak{sl}(3,\mathbb{C})$, (b) $e_1 - e_2$ string through $2e_1$ for $\mathfrak{sp}(2,\mathbb{C})$

Also we transfer the restriction to $\mathfrak{h}$ of the Killing form to a bilinear form on the dual $\mathfrak{h}^*$ by the definition

$$\langle \varphi, \psi \rangle = B(H_\varphi, H_\psi) = \varphi(H_\psi) = \psi(H_\varphi) \tag{2.28}$$

for φ and ψ in $\mathfrak{h}^*$. Here H_φ and H_ψ are defined as in Proposition 2.17d.

Proposition 2.29. Let α be in Δ, and let β be in $\Delta \cup \{0\}$.

(a) The α string containing β has the form $\beta + n\alpha$ for $-p \le n \le q$ with $p \ge 0$ and $q \ge 0$. There are no gaps. Furthermore

$$p - q = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle},$$

and $\dfrac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ is in $\mathbb{Z}$.

(b) If $\beta+n\alpha$ is never 0, define $\mathfrak{sl}_\alpha$ to be the isomorphic copy of $\mathfrak{sl}(2,\mathbb{C})$ spanned by H'_α, E'_α, and $E'_{-\alpha}$ as in (2.26), and let $\mathfrak{g}'=\bigoplus_{n\in\mathbb{Z}}\mathfrak{g}_{\beta+n\alpha}$. Then the representation of $\mathfrak{sl}_\alpha$ on $\mathfrak{g}'$ by ad is irreducible.

PROOF. If $\beta+n\alpha=0$ for some n, then conclusion (a) follows from Proposition 2.21, and there is nothing to prove for (b). Thus we may assume that $\beta+n\alpha$ is never 0, and we shall prove (a) and (b) together.

By Proposition 2.21 the transformation ad H'_α is diagonable on $\mathfrak{g}'$ with distinct eigenvalues, and these eigenvalues are

$$\begin{aligned}(\beta+n\alpha)(H'_\alpha)&=\frac{2}{\langle\alpha,\alpha\rangle}\,(\beta+n\alpha)(H_\alpha)\\&=\frac{2}{\langle\alpha,\alpha\rangle}\,(\langle\beta,\alpha\rangle+n\langle\alpha,\alpha\rangle)\\&=\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}+2n.\end{aligned}\tag{2.30}$$

Thus any ad H'_α invariant subspace of $\mathfrak{g}'$ is a sum of certain $\mathfrak{g}_{\beta+n\alpha}$'s. Hence the same thing is true of any $\operatorname{ad}(\mathfrak{sl}_\alpha)$ invariant subspace.

Let V be an irreducible such subspace, and let $-p$ and q be the smallest and largest n's appearing for V. Theorem 1.66 shows that the eigenvalues of ad $h=$ ad H'_α in V are $N-2i$ with $0\le i\le N$, where $N=\dim V-1$. Since these eigenvalues jump by 2's, (2.30) shows that all n's between $-p$ and q are present. Also (2.30) gives

$$N=\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}+2q$$

and

$$-N=\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}-2p.$$

Adding, we obtain

$$p-q=\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}.\tag{2.31}$$

Theorem 1.67 shows that $\mathfrak{g}'$ is the direct sum of irreducible subspaces under $\mathfrak{sl}_\alpha$. If V' is another irreducible subspace, let $-p'$ and q' be the smallest and largest n's appearing for V'. Then (2.31), applied to V', gives

$$p'-q'=\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle},$$

so that

$$p' - q' = p - q. \tag{2.32}$$

On the other hand, all the n's from $-p$ to q are accounted for by V, and we must therefore have either $-p' > q$ or $q' < -p$. By symmetry we may assume that $-p' > q$. This inequality implies that

$$p' < -q \tag{2.33}$$

and that $q' \geq -p' > q \geq -p$. From the latter inequality we obtain

$$-q' < p. \tag{2.34}$$

Adding (2.33) and (2.34), we obtain a contradiction with (2.32), and the proposition follows.

Corollary 2.35. If α and β are in $\Delta \cup \{0\}$ and $\alpha + \beta \neq 0$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

PROOF. Without loss of generality, let $\alpha \neq 0$. Proposition 2.5c shows that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}. \tag{2.36}$$

We are to prove that equality holds in (2.36) We consider cases.

If β is an integral multiple of α and is not equal to $-\alpha$, then Proposition 2.21 shows that β must be α or 0. If $\beta = \alpha$, then $\mathfrak{g}_{\alpha+\beta} = 0$ by Proposition 2.21, and hence equality must hold in (2.36). If $\beta = 0$, then the equality $[\mathfrak{h}, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$ says that equality holds in (2.36).

If β is not an integral multiple of α, then Proposition 2.29b is applicable and shows that $\mathfrak{sl}_\alpha$ acts irreducibly on $\mathfrak{g}' = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{\beta+n\alpha}$. Making the identification (2.27) and matching data with Theorem 1.66, we see that the root vectors $E_{\beta+n\alpha}$, except for constant factors, are the vectors v_i of Theorem 1.66. The only i for which $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ maps v_i to 0 is $i = 0$, and v_0 corresponds to $E_{\beta+q\alpha}$. Thus $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ forces $q = 0$ and says that $\beta+\alpha$ is not a root. In this case, $\mathfrak{g}_{\alpha+\beta} = 0$, and equality must hold in (2.36).

Corollary 2.37. Let α and β be roots such that $\beta + n\alpha$ is never 0 for $n \in \mathbb{Z}$. Let E_α, $E_{-\alpha}$, and E_β be any root vectors for α, $-\alpha$, and β, respectively, and let p and q be the integers in Proposition 2.29a. Then

$$[E_{-\alpha}, [E_\alpha, E_\beta]] = \frac{q(1+p)}{2}\alpha(H_\alpha)B(E_\alpha, E_{-\alpha})E_\beta.$$

PROOF. Both sides are linear in E_α and $E_{-\alpha}$, and we may therefore normalize them as in Corollary 2.25 so that $B(E_\alpha, E_{-\alpha}) = 1$. If we then make the identification (2.27) of the span of $\{H_\alpha,\ E_\alpha,\ E_{-\alpha}\}$ with $\mathfrak{sl}(2,\mathbb{C})$, we can reinterpret the desired formula as

$$\frac{\langle\alpha,\alpha\rangle}{2}\,[f,[e,E_\beta]] \overset{?}{=} \frac{q(1+p)}{2}\,\alpha(H_\alpha)E_\beta,$$

i.e., as

$$[f,[e,E_\beta]] \overset{?}{=} q(1+p)E_\beta.$$

From Proposition 2.29b, the action of the span of $\{h,\ e,\ f\}$ on $\mathfrak{g}'$ is irreducible. The vector $E_{\beta+q\alpha}$ corresponds to a multiple of the vector v_0 in Theorem 1.66. Since E_β is a multiple of $(\operatorname{ad} f)^q E_{\beta+q\alpha}$, E_β corresponds to a multiple of v_q. By (d) and then (c) in Theorem 1.66, we obtain

$$(\operatorname{ad} f)(\operatorname{ad} e)E_\beta = q(N-q+1)E_\beta,$$

where $N = \dim\mathfrak{g}' - 1 = (q+p+1)-1$. Then $q(N-q+1) = q(1+p)$, and the result follows.

Corollary 2.38. Let V be the $\mathbb{R}$ linear span of Δ in $\mathfrak{h}^*$. Then V is a real form of the vector space $\mathfrak{h}^*$, and the restriction of the bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle$ to $V\times V$ is a positive-definite inner product. Moreover, if $\mathfrak{h}_0$ denotes the $\mathbb{R}$ linear span of all H_α for $\alpha\in\Delta$, then $\mathfrak{h}_0$ is a real form of the vector space $\mathfrak{h}$, the members of V are exactly those linear functionals that are real on $\mathfrak{h}_0$, and restriction of the operation of those linear functionals from $\mathfrak{h}$ to $\mathfrak{h}_0$ is an $\mathbb{R}$ isomorphism of V onto $\mathfrak{h}_0^*$.

REMARK. The proof will make use of Corollary 2.24, which was the only result so far that used any properties of the Killing form other than that B is a nondegenerate symmetric invariant bilinear form. The present corollary will show that B is positive definite on $\mathfrak{h}_0$, and then Corollary 2.24 will no longer be needed. The remaining theory for complex semisimple Lie algebras in this chapter goes through if B is replaced by any nondegenerate symmetric invariant bilinear form that is positive definite on $\mathfrak{h}_0$. Because of Theorem 2.15, once such a form B is positive definite on the real form $\mathfrak{h}_0$ of the Cartan subalgebra $\mathfrak{h}$, it is positive definite on the corresponding real form of any other Cartan subalgebra.

PROOF. Combining Corollary 2.24 with the definition (2.28), we obtain

$$\langle\varphi,\psi\rangle = B(H_\varphi,H_\psi) = \sum_{\beta\in\Delta}\beta(H_\varphi)\beta(H_\psi) = \sum_{\beta\in\Delta}\langle\beta,\varphi\rangle\langle\beta,\psi\rangle \tag{2.39}$$

for all φ and ψ in $\mathfrak{h}^*$. Let α be a root, and let p_β and q_β be the integers p and q associated to the α string containing β in Proposition 2.29a. Specializing (2.39) to $\varphi = \psi = \alpha$ gives

$$\langle\alpha, \alpha\rangle = \sum_{\beta\in\Delta}\langle\beta, \alpha\rangle^2 = \sum_{\beta\in\Delta}[(p_\beta - q_\beta)\tfrac{1}{2}\langle\alpha, \alpha\rangle]^2.$$

Since $\langle\alpha, \alpha\rangle \neq 0$ according to Lemma 2.18c, we obtain

$$\langle\alpha, \alpha\rangle = \frac{4}{\sum_{\beta\in\Delta}(p_\beta - q_\beta)^2},$$

and therefore $\langle\alpha, \alpha\rangle$ is rational. By Lemma 2.18b,

$$\beta(H_\alpha) \text{ is rational for all } \alpha \text{ and } \beta \text{ in } \Delta. \tag{2.40}$$

Let $\dim_{\mathbb{C}} \mathfrak{h} = l$. By Proposition 2.17e we can choose l roots $\alpha_1, \dots, \alpha_l$ such that $H_{\alpha_1}, \dots, H_{\alpha_l}$ is a basis of $\mathfrak{h}$ over $\mathbb{C}$. Let $\omega_1, \dots, \omega_l$ be the dual basis of $\mathfrak{h}^*$ satisfying $\omega_i(H_{\alpha_j}) = \delta_{ij}$, and let V be the real vector space of all members of $\mathfrak{h}^*$ that are real on all of $H_{\alpha_1}, \dots, H_{\alpha_l}$. Then $V = \bigoplus_{j=1}^l \mathbb{R}\omega_j$, and it follows that V is a real form of the vector space $\mathfrak{h}^*$. By (2.40) all roots are in V. Since $\alpha_1, \dots, \alpha_l$ are already linearly independent over $\mathbb{R}$, we conclude that V is the $\mathbb{R}$ linear span of the roots.

If φ is in V, then $\varphi(H_\beta)$ is real for each root β. Since (2.39) gives

$$\langle\varphi, \varphi\rangle = \sum_{\beta\in\Delta}\langle\beta, \varphi\rangle^2 = \sum_{\beta\in\Delta}\varphi(H_\beta)^2,$$

we see that the restriction of $\langle\,\cdot\,,\,\cdot\,\rangle$ to $V \times V$ is a positive-definite inner product.

Now let $\mathfrak{h}_0$ denote the $\mathbb{R}$ linear span of all H_α for $\alpha \in \Delta$. Since $\varphi \mapsto H_\varphi$ is an isomorphism of $\mathfrak{h}^*$ with $\mathfrak{h}$ carrying V to $\mathfrak{h}_0$, it follows that $\mathfrak{h}_0$ is a real form of $\mathfrak{h}$. We know that the real linear span of the roots (namely V) has real dimension l, and consequently the real linear span of all H_α for $\alpha \in \Delta$ has real dimension l. Since $H_{\alpha_1}, \dots, H_{\alpha_l}$ is linearly independent over $\mathbb{R}$, it is a basis of $\mathfrak{h}_0$ over $\mathbb{R}$. Hence V is the set of members of $\mathfrak{h}^*$ that are real on all of $\mathfrak{h}_0$. Therefore restriction from $\mathfrak{h}$ to $\mathfrak{h}_0$ is a vector-space isomorphism of V onto $\mathfrak{h}_0^*$.

Let $|\cdot|^2$ denote the norm squared associated to the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathfrak{h}_0^* \times \mathfrak{h}_0^*$. Let α be a root. Relative to the inner product, we introduce the **root reflection**

$$s_\alpha(\varphi) = \varphi - \frac{2\langle\varphi, \alpha\rangle}{|\alpha|^2}\alpha \qquad \text{for } \varphi \in \mathfrak{h}_0^*.$$

This is an orthogonal transformation on $\mathfrak{h}_0^*$, is -1 on $\mathbb{R}\alpha$, and is $+1$ on the orthogonal complement of α.

Proposition 2.41. For any root α, the root reflection s_α carries Δ into itself.

PROOF. Let β be in Δ, and let p and q be as in Proposition 2.29a. Then

$$s_\alpha\beta = \beta - \frac{2\langle\beta,\alpha\rangle}{|\alpha|^2}\,\alpha = \beta - (p-q)\alpha = \beta + (q-p)\alpha.$$

Since $-p \le q-p \le q$, $\beta + (q-p)\alpha$ is in the α string containing β. Hence $s_\alpha\beta$ is a root or is 0. Since s_α is an orthogonal transformation on $\mathfrak{h}_0^*$, $s_\alpha\beta$ is not 0. Thus s_α carries Δ into Δ.

5. Abstract Root Systems

To examine roots further, it is convenient to abstract the results we have obtained so far. This approach will allow us to work more easily toward a classification of complex semisimple Lie algebras and also to apply the theory of roots in a different situation that will arise in Chapter VI.

An **abstract root system** in a finite-dimensional real inner product space V with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ and norm squared $|\cdot|^2$ is a finite set Δ of nonzero elements of V such that

(i) Δ spans V,

(ii) the orthogonal transformations $s_\alpha(\varphi) = \varphi - \dfrac{2\langle\varphi,\alpha\rangle}{|\alpha|^2}\,\alpha$, for $\alpha \in \Delta$, carry Δ to itself,

(iii) $\dfrac{2\langle\beta,\alpha\rangle}{|\alpha|^2}$ is an integer whenever α and β are in Δ.

An abstract root system is said to be **reduced** if $\alpha \in \Delta$ implies $2\alpha \notin \Delta$. Much of what we saw in §4 can be summarized in the following theorem.

Theorem 2.42. The root system of a complex semisimple Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ forms a reduced abstract root system in $\mathfrak{h}_0^*$.

PROOF. With $V = \mathfrak{h}_0^*$, V is an inner product space spanned by Δ as a consequence of Corollary 2.38. Property (ii) follows from Proposition 2.41, and property (iii) follows from Proposition 2.29a. According to Proposition 2.21, the abstract root system Δ is reduced.

As a consequence of the theorem, the examples of §1 give us many examples of reduced abstract root systems. We recall them here and tell what names we shall use for them:

(2.43)

	Vector Space	Root System	$\mathfrak{g}$
A_n	$V = \left\{ \begin{array}{l} \sum_{i=1}^{n+1} a_i e_i \\ \text{with} \\ \sum a_i e_i = 0 \end{array} \right\}$	$\Delta = \{e_i - e_j \mid i \neq j\}$	$\mathfrak{sl}(n+1, \mathbb{C})$
B_n	$V = \left\{ \sum_{i=1}^{n} a_i e_i \right\}$	$\Delta = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm e_i\}$	$\mathfrak{so}(2n+1, \mathbb{C})$
C_n	$V = \left\{ \sum_{i=1}^{n} a_i e_i \right\}$	$\Delta = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm 2e_i\}$	$\mathfrak{sp}(n, \mathbb{C})$
D_n	$V = \left\{ \sum_{i=1}^{n} a_i e_i \right\}$	$\Delta = \{\pm e_i \pm e_j \mid i \neq j\}$	$\mathfrak{so}(2n, \mathbb{C})$

Some 2-dimensional examples of abstract root systems are given in Figure 2.2. All but $(BC)_2$ are reduced. The system $A_1 \oplus A_1$ arises as the root system for $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$.

We say that two abstract root systems Δ in V and Δ' in V' are **isomorphic** if there is a vector-space isomorphism of V onto V' carrying Δ onto Δ' and preserving the integers $2\langle \beta, \alpha \rangle / |\alpha|^2$ for α and β in Δ. The systems B_2 and C_2 in Figure 2.2 are isomorphic.

An abstract root system Δ is said to be **reducible** if Δ admits a nontrivial disjoint decomposition $\Delta = \Delta' \cup \Delta''$ with every member of Δ' orthogonal to every member of Δ''. We say that Δ is **irreducible** if it admits no such nontrivial decomposition. In Figure 2.2 all the abstract root systems are irreducible except $A_1 \oplus A_1$. The fact that this root system comes from a complex semisimple Lie algebra that is not simple generalizes as in Proposition 2.44 below.

Proposition 2.44. The root system Δ of a complex semisimple Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ is irreducible as an abstract reduced root system if and only if $\mathfrak{g}$ is simple.

PROOF THAT Δ IRREDUCIBLE IMPLIES $\mathfrak{g}$ SIMPLE. Suppose that $\mathfrak{g}$ is a nontrivial direct sum of ideals $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$. Let α be a root, and decompose the corresponding root vector E_α accordingly as $E_\alpha = E'_\alpha + E''_\alpha$. For H in $\mathfrak{h}$, we have

$$0 = [H, E_\alpha] - \alpha(H)E_\alpha = ([H, E'_\alpha] - \alpha(H)E'_\alpha) + ([H, E''_\alpha] - \alpha(H)E''_\alpha).$$

Since $\mathfrak{g}'$ and $\mathfrak{g}''$ are ideals and have 0 intersection, the two terms on the right are separately 0. Thus E'_α and E''_α are both in the root space $\mathfrak{g}_\alpha$. Since $\dim \mathfrak{g}_\alpha = 1$, $E'_\alpha = 0$ or $E''_\alpha = 0$. Thus $\mathfrak{g}_\alpha \subseteq \mathfrak{g}'$ or $\mathfrak{g}_\alpha \subseteq \mathfrak{g}''$. Define

$$\begin{aligned} \Delta' &= \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subseteq \mathfrak{g}'\} \\ \Delta'' &= \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subseteq \mathfrak{g}''\}. \end{aligned} \tag{2.45}$$

What we have just shown about (2.45) is that $\Delta = \Delta' \cup \Delta''$ disjointly. Now with obvious notation we have

$$\alpha'(H_{\alpha''})E_{\alpha'} = [H_{\alpha''}, E_{\alpha'}] \subseteq [H_{\alpha''}, \mathfrak{g}'] = [[E_{\alpha''}, E_{-\alpha''}], \mathfrak{g}'] \subseteq [\mathfrak{g}'', \mathfrak{g}'] = 0,$$

and thus $\alpha'(H_{\alpha''}) = 0$. Hence Δ' and Δ'' are mutually orthogonal.

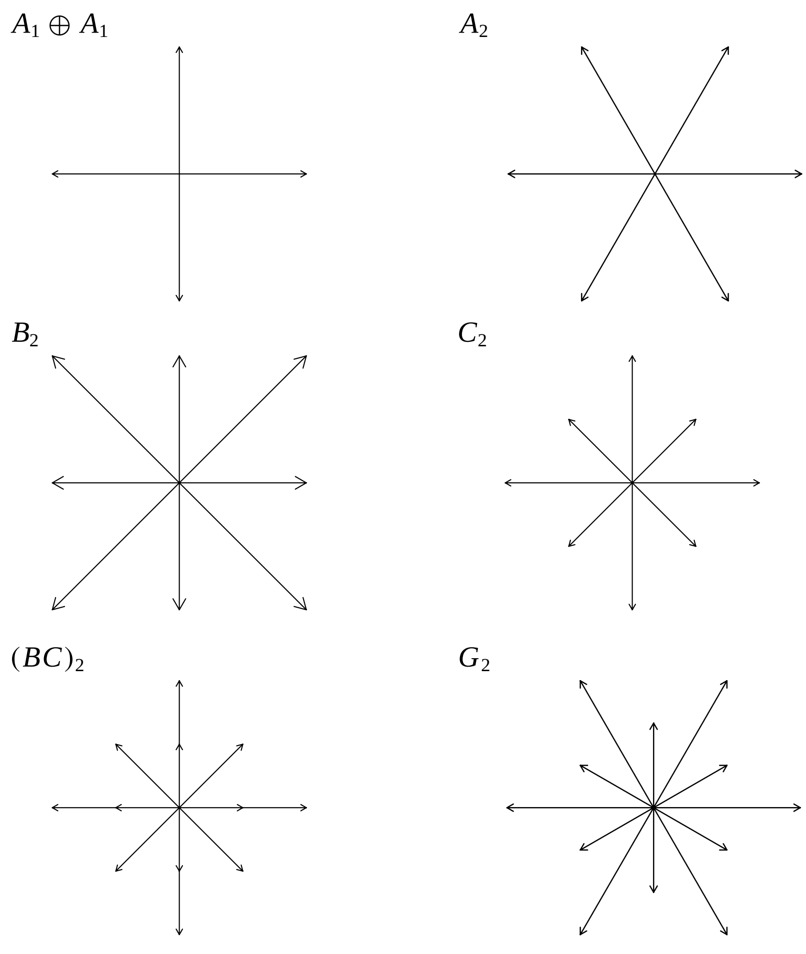


FIGURE 2.2. Abstract root systems with $V = \mathbb{R}^2$

PROOF THAT $\mathfrak{g}$ SIMPLE IMPLIES Δ IRREDUCIBLE. Suppose that $\Delta = \Delta' \cup \Delta''$ exhibits Δ as reducible. Define

$$\mathfrak{g}' = \sum_{\alpha \in \Delta'} \{\mathbb{C}H_\alpha + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}\}$$
$$\mathfrak{g}'' = \sum_{\alpha \in \Delta''} \{\mathbb{C}H_\alpha + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}\}.$$

Then $\mathfrak{g}'$ and $\mathfrak{g}''$ are vector subspaces of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$ as vector spaces. To complete the proof, it is enough to show that $\mathfrak{g}'$ and $\mathfrak{g}''$ are ideals in $\mathfrak{g}$. It is clear that they are Lie subalgebras. For α' in Δ' and α'' in Δ'', we have

$$[H_{\alpha'}, E_{\alpha''}] = \alpha''(H_{\alpha'})E_{\alpha''} = 0 \tag{2.46}$$

by the assumed orthogonality. Also if $[\mathfrak{g}_{\alpha'}, \mathfrak{g}_{\alpha''}] \neq 0$, then $\alpha' + \alpha''$ is a root that is not orthogonal to every member of Δ' (α' for instance) and is not orthogonal to every member of Δ'' (α'' for instance), in contradiction with the given orthogonal decomposition of Δ. We conclude that

$$[\mathfrak{g}_{\alpha'}, \mathfrak{g}_{\alpha''}] = 0. \tag{2.47}$$

Combining (2.46) and (2.47), we see that $[\mathfrak{g}', \mathfrak{g}_{\alpha''}] = 0$. Since $[\mathfrak{g}', \mathfrak{h}] \subseteq \mathfrak{g}'$ and since $\mathfrak{g}'$ is a subalgebra, $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$. Similarly $\mathfrak{g}''$ is an ideal. This completes the proof.

EXAMPLE. Let $\mathfrak{g} = \mathfrak{so}(4, \mathbb{C})$ with notation as in §1. The root system is $\Delta = \{\pm e_1 \pm e_2\}$. If we put $\Delta' = \{\pm(e_1 - e_2)\}$ and $\Delta'' = \{\pm(e_1 + e_2)\}$, then $\Delta = \Delta' \cup \Delta''$ exhibits Δ as reducible. By Proposition 2.44, $\mathfrak{so}(4, \mathbb{C})$ is not simple. The root system is isomorphic to $A_1 \oplus A_1$.

We extend our earlier definition of **root string** to the context of an abstract root system Δ. For $\alpha \in \Delta$ and $\beta \in \Delta \cup \{0\}$, the α **string containing** β is the set of all members of $\Delta \cup \{0\}$ of the form $\beta + n\alpha$ with $n \in \mathbb{Z}$. Figure 2.1 in §4 showed examples of root strings. In the system G_2 as pictured in Figure 2.2, there are root strings containing four roots.

If α is a root and $\frac{1}{2}\alpha$ is not a root, we say that α is **reduced**.

Proposition 2.48. Let Δ be an abstract root system in the inner product space V.

(a) If α is in Δ, then $-\alpha$ is in Δ.

(b) If α is in Δ and is reduced, then the only members of $\Delta \cup \{0\}$ proportional to α are $\pm\alpha$, $\pm 2\alpha$, and 0, and $\pm 2\alpha$ cannot occur if Δ is reduced.

(c) If α is in Δ and β is in $\Delta \cup \{0\}$, then

$$\frac{2\langle \beta, \alpha\rangle}{|\alpha|^2} = 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \text{or}\ \pm 4,$$

and ± 4 occurs only in a nonreduced system with $\beta = \pm 2\alpha$.

(d) If α and β are nonproportional members of Δ such that $|\alpha| \le |\beta|$, then $\dfrac{2\langle \beta, \alpha\rangle}{|\beta|^2}$ equals 0 or $+1$ or -1.

(e) If α and β are in Δ with $\langle \alpha, \beta\rangle > 0$, then $\alpha - \beta$ is a root or 0. If α and β are in Δ with $\langle \alpha, \beta\rangle < 0$, then $\alpha + \beta$ is a root or 0.

(f) If α and β are in Δ and neither $\alpha + \beta$ nor $\alpha - \beta$ is in $\Delta \cup \{0\}$, then $\langle \alpha, \beta\rangle = 0$.

(g) If α is in Δ and β is in $\Delta \cup \{0\}$, then the α string containing β has the form $\beta + n\alpha$ for $-p \le n \le q$ with $p \ge 0$ and $q \ge 0$. There are no gaps. Furthermore $p - q = \dfrac{2\langle \beta, \alpha\rangle}{|\alpha|^2}$. The α string containing β contains at most four roots.

PROOF.

(a) This follows since $s_\alpha(\alpha) = -\alpha$.

(b) Let α be in Δ, and let $c\alpha$ be in $\Delta \cup \{0\}$. We may assume that $c \ne 0$. Then $2\langle c\alpha, \alpha\rangle/|\alpha|^2$ and $2\langle \alpha, c\alpha\rangle/|c\alpha|^2$ are both integers, from which it follows that $2c$ and $2/c$ are integers. Since $c \ne \pm\frac{1}{2}$, the only possibilities are $c = \pm 1$ and $c = \pm 2$, as asserted. If Δ is reduced, $c = \pm 2$ cannot occur.

(c) We may assume that $\beta \ne 0$. From the Schwarz inequality we have

$$\left| \frac{2\langle \alpha, \beta\rangle}{|\alpha|^2} \frac{2\langle \alpha, \beta\rangle}{|\beta|^2} \right| \le 4$$

with equality only if $\beta = c\alpha$. The case of equality is handled by (b). If strict equality holds, then $\dfrac{2\langle \alpha, \beta\rangle}{|\alpha|^2}$ and $\dfrac{2\langle \alpha, \beta\rangle}{|\beta|^2}$ are two integers whose product is ≤ 3 in absolute value. The result follows in either case.

(d) We have an inequality of integers

$$\left| \frac{2\langle \alpha, \beta\rangle}{|\alpha|^2} \right| \ge \left| \frac{2\langle \alpha, \beta\rangle}{|\beta|^2} \right|,$$

and the proof of (c) shows that the product of the two sides is ≤ 3. Therefore the smaller side is 0 or 1.

(e) We may assume that α and β are not proportional. For the first statement, assume that $|\alpha| \le |\beta|$. Then $s_\beta(\alpha) = \alpha - \dfrac{2\langle\alpha,\beta\rangle}{|\beta|^2}\beta$ must be $\alpha-\beta$, by (d). So $\alpha-\beta$ is in Δ. If $|\beta| \le |\alpha|$ instead, we find that $s_\alpha(\beta) = \beta-\alpha$ is in Δ, and then $\alpha-\beta$ is in Δ as a consequence of (a). For the second statement we apply the first statement to $-\alpha$.

(f) This is immediate from (e).

(g) Let $-p$ and q be the smallest and largest values of n such that $\beta+n\alpha$ is in $\Delta\cup\{0\}$. If the string has a gap, we can find r and s with $r < s-1$ such that $\beta+r\alpha$ is in $\Delta\cup\{0\}$, $\beta+(r+1)\alpha$ and $\beta+(s-1)\alpha$ are not in $\Delta\cup\{0\}$, and $\beta+s\alpha$ is in $\Delta\cup\{0\}$. By (e),

$$\langle\beta+r\alpha,\alpha\rangle \ge 0 \qquad \text{and} \qquad \langle\beta+s\alpha,\alpha\rangle \le 0.$$

Subtracting these inequalities, we obtain $(r-s)|\alpha|^2 \ge 0$, and thus $r \ge s$, contradiction. We conclude that there are no gaps. Next

$$s_\alpha(\beta+n\alpha) = \beta+n\alpha - \frac{2\langle\beta+n\alpha,\alpha\rangle}{|\alpha|^2}\alpha = \beta - \left(n+\frac{2\langle\beta,\alpha\rangle}{|\alpha|^2}\right)\alpha,$$

and thus $-p \le n \le q$ implies $-q \le n + \dfrac{2\langle\beta,\alpha\rangle}{|\alpha|^2} \le p$. Taking $n=q$ and then $n=-p$, we obtain in turn

$$\frac{2\langle\beta,\alpha\rangle}{|\alpha|^2} \le p-q \quad \text{and then} \quad p-q \le \frac{2\langle\beta,\alpha\rangle}{|\alpha|^2}.$$

Thus $2\langle\beta,\alpha\rangle/|\alpha|^2 = p-q$. Finally, to investigate the length of the string, we may assume $q=0$. The length of the string is then $p+1$, with $p = 2\langle\beta,\alpha\rangle/|\alpha|^2$. The conclusion that the string has at most four roots then follows from (c) and (b).

We now introduce a notion of positivity in V that extends the notion in the examples in §1. The intention is to single out a subset of nonzero elements of V as **positive**, writing $\varphi > 0$ if φ is a positive element. The only properties of positivity that we need are that

(i) for any nonzero $\varphi \in V$, exactly one of φ and $-\varphi$ is positive,

(ii) the sum of positive elements is positive, and any positive multiple of a positive element is positive.

The way in which such a notion of positivity is introduced is not important, and we shall give a sample construction shortly.

We say that $\varphi > \psi$ or $\psi < \varphi$ if $\varphi - \psi$ is positive. Then $>$ defines a simple ordering on V that is preserved under addition and under multiplication by positive scalars.

One way to define positivity is by means of a **lexicographic ordering**. Fix a spanning set $\varphi_1, \dots, \varphi_m$ of V, and define positivity as follows: We say that $\varphi > 0$ if there exists an index k such that $\langle \varphi, \varphi_i \rangle = 0$ for $1 \le i \le k-1$ and $\langle \varphi, \varphi_k \rangle > 0$.

A lexicographic ordering sometimes arises disguised in a kind of dual setting. To use notation consistent with applications, think of V as the vector space dual of a space $\mathfrak{h}_0$, and fix a spanning set $H_1, \dots, H_m$ for $\mathfrak{h}_0$. Then we say that $\varphi > 0$ if there exists an index k such that $\varphi(H_i) = 0$ for $1 \le i \le k-1$ and $\varphi(H_k) > 0$.

Anyway, we fix a notion of positivity and the resulting ordering for V. We say that a root α is **simple** if $\alpha > 0$ and if α does not decompose as $\alpha = \beta_1 + \beta_2$ with β_1 and β_2 both positive roots. A simple root is necessarily reduced.

Proposition 2.49. With $l = \dim V$, there are l simple roots $\alpha_1, \dots, \alpha_l$, and they are linearly independent. If β is a root and is written as $\beta = x_1\alpha_1 + \cdots + x_l\alpha_l$, then all the x_j have the same sign (if 0 is allowed to be positive or negative), and all the x_j are integers.

REMARKS. Once this proposition has been proved, any positive root α can be written as $\alpha = \sum_{i=1}^{l} n_i\alpha_i$ with each n_i an integer ≥ 0. The integer $\sum_{i=1}^{l} n_i$ is called the **level** of α relative to $\{\alpha_1, \dots, \alpha_l\}$ and is sometimes used in inductive proofs. The first example of such a proof will be with Proposition 2.54 below.

(2.50)

	Positive Roots	Simple Roots
A_n	$e_i - e_j,\ i < j$	$e_1 - e_2,\ e_2 - e_3, \dots, e_n - e_{n+1}$
B_n	$e_i \pm e_j$ with $i < j$ e_i	$e_1 - e_2,\ e_2 - e_3, \dots, e_{n-1} - e_n,\ e_n$
C_n	$e_i \pm e_j$ with $i < j$ $2e_i$	$e_1 - e_2,\ e_2 - e_3, \dots, e_{n-1} - e_n,\ 2e_n$
D_n	$e_i \pm e_j$ with $i < j$	$e_1 - e_2, \dots, e_{n-2} - e_{n-1},\ e_{n-1} - e_n,\ e_{n-1} + e_n$

Before coming to the proof, let us review the examples in (2.43), which came from the complex semisimple Lie algebras in §1. In (2.50) we recall the choice of positive roots we made in §1 for each example and tell what the corresponding simple roots are.

Lemma 2.51. If α and β are distinct simple roots, then $\alpha - \beta$ is not a root. Hence $\langle \alpha, \beta \rangle \leq 0$.

PROOF. Assuming the contrary, suppose that $\alpha - \beta$ is a root. If $\alpha - \beta$ is positive, then $\alpha = (\alpha - \beta) + \beta$ exhibits α as a nontrivial sum of positive roots. If $\alpha - \beta$ is negative, then $\beta = (\beta - \alpha) + \alpha$ exhibits β as a nontrivial sum of positive roots. In either case we have a contradiction. Thus $\alpha - \beta$ is not a root, and Proposition 2.48e shows that $\langle \alpha, \beta \rangle \leq 0$.

PROOF OF PROPOSITION 2.49. Let $\beta > 0$ be in Δ. If β is not simple, write $\beta = \beta_1 + \beta_2$ with β_1 and β_2 both positive in Δ. Then decompose β_1 and/or β_2, and then decompose each of their components if possible. Continue in this way. We can list the decompositions as tuples (β, β_1, component of β_1, etc.) with each entry a component of the previous entry. The claim is that no tuple has more entries than there are positive roots, and therefore the decomposition process must stop. In fact, otherwise some tuple would have the same $\gamma > 0$ in it at least twice, and we would have $\gamma = \gamma + \alpha$ with α a nonempty sum of positive roots, contradicting the properties of an ordering. Thus β is exhibited as $\beta = x_1\alpha_1 + \cdots + x_m\alpha_m$ with all x_j positive integers or 0 and with all α_j simple. Thus the simple roots span in the fashion asserted.

Finally we prove linear independence. Renumbering the α_j's, suppose that

$$x_1\alpha_1 + \cdots + x_s\alpha_s - x_{s+1}\alpha_{s+1} - \cdots - x_m\alpha_m = 0$$

with all $x_j \geq 0$ in $\mathbb{R}$. Put $\beta = x_1\alpha_1 + \cdots + x_s\alpha_s$. Then

$$0 \leq \langle \beta, \beta \rangle = \left\langle \sum_{j=1}^{s} x_j\alpha_j, \sum_{k=s+1}^{m} x_k\alpha_k \right\rangle = \sum_{j,k} x_j x_k \langle \alpha_j, \alpha_k \rangle \leq 0.$$

the last inequality holding by Lemma 2.51. We conclude that $\langle \beta, \beta \rangle = 0$, $\beta = 0$, and all the x_j's equal 0 since a positive combination of positive roots cannot be 0.

For the remainder of this section, we fix an abstract root system Δ, and we assume that Δ is reduced. Fix also an ordering coming from a notion of positivity as above, and let Π be the set of simple roots. We shall associate a "Cartan matrix" to the system Π and note some of the properties of this matrix. An "abstract Cartan matrix" will be any square matrix with this list of properties. Working with an abstract Cartan matrix is made easier by associating to the matrix a kind of graph known as an "abstract Dynkin diagram."

Enumerate Π as $\Pi = \{\alpha_1, \dots, \alpha_l\}$, where $l = \dim V$. The l-by-l matrix $A = (A_{ij})$ given by

$$A_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{|\alpha_i|^2}$$

is called the **Cartan matrix** of Δ and Π. The Cartan matrix depends on the enumeration of Π, and distinct enumerations evidently lead to Cartan matrices that are conjugate to one another by a permutation matrix.

For the examples in Figure 2.2 with $\dim V = 2$, the Cartan matrices are of course 2-by-2 matrices. For all the examples except G_2, an enumeration of the simple roots is given in (2.50). For G_2 let us agree to list the short simple root first. Then the Cartan matrices are as follows:

$$\begin{array}{cc}
A_1 \oplus A_1 & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\
A_2 & \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \\
B_2 & \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \\
C_2 & \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \\
G_2 & \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}
\end{array}$$

Proposition 2.52. The Cartan matrix $A = (A_{ij})$ of Δ relative to the set Π of simple roots has the following properties:

(a) A_{ij} is in $\mathbb{Z}$ for all i and j,
(b) $A_{ii} = 2$ for all i,
(c) $A_{ij} \le 0$ for $i \ne j$,
(d) $A_{ij} = 0$ if and only if $A_{ji} = 0$,
(e) there exists a diagonal matrix D with positive diagonal entries such that DAD^{-1} is symmetric positive definite.

PROOF. Properties (a), (b), and (d) are trivial, and (c) follows from Lemma 2.51. Let us prove (e). Put

$$D = \mathrm{diag}(|\alpha_1|, \dots, |\alpha_l|), \tag{2.53}$$

so that $DAD^{-1} = \left(2\left\langle \frac{\alpha_i}{|\alpha_i|}, \frac{\alpha_j}{|\alpha_j|}\right\rangle\right)$. This is symmetric, and we can discard the 2 in checking positivity. But $(\langle\varphi_i, \varphi_j\rangle)$ is positive definite whenever $\{\varphi_i\}$ is a basis, since

$$\begin{pmatrix} c_1 & \cdots & c_l \end{pmatrix} \big(\langle\varphi_i, \varphi_j\rangle\big) \begin{pmatrix} c_1 \\ \vdots \\ c_l \end{pmatrix} = \Big|\sum_i c_i\varphi_i\Big|^2.$$

The normalized simple roots may be taken as the basis φ_i of V, according to Proposition 2.49, and the result follows.

A square matrix A satisfying properties (a) through (e) in Proposition 2.52 will be called an **abstract Cartan matrix**. Two abstract Cartan matrices are **isomorphic** if one is conjugate to the other by a permutation matrix.

Proposition 2.54. The abstract reduced root system Δ is reducible if and only if, for some enumeration of the indices, the Cartan matrix is block diagonal with more than one block.

PROOF. Suppose that $\Delta = \Delta' \cup \Delta''$ disjointly with every member of Δ' orthogonal to every member of Δ''. We enumerate the simple roots by listing all those in Δ' before all those in Δ'', and then the Cartan matrix is block diagonal.

Conversely suppose that the Cartan matrix is block diagonal, with the simple roots $\alpha_1, \dots, \alpha_s$ leading to one block and the simple roots $\alpha_{s+1}, \dots, \alpha_l$ leading to another block. Let Δ' be the set of all roots whose expansion in terms of the basis $\alpha_1, \dots, \alpha_l$ involves only $\alpha_1, \dots, \alpha_s$, and let Δ'' be the set of all roots whose expansion involves only $\alpha_{s+1}, \dots, \alpha_l$. Then Δ' and Δ'' are nonempty and are orthogonal to each other, and it is enough to show that their union is Δ. Let $\alpha \in \Delta$ be given, and write $\alpha = \sum_{i=1}^{l} n_i\alpha_i$. We are to show that either $n_i = 0$ for $i > s$ or $n_i = 0$ for $i \le s$. Proposition 2.49 says that all the n_i are integers and they have the same sign. Without loss of generality we may assume that α is positive, so that all n_i are ≥ 0.

We proceed by induction on the level $\sum_{i=1}^{l} n_i$. If the sum is 1, then $\alpha = \alpha_j$ for some j. Certainly either $n_i = 0$ for $i > s$ or $n_i = 0$ for $i \leq s$. Assume the result for level $n-1$, and let the level be $n > 1$ for α. We have

$$0 < |\alpha|^2 = \sum_{i=1}^{l} n_i \langle \alpha, \alpha_i \rangle,$$

and therefore $\langle \alpha, \alpha_j \rangle > 0$ for some j. To fix the notation, let us say that $1 \leq j \leq s$. By Proposition 2.48e, $\alpha - \alpha_j$ is a root, evidently of level $n-1$. By inductive hypothesis, $\alpha - \alpha_j$ is in Δ' or Δ''. If $\alpha - \alpha_j$ is in Δ', then α is in Δ', and the induction is complete. So we may assume that $\alpha - \alpha_j$ is in Δ''. Then $\langle \alpha - \alpha_j, \alpha_j \rangle = 0$. By Proposition 2.48g, the α_j string containing $\alpha - \alpha_j$ has $p = q$, and this number must be ≥ 1 since α is a root. Hence $\alpha - 2\alpha_j$ is in $\Delta \cup \{0\}$. We cannot have $\alpha - 2\alpha_j = 0$ since Δ is reduced, and we conclude that the coefficient of α_j in $\alpha - \alpha_j$ is > 0, in contradiction with the assumption that $\alpha - \alpha_j$ is in Δ''. Thus $\alpha - \alpha_j$ could not have been in Δ'', and the induction is complete.

Motivated by Proposition 2.54, we say that an abstract Cartan matrix is **reducible** if, for some enumeration of the indices, the matrix is block diagonal with more than one block. Otherwise the abstract Cartan matrix is said to be **irreducible**.

If we have several abstract Cartan matrices, we can arrange them as the blocks of a block-diagonal matrix, and the result is a new abstract Cartan matrix. The converse direction is addressed by the following proposition.

Proposition 2.55. After a suitable enumeration of the indices, any abstract Cartan matrix may be written in block-diagonal form with each block an irreducible abstract Cartan matrix.

PROOF. Call two indices i and j equivalent if there exists a sequence of integers $i = k_0, k_1, \ldots, k_{r-1}, k_r = j$ such that $A_{k_{s-1}k_s} \neq 0$ for $1 \leq s \leq r$. Enumerate the indices so that the members of each equivalence class appear together, and then the abstract Cartan matrix will be in block-diagonal form with each block irreducible.

To our set Π of simple roots for the reduced abstract root system Δ, let us associate a kind of graph known as a "Dynkin diagram." We associate to each simple root α_i a vertex of a graph, and we attach to that vertex a weight proportional to $|\alpha_i|^2$. The vertices of the graph are connected by edges as follows. If two vertices are given, say corresponding to distinct simple roots α_i and α_j, we connect those vertices by $A_{ij}A_{ji}$ edges. The resulting graph is called the **Dynkin diagram** of Π. It follows from Proposition 2.54 that Δ is irreducible if and only if the Dynkin diagram is connected. Figure 2.3 gives the Dynkin diagrams for the root systems A_n, B_n, C_n, and D_n when the simple roots are chosen as in (2.50). Figure 2.3 shows also the Dynkin diagram for the root system G_2 of Figure 2.1 when the two simple roots are chosen so that $|\alpha_1| < |\alpha_2|$.

Let us indicate how we can determine the Dynkin diagram almost completely from the Cartan matrix. The key is the following lemma.

Lemma 2.56. Let A be an abstract Cartan matrix in block-diagonal form with each block an irreducible abstract Cartan matrix. Then the associated diagonal matrix D given in the defining property (e) of an abstract Cartan matrix is unique up to a multiplicative scalar on each block.

PROOF. Suppose that D and D' are two diagonal matrices with positive diagonal entries such that $P = DAD^{-1}$ and $P' = D'AD'^{-1}$ are symmetric positive definite. Then P and $P' = (D'D^{-1})P(D'D^{-1})^{-1}$ are both symmetric. Write $D'D^{-1} = \operatorname{diag}(b_1, \ldots, b_l)$. For any i and j, we have

$$b_i P_{ij} b_j^{-1} = P'_{ij} = P'_{ji} = b_j P_{ji} b_i^{-1} = b_j P_{ij} b_i^{-1}.$$

Thus either $P_{ij} = 0$ or $b_i = b_j$, i.e.,

$$A_{ij} = 0 \quad \text{or} \quad b_i = b_j. \tag{2.57}$$

If i and j are in the same block of A, then there exists a sequence of integers $i = k_0, k_1, \ldots, k_{r-1}, k_r = j$ such that $A_{k_{s-1}k_s} \neq 0$ for $1 \leq s \leq r$. From (2.57) we obtain

$$b_i = b_{k_0} = b_{k_1} = \cdots = b_{k_{r-1}} = b_{k_r} = b_j.$$

Thus the diagonal entries of D' are proportional to the diagonal entries of D within each block for A.

Returning to a Cartan matrix arising from the abstract reduced root system Δ and the set Π of simple roots, we note that the numbers $A_{ij}A_{ji}$ available from the Cartan matrix determine the numbers of edges between vertices in the Dynkin diagram. But the Cartan matrix also almost completely determines the weights in the Dynkin diagram. In fact, (2.53) says that the square roots of the weights are the diagonal entries of the matrix

D of Proposition 2.52e. Lemma 2.56 says that D is determined by the properties of A up to a multiplicative scalar on each irreducible block, and irreducible blocks correspond to connected components of the Dynkin diagram. Thus by using A, we can determine the weights in the Dynkin diagram up to a proportionality constant on each connected component. These proportionality constants are the only ambiguity in obtaining the Dynkin diagram from the Cartan matrix.

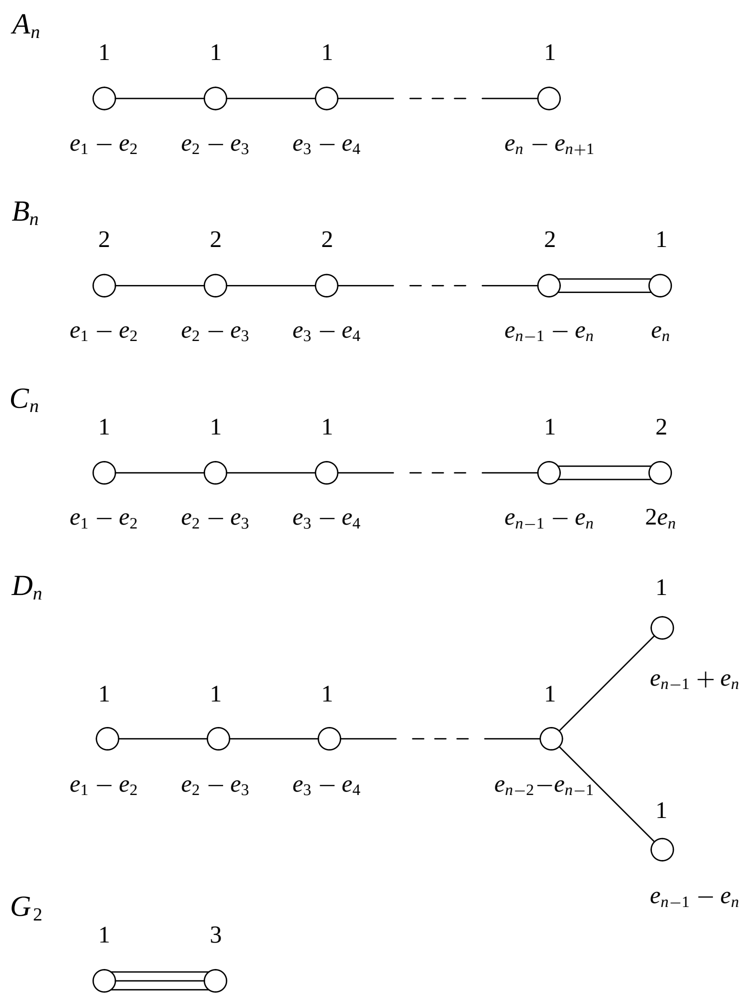


FIGURE 2.3. Dynkin diagrams for A_n, B_n, C_n, D_n, G_2

The same considerations allow us to associate an "abstract Dynkin diagram" to an abstract Cartan matrix A. If A has size l-by-l, the **abstract Dynkin diagram** is a graph with l vertices, the i^{th} and j^{th} vertices being connected by $A_{ij}A_{ji}$ edges. If D is the matrix given in defining property (e) of an abstract Cartan matrix in Proposition 2.52, then we assign a weight to the vertex i equal to the square of the i^{th} diagonal entry of D. Then A by itself determines the abstract Dynkin diagram up to a proportionality constant for the weights on each connected component.

Finally let us observe that we can recover an abstract Cartan matrix A from its abstract Dynkin diagram. Let the system of weights be $\{w_i\}$. First suppose there are no edges from the i^{th} vertex to the j^{th} vertex. Then $A_{ij}A_{ji} = 0$. Since $A_{ij} = 0$ if and only if $A_{ji} = 0$, we obtain $A_{ij} = A_{ji} = 0$. Next suppose there exist edges between the i^{th} vertex and the j^{th} vertex. Then the number of edges tells us $A_{ij}A_{ji}$, while the symmetry of DAD^{-1} says that

$$w_i^{1/2} A_{ij} w_j^{-1/2} = w_j^{1/2} A_{ji} w_i^{-1/2},$$

i.e., that

$$\frac{A_{ij}}{A_{ji}} = \frac{w_j}{w_i}.$$

Since A_{ij} and A_{ji} are < 0, the number of edges and the ratio of weights together determine A_{ij} and A_{ji}.

6. Weyl Group

Schematically we can summarize our work so far in this chapter as constructing a two-step passage

(2.58)
$$\begin{matrix}\text{complex}\\ \text{semisimple}\\ \text{Lie algebra}\end{matrix} \xrightarrow[\text{Cartan subalgebra}]{\text{choice of}} \begin{matrix}\text{abstract reduced}\\ \text{root system}\end{matrix} \xrightarrow[\text{ordering}]{\text{choice of}} \begin{matrix}\text{abstract}\\ \text{Cartan}\\ \text{matrix.}\end{matrix}$$

Each step of the passage relies on a certain choice, and that choice is listed as part of the arrow. For this two-step passage to be especially useful, we should show that each step is independent of its choice, at least up to isomorphism. Then we will have a well defined way of passing from a complex semisimple Lie algebra first to an abstract reduced root system and then to an abstract Cartan matrix.

We can ask for even more. Once (2.58) is shown to be well defined independently of the choices, we can try to show that each step is one-one, up to isomorphism. In other words, two complex semisimple Lie algebras with isomorphic abstract reduced root systems are to be isomorphic, and two abstract reduced root systems leading to isomorphic abstract Cartan matrices are to be isomorphic. Then we can detect isomorphisms of complex semisimple Lie algebras by using Dynkin diagrams.

Finally we can ask that each step of the two-step passage be onto. In other words, every abstract reduced root system, up to isomorphism, is to come from a complex semisimple Lie algebra, and every abstract Cartan matrix is to come from an abstract reduced root system. Then a classification of abstract Cartan matrices will achieve a classification of complex semisimple Lie algebras.

We begin these steps in this section, starting by showing that each step in (2.58) is well defined, independently of the choices, up to isomorphism. For the first step, from the complex semisimple Lie algebra to the abstract reduced root system, the tool is Theorem 2.15, which says that any two Cartan subalgebras of our complex semisimple Lie algebra $\mathfrak{g}$ are conjugate via $\operatorname{Int}\mathfrak{g}$. It is clear that we can follow the effect of this conjugating automorphism through to its effect on roots and obtain an isomorphism of the associated root systems.

For the second step, from the abstract reduced root system to the abstract Cartan matrix up to isomorphism (or equivalently to the set Π of simple roots), the tool is the "Weyl group," which we study in this section.

Thus let Δ be an abstract root system in a finite-dimensional inner product space V. It will not be necessary to assume that Δ is reduced. We let $W = W(\Delta)$ be the subgroup of the orthogonal group on V generated by the reflections s_α for $\alpha \in \Delta$. This is the **Weyl group** of Δ. In the special case that Δ is the root system of a complex semisimple Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$, we sometimes write $W(\mathfrak{g}, \mathfrak{h})$ for the Weyl group.

We immediately see that W is a finite group of orthogonal transformations of V. In fact, any w in W maps the finite set Δ to itself. If w fixes each element of Δ, then w fixes a spanning set of V and hence fixes V. The assertion follows.

In addition, we have the formula

$$s_{r\alpha} = r s_\alpha r^{-1} \tag{2.59}$$

for any orthogonal transformation r of V. In fact,

$$s_{r\alpha}(r\varphi) = r\varphi - \frac{2\langle r\varphi, r\alpha\rangle}{|r\alpha|^2} r\alpha = r\varphi - \frac{2\langle\varphi, \alpha\rangle}{|\alpha|^2} r\alpha = r(s_\alpha\varphi).$$

As a consequence of (2.59), if r is in W and $r\alpha = \beta$, then

$$s_\beta = r s_\alpha r^{-1}. \tag{2.60}$$

EXAMPLES.

1) The root systems of types A_n, B_n, C_n, and D_n are described in (2.43). For A_n, $W(\Delta)$ consists of all permutations of $e_1, \dots, e_{n+1}$. For B_n and C_n, $W(\Delta)$ is generated by all permutations of $e_1, \dots, e_n$ and all sign changes (of the coefficients of $e_1, \dots, e_n$). For D_n, $W(\Delta)$ is generated by all permutations of $e_1, \dots, e_n$ and all even sign changes.

2) The nonreduced abstract root system $(BC)_2$ is pictured in Figure 2.2. For it, $W(\Delta)$ has order 8 and is the same group as for B_2 and C_2. The group contains the 4 rotations through multiples of angles $\pi/2$, together with the 4 reflections defined by sending a root to its negative and leaving the orthogonal complement fixed.

3) The reduced abstract root system G_2 is pictured in Figure 2.2. For it, $W(\Delta)$ has order 12 and consists of the 6 rotations through multiples of angles $\pi/3$, together with the 6 reflections defined by sending a root to its negative and leaving the orthogonal complement fixed.

Introduce a notion of positivity within V, such as from a lexicographic ordering, and let Δ^+ be the set of positive roots. The set Δ^+ determines a set $\Pi = \{\alpha_1, \dots, \alpha_l\}$ of simple roots, and in turn Π can be used to pick out the members of Δ^+ from Δ, since Proposition 2.49 says that the positive roots are those of the form $\alpha = \sum_i n_i\alpha_i$ with all $n_i \geq 0$.

Now suppose that $\Pi = \{\alpha_1, \dots, \alpha_l\}$ is any set of l independent reduced elements α_i such that every expression of a member α of Δ as $\sum_i c_i\alpha_i$ has all nonzero c_i of the same sign. We call Π a **simple system**. Given a simple system Π, we can define Δ^+ to be all roots of the form $\sum_i c_i\alpha_i$ with all $c_i \geq 0$. The claim is that Δ^+ is the set of positive roots in some lexicographic ordering. In fact, we can use the dual basis to $\{\alpha_i\}$ to get such an ordering. In more detail if $\langle\alpha_i, \omega_j\rangle = \delta_{ij}$ and if j is the first index with $\langle\alpha, \omega_j\rangle$ nonzero, then the fact that $\langle\alpha, \omega_j\rangle = c_j$ is positive implies that α is positive.

Thus we have an abstract characterization of the possible Π's that can arise as sets of simple roots: they are all possible simple systems.

Lemma 2.61. Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be a simple system, and let $\alpha > 0$ be in Δ. Then

$$s_{\alpha_i}(\alpha) \text{ is } \begin{cases} = -\alpha & \text{if } \alpha = \alpha_i \text{ or } \alpha = 2\alpha_i \\ > 0 & \text{otherwise.} \end{cases}$$

PROOF. If $\alpha = \sum c_j \alpha_j$, then

$$s_{\alpha_i}(\alpha) = \sum_{j=1}^{l} c_j \alpha_j - \frac{2\langle \alpha, \alpha_i \rangle}{|\alpha_i|^2} \alpha_i.$$

If at least one c_j is > 0 for $j \neq i$, then $s_{\alpha_i}(\alpha)$ has the same coefficient for α_j that α does, and $s_{\alpha_i}(\alpha)$ must be positive. The only remaining case is that α is a multiple of α_i, and then α must be α_i or $2\alpha_i$, by Proposition 2.48b.

Proposition 2.62. Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be a simple system. Then $W(\Delta)$ is generated by the root reflections s_{α_i} for α_i in Π. If α is any reduced root, then there exist $\alpha_j \in \Pi$ and $s \in W(\Delta)$ such that $s\alpha_j = \alpha$.

PROOF. We begin by proving a seemingly sharper form of the second assertion. Let $W' \subseteq W$ be the group generated by the s_{α_i} for $\alpha_i \in \Pi$. We prove that any reduced root $\alpha > 0$ is of the form $s\alpha_j$ with $s \in W'$. Writing $\alpha = \sum n_j \alpha_j$, we proceed by induction on $\text{level}(\alpha) = \sum n_j$. The case of level one is the case of $\alpha = \alpha_i$ in Π, and we can take $s = 1$. Assume the assertion for level $< \text{level}(\alpha)$, let $\text{level}(\alpha)$ be > 1, and write $\alpha = \sum n_j \alpha_j$. Since

$$0 < |\alpha|^2 = \sum n_j \langle \alpha, \alpha_j \rangle,$$

we must have $\langle \alpha, \alpha_i \rangle > 0$ for some $i = i_0$. By our assumptions, α is neither α_{i_0} nor $2\alpha_{i_0}$. Then $\beta = s_{\alpha_{i_0}}(\alpha)$ is > 0 by Lemma 2.61 and has

$$\beta = \sum_{j \neq i_0} n_j \alpha_j + \left(c_{i_0} - \frac{2\langle \alpha, \alpha_{i_0} \rangle}{|\alpha_{i_0}|^2} \right) \alpha_{i_0}.$$

Since $\langle \alpha, \alpha_{i_0} \rangle > 0$, $\text{level}(\beta) < \text{level}(\alpha)$. By inductive hypothesis, $\beta = s'\alpha_j$ for some $s' \in W'$ and some index j. Then $\alpha = s_{\alpha_{i_0}}\beta = s_{\alpha_{i_0}} s' \alpha_j$ with $s_{\alpha_{i_0}} s'$ in W'. This completes the induction.

If $\alpha < 0$, then we can write $-\alpha = s\alpha_j$, and it follows that $\alpha = s s_{\alpha_j} \alpha_j$. Thus each reduced member α of Δ is of the form $s'\alpha_j$ for some $s' \in W'$ and some $\alpha_j \in \Pi$.

To complete the proof, we show that each s_α, for $\alpha \in \Delta$, is in W'. There is no loss of generality in assuming that α is reduced. Write $\alpha = s\alpha_j$ with $s \in W'$. Then (2.60) shows that $s_\alpha = s s_{\alpha_j} s^{-1}$, which is in W'. Since W is generated by the reflections s_α for $\alpha \in \Delta$, $W \subseteq W'$ and $W = W'$.

Theorem 2.63. If Π and Π' are two simple systems for Δ, then there exists one and only one element $s \in W$ such that $s\Pi = \Pi'$.

PROOF OF EXISTENCE. Let Δ^+ and $\Delta^{+\prime}$ be the sets of positive roots in question. We have $|\Delta^+| = |\Delta^{+\prime}| = \frac{1}{2}|\Delta|$, which we write as q. Also $\Delta^+ = \Delta^{+\prime}$ if and only if $\Pi = \Pi'$, and $\Delta^+ \neq \Delta^{+\prime}$ implies $\Pi \nsubseteq \Delta^{+\prime}$ and $\Pi' \nsubseteq \Delta^+$. Let $r = |\Delta^+ \cap \Delta^{+\prime}|$. We induct downward on r, the case $r = q$ being handled by using $s = 1$. Let $r < q$. Choose $\alpha_i \in \Pi$ with $\alpha_i \notin \Delta^{+\prime}$, so that $-\alpha_i \in \Delta^{+\prime}$. If β is in $\Delta^+ \cap \Delta^{+\prime}$, then $s_{\alpha_i}\beta$ is in Δ^+ by Lemma 2.61. Thus $s_{\alpha_i}\beta$ is in $\Delta^+ \cap s_{\alpha_i}\Delta^{+\prime}$. Also $\alpha_i = s_{\alpha_i}(-\alpha_i)$ is in $\Delta^+ \cap s_{\alpha_i}\Delta^{+\prime}$. Hence $|\Delta^+ \cap s_{\alpha_i}\Delta^{+\prime}| \geq r + 1$. Now $s_{\alpha_i}\Delta^{+\prime}$ corresponds to the simple system $s_{\alpha_i}\Pi'$, and by inductive hypothesis we can find $t \in W$ with $t\Pi = s_{\alpha_i}\Pi'$. Then $s_{\alpha_i}t\Pi = \Pi'$, and the induction is complete.

PROOF OF UNIQUENESS. We may assume that $s\Pi = \Pi$, and we are to prove that $s = 1$. Write $\Pi = \{\alpha_1, \ldots, \alpha_l\}$, and abbreviate s_{α_j} as s_j. For $s = s_{i_m} \cdots s_{i_1}$, we prove by induction on m that $s\Pi = \Pi$ implies $s = 1$. If $m = 1$, then $s = s_{i_1}$ and $s\alpha_{i_1} < 0$. If $m = 2$, we obtain $s_{i_2}\Pi = s_{i_1}\Pi$, whence $-\alpha_{i_2}$ is in $s_{i_1}\Pi$ and so $-\alpha_{i_2} = -\alpha_{i_1}$, by Lemma 2.61; hence $s = 1$. Thus assume inductively that

$$(2.64) \qquad t\Pi = \Pi \text{ with } t = s_{j_r} \cdots s_{j_1} \text{ and } r < m \quad \text{implies} \quad t = 1,$$

and let $s = s_{i_m} \cdots s_{i_1}$ satisfy $s\Pi = \Pi$ with $m > 2$.

Put $s' = s_{i_{m-1}} \cdots s_{i_1}$, so that $s = s_{i_m}s'$. Then $s' \neq 1$ by (2.64) for $t = s_{i_m}$. Also $s'\alpha_j < 0$ for some j by (2.64) applied to $t = s'$. The latter fact, together with

$$s_{i_m}s'\alpha_j = s\alpha_j > 0.$$

says that $-\alpha_{i_m} = s'\alpha_j$, by Lemma 2.61. Also if $\beta > 0$ and $s'\beta < 0$, then $s'\beta = -c\alpha_{i_m} = s'(c\alpha_j)$, so that $\beta = c\alpha_j$ with $c = 1$ or 2. Thus s' satisfies

(i) $s'\alpha_j = -\alpha_{i_m}$,

(ii) $s'\beta > 0$ for every positive $\beta \in \Delta$ other than α_j and $2\alpha_j$.

Now $s_{i_{m-1}} \cdots s_{i_1}\alpha_j = -\alpha_{i_m} < 0$ by (i). Choose k so that $t = s_{i_{k-1}} \cdots s_{i_1}$ satisfies $t\alpha_j > 0$ and $s_{i_k}t\alpha_j < 0$. Then $t\alpha_j = \alpha_{i_k}$. By (2.60), $ts_jt^{-1} = s_{i_k}$. Hence $ts_j = s_{i_k}t$.

Put $t' = s_{i_{m-1}} \cdots s_{i_{k+1}}$, so that $s' = t's_{i_k}t = t'ts_j$. Then $t't = s's_j$. Now $\alpha > 0$ and $\alpha \neq c\alpha_j$ imply $s_j\alpha = \beta > 0$ with $\beta \neq c\alpha_j$. Thus

$$t't\alpha = s's_j\alpha = s'\beta > 0 \qquad \text{by (ii)}$$

and

$$t't\alpha_j = s'(-\alpha_j) = \alpha_{i_m} > 0 \qquad \text{by (i).}$$

Hence $t't\Pi = \Pi$. Now $t't$ is a product of $m-2$ s_j's. By inductive hypothesis, $t't = 1$. Then $s's_j = 1$, $s' = s_j$, and $s = s_{i_m}s' = s_{i_m}s_j$. Since (2.64) has been proved for $r = 2$, we conclude that $s = 1$. This completes the proof.

Corollary 2.65. In the second step of the two-step passage (2.58), the resulting Cartan matrix is independent of the choice of positive system, up to permutation of indices.

PROOF. Let Π and Π' be the simple systems that result from two different positive systems. By Theorem 2.63, $\Pi' = s\Pi$ for some $s \in W(\Delta)$. Then we can choose enumerations $\Pi = \{\alpha_1, \dots, \alpha_l\}$ and $\Pi' = \{\beta_1, \dots, \beta_l\}$ so that $\beta_j = s\alpha_j$, and we have

$$\frac{2\langle\beta_i, \beta_j\rangle}{|\beta_i|^2} = \frac{2\langle s\alpha_i, s\alpha_j\rangle}{|s\alpha_i|^2} = \frac{2\langle\alpha_i, \alpha_j\rangle}{|\alpha_i|^2}$$

since s is orthogonal. Hence the resulting Cartan matrices match.

Consequently our use of the root-system names A_n, B_n, etc., with the Dynkin diagrams in Figure 2.3 was legitimate. The Dynkin diagram is not changed by changing the positive system (except that the names of roots attached to vertices change).

This completes our discussion of the fact that the steps in the passages (2.58) are well defined independently of the choices.

Let us take a first look at the uniqueness questions associated with (2.58). We want to see that each step in (2.58) is one-one, up to isomorphism. The following proposition handles the second step.

Proposition 2.66. The second step in the passage (2.58) is one-one, up to isomorphism. That is, the Cartan matrix determines the reduced root system up to isomorphism.

PROOF. First let us see that the Cartan matrix determines the set of simple roots, up to a linear transformation of V that is a scalar multiple of an orthogonal transformation on each irreducible component. In fact, we may assume that Δ is already irreducible, and we let $\alpha_1, \dots, \alpha_l$ be the simple roots. Lemma 2.56 and (2.53) show that the Cartan matrix determines $|\alpha_1|, \dots, |\alpha_l|$ up to a single proportionality constant. Suppose $\beta_1, \dots, \beta_l$ is another simple system for the same Cartan matrix. Normalizing, we may assume that $|\alpha_j| = |\beta_j|$ for all j. From the Cartan matrix we obtain

$\frac{2\langle\alpha_i,\alpha_j\rangle}{|\alpha_i|^2} = \frac{2\langle\beta_i,\beta_j\rangle}{|\beta_i|^2}$ for all i and j and hence $\langle\alpha_i, \alpha_j\rangle = \langle\beta_i, \beta_j\rangle$ for all i and j. In other words the linear transformation L defined by $L\alpha_i = \beta_i$ preserves inner products on a basis; it is therefore orthogonal.

To complete the proof, we want to see that the set $\{\alpha_1, \ldots, \alpha_l\}$ of simple roots determines the set of roots. Let W' be the group generated by the root reflections in the simple roots, and let $\Delta' = \bigcup_{j=1}^{l} W'\alpha_j$. Proposition 2.62 shows that $\Delta' = \Delta$ and that $W' = W(\Delta)$. The result follows.

Before leaving the subject of Weyl groups, we prove some further handy results. For the first result let us fix a system Δ^+ of positive roots and the corresponding simple system Π. We say that a member λ of V is **dominant** if $\langle\lambda, \alpha\rangle \geq 0$ for all $\alpha \in \Delta^+$. It is enough that $\langle\lambda, \alpha_i\rangle \geq 0$ for all $\alpha_i \in \Pi$.

Proposition 2.67. If λ is in V, then there exists a simple system Π for which λ is dominant.

PROOF. We may assume $\lambda \neq 0$. Put $\varphi_1 = \lambda$ and extend to an orthogonal basis $\varphi_1, \ldots, \varphi_l$ of V. Use this basis to define a lexicographic ordering and thereby to determine a simple system Π. Then λ is dominant relative to Π.

Corollary 2.68. If λ is in V and if a positive system Δ^+ is specified, then there is some element w of the Weyl group such that $w\lambda$ is dominant.

PROOF. This follows from Proposition 2.67 and Theorem 2.63.

For the remaining results we assume that Δ is reduced. Fix a positive system Δ^+, and let δ be half the sum of the members of Δ^+.

Proposition 2.69. Fix a positive system Δ^+ for the reduced abstract root system Δ. If α is a simple root, then $s_\alpha(\delta) = \delta - \alpha$ and $2\langle\delta, \alpha\rangle/|\alpha|^2 = 1$.

PROOF. By Lemma 2.61, s_α permutes the positive roots other than α and sends α to $-\alpha$. Therefore

$$s_\alpha(2\delta) = s_\alpha(2\delta - \alpha) + s_\alpha(\alpha) = (2\delta - \alpha) - \alpha = 2(\delta - \alpha),$$

and $s_\alpha(\delta) = \delta - \alpha$. Using the definition of s_α, we then see that

$$2\langle\delta, \alpha\rangle/|\alpha|^2 = 1.$$

For w in $W(\Delta)$, let $l(w)$ be the number of roots $\alpha > 0$ such that $w\alpha < 0$; $l(w)$ is called the **length** of the Weyl group element w relative to Π. In terms of a simple system $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ and its associated positive system Δ^+, let us abbreviate s_{α_j} as s_j.

Proposition 2.70. Fix a simple system $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ for the reduced abstract root system Δ. Then $l(w)$ is the smallest integer k such that w can be written as a product $w = s_{i_k} \cdots s_{i_1}$ of k reflections in simple roots.

REMARKS. Proposition 2.62 tells us that w has at least one expansion as a product of reflections in simple roots. Therefore the smallest integer k cited in the proposition exists. We prove Proposition 2.70 after first giving a lemma.

Lemma 2.71. Fix a simple system $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ for the reduced abstract root system Δ. If γ is a simple root and w is in $W(\Delta)$, then

$$l(ws_\gamma) = \begin{cases} l(w) - 1 & \text{if } w\gamma < 0 \\ l(w) + 1 & \text{if } w\gamma > 0. \end{cases}$$

PROOF. If α is a positive root other than γ, then Lemma 2.61 shows that $s_\gamma \alpha > 0$, and hence the correspondence $s_\gamma \alpha \leftrightarrow \alpha$ gives

$$\#\{\beta > 0 \mid \beta \neq \gamma \text{ and } ws_\gamma\beta < 0\} = \#\{\alpha > 0 \mid \alpha \neq \gamma \text{ and } w\alpha < 0\}.$$

To obtain $l(ws_\gamma)$, we add 1 to the left side if $w\gamma > 0$ and leave the left side alone if $w\gamma < 0$. To obtain $l(w)$, we add 1 to the right side if $w\gamma < 0$ and leave the right side alone if $w\gamma > 0$. The lemma follows.

PROOF OF PROPOSITION 2.70. Write $w = s_{i_k} \cdots s_{i_1}$ as a product of k reflections in simple roots. Then Lemma 2.71 implies that $l(w) \le k$.

To get the equality asserted by the proposition, we need to show that if w sends exactly k positive roots into negative roots, then w can be expressed as a product of k factors $w = s_{i_k} \cdots s_{i_1}$. We do so by induction on k. For $k = 0$, this follows from the uniqueness in Theorem 2.63. Inductively assume the result for $k - 1$. If $k > 0$ and $l(w) = k$, then w must send some simple root α_j into a negative root. Set $w' = ws_j$. By Lemma 2.71, $l(w') = k - 1$. By inductive hypothesis, w' has an expansion $w' = s_{i_{k-1}} \cdots s_{i_1}$. Then $w = s_{i_{k-1}} \cdots s_{i_1} s_j$, and the induction is complete.

Proposition 2.72 (Chevalley's Lemma). Let the abstract root system Δ be reduced. Fix v in V, and let $W_0 = \{w \in W \mid wv = v\}$. Then W_0 is generated by the root reflections s_α such that $\langle v, \alpha\rangle = 0$.

PROOF. Choose an ordering with v first, so that $\langle \beta, v\rangle > 0$ implies $\beta > 0$. Arguing by contradiction, choose $w \in W_0$ with $l(w)$ as small as possible so that w is not a product of elements s_α with $\langle v, \alpha\rangle = 0$. Then

$l(w) > 0$ by the uniqueness in Theorem 2.63. Let $\gamma > 0$ be a simple root such that $w\gamma < 0$. If $\langle v, \gamma \rangle > 0$, then

$$\langle v, w\gamma \rangle = \langle wv, w\gamma \rangle = \langle v, \gamma \rangle > 0,$$

in contradiction with the condition $w\gamma < 0$. Hence $\langle v, \gamma \rangle = 0$. That is, s_γ is in W_0. But then ws_γ is in W_0 with $l(ws_\gamma) < l(w)$, by Lemma 2.71. By assumption ws_γ is a product of the required root reflections, and therefore so is w.

Corollary 2.73. Let the abstract root system Δ be reduced. Fix v in V, and suppose that some element $w \neq 1$ of $W(\Delta)$ fixes v. Then some root is orthogonal to v.

PROOF. By Proposition 2.72, w is the product of root reflections s_α such that $\langle v, \alpha \rangle = 0$. Since $w \neq 1$, there must be such a root reflection.

7. Classification of Abstract Cartan Matrices

In this section we shall classify abstract Cartan matrices, and then we shall show that every abstract Cartan matrix arises from a reduced abstract root system. These results both contribute toward an understanding of the two-step passage (2.58), the second result showing that the second step of the passage is onto.

Recall that an abstract Cartan matrix is a square matrix satisfying properties (a) through (e) in Proposition 2.52. We continue to regard two such matrices as isomorphic if one can be obtained from the other by permuting the indices.

To each abstract Cartan matrix, we saw in §5 how to associate an abstract Dynkin diagram, the only ambiguity being a proportionality constant for the weights on each component of the diagram. We shall work simultaneously with a given abstract Cartan matrix and its associated abstract Dynkin diagram. Operations on the abstract Cartan matrix will correspond to operations on the abstract Dynkin diagram, and the diagram will thereby give us a way of visualizing what is happening. Our objective is to classify irreducible abstract Cartan matrices, since general abstract Cartan matrices can be obtaining by using irreducible such matrices as blocks. But we do not assume irreducibility yet.

We first introduce two operations on abstract Dynkin diagrams. Each operation will have a counterpart for abstract Cartan matrices, and we shall see that the counterpart carries abstract Cartan matrices to abstract Cartan matrices. Therefore each of our operations sends abstract Dynkin diagrams to abstract Dynkin diagrams:

1) Remove the i^{th} vertex from the abstract Dynkin diagram, and remove all edges attached to that vertex.

2) Suppose that the i^{th} and j^{th} vertices are connected by a single edge. Then the weights attached to the two vertices are equal. Collapse the two vertices to a single vertex and give it the common weight, remove the edge that joins the two vertices, and retain all other edges issuing from either vertex.

For Operation #1, the corresponding operation on a Cartan matrix A is to remove the i^{th} row and column from A. It is clear that the new matrix satisfies the defining properties of an abstract Cartan matrix given in Proposition 2.52. This fact allows us to prove the following proposition.

Proposition 2.74. Let A be an abstract Cartan matrix. If $i \neq j$, then

(a) $A_{ij}A_{ji} < 4$,
(b) A_{ij} is 0 or -1 or -2 or -3.

PROOF.
(a) Let the diagonal matrix D of defining property (e) be given by $D = \operatorname{diag}(d_1, \ldots, d_l)$. Using Operation #1, remove all but the i^{th} and j^{th} rows and columns from the abstract Cartan matrix A. Then

$$\begin{pmatrix} d_i & 0 \\ 0 & d_j \end{pmatrix} \begin{pmatrix} 2 & A_{ij} \\ A_{ji} & 2 \end{pmatrix} \begin{pmatrix} d_i^{-1} & 0 \\ 0 & d_j^{-1} \end{pmatrix}$$

is positive definite. So its determinant is > 0, and $A_{ij}A_{ji} < 4$.
(b) If $A_{ij} \neq 0$, then $A_{ji} \neq 0$, by defining property (d) in Proposition 2.52. Since A_{ij} and A_{ji} are integers ≤ 0, the result follows from (a).

We shall return presently to the verification that Operation #2 is a legitimate one on abstract Dynkin diagrams. First we derive some more subtle consequences of the use of Operation #1.

Let A be an l-by-l abstract Cartan matrix, and let $D = \operatorname{diag}(d_1, \ldots, d_l)$ be a diagonal matrix of the kind in defining condition (e) of Proposition 2.52. We shall define vectors $\alpha_i \in \mathbb{R}^l$ for $1 \leq i \leq l$ that will play the role of simple roots. Let us write $DAD^{-1} = 2Q$. Here $Q = (Q_{ij})$ is symmetric positive definite with 1's on the diagonal. Let $Q^{1/2}$ be its positive-definite square root. Define vectors $\varphi \in \mathbb{R}^l$ for $1 \le i \le l$ by $\varphi_i = Q^{1/2}e_i$, where e_i is the i^{th} standard basis vector of $\mathbb{R}^l$. Then

$$\langle \varphi_j, \varphi_i \rangle = \langle Q^{1/2}e_j, Q^{1/2}e_i \rangle = \langle Qe_j, e_i \rangle = Q_{ij},$$

and in particular φ_i is a unit vector. Put

$$\alpha_i = d_i \varphi_i, \tag{2.75}$$

so that

$$d_i = |\alpha_i|. \tag{2.76}$$

Then

$$\begin{aligned} A_{ij} &= 2(D^{-1}QD)_{ij} = 2d_i^{-1}Q_{ij}d_j \\ &= 2d_i^{-1}d_j\langle \varphi_j, \varphi_i \rangle = 2d_i^{-1}d_j\langle d_j^{-1}\alpha_j, d_i^{-1}\alpha_i \rangle \\ &= \frac{2\langle \alpha_i, \alpha_j \rangle}{|\alpha_i|^2}. \end{aligned} \tag{2.77}$$

The vectors α_i are linearly independent since $\det A \neq 0$.

We shall find it convenient to refer to a vertex of the abstract Dynkin diagram either by its index i or by the associated vector α_i, depending on the context. We may write A_{ij} or $A_{\alpha_i, \alpha_{i+1}}$ for an entry of the abstract Cartan matrix.

Proposition 2.78. The abstract Dynkin diagram associated to the l-by-l abstract Cartan matrix A has the following properties:

(a) there are at most l pairs of vertices $i < j$ with at least one edge connecting them,
(b) there are no loops,
(c) at most three edges issue from any point of the diagram.

PROOF.
(a) With α_i as in (2.75), put $\alpha = \sum_{i=1}^{l} \dfrac{\alpha_i}{|\alpha_i|}$. Then

$$\begin{aligned} 0 < |\alpha|^2 &= \sum_{i,j} \left\langle \frac{\alpha_i}{|\alpha_i|}, \frac{\alpha_j}{|\alpha_j|} \right\rangle \\ &= \sum_{i} \left\langle \frac{\alpha_i}{|\alpha_i|}, \frac{\alpha_i}{|\alpha_i|} \right\rangle + 2\sum_{i<j} \left\langle \frac{\alpha_i}{|\alpha_i|}, \frac{\alpha_j}{|\alpha_j|} \right\rangle \\ &= l + \sum_{i<j} \frac{2\langle \alpha_i, \alpha_j \rangle}{|\alpha_i||\alpha_j|} \\ &= l - \sum_{i<j} \sqrt{A_{ij}A_{ji}}. \end{aligned} \tag{2.79}$$

By Proposition 2.74, $\sqrt{A_{ij}A_{ji}}$ is 0 or 1 or $\sqrt{2}$ or $\sqrt{3}$. When nonzero, it is therefore ≥ 1. Therefore the right side of (2.79) is

$$\leq l - \sum_{\substack{i<j,\\ \text{connected}}} 1.$$

Hence the number of connected pairs of vertices is $< l$.

(b) If there were a loop, we could use Operation #1 to remove all vertices except those in a loop. Then (a) would be violated for the loop.

(c) Fix $\alpha = \alpha_i$ as in (2.75). Consider the vertices that are connected by edges to the i^{th} vertex. Write $\beta_1, \dots, \beta_r$ for the α_j's associated to these vertices, and let there be $l_1, \dots, l_r$ edges to the i^{th} vertex. Let U be the $(r+1)$-dimensional vector subspace of $\mathbb{R}^l$ spanned by $\beta_1, \dots, \beta_r, \alpha$. Then $\langle \beta_i, \beta_j \rangle = 0$ for $i \neq j$ by (b), and hence $\{\beta_k/|\beta_k|\}_{k=1}^r$ is an orthonormal set. Adjoin $\delta \in U$ to this set to make an orthonormal basis of U. Then $\langle \alpha, \delta \rangle \neq 0$ since $\{\beta_1, \dots, \beta_r, \alpha\}$ is linearly independent. By Parseval's equality,

$$|\alpha|^2 = \sum_k \left\langle \alpha, \frac{\beta_k}{|\beta_k|} \right\rangle^2 + \langle \alpha, \delta \rangle^2 > \sum_k \left\langle \alpha, \frac{\beta_k}{|\beta_k|} \right\rangle^2$$

and hence

$$1 > \sum_k \frac{\langle \alpha, \beta_k \rangle^2}{|\alpha|^2 |\beta_k|^2} = \tfrac{1}{4} \sum_k l_k.$$

Thus $\sum_k l_k < 4$. This completes the proof.

We turn to Operation #2, which we have described in terms of abstract Dynkin diagrams. Let us describe the operation in terms of abstract Cartan matrices. We assume that $A_{ij} = A_{ji} = -1$, and we have asserted that the weights attached to the i^{th} and j^{th} vertices, say w_i and w_j, are equal. The weights are given by $w_i = d_i^2$ and $w_j = d_j^2$. The symmetry of DAD^{-1} implies that

$$d_i A_{ij} d_j^{-1} = d_j A_{ji} d_i^{-1},$$

hence that $d_i^2 = d_j^2$ and $w_i = w_j$. Thus

$$A_{ij} = A_{ji} = -1 \qquad \text{implies} \qquad w_i = w_j. \tag{2.80}$$

Under the assumption that $A_{ij} = A_{ji} = -1$, Operation #2 replaces the abstract Cartan matrix A of size l by a square matrix of size $l-1$, collapsing the i^{th} and j^{th} indices. The replacement row is the sum of the i^{th} and j^{th} rows of A in entries $k \notin \{i, j\}$, and similarly for the replacement column. The 2-by-2 matrix from the i^{th} and j^{th} indices is $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ within A and gets replaced by the 1-by-1 matrix (2).

Proposition 2.81. Operation #2 replaces the abstract Cartan matrix A by another abstract Cartan matrix.

PROOF. Without loss of generality, let the indices i and j be $l-1$ and l. Define E to be the $(l-1)$-by-l matrix

$$E = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & \ddots & & \vdots & \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1_{l-2} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The candidate for a new Cartan matrix is EAE^t, and we are to verify the five axioms in Proposition 2.52. The first four are clear, and we have to check (e). Let P be the positive-definite matrix $P = DAD^{-1}$, and define

$$D' = EDE^t \operatorname{diag}(1, \dots, 1, \tfrac{1}{2}),$$

which is square of size $l-1$. Remembering from (2.80) that the weights w_i satisfy $w_i = d_i^2$ and that $w_{l-1} = w_l$, we see that $d_{l-1} = d_l$. Write d for the common value of d_{l-1} and d_l. In block form, D is then of the form

$$D = \begin{pmatrix} D_0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix}.$$

Therefore D' in block form is given by

$$\begin{aligned} D' &= \begin{pmatrix} 1_{l-2} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} D_0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix} \begin{pmatrix} 1_{l-2} & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1_{l-2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} D_0 & 0 \\ 0 & d \end{pmatrix}. \end{aligned}$$

Meanwhile

$$\begin{aligned} E^t \operatorname{diag}(1, \dots, 1, \tfrac{1}{2}) E &= \begin{pmatrix} 1_{l-2} & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1_{l-2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1_{l-2} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1_{l-2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \end{aligned}$$

and it follows that $E^t\mathrm{diag}(1,\dots,1,\frac{1}{2})E$ commutes with D. Since

$$EE^t\mathrm{diag}(1,\dots,1,\tfrac{1}{2}) = 1,$$

we therefore have

$$D'E = EDE^t\mathrm{diag}(1,\dots,1,\tfrac{1}{2})E = EE^t\mathrm{diag}(1,\dots,1,\tfrac{1}{2})ED = ED.$$

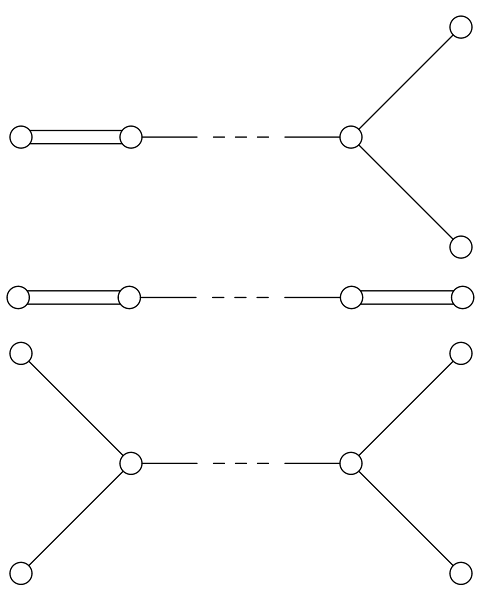
The same computation gives also $D'^{-1}E = ED^{-1}$, whose transpose is $E^tD'^{-1} = D^{-1}E^t$. Thus

$$D'(EAE^t)D'^{-1} = (D'E)A(E^tD'^{-1}) = EDAD^{-1}E^t = EPE^t,$$

and the right side is symmetric and positive semidefinite. To see that it is definite, let $\langle EPE^tv, v\rangle = 0$. Then $\langle PE^tv, E^tv\rangle = 0$. Since P is positive definite, $E^tv = 0$. But E^t is one-one, and therefore $v = 0$. We conclude that EPE^t is definite.

Now we specialize to irreducible abstract Cartan matrices, which correspond to connected abstract Dynkin diagrams. In five steps, we can obtain the desired classification.

1) No abstract Dynkin diagram contains a configuration



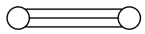
or

or

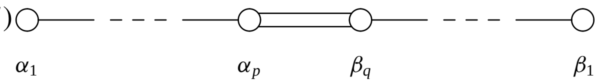
In fact, otherwise Operation #2 would allow us to collapse all the single-line part in the center to a single vertex, in violation of Proposition 2.78c.

2) The following are the only possibilities left for a connected abstract Dynkin diagram:

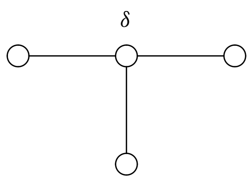
2a) There is a triple line. By Proposition 2.78c the only possibility is

(G_2) 

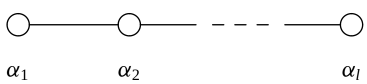
2b) There is a double line, but there is no triple line. Then Step 1 shows that the diagram is

(B, C, F) 

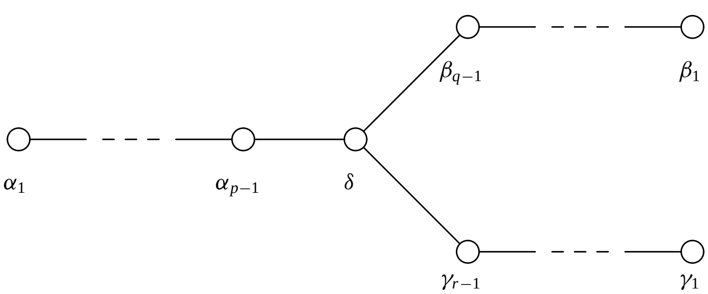
2c) There are only single lines. Call



a **triple point**. If there is no triple point, then the absence of loops implies that the diagram is

(A) 

If there is a triple point, then there is only one, by Step 1, and the diagram is

(D, E) 

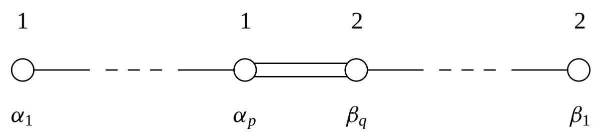
3) The following are the possibilities for weights:

3a) If the i^{th} and j^{th} vertices are connected by a single line, then $A_{ij} = A_{ji} = -1$. By (2.80) the weights satisfy $w_i = w_j$. Thus in the cases (A) and (D, E) of Step 2, all the weights are equal, and we may take them to be 1. In this situation we shall omit the weights from the diagram.

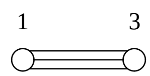
3b) In the case (B, C, F) of Step 2, let $\alpha = \alpha_p$ and $\beta = \beta_q$. Also let us use α and β to denote the corresponding vertices. Possibly reversing the roles of α and β, we may assume that $A_{\alpha\beta} = -2$ and $A_{\beta\alpha} = -1$. Then

$$\begin{pmatrix} |\alpha| & 0 \\ 0 & |\beta| \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} |\alpha|^{-1} & 0 \\ 0 & |\beta|^{-1} \end{pmatrix}$$

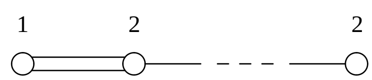
is symmetric, so that $-2|\alpha||\beta|^{-1} = -1|\beta||\alpha|^{-1}$ and $|\beta|^2 = 2|\alpha|^2$. Apart from a proportionality constant, we obtain the diagram

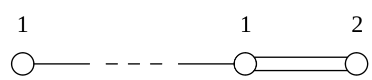


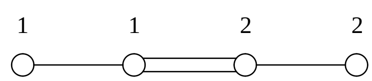
3c) In the case (G_2) of Step 2, similar reasoning leads us to the diagram



4) In case (B, C, F) of Step 2, the only possibilities are

(B)


(C)


(F_4)


Let us prove this assertion. In the notation of Step 3b, it is enough to show that

(2.82) $$(p-1)(q-1) < 2.$$

This inequality will follow by applying the Schwarz inequality to

$$\alpha = \sum_{i=1}^{p} i\alpha_i \qquad \text{and} \qquad \beta = \sum_{j=1}^{q} j\beta_j.$$

Since $|\alpha_1|^2 = \cdots = |\alpha_p|^2$, we have

$$-1 = A_{\alpha_i,\alpha_{i+1}} = \frac{2\langle\alpha_i, \alpha_{i+1}\rangle}{|\alpha_i|^2} = \frac{2\langle\alpha_i, \alpha_{i+1}\rangle}{|\alpha_p|^2}.$$

Thus

$$2\langle\alpha_i, \alpha_{i+1}\rangle = -|\alpha_p|^2.$$

Similarly
$$2\langle\beta_j, \beta_{j+1}\rangle = -|\beta_q|^2.$$

Also

$$2 = A_{\alpha_p,\beta_q} A_{\beta_q,\alpha_p} = \frac{4\langle\alpha_p, \beta_q\rangle^2}{|\alpha_p|^2|\beta_q|^2}$$

and hence

$$\langle\alpha_p, \beta_q\rangle^2 = \tfrac{1}{2}|\alpha_p|^2|\beta_q|^2.$$

Then

$$\langle\alpha, \beta\rangle = \sum_{i,j} \langle i\alpha_i, j\beta_j\rangle = pq\langle\alpha_p, \beta_q\rangle,$$

while

$$\begin{aligned}
|\alpha|^2 &= \sum_{i,j} \langle i\alpha_i, j\alpha_j\rangle = \sum_{i=1}^{p} i^2\langle\alpha_i, \alpha_i\rangle + 2\sum_{i=1}^{p-1} i(i+1)\langle\alpha_i, \alpha_{i+1}\rangle \\
&= |\alpha_p|^2\Big(\sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1)\Big) = |\alpha_p|^2\Big(p^2 - \sum_{i=1}^{p-1} i\Big) \\
&= |\alpha_p|^2(p^2 - \tfrac{1}{2}(p-1)p) = |\alpha_p|^2(\tfrac{1}{2}p(p+1)).
\end{aligned}$$

Similarly

$$|\beta|^2 = |\beta_q|^2(\tfrac{1}{2}q(q+1)).$$

Since α and β are nonproportional, the Schwarz inequality gives $\langle\alpha, \beta\rangle^2 < |\alpha|^2|\beta|^2$. Thus

$$\tfrac{1}{2}p^2q^2|\alpha_p|^2|\beta_q|^2 = p^2q^2\langle\alpha_p, \beta_q\rangle^2 < |\alpha_p|^2|\beta_q|^2(\tfrac{1}{4}p(p+1)q(q+1)).$$

Hence $2pq < (p+1)(q+1)$ and $pq < p+q+1$, and (2.82) follows.

5) In case (D, E) of Step 2, we may take $p \geq q \geq r$, and then the only possibilities are

$$(D) \qquad r = 2, \; q = 2, \; p \text{ arbitrary} \geq 2$$

$$(E) \qquad r = 2, \; q = 3, \; p = 3 \text{ or } 4 \text{ or } 5.$$

Let us prove this assertion. In the notation of Step 2c, it is enough to show that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1. \tag{2.83}$$

This inequality will follow by applying Parseval's equality to

$$\alpha = \sum_{i=1}^{p-1} i\alpha_i, \quad \beta = \sum_{j=1}^{q-1} j\beta_j, \quad \gamma = \sum_{k=1}^{r-1} k\gamma_k, \quad \text{and} \quad \delta.$$

As in Step 4 (but with p replaced by $p-1$), we have

$$2\langle\alpha_i, \alpha_{i+1}\rangle = -|\delta|^2 \qquad \text{and} \qquad |\alpha|^2 = |\delta|^2(\tfrac{1}{2}p(p-1)),$$

and similarly for β and γ. Also

$$\langle\alpha, \delta\rangle = \langle(p-1)\alpha_{p-1}, \delta\rangle = (p-1)(-\tfrac{1}{2}|\delta|^2) = -\tfrac{1}{2}(p-1)|\delta|^2$$

and similarly for β and γ. The span U of $\{\alpha, \beta, \gamma, \delta\}$ is 4-dimensional since these four vectors are linear combinations of disjoint subsets of members of a basis. Within this span the set

$$\left\{\frac{\alpha}{|\alpha|}, \frac{\beta}{|\beta|}, \frac{\gamma}{|\gamma|}\right\}$$

is orthonormal. Adjoin ε to this set to obtain an orthonormal basis of U. Since δ is independent of $\{\alpha, \beta, \gamma\}$, we have $\langle\delta, \varepsilon\rangle \neq 0$. By the Bessel inequality

$$|\delta|^2 \geq \left\langle\delta, \frac{\alpha}{|\alpha|}\right\rangle^2 + \left\langle\delta, \frac{\beta}{|\beta|}\right\rangle^2 + \left\langle\delta, \frac{\gamma}{|\gamma|}\right\rangle^2 + \langle\delta, \varepsilon\rangle^2,$$

with the last term > 0. Thus

$$\begin{aligned}
1 &> \left(\frac{\langle\alpha, \delta\rangle}{|\alpha||\delta|}\right)^2 + \left(\frac{\langle\beta, \delta\rangle}{|\beta||\delta|}\right)^2 + \left(\frac{\langle\gamma, \delta\rangle}{|\gamma||\delta|}\right)^2 \\
&= \left(\frac{p-1}{2}\right)^2 \frac{1}{\frac{1}{2}p(p-1)} + \left(\frac{q-1}{2}\right)^2 \frac{1}{\frac{1}{2}q(q-1)} + \left(\frac{r-1}{2}\right)^2 \frac{1}{\frac{1}{2}r(r-1)} \\
&= \frac{1}{2}\,\frac{p-1}{p} + \frac{1}{2}\,\frac{q-1}{q} + \frac{1}{2}\,\frac{r-1}{r}.
\end{aligned}$$

Thus $2 > 3 - \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)$, and (2.83) follows.

Theorem 2.84 (classification). Up to isomorphism the connected abstract Dynkin diagrams are exactly those in Figure 2.4, specifically A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$, D_n for $n \geq 4$, E_6, E_7, E_8, F_4, and G_2.

REMARKS.

1) The subscripts refer to the numbers of vertices in the various diagrams.

2) The names A_n, B_n, C_n, D_n, and G_2 are names of root systems, and Corollary 2.65 shows that the associated Dynkin diagrams are independent of the ordering. As yet, the names E_6, E_7, E_8, and F_4 are attached only to abstract Dynkin diagrams. At the end of this section, we show that these diagrams come from root systems, and then we may use these names unambiguously for the root systems.

PROOF. We have seen that any connected abstract Dynkin diagram has to be one of the ones in this list, up to isomorphism. Also we know that A_n, B_n, C_n, D_n, and G_2 come from abstract reduced root systems and are therefore legitimate Dynkin diagrams. To check that E_6 E_7, E_8, and F_4 are legitimate Dynkin diagrams, we write down the candidates for abstract Cartan matrices and observe the first four defining properties of an abstract Cartan matrix by inspection. For property (e) we exhibit vectors $\{\alpha_i\}$ for each case such that the matrix in question has entries $A_{ij} = \dfrac{2\langle\alpha_i, \alpha_j\rangle}{|\alpha_i|^2}$, and then property (e) follows.

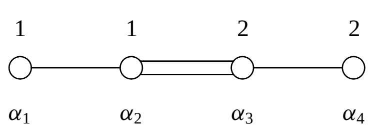
For F_4, the matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \tag{2.85a}$$

and the vectors are the following members of $\mathbb{R}^4$:

$$\begin{aligned} \alpha_1 &= \tfrac{1}{2}(e_1 - e_2 - e_3 - e_4) \\ \alpha_2 &= e_4 \\ \alpha_3 &= e_3 - e_4 \\ \alpha_4 &= e_2 - e_3. \end{aligned} \tag{2.85b}$$

For reference we note that these vectors are attached to the vertices of the Dynkin diagram as follows:

(2.85c) 

For E_8, the matrix is

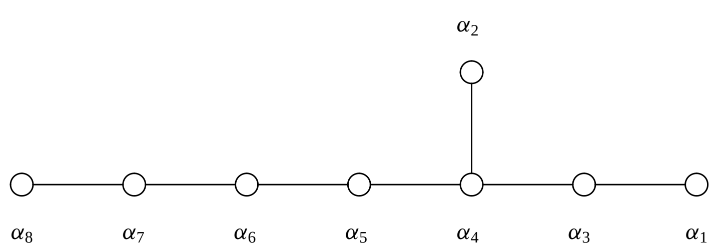
$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \tag{2.86a}$$

and the vectors are the following members of $\mathbb{R}^8$:

$$\begin{aligned} \alpha_1 &= \tfrac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1) \\ \alpha_2 &= e_2 + e_1 \\ \alpha_3 &= e_2 - e_1 \\ \alpha_4 &= e_3 - e_2 \\ \alpha_5 &= e_4 - e_3 \\ \alpha_6 &= e_5 - e_4 \\ \alpha_7 &= e_6 - e_5 \\ \alpha_8 &= e_7 - e_6. \end{aligned} \tag{2.86b}$$

For reference we note that these vectors are attached to the vertices of the Dynkin diagram as follows:

(2.86c)



For E_7 or E_6, the matrix is the first 7 or 6 rows and columns of (2.86a), and the vectors are the first 7 or 6 of the vectors (2.86b).

This completes the classification of abstract Cartan matrices. The corresponding Dynkin diagrams are tabulated in Figure 2.4.

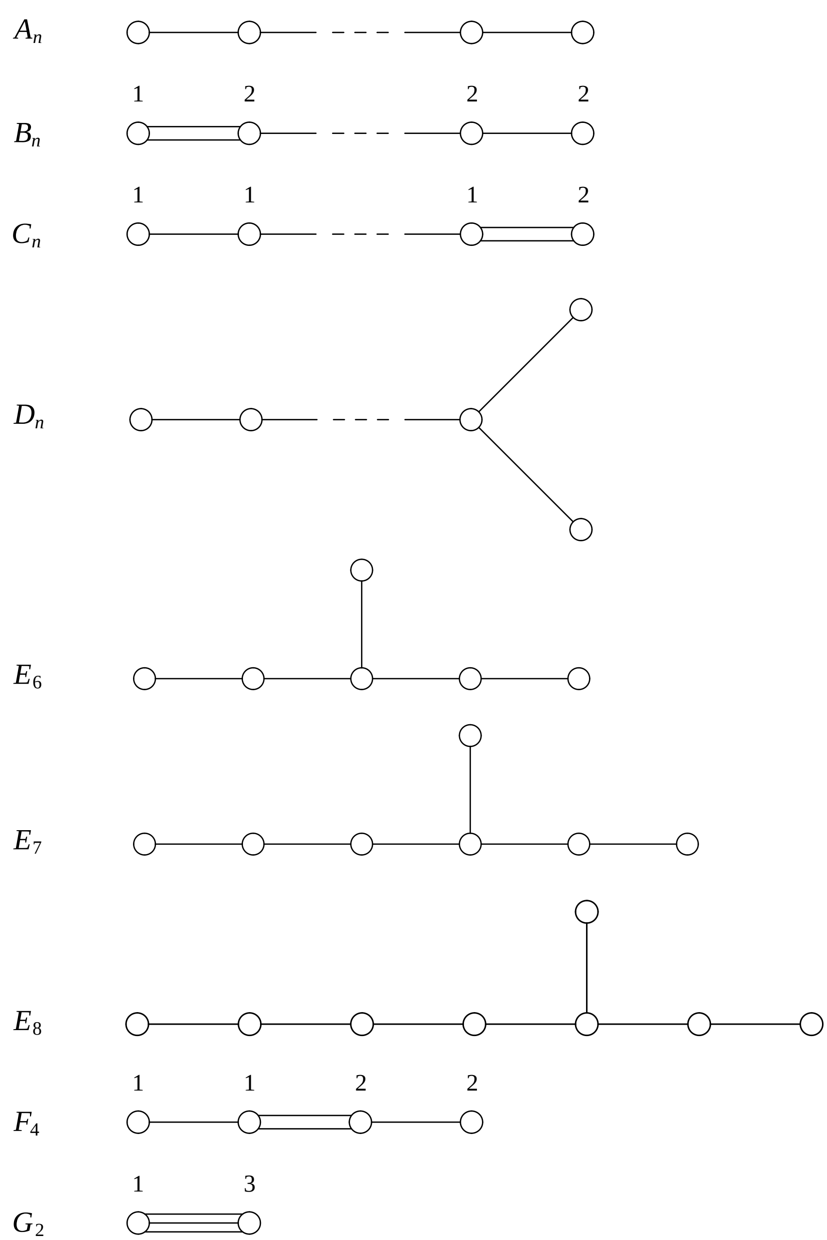


FIGURE 2.4. Classification of Dynkin diagrams

Actually we can see without difficulty that E_6, E_7, E_8, and F_4 are not just abstract Cartan matrices but actually come from abstract reduced root systems. As we remarked in connection with Theorem 2.84, we can then use the same names for the abstract root systems as for the Cartan matrices. The fact that E_6, E_7, E_8, and F_4 come from abstract reduced root systems enables us to complete our examination of the second step of the passage (2.58) from complex semisimple Lie algebras to abstract Cartan matrices.

Proposition 2.87. The second step in the passage (2.58) is onto. That is, every abstract Cartan matrix comes from a reduced root system.

Proof. In the case of F_4, we take $V = \mathbb{R}^4$, and we let

$$\Delta = \begin{cases} \pm e_i & \\ \pm e_i \pm e_j & \text{for } i \neq j \\ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) & \end{cases} \tag{2.88}$$

with all possible signs allowed. We have to check the axioms for an abstract root system. Certainly the roots span $\mathbb{R}^4$, and it is a simple matter to check that $2\langle\beta, \alpha\rangle/|\alpha|^2$ is always an integer. The problem is to check that the root reflections carry roots to roots. The case that needs attention is $s_\alpha\beta$ with α of the third kind. If β is of the first kind, then $s_\alpha\beta = \pm s_\beta\alpha$, and there is no difficulty. If β is of the second kind, there is no loss of generality in assuming that $\beta = e_1 + e_2$. Then $s_\alpha\beta = \beta$ unless the coefficients of e_1 and e_2 in α are equal. In this case $s_\alpha\beta$ gives plus or minus the e_3, e_4 part of β, but without the factor of $\frac{1}{2}$.

Now suppose that α and β are both of the third kind. We need to consider $s_\alpha\beta$ when one or three of the signs in α and β match. In either case there is one exceptional sign, say as coefficient of e_i. Then $s_\alpha\beta = \pm e_i$, and hence the root reflections carry Δ to itself.

Therefore Δ is an abstract reduced root system. The vectors α_i in (2.85b) are the simple roots relative to the lexicographic ordering obtained from the ordered basis e_1, e_2, e_3, e_4, and then (2.85a) is the Cartan matrix.

In the case of E_8, we take $V = \mathbb{R}^8$, and we let

$$\Delta = \begin{cases} \pm e_i \pm e_j & \text{for } i \neq j \\ \frac{1}{2}\sum_{i=1}^8 (-1)^{n(i)} e_i & \text{with } \sum_{i=1}^8 (-1)^{n(i)} \text{ even.} \end{cases} \tag{2.89}$$

For the first kind of root, all possible signs are allowed. Again we have to check the axioms for an abstract root system, and again the problem is to check that the root reflections carry roots to roots. This time all roots have the same length. Thus when α and β are nonorthogonal and nonproportional, we have $s_\alpha\beta = \pm s_\beta\alpha$. Hence matters come down to checking the case that α and β are both of the second kind.

In this case we need to consider $s_\alpha\beta$ when two or six of the signs in α and β match. In either case there are two exceptional signs, say as coefficients of e_i and e_j. We readily check that $s_\alpha\beta = \pm e_i \pm e_j$ for a suitable choice of signs, and hence the root reflections carry Δ to itself.

Therefore Δ is an abstract reduced root system. The vectors α_i in (2.86b) are the simple roots relative to the lexicographic ordering obtained from the ordered basis $e_8, e_7, e_6, e_5, e_4, e_3, e_2, e_1$, and then (2.86a) is the Cartan matrix.

In the case of E_7, we take V to be the subspace of the space for E_8 orthogonal to $e_8 + e_7$, and we let Δ be the set of roots for E_8 that are in this space. Since E_8 is a root system, it follows that E_7 is a root system. All the α_i for E_8 except α_8 are roots for E_7, and they must remain simple. Since there are 7 such roots, we see that $\alpha_1, \ldots, \alpha_7$ must be all of the simple roots. The associated Cartan matrix is then the part of (2.86a) that excludes α_8.

In the case of E_6, we take V to be the subspace of the space for E_8 orthogonal to $e_8 + e_7$ and $e_8 + e_6$, and we let Δ be the set of roots for E_8 that are in this space. Since E_8 is a root system, it follows that E_6 is a root system. All the α_i for E_8 except α_7 and α_8 are roots for E_6, and they must remain simple. Since there are 6 such roots, we see that $\alpha_1, \ldots, \alpha_6$ must be all of the simple roots. The associated Cartan matrix is then the part of (2.86a) that excludes α_7 and α_8.

8. Classification of Nonreduced Abstract Root Systems

In this section we digress from considering the two-step passage (2.58) from complex semisimple Lie algebras to abstract Cartan matrices. Our topic will be nonreduced abstract root systems. Abstract root systems that are not necessarily reduced arise in the structure theory of real semisimple Lie algebras, as presented in Chapter VI; the root systems in question are the systems of "restricted roots" of the Lie algebra. In order not to attach special significance later to those real semisimple Lie algebras whose systems of restricted roots turn out to be reduced, we shall give a classification now of nonreduced abstract root systems. There is no loss of generality in assuming that such a system is irreducible.

An example arises by forming the union of the root systems B_n and C_n

given in (2.43). The union is called $(BC)_n$ and is given as follows:

(2.90)
$$(BC)_n \quad V = \left\{ \sum_{i=1}^n a_i e_i \right\} \quad \Delta = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm e_i\} \cup \{\pm 2e_i\}.$$

A diagram of all of the roots of $(BC)_2$ appears in Figure 2.2.

In contrast with Proposition 2.66, the simple roots of an abstract root system that is not necessarily reduced do not determine the root system. For example, if B_n and $(BC)_n$ are taken to have the sets of positive roots as in (2.50), then they have the same sets of simple roots. Thus it is not helpful to associate an unadorned abstract Cartan matrix and Dynkin diagram to such a system. But we can associate the slightly more complicated diagram in Figure 2.5 to $(BC)_n$, and it conveys useful unambiguous information.

$(BC)_n$

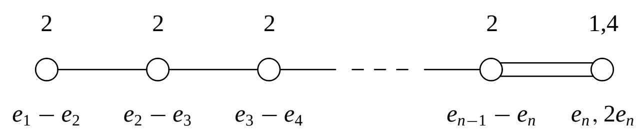


FIGURE 2.5. Substitute Dynkin diagram for $(BC)_n$

Now let Δ be any abstract root system in an inner product space V. Recall that if α is a root and $\frac{1}{2}\alpha$ is not a root, we say that α is **reduced**.

Lemma 2.91. The reduced roots $\alpha \in \Delta$ form a reduced abstract root system Δ_s in V. The roots $\alpha \in \Delta$ such that $2\alpha \notin \Delta$ form a reduced abstract root system Δ_l in V. The Weyl groups of Δ, Δ_s, and Δ_l coincide.

PROOF. It follows immediately from the definitions that Δ_s and Δ_l are abstract root systems. Also it is clear that Δ_s and Δ_l are reduced. The reflections for Δ, Δ_s, and Δ_l coincide, and hence the Weyl groups coincide.

Proposition 2.92. Up to isomorphism the only irreducible abstract root systems Δ that are not reduced are of the form $(BC)_n$ for $n \geq 1$.

PROOF. We impose a lexicographic ordering, thereby fixing a system of simple roots. Also we form Δ_s as in Lemma 2.91. Since Δ is not reduced, there exists a root α such that 2α is a root. By Proposition 2.62, α is conjugate via the Weyl group to a simple root. Thus there exists a simple root β such that 2β is a root. Evidently β is simple in Δ_s, and Δ_s

is irreducible. Let $\gamma \neq \beta$ be any simple root of Δ_s such that $\langle \beta, \gamma \rangle \neq 0$. Then

$$\frac{2\langle \gamma, \beta \rangle}{|\beta|^2} \qquad \text{and} \qquad \frac{2\langle \gamma, 2\beta \rangle}{|2\beta|^2} = \frac{1}{2}\, \frac{2\langle \gamma, \beta \rangle}{|\beta|^2}$$

are negative integers, and it follows that $2\langle \gamma, \beta \rangle / |\beta|^2 = -2$. Referring to the classification in Theorem 2.84, we see that Δ_s is of type B_n, with β as the unique short simple root. Any Weyl group conjugate β' of β has $2\beta'$ in Δ, and the roots β' with $2\beta'$ in Δ are exactly those with $|\beta'| = |\beta|$. The result follows.

9. Serre Relations

We return to our investigation of the two-step passage (2.58), first from complex semisimple Lie algebras to reduced abstract root systems and then from reduced abstract root systems to abstract Cartan matrices. We have completed our investigation of the second step, showing that that step is independent of the choice of ordering up to isomorphism, is one-one up to isomorphism, and is onto. Moreover, we have classified the abstract Cartan matrices.

For the remainder of this chapter we concentrate on the first step. Theorem 2.15 enabled us to see that the passage from complex semisimple Lie algebras to reduced abstract root systems is well defined up to isomorphism, and we now want to see that it is one-one and onto, up to isomorphism. First we show that it is one-one. Specifically we shall show that an isomorphism between the root systems of two complex semisimple Lie algebras lifts to an isomorphism between the Lie algebras themselves. More than one such isomorphism of Lie algebras exists, and we shall impose additional conditions so that the isomorphism exists and is unique. The result, known as the Isomorphism Theorem, will be the main result of the next section and will be the cornerstone of our development of structure theory for real semisimple Lie algebras and Lie groups in Chapter VI. The technique will be to use generators and relations, realizing any complex semisimple Lie algebra as the quotient of a "free Lie algebra" by an ideal generated by some "relations."

Thus let $\mathfrak{g}$ be a complex semisimple Lie algebra, fix a Cartan subalgebra $\mathfrak{h}$, let Δ be the set of roots, let B be a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ that is positive definite on the real form of $\mathfrak{h}$ where the roots are real, let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a simple system, and let $A =$

$(A_{ij})_{i,j=1}^{l}$ be the Cartan matrix. For $1 \le i \le l$, let

$$
\begin{aligned}
h_i &= \frac{2}{|\alpha_i|^2} H_{\alpha_i} \\
e_i &= \text{nonzero root vector for } \alpha_i \\
f_i &= \text{nonzero root vector for } -\alpha_i \text{ with } B(e_i, f_i) = 2/|\alpha_i|^2.
\end{aligned}
\tag{2.93}
$$

Proposition 2.94. The set $X = \{h_i, e_i, f_i\}_{i=1}^{l}$ generates $\mathfrak{g}$ as a Lie algebra.

REMARK. We call X a set of **standard generators** of $\mathfrak{g}$ relative to $\mathfrak{h}$, Δ, B, Π, and $A = (A_{ij})_{i,j=1}^{l}$.

PROOF. The linear span of the h_i's is all of $\mathfrak{h}$ since the α_i form a basis of $\mathfrak{h}^*$. Let α be a positive root, and let e_α be a nonzero root vector. If $\alpha = \sum_i n_i \alpha_i$, we show by induction on the level $\sum_i n_i$ that e_α is a multiple of an iterated bracket of the e_i's. If the level is 1, then $\alpha = \alpha_j$ for some j, and e_α is a multiple of e_j. Assume the result for level $< n$ and let the level of α be $n > 1$. Since

$$0 < |\alpha|^2 = \sum_i n_i \langle \alpha, \alpha_i \rangle,$$

we must have $\langle \alpha, \alpha_j \rangle > 0$ for some j. By Proposition 2.48e, $\beta = \alpha - \alpha_j$ is a root, and Proposition 2.49 shows that β is positive. If e_β is a nonzero root vector for β, then the induction hypothesis shows that e_β is a multiple of an iterated bracket of the e_i's. Corollary 2.35 shows that e_α is a multiple of $[e_\beta, e_j]$, and the induction is complete.

Thus all the root spaces for positive roots are in the Lie subalgebra of $\mathfrak{g}$ generated by X. A similar argument with negative roots, using the f_i's, shows that the root spaces for the negative roots are in this Lie subalgebra, too. Therefore X generates all of $\mathfrak{g}$.

Proposition 2.95. The set $X = \{h_i, e_i, f_i\}_{i=1}^{l}$ satisfies the following properties within $\mathfrak{g}$:

(a) $[h_i, h_j] = 0$,
(b) $[e_i, f_j] = \delta_{ij} h_i$,
(c) $[h_i, e_j] = A_{ij} e_j$,
(d) $[h_i, f_j] = -A_{ij} f_j$,
(e) $(\operatorname{ad} e_i)^{-A_{ij}+1} e_j = 0$ when $i \ne j$,
(f) $(\operatorname{ad} f_i)^{-A_{ij}+1} f_j = 0$ when $i \ne j$.

REMARK. Relations (a) through (f) are called the **Serre relations** for $\mathfrak{g}$. We shall refer to them by letter.

PROOF.
(a) The subalgebra $\mathfrak{h}$ is abelian.
(b) For $i = j$, we use Lemma 2.18a. When $i \neq j$, $\alpha_i - \alpha_j$ cannot be a root, by Proposition 2.49.
(c, d) We observe that $[h_i, e_j] = \alpha_j(h_i)e_j = \frac{2}{|\alpha_i|^2}\alpha_j(H_{\alpha_i})e_j = A_{ij}e_j$, and we argue similarly for $[h_i, f_j]$.
(e, f) When $i \neq j$, the α_i string containing α_j is

$$\alpha_j, \alpha_j + \alpha_i, \dots, \alpha_j + q\alpha_i \qquad \text{since } \alpha_j - \alpha_i \notin \Delta.$$

Thus $p = 0$ for the root string, and

$$-q = p - q = \frac{2\langle \alpha_j, \alpha_i \rangle}{|\alpha_i|^2} = A_{ij}.$$

Hence $1 - A_{ij} = q + 1$, and $\alpha_j + (1 - A_{ij})\alpha_i$ is not a root. Then (e) follows, and (f) is proved similarly.

Now we look at (infinite-dimensional) complex Lie algebras with no relations. A **free Lie algebra** on a set X is a pair $(\mathfrak{F}, \iota)$ consisting of a Lie algebra $\mathfrak{F}$ and a function $\iota : X \to \mathfrak{F}$ with the following universal mapping property: Whenever $\mathfrak{l}$ is a complex Lie algebra and $\varphi : X \to \mathfrak{l}$ is a function, then there exists a unique Lie algebra homomorphism $\widetilde{\varphi}$ such that the diagram

$$\begin{tikzcd} & \mathfrak{F} \arrow[dr, dashed, "\widetilde{\varphi}"] & \\ X \arrow[ur, "\iota"] \arrow[rr, "\varphi"'] & & \mathfrak{l} \end{tikzcd}$$

commutes.

Proposition 2.96. If X is a nonempty set, then there exists a free Lie algebra $\mathfrak{F}$ on X, and the image of X in $\mathfrak{F}$ generates $\mathfrak{F}$. Any two free Lie algebras on X are canonically isomorphic.

REMARK. The proof is elementary but uses the Poincaré–Birkhoff–Witt Theorem, which will be not be proved until Chapter III. We therefore postpone the proof of Proposition 2.96 until that time.

Now we can express our Lie algebra in terms of generators and relations. With $\mathfrak{g}$, $\mathfrak{h}$, Δ, B, Π, and $A = (A_{ij})_{i,j=1}^{l}$ as before, let $\mathfrak{F}$ be the free Lie algebra on the set $X = \{h_i, e_i, f_i\}_{i=1}^{l}$, and let $\mathfrak{R}$ be the ideal in $\mathfrak{F}$ generated by the Serre relations (a) through (f), i.e., generated by the differences of the left sides and right sides of all equalities (a) through (f) in Proposition 2.95. We set up the diagram

(2.97)
$$\begin{tikzcd} & \mathfrak{F} \arrow[dr, dashed] & \\ X \arrow[ur] \arrow[rr] & & \mathfrak{g} \end{tikzcd}$$

and obtain a Lie algebra homomorphism of $\mathfrak{F}$ into $\mathfrak{g}$. This homomorphism carries $\mathfrak{R}$ to 0 as a consequence of Proposition 2.95, and therefore it descends to a Lie algebra homomorphism

$$\mathfrak{F}/\mathfrak{R} \longrightarrow \mathfrak{g}$$

that is onto $\mathfrak{g}$ by Proposition 2.94 and is one-one on the linear span of $X = \{h_i, e_i, f_i\}_{i=1}^{l}$. We call this map the **canonical homomorphism** of $\mathfrak{F}/\mathfrak{R}$ onto $\mathfrak{g}$ relative to $\{h_i, e_i, f_i\}_{i=1}^{l}$.

Theorem 2.98 (Serre). Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $X = \{h_i, e_i, f_i\}_{i=1}^{l}$ be a set of standard generators. Let $\mathfrak{F}$ be the free Lie algebra on $3l$ generators h_i, e_i, f_i with $1 \le i \le l$, and let $\mathfrak{R}$ be the ideal generated in $\mathfrak{F}$ by the Serre relations (a) through (f). Then the canonical homomorphism of $\mathfrak{F}/\mathfrak{R}$ onto $\mathfrak{g}$ is an isomorphism.

REMARK. The proof will be preceded by two lemmas that will play a role both here and in §11.

Lemma 2.99. Let $A = A = (A_{ij})_{i,j=1}^{l}$ be an abstract Cartan matrix, let $\mathfrak{F}$ be the free Lie algebra on $3l$ generators h_i, e_i, f_i with $1 \le i \le l$, and let $\widetilde{\mathfrak{R}}$ be the ideal generated in $\mathfrak{F}$ by the Serre relations (a) through (d). Define $\widetilde{\mathfrak{g}} = \mathfrak{F}/\widetilde{\mathfrak{R}}$, and write h_i, e_i, f_i also for the images of the generators in $\widetilde{\mathfrak{g}}$. In $\widetilde{\mathfrak{g}}$, put

$$\begin{aligned} \widetilde{\mathfrak{h}} &= \operatorname{span}\{h_i\}, \quad \text{an abelian Lie subalgebra} \\ \widetilde{\mathfrak{e}} &= \text{Lie subalgebra generated by all } e_i \\ \widetilde{\mathfrak{f}} &= \text{Lie subalgebra generated by all } f_i. \end{aligned}$$

Then

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{e}} \oplus \widetilde{\mathfrak{f}}.$$

PROOF. Proposition 2.96 shows that X generates $\mathfrak{F}$, and consequently the image of X in $\widetilde{\mathfrak{g}}$ generates $\widetilde{\mathfrak{g}}$. Therefore $\widetilde{\mathfrak{g}}$ is spanned by iterated brackets of elements from X. In $\widetilde{\mathfrak{g}}$, each generator from X is an eigenvector under $\operatorname{ad} h_i$, by Serre relations (a), (c), and (d). Hence so is any iterated bracket, the eigenvalue for an iterated bracket being the sum of the eigenvalues from the factors.

To see that

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{h}} + \widetilde{\mathfrak{e}} + \widetilde{\mathfrak{f}}, \tag{2.100}$$

we observe that X is contained in the right side of (2.100). Thus it is enough to see that the right side is invariant under the operation $\operatorname{ad} x$ for each $x \in X$. Each of $\widetilde{\mathfrak{h}}$, $\widetilde{\mathfrak{e}}$, $\widetilde{\mathfrak{f}}$ is invariant under $\operatorname{ad} h_i$, from the previous paragraph. Also $\widetilde{\mathfrak{h}} + \widetilde{\mathfrak{e}}$ is invariant under $\operatorname{ad} e_i$. We prove that $(\operatorname{ad} f_i)\widetilde{\mathfrak{e}} \subseteq \widetilde{\mathfrak{h}} + \widetilde{\mathfrak{e}}$. We do so by treating the iterated brackets that span $\widetilde{\mathfrak{e}}$, proceeding inductively on the number of factors. When we have one factor, Serre relation (b) gives us

$$(\operatorname{ad} f_i)e_j = -\delta_{ij} h_i \in \widetilde{\mathfrak{h}} + \widetilde{\mathfrak{e}}.$$

When we have more than one factor, let the iterated bracket from $\widetilde{\mathfrak{e}}$ be $[x, y]$ with n factors, where x and y have $< n$ factors. Then $(\operatorname{ad} f_i)x$ and $(\operatorname{ad} f_i)y$ are in $\widetilde{\mathfrak{h}} + \widetilde{\mathfrak{e}}$ by inductive hypothesis, and hence

$$(\operatorname{ad} f_i)[x, y] = [(\operatorname{ad} f_i)x, y] + [x, (\operatorname{ad} f_i)y] \in [\widetilde{\mathfrak{h}} + \widetilde{\mathfrak{e}}, \widetilde{\mathfrak{e}}] + [\widetilde{\mathfrak{e}}, \widetilde{\mathfrak{h}} + \widetilde{\mathfrak{e}}] \subseteq \widetilde{\mathfrak{e}}.$$

Therefore $(\operatorname{ad} x)\widetilde{\mathfrak{e}} \subseteq \widetilde{\mathfrak{h}} + \widetilde{\mathfrak{e}} + \widetilde{\mathfrak{f}}$ for each $x \in X$. Similarly $(\operatorname{ad} x)\widetilde{\mathfrak{f}} \subseteq \widetilde{\mathfrak{h}} + \widetilde{\mathfrak{e}} + \widetilde{\mathfrak{f}}$ for each $x \in X$, and we obtain (2.100).

Now let us prove that the sum (2.100) is direct. As we have seen, each term on the right side of (2.100) is spanned by simultaneous eigenvectors for $\operatorname{ad} \widetilde{\mathfrak{h}}$. Let us be more specific. As a result of Serre relation (c), an iterated bracket in $\widetilde{\mathfrak{e}}$ involving $e_{j_1}, \dots, e_{j_k}$ has eigenvalue under $\operatorname{ad} h_i$ given by

$$A_{ij_1} + \cdots + A_{ij_k} = \sum_{j=1}^{l} m_j A_{ij} \qquad \text{with } m_j \geq 0 \text{ an integer.}$$

If an eigenvalue for $\widetilde{\mathfrak{e}}$ coincides for all i with an eigenvalue for $\widetilde{\mathfrak{h}} + \widetilde{\mathfrak{f}}$, we obtain an equation $\sum_{j=1}^{l} m_j A_{ij} = -\sum_{j=1}^{l} n_j A_{ij}$ for all i with $m_j \geq 0$, $n_j \geq 0$, and not all m_j equal to 0. Consequently $\sum_{i=1}^{l}(m_j + n_j)A_{ij} = 0$ for all i. Since (A_{ij}) is nonsingular, $m_j + n_j = 0$ for all j. Then $m_j = n_j = 0$ for all j, contradiction. Therefore the sum (2.100) is direct.

Lemma 2.101. Let $A = (A_{ij})_{i,j=1}^{l}$ be an abstract Cartan matrix, let $\mathfrak{F}$ be the free Lie algebra on $3l$ generators h_i, e_i, f_i with $1 \leq i \leq l$, and let $\mathfrak{R}$ be the ideal generated in $\mathfrak{F}$ by the Serre relations (a) through (f). Define $\mathfrak{g}' = \mathfrak{F}/\mathfrak{R}$, and suppose that $\operatorname{span}\{h_i\}_{i=1}^{l}$ maps one-one from $\mathfrak{F}$ into $\mathfrak{g}'$. Write h_i also for the images of the generators h_i in $\mathfrak{g}'$. Then $\mathfrak{g}'$ is a (finite-dimensional) complex semisimple Lie algebra, the subspace $\mathfrak{h}' = \operatorname{span}\{h_i\}$ is a Cartan subalgebra, the linear functionals $\alpha_j \in \mathfrak{h}'^*$ given by $\alpha_j(h_i) = A_{ij}$ form a simple system within the root system, and the Cartan matrix relative to this simple system is exactly A.

PROOF. Use the notation e_i and f_i also for the images of the generators e_i and f_i in $\mathfrak{g}'$. Let us observe that under the quotient map from $\mathfrak{F}$ to $\mathfrak{g}'$, all the e_i's and f_i's map to nonzero elements in $\mathfrak{g}'$. In fact, $\{h_i\}$ maps to a linearly independent set by hypothesis, and hence the images of the h_i's are nonzero. Then Serre relation (b) shows that $[e_i, f_i] = h_i \neq 0$ in $\mathfrak{g}'$, and hence e_i and f_i are nonzero in $\mathfrak{g}'$, as asserted..

Because the h_i are linearly independent in $\mathfrak{g}$, we can define $\alpha_j \in \mathfrak{h}'^*$ by $\alpha_j(h_i) = A_{ij}$. These linear functionals are a basis of $\mathfrak{h}'^*$. For $\varphi \in \mathfrak{h}'^*$, put

$$\mathfrak{g}'_\varphi = \{x \in \mathfrak{g}' \mid (\operatorname{ad} h)x = \varphi(h)x \text{ for all } h \in \mathfrak{h}'\}.$$

We call φ a **root** if $\varphi \neq 0$ and $\mathfrak{g}'_\varphi \neq 0$, and we call $\mathfrak{g}'_\varphi$ the corresponding **root space**. The Lie algebra $\mathfrak{g}'$ is a quotient of the Lie algebra $\widetilde{\mathfrak{g}}$ of Lemma 2.99, and it follows from Lemma 2.99 that

$$\mathfrak{g}' = \mathfrak{h}' \oplus \bigoplus_{\varphi = \text{root}} \mathfrak{g}'_\varphi$$

and that all roots are of the form $\varphi = \sum n_j \alpha_j$ with all nonzero n_j given as integers of the same sign. Let Δ' be the set of all roots, Δ'^+ the set of all roots with all $n_j \geq 0$, and Δ'^- the set of all roots with all $n_j \leq 0$. We have just established that

$$\Delta' = \Delta'^+ \cup \Delta'^-. \tag{2.102}$$

Let us show that $\mathfrak{g}'_\varphi$ is finite dimensional for each root φ. First consider $\varphi = \sum n_j \alpha_j$ in Δ'^+. Lemma 2.99 shows that $\mathfrak{g}'_\varphi$ is spanned by the images of all iterated brackets of e_i's in $\widetilde{\mathfrak{g}}$ involving n_j instances of e_j, and there are only finitely many such iterated brackets. Therefore $\mathfrak{g}'_\varphi$ is finite dimensional when φ is in Δ'^+. Similarly $\mathfrak{g}'_\varphi$ is finite dimensional when φ is in Δ'^-, and it follows from (2.102) that $\mathfrak{g}'_\varphi$ is finite dimensional for each root φ.

The vectors e_i and f_i, which we have seen are nonzero, are in the respective spaces $\mathfrak{g}'_{\alpha_i}$ and $\mathfrak{g}'_{-\alpha_i}$, and hence each α_i and $-\alpha_i$ is a root. For these roots the root spaces have dimension 1.

Next let us show for each $\varphi \in \mathfrak{h}'^*$ that

$$\dim \mathfrak{g}'_{\varphi} = \dim \mathfrak{g}'_{-\varphi} \quad \text{and hence} \quad \Delta'^{-} = -\Delta'^{+}. \tag{2.103}$$

In fact, we set up the diagram

$$\begin{tikzcd} & \mathfrak{F} \arrow[dr, dashed, "\widetilde{\eta}"] & \\ X \arrow[ur, "\iota"] \arrow[rr, "\eta"'] & & \mathfrak{F} \end{tikzcd}$$

where η is the function $\eta(e_i) = f_i$, $\eta(f_i) = e_i$, and $\eta(h_i) = -h_i$. By the universal mapping property of $\mathfrak{F}$, η extends to a Lie algebra homomorphism $\widetilde{\eta}$ of $\mathfrak{F}$ into itself. If we next observe that $\widetilde{\eta}^2$ is an extension of the inclusion ι of X into $\mathfrak{F}$ in the diagram

$$\begin{tikzcd} & \mathfrak{F} \arrow[dr, dashed, "\widetilde{\eta}^2"] & \\ X \arrow[ur, "\iota"] \arrow[rr, "\iota"'] & & \mathfrak{F} \end{tikzcd}$$

then we conclude from the uniqueness of the extension that $\widetilde{\eta}^2 = 1$. We readily check that $\widetilde{\eta}(\mathfrak{R}) \subseteq \mathfrak{R}$, and hence $\widetilde{\eta}$ descends to a homomorphism $\widetilde{\eta} : \mathfrak{g}' \to \mathfrak{g}'$ that is -1 on $\mathfrak{h}'$ and interchanges e_i with f_i for all i. Moreover $\widetilde{\eta}^2 = 1$. Since $\widetilde{\eta}$ is -1 on $\mathfrak{h}'$ and is invertible, we see that $\widetilde{\eta}(\mathfrak{g}'_{\varphi}) = \mathfrak{g}'_{-\varphi}$ for all $\varphi \in \mathfrak{h}'^*$, and then (2.103) follows.

We shall introduce an inner product on the real form of $\mathfrak{h}'^*$ given by $\mathfrak{h}'^*_0 = \sum \mathbb{R}\alpha_i$. We saw in (2.75) and (2.77) how to construct vectors $\beta_i \in \mathbb{R}^l$ for $1 \le i \le l$ such that

$$A_{ij} = 2\langle \beta_i, \beta_j \rangle / |\beta_i|^2. \tag{2.104}$$

We define a linear map $\mathbb{R}^l \to \mathfrak{h}'^*_0$ by $\beta_i \mapsto \alpha_i$, and we carry the inner product from $\mathbb{R}^l$ to $\mathfrak{h}'^*_0$. Then we have

$$\alpha_j(h_i) = A_{ij} = \frac{2\langle \beta_i, \beta_j \rangle}{|\beta_i|^2} = \frac{2\langle \alpha_i, \alpha_j \rangle}{|\alpha_i|^2} = \alpha_j\left(\frac{2H_{\alpha_i}}{|\alpha_i|^2}\right)$$

for all j, and it follows that

$$h_i = \frac{2H_{\alpha_i}}{|\alpha_i|^2} \tag{2.105}$$

in this inner product.

Next we define a Weyl group. For $1 \le i \le l$, let $s_{\alpha_i} : \mathfrak{h}_0'^* \to \mathfrak{h}_0'^*$ be the linear transformation given by

$$s_{\alpha_i}(\varphi) = \varphi - \varphi(h_i)\alpha_i = \varphi - \frac{2\langle \varphi, \alpha_i \rangle}{|\alpha_i|^2}\,\alpha_i.$$

This is an orthogonal transformation on $\mathfrak{h}_0'^*$. Let W' be the group of orthogonal transformations generated by the s_{α_i}, $1 \le i \le l$.

Let us prove that W' is a finite group. From the correspondence of reduced abstract root systems to abstract Cartan matrices established in §7, we know that the members $\beta_i \in \mathbb{R}^l$ in (2.104) have reflections generating a finite group W such that $\Delta = \bigcup_{i=1}^{l} W\beta_i$ is the reduced abstract root system associated to the abstract Cartan matrix A. Under the isomorphism $\beta_i \mapsto \alpha_i$, W is identified with W', and Δ is identified with the subset $\bigcup_{i=1}^{l} W\alpha_i$ of $\mathfrak{h}_0'^*$. Since $W \cong W'$, W' is finite.

We now work toward the conclusion that $\mathfrak{g}'$ is finite dimensional. Fix i, and let $\mathfrak{sl}_i$ be the span of $\{h_i, e_i, f_i\}$ within $\mathfrak{g}'$. This is a Lie subalgebra of $\mathfrak{g}'$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. We shall first show that every element of $\mathfrak{g}'$ lies in a finite-dimensional subspace invariant under $\mathfrak{sl}_i$.

If $j \ne i$, consider the subspace of $\mathfrak{g}'$ spanned by

$$f_j, (\operatorname{ad} f_i) f_j, \ldots, (\operatorname{ad} f_i)^{-A_{ij}} f_j.$$

These vectors are eigenvectors for $\operatorname{ad} h_i$ with respective eigenvalues

$$\alpha_j(h_i),\ \alpha_j(h_i) - 2,\ \ldots, \alpha_j(h_i) + 2A_{ij},$$

and hence the subspace is invariant under $\operatorname{ad} h_i$. It is invariant under $\operatorname{ad} f_i$ since $(\operatorname{ad} f_i)^{-A_{ij}+1} f_j = 0$ by Serre relation (f). Finally it is invariant under $\operatorname{ad} e_i$ by induction, starting from the fact that $(\operatorname{ad} e_i) f_j = 0$ (Serre relation (b)). Thus the subspace is invariant under $\mathfrak{sl}_i$.

Similarly for $j \ne i$, the subspace of $\mathfrak{g}'$ spanned by

$$e_j, (\operatorname{ad} e_i)e_j, \ldots, (\operatorname{ad} e_i)^{-A_{ij}} e_j$$

is invariant under $\mathfrak{sl}_i$, by Serre relations (e) and (b). And also $\operatorname{span}\{h_i, e_i, f_i\}$ is invariant under $\mathfrak{sl}_i$. Therefore a generating subset of $\mathfrak{g}'$ lies in a finite-dimensional subspace invariant under $\mathfrak{sl}_i$.

Now consider the set of all elements in $\mathfrak{g}'$ that lie in some finite-dimensional space invariant under $\mathfrak{sl}_i$. Say r and s are two such elements, lying in spaces R and S. Form the finite-dimensional subspace $[R, S]$ generated by all brackets from R and S. If x is in $\mathfrak{sl}_i$, then

$$(\operatorname{ad} x)[R, S] \subseteq [(\operatorname{ad} x)R, S] + [R, (\operatorname{ad} x)S] \subseteq [R, S],$$

and hence $[r, s]$ is such an element of $\mathfrak{g}'$. We conclude that every element of $\mathfrak{g}'$ lies in a finite-dimensional subspace invariant under $\mathfrak{sl}_i$.

Continuing toward the conclusion that $\mathfrak{g}'$ is finite dimensional, let us introduce an analog of the root string analysis done in §4. Fix i, let φ be in $\Delta' \cup \{0\}$, and consider the subspace $\bigoplus_{n\in\mathbb{Z}} \mathfrak{g}'_{\varphi+n\alpha_i}$ of $\mathfrak{g}'$. This is invariant under $\mathfrak{sl}_i$, and what we have just shown implies that every member of it lies in a finite-dimensional subspace invariant under $\mathfrak{sl}_i$. By Corollary 1.73 it is the direct sum of irreducible invariant subspaces. Let U be one of the irreducible summands. Since U is invariant under $\operatorname{ad} h_i$, we have

$$U = \bigoplus_{n=-p}^{q} (U \cap \mathfrak{g}'_{\varphi+n\alpha_i})$$

with $U \cap \mathfrak{g}'_{\varphi-p\alpha_i} \neq 0$ and $U \cap \mathfrak{g}'_{\varphi+q\alpha_i} \neq 0$. By Corollary 1.72,

$$(\varphi + q\alpha_i)(h_i) = -(\varphi - p\alpha_i)(h_i)$$

and hence

$$p - q = \varphi(h_i). \tag{2.106}$$

Moreover Theorem 1.66 shows that $U \cap \mathfrak{g}'_{\varphi+n\alpha_i}$ has dimension 1 for $-p \leq n \leq q$ and has dimension 0 otherwise.

In our direct sum decomposition of $\bigoplus_{n\in\mathbb{Z}} \mathfrak{g}'_{\varphi+n\alpha_i}$ into irreducible subspaces U, suppose that the root space $\mathfrak{g}'_{\varphi+n\alpha_i}$ has dimension m. Then it meets a collection of exactly m such U's, say $U_1, \ldots, U_m$. The root space

$$\mathfrak{g}'_{s_{\alpha_i}(\varphi+n\alpha_i)} = \mathfrak{g}'_{\varphi-(n+\varphi(h_i))\alpha_i}$$

must meet the same $U_1, \ldots, U_m$ since (2.106) shows that

$$-p \leq n \leq q \quad \text{implies} -p \leq -n - \varphi(h_i) = -n + q - p \leq q.$$

We conclude that

$$\dim \mathfrak{g}'_\varphi = \dim \mathfrak{g}'_{s_{\alpha_i}\varphi}. \tag{2.107}$$

From (2.107), we see that $W'\Delta' \subseteq \Delta'$. Since W' mirrors for $\mathfrak{h}_0'^*$ the action of W on $\mathbb{R}^l$, the linear extension of the map $\beta_i \mapsto \alpha_i$ carries Δ into Δ'. Since $\dim \mathfrak{g}'_{\alpha_i} = 1$ for all i, we see that $\dim \mathfrak{g}'_\varphi = 1$ for every root φ in the finite set $\bigcup_{i=1}^l W'\alpha_i$.

To complete the proof of finite dimensionality of $\mathfrak{g}'$, we show that every root lies in $\bigcup_{i=1}^l W'\alpha_i$. Certainly $\bigcup_{i=1}^l W'\alpha_i$ is closed under negatives, since it is generated by the α_i's and contains the $-\alpha_i$'s. Arguing by contradiction, assume that $\bigcup_{i=1}^l W'\alpha_i$ does not exhaust Δ'. By (2.103) there is some $\alpha = \sum_{j=1}^l n_j\alpha_j$ in Δ'^+ not in $\bigcup_{i=1}^l W'\alpha_i$, and we may assume that $\sum_{j=1}^l n_j$ is as small as possible. From

$$0 < |\alpha|^2 = \sum_{j=1}^l n_j\langle\alpha, \alpha_j\rangle,$$

we see that there is some k such that $n_k > 0$ and $\langle\alpha, \alpha_k\rangle > 0$. Then

$$s_{\alpha_k}(\alpha) = \alpha - \frac{2\langle\alpha, \alpha_k\rangle}{|\alpha_k|^2}\alpha_k = \sum_{j\neq k} n_j\alpha_j + \Big(n_k - \frac{2\langle\alpha, \alpha_k\rangle}{|\alpha_k|^2}\Big)\alpha_k.$$

We must have $n_j > 0$ for some $j \neq k$ since otherwise $\alpha = n_k\alpha_k$, from which we obtain $n_k = 1$ since $[e_k, e_k] = 0$. Thus $s_{\alpha_k}(\alpha)$ is in Δ'^+. Since the sum of coefficients for $s_{\alpha_k}(\alpha)$ is less than $\sum_{j=1}^l n_j$, we conclude by minimality that $s_{\alpha_k}(\alpha)$ is in $\bigcup_{i=1}^l W'\alpha_i$. But then so is α, contradiction. We conclude that $\Delta' = \bigcup_{i=1}^l W'\alpha_i$ and hence that Δ' is finite and $\mathfrak{g}'$ is finite dimensional.

Now that $\mathfrak{g}'$ is finite dimensional, we prove that it is semisimple and has the required structure. In fact, $\operatorname{rad}\mathfrak{g}'$ is $\operatorname{ad}\mathfrak{h}'$ invariant and therefore satisfies

$$\operatorname{rad}\mathfrak{g}' = (\mathfrak{h}' \cap \operatorname{rad}\mathfrak{g}') \oplus \bigoplus_{\varphi\in\Delta'} (\mathfrak{g}'_\varphi \cap \operatorname{rad}\mathfrak{g}').$$

Suppose $h \neq 0$ is in $\mathfrak{h} \cap \operatorname{rad}\mathfrak{g}'$. Choose j with $\alpha_j(h) \neq 0$. Since $\operatorname{rad}\mathfrak{g}'$ is an ideal, $e_j = \alpha_j(h)^{-1}[h, e_j]$ and $f_j = -\alpha_j(h)^{-1}[h, f_j]$ are in $\operatorname{rad}\mathfrak{g}'$, and so is $h_j = [e_j, f_j]$. Thus $\operatorname{rad}\mathfrak{g}'$ contains the semisimple subalgebra $\mathfrak{sl}_j$, contradiction. We conclude that $\mathfrak{h}' \cap \operatorname{rad}\mathfrak{g}' = 0$.

Since the root spaces are 1-dimensional, we obtain

$$\operatorname{rad}\mathfrak{g}' = \bigoplus_{\varphi\in\Delta_0'} \mathfrak{g}'_\varphi$$

for some subset Δ'_0 of Δ'. The Lie algebra $\mathfrak{g}'/\operatorname{rad}\mathfrak{g}'$ is semisimple, according to Proposition 1.14, and we can write it as

$$\mathfrak{g}'/\operatorname{rad}\mathfrak{g}' = \mathfrak{h} \oplus \bigoplus_{\varphi\in\Delta'-\Delta'_0} \mathfrak{g}'_\varphi \mod (\operatorname{rad}\mathfrak{g}').$$

From this decomposition we see that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}'/\operatorname{rad}\mathfrak{g}'$ and that the root system is $\Delta' - \Delta'_0$. On the other hand, no α_j is in Δ'_0 since $\mathfrak{sl}_j$ is semisimple. Thus $\Delta' - \Delta'_0$ contains each α_j, and these are the simple roots. We have seen that the simple roots determine Δ' as the corresponding abstract root system. Thus Δ'_0 is empty. It follows that $\mathfrak{g}'$ is semisimple, and then the structural conclusions about $\mathfrak{g}'$ are obvious. This completes the proof of Lemma 2.101.

PROOF OF THEOREM 2.98. In the diagram (2.97), X maps to a linearly independent subset of $\mathfrak{g}$, and hence the embedded subset X of $\mathfrak{F}$ maps to a linearly independent subset of $\mathfrak{g}$. Since the map $\mathfrak{F} \to \mathfrak{g}$ factors through $\mathfrak{g}' = \mathfrak{F}/\mathfrak{R}$, $\operatorname{span}\{h_i\}_{i=1}^l$ maps one-one from $\mathfrak{F}$ to $\mathfrak{g}'$ and one-one from $\mathfrak{g}'$ to $\mathfrak{g}$. Since $\operatorname{span}\{h_i\}_{i=1}^l$ maps one-one from $\mathfrak{F}$ to $\mathfrak{g}'$, Lemma 2.101 is applicable and shows that $\mathfrak{g}'$ is finite-dimensional semisimple and that $\mathfrak{h}' = \operatorname{span}\{h_i\}_{i=1}^l$ is a Cartan subalgebra.

The map $\mathfrak{F} \to \mathfrak{g}$ is onto by Proposition 2.94, and hence the map $\mathfrak{g}' \to \mathfrak{g}$ is onto. Thus $\mathfrak{g}$ is isomorphic with a quotient of $\mathfrak{g}'$. If $\mathfrak{a}$ is a simple ideal in $\mathfrak{g}'$, it follows from Proposition 2.13 that $\mathfrak{h}' \cap \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}'$. Since $\mathfrak{h}'$ maps one-one under the quotient map from $\mathfrak{g}'$ to $\mathfrak{g}$, $\mathfrak{h}' \cap \mathfrak{a}$ does not map to 0. Thus $\mathfrak{a}$ does not map to 0. Hence the map of $\mathfrak{g}'$ onto $\mathfrak{g}$ has 0 kernel and is an isomorphism.

10. Isomorphism Theorem

Theorem 2.98 enables us to lift isomorphisms of reduced root systems to isomorphisms of complex semisimple Lie algebras with little effort. The result is as follows.

Theorem 2.108 (Isomorphism Theorem). Let $\mathfrak{g}$ and $\mathfrak{g}'$ be complex semisimple Lie algebras with respective Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ and respective root systems Δ and Δ'. Suppose that a vector space isomorphism $\varphi : \mathfrak{h} \to \mathfrak{h}'$ is given with the property that its transpose $\varphi^t : \mathfrak{h}'^* \to \mathfrak{h}^*$ has $\varphi^t(\Delta') = \Delta$. For α in Δ, write α' for the member $(\varphi^t)^{-1}(\alpha)$ of Δ'. Fix a simple system Π for Δ. For each α in Π, select nonzero root vectors

$E_\alpha \in \mathfrak{g}$ for α and $E_{\alpha'} \in \mathfrak{g}'$ for α'. Then there exists one and only one Lie algebra isomorphism $\widetilde{\varphi} : \mathfrak{g} \to \mathfrak{g}'$ such that $\widetilde{\varphi}|_{\mathfrak{h}} = \varphi$ and $\widetilde{\varphi}(E_\alpha) = E_{\alpha'}$ for all $\alpha \in \Pi$.

PROOF OF UNIQUENESS. If $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ are two such isomorphisms, then $\widetilde{\varphi}_0 = \widetilde{\varphi}_2^{-1}\widetilde{\varphi}_1$ is an automorphism of $\mathfrak{g}$ fixing $\mathfrak{h}$ and the root vectors for the simple roots. If $\{h_i, e_i, f_i\}$ is a triple associated to the simple root α_i by (2.93), then $\widetilde{\varphi}_0(f_i)$ must be a root vector for $-\alpha_i$ and hence must be a multiple of f_i, say $c_i f_i$. Applying $\widetilde{\varphi}_0$ to the relation $[e_i, f_i] = h_i$, we see that $c_i = 1$. Therefore $\widetilde{\varphi}_0$ fixes all h_i, e_i, and f_i. By Proposition 2.94, $\widetilde{\varphi}_0$ is the identity on $\mathfrak{g}$.

PROOF OF EXISTENCE. The linear map $(\varphi^t)^{-1}$ is given by $(\varphi^t)^{-1}(\alpha) = \alpha' = \alpha \circ \varphi^{-1}$. By assumption this map carries Δ to Δ', hence root strings to root strings. Proposition 2.29a therefore gives

$$\frac{2\langle\beta,\alpha\rangle}{|\alpha|^2} = \frac{2\langle\beta',\alpha'\rangle}{|\alpha'|^2} \qquad \text{for all } \alpha, \beta \in \Delta. \tag{2.109}$$

Write $\Pi = \{\alpha_1, \ldots, \alpha_l\}$, and let $\Pi' = (\varphi^t)^{-1}(\Pi) = \{\alpha'_1, \ldots, \alpha'_l\}$. Define h_i and h'_i to be the respective members of $\mathfrak{h}$ and $\mathfrak{h}'$ with $\alpha_j(h_i) = 2\langle\beta,\alpha\rangle/|\alpha|^2$ and $\alpha'_j(h'_i) = 2\langle\beta',\alpha'\rangle/|\alpha'|^2$. These are the elements of the Cartan subalgebras appearing in (2.93). By (2.109), $\alpha'_j(h'_i) = \alpha_j(h_i)$ and hence $(\varphi^t)^{-1}(\alpha_j)(h'_i) = \alpha_j(h_i)$ and $\alpha_j(\varphi^{-1}(h'_i)) = \alpha_j(h_i)$. Therefore

$$\varphi(h_i) = h'_i \qquad \text{for all } i. \tag{2.110}$$

Take e_i in (2.93) to be E_{α_i}, and let $e'_i = E_{\alpha'_i}$. Define $f_i \in \mathfrak{g}$ to be a root vector for $-\alpha_i$ with $[e_i, f_i] = h_i$, and define $f'_i \in \mathfrak{g}'$ to be a root vector for $-\alpha'_i$ with $[e'_i, f'_i] = h'_i$. Then $X = \{h_i, e_i, f_i\}_{i=1}^l$ and $X' = \{h'_i, e'_i, f'_i\}_{i=1}^l$ are standard sets of generators for $\mathfrak{g}$ and $\mathfrak{g}'$ as in (2.93) and Proposition 2.94.

Let $\mathfrak{F}$ and $\mathfrak{F}'$ be the free Lie algebras on X and X', and let $\mathfrak{R}$ and $\mathfrak{R}'$ be the ideals in $\mathfrak{F}$ and $\mathfrak{F}'$ generated by the Serre relations (a) through (f) in Theorem 2.98. Let us define $\psi : X \to \mathfrak{F}'$ by $\psi(h_i) = h'_i$, $\psi(e_i) = e'_i$, and $\psi(f_i) = f'_i$. Setting up the diagram

$$\begin{tikzcd} & \mathfrak{F} \arrow[dr, dashed, "\widetilde{\psi}"] & \\ X \arrow[ur] \arrow[rr, "\psi"'] & & \mathfrak{F}' \end{tikzcd}$$

we see from the universal mapping property of $\mathfrak{F}$ that ψ extends to a Lie algebra homomorphism $\widetilde{\psi} : \mathfrak{F} \to \mathfrak{F}'$. By (2.109), $\widetilde{\psi}(\mathfrak{R}) \subseteq \mathfrak{R}'$. Therefore $\widetilde{\psi}$ descends to Lie algebra homomorphism $\mathfrak{F}/\mathfrak{R} \to \mathfrak{F}'/\mathfrak{R}'$, and we denote this homomorphism by $\widetilde{\psi}$ as well.

Meanwhile the canonical maps $\widetilde{\varphi}_1 : \mathfrak{F}/\mathfrak{R} \to \mathfrak{g}$ and $\widetilde{\varphi}_2 : \mathfrak{F}'/\mathfrak{R}' \to \mathfrak{g}'$, which are isomorphisms by Theorem 2.98, satisfy

$$\begin{aligned} \widetilde{\varphi}_1^{-1}(h_i) &= h_i \mod \mathfrak{R} \qquad &\text{and} \qquad \widetilde{\varphi}_1^{-1}(E_{\alpha_i}) &= e_i \mod \mathfrak{R}, \\ \widetilde{\varphi}_2(h_i' \bmod \mathfrak{R}') &= h_i' \qquad &\text{and} \qquad \widetilde{\varphi}_2(e_i' \bmod \mathfrak{R}') &= E_{\alpha_i'}. \end{aligned}$$

Therefore $\widetilde{\varphi} = \widetilde{\varphi}_2 \circ \widetilde{\psi} \circ \widetilde{\varphi}_1^{-1}$ is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{g}'$ with $\widetilde{\varphi}(h_i) = h_i'$ and $\widetilde{\varphi}(E_{\alpha_i}) = E_{\alpha_i'}$ for all i. By (2.110), $\widetilde{\varphi}|_{\mathfrak{h}} = \varphi$.

To see that $\widetilde{\varphi}$ is an isomorphism, we observe that $\widetilde{\varphi} : \mathfrak{h} \to \mathfrak{h}'$ is an isomorphism. By the same argument as in the last paragraph of §9, it follows that $\widetilde{\varphi} : \mathfrak{g} \to \mathfrak{g}'$ is one-one. Finally

$$\dim \mathfrak{g} = \dim \mathfrak{h} + |\Delta| = \dim \mathfrak{h}' + |\Delta'| = \dim \mathfrak{g}',$$

and we conclude that $\widetilde{\varphi}$ is an isomorphism.

EXAMPLES.

1) One-oneness of first step in (2.58). We are to show that if $\mathfrak{g}$ and $\mathfrak{g}'$ are two complex semisimple Lie algebras with isomorphic root systems, then $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic. To do so, we apply Theorem 2.108, mapping the root vector E_α for each simple root α to any nonzero root vector for the corresponding simple root for $\mathfrak{g}'$. We conclude that the first step of the two-step passage (2.58) is one-one, up to isomorphism.

2) Automorphisms of Dynkin diagram. Let $\mathfrak{g}$, $\mathfrak{h}$, Δ, and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be arbitrary. Suppose that σ is an automorphism of the Dynkin diagram, i.e., a permutation of the indices $1, \dots, l$ such that the Cartan matrix satisfies $A_{ij} = A_{\sigma(i)\sigma(j)}$. Define $\varphi : \mathfrak{h} \to \mathfrak{h}$ to be the linear extension of the map $h_i \to h_{\sigma(i)}$, and apply Theorem 2.108. The result is an automorphism $\widetilde{\varphi}$ of $\mathfrak{g}$ that normalizes $\mathfrak{h}$, maps the set of positive roots to itself, and has the effect σ on the Dynkin diagram.

3) An automorphism constructed earlier. With $\mathfrak{g}$, $\mathfrak{h}$, and Δ given, define $\varphi = -1$ on $\mathfrak{h}$. Then Δ gets carried to Δ, and hence φ extends to an automorphism $\widetilde{\varphi}$ of $\mathfrak{g}$. This automorphism has already been constructed directly (as $\widetilde{\eta}$ in the course of the proof of Lemma 2.101).

11. Existence Theorem

We have now shown that the first step in the passage (2.58), i.e., the step from complex semisimple Lie algebras to abstract reduced root systems, is well defined independently of the choice of Cartan subalgebra and is one-one up to isomorphism. To complete our discussion of (2.58), we show that this step is onto, i.e., that any reduced abstract root system is the root system of a complex semisimple Lie algebra.

The Existence Theorem accomplishes this step, actually showing that any abstract Cartan matrix comes via the two steps of (2.58) from a complex semisimple Lie algebra. However, the theorem does not substitute for our case-by-case argument in §7 that the second step of (2.58) is onto. The fact that the second step is onto was used critically in the proof of Lemma 2.101 to show that W' is a finite group.

The consequence of the Existence Theorem is that there exist complex simple Lie algebras with root systems of the five exceptional types E_6, E_7, E_8, F_4, and G_2. We shall have occasion to use these complex Lie algebras in Chapter VI and then shall refer to them as complex simple Lie algebras of types E_6, etc.

Theorem 2.111 (Existence Theorem). If $A = (A_{ij})_{i,j=1}^{l}$ is an abstract Cartan matrix, then there exists a complex semisimple Lie algebra $\mathfrak{g}$ whose root system has A as Cartan matrix.

PROOF. Let $\mathfrak{F}$ be the free Lie algebra on the set $X = \{h_i, e_i, f_i\}_{i=1}^{l}$, and let $\mathfrak{R}$ be the ideal in $\mathfrak{F}$ generated by the Serre relations (a) through (f) given in Proposition 2.95. Put $\mathfrak{g} = \mathfrak{F}/\mathfrak{R}$. According to Lemma 2.101, $\mathfrak{g}$ will be the required Lie algebra if it is shown that $\operatorname{span}\{h_i\}_{i=1}^{l}$ maps one-one from $\mathfrak{F}$ to its image in $\mathfrak{F}/\mathfrak{R}$.

We shall establish this one-one behavior by factoring the quotient map into two separate maps and showing that $\operatorname{span}\{h_i\}_{i=1}^{l}$ maps one-one in each case. The first map is from $\mathfrak{F}$ to $\mathfrak{F}/\widetilde{\mathfrak{R}}$, where $\widetilde{\mathfrak{R}}$ is the ideal in $\mathfrak{F}$ generated by the Serre relations (a) through (d). Write h_i, e_i, f_i also for the images of the generators in $\mathfrak{F}/\widetilde{\mathfrak{R}}$. Define $\widetilde{\mathfrak{h}}$, $\widetilde{\mathfrak{e}}$, and $\widetilde{\mathfrak{e}}$ as in the statement of Lemma 2.99. The lemma says that

$$\mathfrak{F}/\widetilde{\mathfrak{R}} = \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{e}} \oplus \widetilde{\mathfrak{f}}, \tag{2.112}$$

but it does not tell us how large $\widetilde{\mathfrak{h}}$ is.

To get at the properties of the first map, we introduce an l-dimensional complex vector space V with basis $\{v_1, \ldots, v_l\}$, and we let $T(V)$ be the tensor algebra over V. (Appendix A gives the definition and elementary properties of $T(V)$.) We drop tensor signs in writing products within $T(V)$ in order to simplify the notation. In view of the diagram

$$\begin{tikzcd} & \mathfrak{F} \arrow[dr, dashed, "\widetilde{\psi}"] & \\ X \arrow[ur, "\iota"] \arrow[rr, "\psi"'] & & \operatorname{End}_{\mathbb{C}}(T(V)) \end{tikzcd}$$

we can construct a homomorphism $\widetilde{\psi} : \mathfrak{F} \to \operatorname{End}_{\mathbb{C}}(T(V))$ by telling how x acts in $T(V)$ for each x in X. Dropping the notation ψ from the action, we define

$$\begin{aligned} h_i(1) &= 0 \\ h_i(v_{j_1}\cdots v_{j_k}) &= -(A_{ij_1}+\cdots+A_{ij_k})v_{j_1}\cdots v_{j_k} \\ f_i(1) &= v_i \\ f_i(v_{j_1}\cdots v_{j_k}) &= v_i v_{j_1}\cdots v_{j_k} \\ e_i(1) &= 0 \\ e_i(v_j) &= 0 \\ e_i(v_{j_1}\cdots v_{j_k}) &= v_{j_1}\cdot e_i(v_{j_2}\cdots v_{j_k}) - \delta_{ij_1}(A_{ij_2}+\cdots+A_{ij_k})v_{j_2}\cdots v_{j_k}. \end{aligned}$$

(The last three lines, defining the action of e_i, are made recursively on the order of the tensor.)

We show that this homomorphism defined on $\mathfrak{F}$ descends to a homomorphism $\mathfrak{F}/\widetilde{\mathfrak{R}} \to \operatorname{End}_{\mathbb{C}}(T(V))$ by showing that the generators of $\widetilde{\mathfrak{R}}$ act by 0. We check the generators of types (a), (d), (b), and (c) in turn.

For (a) the generator is $[h_i, h_j]$. The span of the h_i's acts diagonally, and thus

$$\widetilde{\psi}[h_i, h_j] = [\widetilde{\psi}(h_i), \widetilde{\psi}(h_j)] = \widetilde{\psi}(h_i)\widetilde{\psi}(h_j) - \widetilde{\psi}(h_j)\widetilde{\psi}(h_i) = 0.$$

For (d) the generator is $[h_i, f_j] + A_{ij} f_j$, and we have

$$\widetilde{\psi}([h_i, f_j] + A_{ij} f_j) = \widetilde{\psi}(h_i)\widetilde{\psi}(f_j) - \widetilde{\psi}(f_j)\widetilde{\psi}(h_i) + A_{ij}\widetilde{\psi}(f_j).$$

On 1, the right side gives

$$\widetilde{\psi}(h_i)v_j - 0 + A_{ij}v_j = 0.$$

On $v_{j_1}\cdots v_{j_k}$, the right side gives

$$\begin{aligned}&-(A_{ij}+A_{ij_1}+\cdots+A_{ij_k})v_jv_{j_1}\cdots v_{j_k}\\&\qquad+(A_{ij_1}+\cdots+A_{ij_k})v_jv_{j_1}\cdots v_{j_k}+A_{ij}v_jv_{j_1}\cdots v_{j_k}=0.\end{aligned}$$

For (b) the generator is $[e_i,\ f_j]-\delta_{ij}h_i$, and we have

$$\widetilde{\psi}([e_i,\ f_j]-\delta_{ij}h_i)=\widetilde{\psi}(e_i)\widetilde{\psi}(f_j)-\widetilde{\psi}(f_j)\widetilde{\psi}(e_i)-\delta_{ij}\widetilde{\psi}(h_i).$$

On 1, each term on the right side acts as 0. On a monomial $v_{j_2}\cdots v_{j_k}$, the right side gives

$$e_i(v_jv_{j_2}\cdots v_{j_k})-v_j\cdot e_i(v_{j_2}\cdots v_{j_k})+\delta_{ij}(A_{ij_2}+\cdots+A_{ij_k})v_{j_2}\cdots v_{j_k},$$

and this is 0 by the recursive definition of the action of e_i.

For (c) the generator is $[h_i,e_j]-A_{ij}e_j$. Let us observe by induction on k that

$$h_ie_j(v_{j_1}\cdots v_{j_k})=-(A_{ij_1}+\cdots+A_{ij_k}-A_{ij})e_j(v_{j_1}\cdots v_{j_k}). \tag{2.113}$$

Formula (2.113) is valid for $k=0$ and $k=1$ since e_j acts as 0 on monomials of degrees 0 and 1. For general k, the recursive definition of the action of e_i and the inductive hypothesis combine to show that the left side of (2.113) is

$$\begin{aligned}h_ie_j(v_{j_1}\cdots v_{j_k})&=h_i(v_{j_1}\cdot e_j(v_{j_2}\cdots v_{j_k}))-\delta_{jj_1}(A_{jj_2}+\cdots+A_{jj_k})h_i(v_{j_2}\cdots v_{j_k})\\&=-(A_{ij_1}+\cdots+A_{ij_k}-A_{ij})v_{j_1}\cdot e_j(v_{j_2}\cdots v_{j_k})\\&\quad+\delta_{jj_1}(A_{jj_2}+\cdots+A_{jj_k})(A_{ij_2}+\cdots+A_{ij_k})v_{j_2}\cdots v_{j_k},\end{aligned}$$

and that the right side of (2.113) is

$$\begin{aligned}&-(A_{ij_1}+\cdots+A_{ij_k}-A_{ij})e_j(v_{j_1}\cdots v_{j_k})\\&\quad=-(A_{ij_1}+\cdots+A_{ij_k}-A_{ij})v_{j_1}\cdot e_j(v_{j_2}\cdots v_{j_k})\\&\qquad+(A_{ij_1}+\cdots+A_{ij_k}-A_{ij})\delta_{jj_1}(A_{jj_2}+\cdots+A_{jj_k})v_{j_2}\cdots v_{j_k}.\end{aligned}$$

Subtraction shows that the difference of the left side and the right side of (2.113) is

$$=-\delta_{jj_1}(A_{ij_1}-A_{ij})(A_{jj_2}+\cdots+A_{jj_k})v_{j_2}\cdots v_{j_k}=0.$$

The induction is complete, and (2.113) is established. Returning to our generator, we have

$$\widetilde{\psi}([h_i, e_j] - A_{ij}e_j) = \widetilde{\psi}(h_i)\widetilde{\psi}(e_j) - \widetilde{\psi}(e_j)\widetilde{\psi}(h_i) - A_{ij}\widetilde{\psi}(e_j).$$

On 1, each term on the right side acts as 0. On $v_{j_1}\cdots v_{j_k}$, (2.113) shows that the effect of the right side is

$$\begin{aligned} &= -(A_{ij_1} + \cdots + A_{ij_k} - A_{ij})e_j(v_{j_1}\cdots v_{j_k}) \\ &\quad + (A_{ij_1} + \cdots + A_{ij_k})e_j(v_{j_1}\cdots v_{j_k}) - A_{ij}e_j(v_{j_1}\cdots v_{j_k}) = 0. \end{aligned}$$

Thus $\widetilde{\psi}$ descends to $\mathfrak{F}/\widetilde{\mathfrak{R}}$.

Now we can prove that $\operatorname{span}\{h_i\}_{i=1}^l$ maps one-one from $\mathfrak{F}$ to $\mathfrak{F}/\widetilde{\mathfrak{R}}$. If a nontrivial $\sum c_i h_i$ maps to 0, then we have

$$0 = \Big(\sum_i c_i h_i\Big)(v_j) = -\Big(\sum_i c_i A_{ij}\Big)v_j$$

for all j. Hence $\sum_i c_i A_{ij} = 0$ for all j, in contradiction with the linear independence of the rows of (A_{ij}). We conclude that $\operatorname{span}\{h_i\}_{i=1}^l$ maps one-one from $\mathfrak{F}$ to $\mathfrak{F}/\widetilde{\mathfrak{R}}$.

Now we bring in Serre relations (e) and (f), effectively imposing them directly on $\mathfrak{F}/\widetilde{\mathfrak{R}}$ to obtain $\mathfrak{g}$ as quotient. Define $\widetilde{\mathfrak{g}} = \mathfrak{F}/\widetilde{\mathfrak{R}}$. Let $\mathfrak{R}'$ be the ideal in $\widetilde{\mathfrak{g}}$ generated by all

$$(\operatorname{ad} e_i)^{-A_{ij}+1}e_j \quad \text{and all} \quad (\operatorname{ad} f_i)^{-A_{ij}+1}f_j \qquad \text{for } i \neq j.$$

Then indeed $\mathfrak{g} \cong \widetilde{\mathfrak{g}}/\mathfrak{R}'$.

We define subalgebras $\widetilde{\mathfrak{h}}$, $\widetilde{\mathfrak{e}}$, and $\widetilde{\mathfrak{f}}$ of $\widetilde{\mathfrak{g}}$ as in the statement of Lemma 2.99. Let $\widetilde{\mathfrak{e}}'$ be the ideal in $\widetilde{\mathfrak{e}}$ generated by all $(\operatorname{ad} e_i)^{-A_{ij}+1}e_j$, and let $\widetilde{\mathfrak{f}}'$ be the ideal in $\widetilde{\mathfrak{f}}$ generated by all $(\operatorname{ad} f_i)^{-A_{ij}+1}f_j$. Then

$$(\text{generators of } \mathfrak{R}') \subseteq \widetilde{\mathfrak{e}}' + \widetilde{\mathfrak{f}}' \subseteq \widetilde{\mathfrak{e}} + \widetilde{\mathfrak{f}}. \tag{2.114}$$

We shall prove that $\widetilde{\mathfrak{e}}'$ is actually an ideal in $\widetilde{\mathfrak{g}}$. We observe that $\widetilde{\mathfrak{e}}'$ is invariant under all $\operatorname{ad} h_k$ (since the generators of $\widetilde{\mathfrak{e}}'$ are eigenvectors) and all e_k (since $\widetilde{\mathfrak{e}}' \subseteq \widetilde{\mathfrak{e}}$). Thus we are to show that

$$(\operatorname{ad} f_k)(\operatorname{ad} e_i)^{-A_{ij}+1}e_j$$

is in $\widetilde{\mathfrak{e}}'$ if $i \neq j$. In fact, we show it is 0.

If $k \neq i$, then $[f_k, e_i] = 0$ shows that ad f_k commutes with ad e_i. Thus we are led to

$$(\mathrm{ad}\, e_i)^{-A_{ij}+1}[f_k, e_j].$$

If $k \neq j$, this is 0 by Serre relation (b). If $k = j$, it is

$$= -(\mathrm{ad}\, e_i)^{-A_{ij}+1}h_j = A_{ji}(\mathrm{ad}\, e_i)^{-A_{ij}}e_i. \tag{2.115}$$

If $A_{ij} < 0$, then the right side of (2.115) is 0 since $[e_i, e_i] = 0$; if $A_{ij} = 0$, then the right side of (2.115) is 0 because the coefficient A_{ji} is 0.

If $k = i$, we are to consider

$$(\mathrm{ad}\, f_i)(\mathrm{ad}\, e_i)^{-A_{ij}+1}e_j.$$

Now

$$(\mathrm{ad}\, f_i)(\mathrm{ad}\, e_i)^n e_j = -(\mathrm{ad}\, h_i)(\mathrm{ad}\, e_i)^{n-1}e_j + (\mathrm{ad}\, e_i)(\mathrm{ad}\, f_i)(\mathrm{ad}\, e_i)^{n-1}e_j.$$

Since $(\mathrm{ad}\, f_i)e_j = 0$, an easy induction with this equation shows that

$$(\mathrm{ad}\, f_i)(\mathrm{ad}\, e_i)^n e_j = -n(A_{ij} + n - 1)(\mathrm{ad}\, e_i)^{n-1}e_j.$$

For $n = -A_{ij} + 1$, the right side is 0, as asserted. This completes the proof that $\widetilde{\mathfrak{e}}'$ is an ideal in $\widetilde{\mathfrak{g}}$.

Similarly $\widetilde{\mathfrak{f}}'$ is an ideal in $\widetilde{\mathfrak{g}}$, and so is the sum $\widetilde{\mathfrak{e}}' + \widetilde{\mathfrak{f}}'$. From (2.114) we therefore obtain

$$\mathfrak{R}' \subseteq \widetilde{\mathfrak{e}}' + \widetilde{\mathfrak{f}}' \subseteq \widetilde{\mathfrak{e}} + \widetilde{\mathfrak{f}}.$$

In view of the direct sum decomposition (2.112), $\mathfrak{R}' \cap \widetilde{\mathfrak{h}} = 0$. Therefore $\mathrm{span}\{h_i\}_{i=1}^{l}$ maps one-one from $\widetilde{\mathfrak{g}}$ to $\widetilde{\mathfrak{g}}/\mathfrak{R}' \cong \mathfrak{g}$, and the proof of the theorem is complete.

12. Problems

1. According to Problem 13 in Chapter I, the trace form is a multiple of the Killing form for $\mathfrak{sl}(n+1, \mathbb{C})$ if $n \geq 1$, for $\mathfrak{so}(2n+1, \mathbb{C})$ if $n \geq 2$, $\mathfrak{sp}(n, \mathbb{C})$ if $n \geq 3$, and $\mathfrak{so}(2n, \mathbb{C})$ if $n \geq 4$. Find the multiple in each case.
2. Since the Dynkin diagrams of $A_1 \oplus A_1$ and D_2 are isomorphic, the Isomorphism Theorem predicts that $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ is isomorphic with $\mathfrak{so}(4, \mathbb{C})$. Using the explicit root-space decomposition for $\mathfrak{so}(4, \mathbb{C})$ found in §1, exhibit two 3-dimensional ideals in $\mathfrak{so}(4, \mathbb{C})$, proving that they are indeed ideals.

3. Let $\mathfrak{g}$ be the 2-dimensional complex Lie algebra with a basis $\{X, Y\}$ such that $[X, Y] = Y$.
   (a) Identify the regular elements.
   (b) Prove that $\mathbb{C}X$ is a Cartan subalgebra but that $\mathbb{C}Y$ is not.
   (c) Find the weight-space decomposition of $\mathfrak{g}$ relative to the Cartan subalgebra $\mathbb{C}X$.

4. Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be a root-space decomposition for a complex semisimple Lie algebra, and let Δ' be a subset of Δ that forms a root system in $\mathfrak{h}_0^*$.
   (a) Show by example that $\mathfrak{s} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta'} \mathfrak{g}_\alpha$ need not be a subalgebra of $\mathfrak{g}$.
   (b) Suppose that $\Delta' \subseteq \Delta$ is a root subsystem with the following property. Whenever α and β are in Δ' and $\alpha + \beta$ is in Δ, then $\alpha + \beta$ is in Δ'. Prove that $\mathfrak{s} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta'} \mathfrak{g}_\alpha$ is a subalgebra of $\mathfrak{g}$ and that it is semisimple.

5. Exhibit complex semisimple Lie algebras of dimensions 8, 9, and 10. Deduce that there are complex semisimple Lie algebras of every dimension ≥ 8.

6. Using results from §§4–5 but not the classification, show that there are no complex semisimple Lie algebras of dimensions 4, 5, or 7.

7. Let Δ be a root system, and fix a simple system Π. Show that any positive root can be written in the form

$$\alpha = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_k}$$

   with each α_{i_j} in Π and with each partial summand from the left equal to a positive root.

8. Let Δ be a root system, and fix a lexicographic ordering. Show that the largest root α_0 has $\langle\alpha_0, \alpha\rangle \geq 0$ for all positive roots α. If Δ is of type B_n with $n \geq 2$, find a positive root β_0 other than α_0 with $\langle\beta_0, \alpha\rangle \geq 0$ for all positive roots α.

9. Write down the Cartan matrices for A_n, B_n, C_n, and D_n.

10. The root system G_2 is pictured in Figure 2.2. According to Theorem 2.63, there are exactly 12 simple systems for this root system.
    (a) Identify them in Figure 2.2.
    (b) Fix one of them, letting the short simple root be α and the long simple root be β. Identify the positive roots, and express each of them as a linear combination of α and β.

11. (a) Prove that two simple roots in a Dynkin diagram that are connected by a single edge are in the same orbit under the Weyl group.
    (b) For an irreducible root system, prove that all roots of a particular length form a single orbit under the Weyl group.

12. In a reduced root system with a positive system imposed, let α and β be distinct simple roots connected by n edges ($0 \le n \le 3$) in the Dynkin diagram, and let s_α and s_β be the corresponding reflections in the Weyl group. Show that

$$(s_\alpha s_\beta)^k = 1, \quad \text{where } k = \begin{cases} 2 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 4 & \text{if } n = 2 \\ 6 & \text{if } n = 3. \end{cases}$$

13. (a) Prove that any element of order 2 in a Weyl group is the product of commuting root reflections.
    (b) Prove that the only reflections in a Weyl group are the root reflections.

14. Let Δ be an abstract root system in V, and fix an ordering. Suppose that λ is in V and w is in the Weyl group. Prove that if λ and $w\lambda$ are both dominant, then $w\lambda = \lambda$.

15. Verify the following table of values for the number of roots, the dimension of $\mathfrak{g}$, and the order of the Weyl group for the classical irreducible reduced root systems:

Type of Δ	$\lvert\Delta\rvert$	$\dim \mathfrak{g}$	$\lvert W\rvert$
A_n	$n(n+1)$	$n(n+2)$	$(n+1)!$
B_n	$2n^2$	$n(2n+1)$	$n!2^n$
C_n	$2n^2$	$n(2n+1)$	$n!2^n$
D_n	$2n(n-1)$	$n(2n-1)$	$n!2^{n-1}$

16. Verify the following table of values for the number of roots and the dimension of $\mathfrak{g}$ for the exceptional irreducible reduced root systems. These systems are described explicitly in Figure 2.2 and Proposition 2.87:

Type of Δ	$\lvert\Delta\rvert$	$\dim \mathfrak{g}$
E_6	72	78
E_7	126	133
E_8	240	248
F_4	48	52
G_2	12	14

17. If Δ is an abstract root system and α is in Δ, let $\alpha^\vee = 2|\alpha|^{-2}\alpha$. Define $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$.
    (a) Prove that $\Delta^\vee$ is an abstract root system with the same Weyl group as Δ.

(b) If Π is a simple system for Δ, prove that $\Pi^\vee = \{\alpha^\vee \mid \alpha \in \Pi\}$ is a simple system for $\Delta^\vee$.

(c) For any reduced irreducible root system Δ other than B_n and C_n, show from the classification that $\Delta^\vee \cong \Delta$. For B_n and C_n, show that $(B_n)^\vee \cong C_n$ and $(C_n)^\vee \cong B_n$.

18. Let Π be a simple system in a root system Δ, and let Δ^+ be the corresponding set of positive roots.
    (a) Prove that the negatives of the members of Π form another simple system, and deduce that there is a unique member w_0 of the Weyl group sending Δ^+ to $-\Delta^+$.
    (b) Prove that $-w_0$ gives an automorphism of the Dynkin diagram, and conclude that -1 is in the Weyl group for B_n, C_n, E_7, E_8, F_4, and G_2.
    (c) Prove that -1 is not in the Weyl group of A_n for $n \geq 2$.
    (d) Prove that -1 is in the Weyl group of D_n if $n \geq 2$ is even but not if $n \geq 3$ is odd.

19. Using the classification theorems, show that Figure 2.2 exhibits all but two of the root systems in 2-dimensional spaces, up to isomorphism. What are the two that are missing?

20. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra, let Δ be the roots, let W be the Weyl group, and let w be in W. Using the Isomorphism Theorem, prove that there is a member of $\mathrm{Aut}_{\mathbb{C}}\,\mathfrak{g}$ whose restriction to $\mathfrak{h}$ is w.

Problems 21–24 concern the length function $l(w)$ on the Weyl group W. Fix a reduced root system Δ and an ordering, and let $l(w)$ be defined as in §6 before Proposition 2.70.

21. Prove that $l(w) = l(w^{-1})$.

22. (a) Define $\operatorname{sgn} w = (-1)^{l(w)}$. Prove that the function sgn carrying W to $\{\pm 1\}$ is a homomorphism.
    (b) Prove that $\operatorname{sgn} w = \det w$ for all $w \in W$.
    (c) Prove that $l(s_\alpha)$ is odd for any root reflection s_α.

23. For w_1 and w_2 in W, prove that

$$l(w_1 w_2) = l(w_1) + l(w_2) - 2\#\{\beta \in \Delta \mid \beta > 0,\ w_1\beta < 0,\ w_2^{-1}\beta < 0\}.$$

24. If α is a root, prove that $l(ws_\alpha) < l(w)$ if $w\alpha < 0$ and that $l(ws_\alpha) > l(w)$ if $w\alpha > 0$.

Problems 25–30 compute the determinants of all irreducible Cartan matrices.

25. Let M_l be an l-by-l Cartan matrix whose first two rows and columns look like

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & * \end{pmatrix},$$

the other entries in those rows and columns being 0. Let M_{l-1} be the Cartan matrix obtained by deleting the first row and column from M_l, and let M_{l-2} be the Cartan matrix obtained by deleting the first row and column from M_{l-1}. Prove that

$$\det M_l = 2 \det M_{l-1} - \det M_{l-2}.$$

26. Reinterpret the condition on the Cartan matrix M_l in Problem 25 as a condition on the corresponding Dynkin diagram.
27. Calculate explicitly the determinants of the irreducible Cartan matrices of types A_1, A_2, B_2, B_3, C_3, and D_4, showing that they are 2, 3, 2, 2, 2, and 4, respectively.
28. Using the inductive formula in Problem 25 and the initial data in Problem 27, show that the determinants of the irreducible Cartan matrices of types A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$, and D_n for $n \geq 4$ are $n + 1$, 2, 2, and 4, respectively.
29. Using the inductive formula in Problem 25 and the initial data for A_4 and D_5 computed in Problem 28, show that the determinants of the irreducible Cartan matrices of types E_6, E_7, and E_8 are 3, 2, and 1, respectively.
30. Calculate explicitly the determinants of the Cartan matrices for F_4 and G_2, showing that they are both 1.

Problems 31–34 compute the order of the Weyl group for the root systems F_4, E_6, E_7, and E_8. In each case the idea is to identify a transitive group action by the Weyl group, compute the number of elements in an orbit, compute the order of the subgroup fixing an element, and multiply.

31. The root system F_4 is given explicitly in (2.88).
    (a) Show that the long roots form a root system of type D_4.
    (b) By (a) the Weyl group W_D of D_4 is a subgroup of the Weyl group W_F of F_4. Show that every element of W_F leaves the system D_4 stable and therefore carries an ordered system of simple roots for D_4 to another ordered simple system. Conclude that $|W_F / W_D|$ equals the number of symmetries of the Dynkin diagram of D_4 that can be implemented by W_F.
    (c) Show that reflection in e_4 and reflection in $\frac{1}{2}(e_1 - e_2 - e_3 - e_4)$ are members of W_F that permute the standard simple roots of D_4 as given in (2.50), and deduce that $|W_F / W_D| = 6$.
    (d) Conclude that $|W_F| = 2^7 \cdot 3^2$.

32. The root system $\Delta = E_6$ is given explicitly in the proof of Proposition 2.87. Let W be the Weyl group.
    (a) Why is the orbit of $\frac{1}{2}(e_8 - e_7 - e_6 + e_5 + e_4 + e_3 + e_2 + e_1)$ under W equal exactly to Δ?
    (b) Show that the subset of Δ orthogonal to the root in (a) is a root system of type A_5.
    (c) The element -1 is not in the Weyl group of A_5. Why does it follow from this fact and (b) that -1 is not in the Weyl group of E_6?
    (d) Deduce from (b) that the subgroup of W fixing the root in (a) is isomorphic to the Weyl group of A_5.
    (e) Conclude that $|W| = 2^7 \cdot 3^4 \cdot 5$.

33. The root system $\Delta = E_7$ is given explicitly in the proof of Proposition 2.87. Let W be the Weyl group.
    (a) Why is the orbit of $e_8 - e_7$ under W equal exactly to Δ?
    (b) Show that the subset of Δ orthogonal to $e_8 - e_7$ is a root system of type D_6.
    (c) Deduce from (b) that the subgroup of W fixing $e_8 - e_7$ is isomorphic to the Weyl group of D_6.
    (d) Conclude that $|W| = 2^{10} \cdot 3^4 \cdot 5 \cdot 7$.

34. The root system $\Delta = E_8$ is given explicitly in (2.89). Let W be the Weyl group.
    (a) Why is the orbit of $e_8 + e_7$ under W equal exactly to Δ?
    (b) Show that the subset of Δ orthogonal to $e_8 + e_7$ is a root system of type E_7.
    (c) Deduce from (b) that the subgroup of W fixing $e_8 + e_7$ is isomorphic to the Weyl group of E_7.
    (d) Conclude that $|W| = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$.

Problems 35–37 exhibit an explicit isomorphism of $\mathfrak{sl}(4, \mathbb{C})$ with $\mathfrak{so}(6, \mathbb{C})$. Such an isomorphism is predicted by the Isomorphism Theorem since the Dynkin diagrams of A_3 and D_3 are isomorphic.

35. Let $I_{3,3}$ be the 6-by-6 diagonal matrix defined in Example 3 in §I.8, and define $\mathfrak{g} = \{X \in \mathfrak{gl}(6, \mathbb{C}) \mid X^t I_{3,3} + I_{3,3} X = 0\}$. Let $S = \operatorname{diag}(i, i, i, 1, 1, 1)$. For $X \in \mathfrak{g}$, let $Y = SXS^{-1}$. Prove that the map $X \mapsto Y$ is an isomorphism of $\mathfrak{g}$ onto $\mathfrak{so}(6, \mathbb{C})$.

36. Any member of $\mathfrak{sl}(4, \mathbb{C})$ acts on the 6-dimensional complex vector space of alternating tensors of rank 2 by $M(e_i \wedge e_j) = Me_i \wedge e_j + e_i \wedge Me_j$, where

$\{e_i\}_{i=1}^4$ is the standard basis of $\mathbb{C}^4$. Using

$$(e_1 \wedge e_2) \pm (e_3 \wedge e_4), \quad (e_1 \wedge e_3) \pm (e_2 \wedge e_4), \quad (e_1 \wedge e_4) \pm (e_2 \wedge e_3)$$

in some particular order as an ordered basis for the alternating tensors, show that the action of M is given by an element of the Lie algebra of $\mathfrak{g}$ in Problem 35.

37. The previous two problems combine to give a Lie algebra homomorphism of $\mathfrak{sl}(4, \mathbb{C})$ into $\mathfrak{so}(6, \mathbb{C})$. Show that no nonzero element of $\mathfrak{sl}(4, \mathbb{C})$ acts as the 0 operator on alternating tensors, and deduce from the simplicity of $\mathfrak{sl}(4, \mathbb{C})$ that the homomorphism is an isomorphism.

Problems 38–39 exhibit an explicit isomorphism of $\mathfrak{sp}(2, \mathbb{C})$ with $\mathfrak{so}(5, \mathbb{C})$. Such an isomorphism is predicted by the Isomorphism Theorem since the Dynkin diagrams of C_2 and B_2 are isomorphic.

38. The composition of the inclusion $\mathfrak{sp}(2, \mathbb{C}) \hookrightarrow \mathfrak{sl}(4, \mathbb{C})$ followed by the mapping of Problem 36 gives a homomorphism of $\mathfrak{sp}(2, \mathbb{C})$ into the Lie algebra $\mathfrak{g}$ of Problem 35. Show that there is some index i, $1 \le i \le 6$, such that the i^{th} row and column of the image in $\mathfrak{g}$ are always 0.

39. Deduce that the composition of the homomorphism of Problem 38 followed by the isomorphism $\mathfrak{g} \cong \mathfrak{so}(6, \mathbb{C})$ of Problem 35 may be regarded as an isomorphism of $\mathfrak{sp}(2, \mathbb{C})$ with $\mathfrak{so}(5, \mathbb{C})$.

Problems 40–42 give an explicit construction of a simple complex Lie algebra of type G_2.

40. Let Δ be the root system of type B_3 given in a space V as in (2.43). Prove that the orthogonal projection of Δ on the subspace of V orthogonal to $e_1+e_2+e_3$ is a root system of type G_2.

41. Let $\mathfrak{g}$ be a simple complex Lie algebra of type B_3. Let $\mathfrak{h}$ be a Cartan subalgebra, let the root system be as in Problem 40, and let B be the Killing form. Prove that the centralizer of $H_{e_1+e_2+e_3}$ is the direct sum of $\mathbb{C}H_{e_1+e_2+e_3}$ and a simple complex Lie algebra of type A_2 and dimension 8.

42. In Problem 41 normalize root vectors X_α so that $B(X_\alpha, X_{-\alpha}) = 1$. From the two vectors $[X_{e_1}, X_{e_2}] + 2X_{-e_3}$ and $[X_{-e_1}, X_{-e_2}] - 2X_{e_3}$, obtain four more vectors by permuting the indices cyclically. Let $\mathfrak{g}'$ be the 14-dimensional linear span of these six vectors and the A_2 Lie subalgebra of Problem 41. Prove that $\mathfrak{g}'$ is a Lie subalgebra of $\mathfrak{g}$ of type G_2.

Problems 43–48 give an alternative way of viewing the three classes of Lie algebras $\mathfrak{so}(2n+1, \mathbb{C})$, $\mathfrak{sp}(n, \mathbb{C})$, and $\mathfrak{so}(2n, \mathbb{C})$ that stresses their similarities. This point of view is useful in the study of automorphic forms. With A^t denoting the usual transpose of a square matrix A, define the **backwards transpose** ${}^tA$ as transpose about the opposite diagonal from usual or equivalently as $({}^tA)_{ij} = A_{n+1-j,n+1-i}$ if A is an n-by-n matrix. The mapping $A \mapsto {}^tA$ is linear, reverses the order of multiplication, leaves determinant unchanged, sends the identity to itself, maps inverses to inverses, and maps exponentials to exponentials. The n-by-n matrices A^t and ${}^tA$ are related by ${}^tA = LA^tL^{-1}$, where L is 1 along the opposite diagonal from usual (i.e., has $L_{i,n+1-i} = 1$ for $1 \le i \le n$) and is 0 otherwise.

43. Prove the **Principal-axis Theorem** concerning symmetric matrices over any field $\Bbbk$ of characteristic $\ne 2$, namely that if A is a square matrix over $\Bbbk$ with $A^t = A$, then there exists a nonsingular square matrix M over $\Bbbk$ such that M^tAM is diagonal. The proof is to proceed by induction on the size n, replacing a matrix $\begin{pmatrix} a & b \\ b^t & d \end{pmatrix}$ in block form with a of size $(n-1)$-by-$(n-1)$ and d of size 1-by-1 by $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ b^t & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^t & 1 \end{pmatrix}$ if $d \ne 0$ and replacing $\begin{pmatrix} a & b \\ b^t & d \end{pmatrix}$ by $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} a & b \\ b^t & 0 \end{pmatrix}\begin{pmatrix} 1 & y^t \\ 0 & 1 \end{pmatrix}$ if $d = 0$.

44. Prove a version of the result in Problem 43 for skew-symmetric matrices, namely that if A is a square matrix over $\Bbbk$ with $A^t = -A$, then there exists a nonsingular square matrix M over $\Bbbk$ such that M^tAM is block diagonal with diagonal blocks that are 2-by-2 or 1-by-1 and are skew-symmetric. The proof is to proceed by induction on the size as in Problem 43, except that d is a 2-by-2 skew-symmetric matrix chosen to be nonzero after a permutation of the coordinates.

45. Prove concerning square matrices over $\mathbb{C}$:
    (a) If A is nonsingular with $A^t = A$, then there exists a nonsingular square matrix M such that $M^tAM = 1$.
    (b) If A is nonsingular with $A^t = -A$, then the size is even and there exists a nonsingular square matrix M such that $M^tAM = J$, where J is as in §I.8.

46. Let A be an n-by-n nonsingular matrix that is symmetric or skew-symmetric, and define $G_A = \{x \in GL(n,\mathbb{C}) \mid x^{-1} = Ax^tA^{-1} \text{ and } \det x = 1\}$.
    (a) Prove that the linear Lie algebra of G_A is
    $$\mathfrak{g}_A = \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid AX^tA^{-1} + X = 0\}.$$
    (b) Prove that if A and B are nonsingular symmetric n-by-n matrices, then there exists $g \in GL(n,\mathbb{C})$ such that $G_B = gG_Ag^{-1}$.
    (c) Prove that if A and B are nonsingular skew-symmetric n-by-n matrices, then n is even and there exists $g \in GL(n,\mathbb{C})$ such that $G_B = gG_Ag^{-1}$.
    (d) Let $SO'(n,\mathbb{C}) = \{x \in GL(n,\mathbb{C}) \mid x^{-1} = {}^tx \text{ and } \det x = 1\}$. Prove that $SO'(n,\mathbb{C})$ is isomorphic to $SO(n,\mathbb{C})$ as a complex Lie group and that the linear Lie algebra of $SO'(n,\mathbb{C})$ is
    $$\mathfrak{so}'(n,\mathbb{C}) = \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid {}^tX + X = 0\}.$$

(e) Prove that $Sp(n,\mathbb{C}) = \{x \in GL(2n,\mathbb{C}) \mid x^{-1} = I_{n,n}{}^t x I_{n,n},\ \det x = 1\}$, where $I_{n,n}$ is the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ defined in §I.8, and that the definition of the Lie algebra $\mathfrak{sp}(n,\mathbb{C})$ may be written as

$$\mathfrak{sp}(n,\mathbb{C}) = \{X \in \mathfrak{gl}(2n,\mathbb{C}) \mid {}^t X I_{n,n} + I_{n,n} X = 0\},$$

47. Let $\mathfrak{g}$ be $\mathfrak{so}'(n,\mathbb{C})$ or $\mathfrak{sp}(n,\mathbb{C})$ as in Problem 46.
   (a) Show that the diagonal subalgebra of $\mathfrak{g}$ is a Cartan subalgebra.
   (b) Using the formula $[H, E_{ij}] = (e_i(H) - e_j(H))E_{ij}$ valid in $\mathfrak{sl}(N,\mathbb{C})$ for diagonal H, compute the root spaces in $\mathfrak{g}$ and show that the positive roots may be taken to be those whose root vectors are upper triangular matrices.

48. Prove that $SO'(N,\mathbb{C}) \cap GL(N,\mathbb{R})$ is isomorphic to $SO(n+1,n)$ if $N = 2n+1$, or to $SO(n,n)$ if $N = 2n$.