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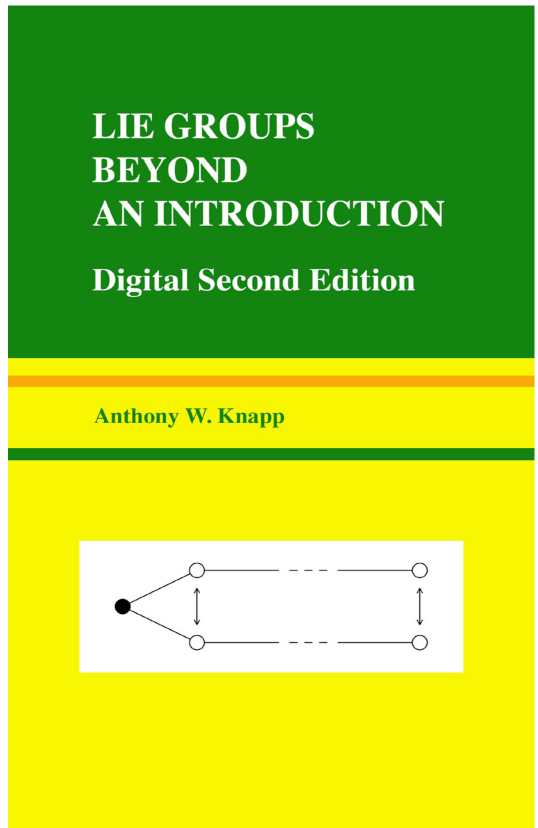

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Pages vii–xviii and 1–812 are the same in the digital and printed second editions. A list of corrections as of June 2023 has been included as pages 813–820 of the digital second edition. The corrections have not been implemented in the text.

Cover: Vogan diagram of $\mathfrak{sl}(2n, \mathbb{R})$. See page 399.

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CHAPTER X

Prehomogeneous Vector Spaces

Abstract. If G is a connected complex Lie group that is the complexification of a compact Lie group U, a "prehomogeneous vector space" for G is a complex finite-dimensional vector V together with a holomorphic representation of G on V such that G has an open orbit in V. The open orbit is necessarily unique. Easy examples include the standard representation of $GL(n, \mathbb{C})$ on $\mathbb{C}^n$, the standard representation of $Sp(n, \mathbb{C})$ on $\mathbb{C}^{2n}$, the action of $K^{\mathbb{C}}$ on $\mathfrak{p}^+$ when G/K is Hermitian, and certain actions obtained from the standard Vogan diagrams of some of the indefinite orthogonal groups.

The question that is to be studied is the decomposition of the symmetric algebra $S(V)$ under U. For any prehomogeneous vector space, the symmetric algebra $S(V)$ embeds in a natural U equivariant fashion into $L^2(U/U_v)$, where U_v is the subgroup of U fixing a point v in V whose G orbit is open. This fact gives a first limitation on what representations can occur in $S(V)$.

A "nilpotent element" e in a finite-dimensional Lie algebra $\mathfrak{g}$ is an element for which $\operatorname{ad} e$ is nilpotent. If $\mathfrak{g}$ is complex semisimple, the Jacobson–Morozov Theorem says that such an e, if nonzero, can be embedded in an "$\mathfrak{sl}_2$ triple" (h, e, f), spanning a copy of $\mathfrak{sl}(2, \mathbb{C})$.

When a complex semisimple Lie algebra is graded as $\bigoplus \mathfrak{g}^k$, $\operatorname{ad} \mathfrak{g}^0$ provides a representation of $\mathfrak{g}^0$ on $\mathfrak{g}^1$, and Vinberg's Theorem says that the result yields a prehomogeneous vector space. All such gradings arise from parabolic subalgebras of $\mathfrak{g}$. The examples above of the action of $K^{\mathbb{C}}$ on $\mathfrak{p}^+$ and of certain actions obtained from indefinite orthogonal groups are prehomogeneous vector spaces of this kind.

For the first of these two examples, the action of K on $S(\mathfrak{p}^+)$ is described by a theorem of Schmid. In the special case of $SU(m, n)$, this theorem reduces to a classical theorem about the action of the product of two unitary groups on the space of polynomials on a matrix space. For the second of these two examples, the action on the symmetric algebra can be analyzed by using this classical theorem in combination with Littlewood's Theorem about restricting respresentations from unitary groups to orthogonal groups.

In the general case of Vinberg's Theorem, if v is suitably chosen in the prehomogeneous vector space V, then $U/(U_v)_0$ fibers by a succession of three compact symmetric spaces, and hence $L^2(U/(U_v)_0)$ can be analyzed by iterating various branching theorems for compact symmetric spaces. This fact gives a second limitation on what representations can occur in $S(V)$.

1. Definitions and Examples

A consequence of Chapter IX is that we are able to use branching theorems to give a representation-theoretic analysis of the L^2 functions on certain compact quotient spaces that arise in the structure theory of noncompact groups. The goal of the present chapter is to develop methods for giving a representation-theoretic analysis of some spaces of holomorphic functions. The discussion will be necessarily incomplete as the topics in the chapter remain an active area of ongoing research.

The context will be as follows. Let G be a connected complex Lie group; usually we shall assume that G is the complexification of a compact Lie group U. A **prehomogeneous vector space** for G is a complex finite-dimensional vector space V together with a holomorphic representation of G on V such that G has an open orbit in V. The representation of G on V yields a holomorphic representation of G on each summand $S^n(V)$ of the symmetric algebra $S(V)$, and, when G is the complexification of U, the same thing is true of the restriction of the representation from G to U. We can lump the representations on the $S^n(V)$ together and think in terms of a single infinite-dimensional representation of G or U on $S(V)$ itself. We do so even though $S(V)$ is not a Hilbert space; we shall complete $S(V)$ to a Hilbert space shortly. The question is what can be said about this infinite-dimensional representation.

Let $\mathfrak{g}$ be the (complex) Lie algebra of G, and let φ be the differential of the representation of G on V. By the Inverse Function Theorem, the condition that the orbit of G through v be open in V can be expressed equivalently as

(i) every member of V is of the form $\varphi(X)v$ with X in $\mathfrak{g}$, or

(ii) the subalgebra $\mathfrak{g}_v$ of $\mathfrak{g}$ annihilating v has $\dim_{\mathbb{C}} V + \dim_{\mathbb{C}} \mathfrak{g}_v = \dim_{\mathbb{C}} \mathfrak{g}$.

EXAMPLES.

1) The standard representation of $G = GL(N, \mathbb{C})$ on $V = \mathbb{C}^N$. The nonzero vectors form an open orbit. The group U may be taken to be $U(N)$, and the representation of U on $S^n(\mathbb{C}^N)$ is irreducible with highest weight ne_1.

2) The standard representation of $G = Sp(N, \mathbb{C})$ on $V = \mathbb{C}^{2N}$. The members of the Lie algebra $\mathfrak{g}$ are of the form $\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$ with B and C symmetric. A count of the dimension of the subspace of members of $\mathfrak{g}$ whose first column is 0 shows that (ii) holds for v equal to the first standard

basis vector, and hence the orbit of that v is open. The group U may be taken to be $Sp(N)$, and the representation of U on $S^n(\mathbb{C}^{2N})$ is irreducible with highest weight ne_1, as a consequence of Example 1 and Theorem 9.76.

3) The action of $K^{\mathbb{C}}$ on $\mathfrak{p}^+$ by Ad when G/K is Hermitian. Let G be a linear semisimple group, let K be a maximal compact subgroup, and let G/K be Hermitian in the sense of §VII.9. In the notation of that section, the complexification $K^{\mathbb{C}}$ of K acts holomorphically on the sum $\mathfrak{p}^+$ of the root spaces for the noncompact roots that are positive in a good ordering. Let $\{\gamma_1, \dots, \gamma_s\}$ be a maximal set of strongly orthogonal positive noncompact roots, and let $E_{\gamma_1}, \dots, E_{\gamma_s}$ be corresponding nonzero root vectors. Let us use (i) above to see that the $K^{\mathbb{C}}$ orbit of $e = \sum_k E_{\gamma_k}$ is open. By way of preliminaries, we show that if β is a compact root, then $\beta + \gamma_i$ and $\beta + \gamma_j$ cannot be roots for two different indices i and j. If, on the contrary, both are roots, then the sum of $\beta + \gamma_i$ and $\beta + \gamma_j$ cannot be a root since $[\mathfrak{p}^+, \mathfrak{p}^+] = 0$ and the difference cannot be a root since γ_i and γ_j are strongly orthogonal. Thus $0 = \langle \beta + \gamma_i, \beta + \gamma_j \rangle = |\beta|^2 + \langle \beta, \gamma_i \rangle + \langle \beta, \gamma_j \rangle$. One of the inner products on the right side must be negative; say $\langle \beta, \gamma_i \rangle < 0$. Then $\langle \beta + \gamma_j, \gamma_i \rangle = \langle \beta, \gamma_i \rangle < 0$, and $\beta + \gamma_i + \gamma_j$ is a root, in contradiction to $[\mathfrak{p}^+, \mathfrak{p}^+] = 0$. We conclude that $\beta + \gamma_i$ and $\beta + \gamma_j$ cannot both be roots. Now let α be any positive noncompact root. We show that a nonzero multiple of the root vector E_α lies in $\operatorname{ad}(\mathfrak{k})e$, $\mathfrak{k}$ being the Lie algebra of $K^{\mathbb{C}}$. If $\alpha = \gamma_i$, then $[H_{\gamma_i}, e]$ is a nonzero multiple of E_{γ_i}. Thus assume that α is not some γ_i. Since $[\mathfrak{p}^+, \mathfrak{p}^+] = 0$, no $\alpha + \gamma_k$ is a root. By maximality of $\{\gamma_1, \dots, \gamma_s\}$, some $\beta = \alpha - \gamma_i$ is a root, necessarily compact. Our preliminary computation shows that $[E_\beta, e]$ is a nonzero multiple of E_α, and we conclude from (i) that the $K^{\mathbb{C}}$ orbit of e is open. The analysis of $S(\mathfrak{p}^+)$ will be discussed in §4.

4) Action of a certain group $L^{\mathbb{C}}$ on a space $\mathfrak{u} \cap \mathfrak{p}$ relative to either of the groups G given by $SO(2m, 2n)_0$ or $SO(2m, 2n+1)_0$ when $m \le n$. Form the standard Vogan diagram associated with the Lie algebra $\mathfrak{g}_0$ of G as in Figure 6.1 or Appendix C, relative to a compact Cartan subalgebra $\mathfrak{h}_0$. There is one simple noncompact root, namely $\alpha = e_m - e_{m+1}$. Write the complexification of $\mathfrak{g}_0$ as $\mathfrak{g} = \bigoplus_{k=-2}^{2} \mathfrak{g}^k$, where $\mathfrak{g}^k$ is the sum of the root spaces for roots whose coefficient of α is k in an expansion in terms of simple roots; include the Cartan subalgebra $\mathfrak{h}$ within $\mathfrak{g}^0$. This direct sum decomposition exhibits $\mathfrak{g}$ as **graded** in the sense that $[\mathfrak{g}^j, \mathfrak{g}^k] \subseteq \mathfrak{g}^{j+k}$. If $\mathfrak{l} = \mathfrak{g}^0$ and $\mathfrak{u} = \sum_{k>0} \mathfrak{g}^k$, then in particular $[\mathfrak{l}, \mathfrak{g}^k] \subseteq \mathfrak{g}^k$ for all k and $\mathfrak{l} \oplus \mathfrak{u}$ is a maximal parabolic subalgebra of $\mathfrak{g}$. For the complexification $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the usual Cartan decomposition of $\mathfrak{g}_0$, we have $\mathfrak{p} = \mathfrak{g}^1 \oplus \mathfrak{g}^{-1}$,

and thus $\mathfrak{u} \cap \mathfrak{p} = \mathfrak{g}^1$ is stable under ad $\mathfrak{l}$. Now let us pass to a group action. The centralizer in $\mathfrak{g}$ of the element $H = H_{e_1+\cdots+e_m}$ is just $\mathfrak{l}$, and iH is in $\mathfrak{k}_0$. By Corollary 4.51 the centralizer in K of iH is a compact connected subgroup L of K, and the complexification $L^{\mathbb{C}}$ of L has Lie algebra $\mathfrak{l}$. The adjoint representation of $L^{\mathbb{C}}$ on $\mathfrak{u} \cap \mathfrak{p}$ is the holomorphic representation of interest to us. We show that $L^{\mathbb{C}}$ acts on $\mathfrak{u} \cap \mathfrak{p}$ with an open orbit. For each noncompact positive root β, choose a root vector E_β normalized as in §VI.7 so that $[E_\beta, \overline{E_\beta}] = H'_\beta$; here the bar denotes the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Let e be the sum of the E_β's for β equal to $e_1 \pm e_{m+1}$, $e_2 \pm e_{m+2}, \ldots, e_m \pm e_{2m}$, let f be the sum of the corresponding elements $\overline{E_\beta}$, and let $h = [e, f]$. We prove that the $L^{\mathbb{C}}$ orbit of e is open by showing that $[\mathfrak{l}, e]$ contains a basis of $\mathfrak{u} \cap \mathfrak{p} = \mathfrak{g}^1$. The strong orthogonality of the roots β we have used makes it so that $\{h, e, f\}$ spans a copy $\mathfrak{s}$ of $\mathfrak{sl}(2, \mathbb{C})$ and so that h is a multiple of the element H above. By Theorem 1.67, $\mathfrak{g}$ is the direct sum of subspaces on which $\mathfrak{s}$ acts irreducibly. Since $\mathfrak{l}$ is the centralizer of h, $\mathfrak{l}$ is spanned by the weight vectors under h of weight 0 from the various irreducible subspaces. Theorem 1.66 then shows that $[\mathfrak{l}, e]$ is the sum of the weight vectors of weight 2, and this includes all the root vectors for the noncompact positive roots. Hence the $L^{\mathbb{C}}$ orbit of e is open. A partial analysis of $S(\mathfrak{u} \cap \mathfrak{p})$ will be discussed in §4.

Proposition 10.1. If V is a prehomogeneous vector space for G, then there is just one open orbit, and that orbit is dense.

PROOF. Fix bases over $\mathbb{C}$ for the vector spaces V and $\mathfrak{g}$, and let Φ and φ be the representations of G and $\mathfrak{g}$ on V. For each v in V, consider $X \mapsto \varphi(X)v$ as a linear transformation from $\mathfrak{g}$ into V, and let A_v be the $(\dim V) \times (\dim \mathfrak{g})$ matrix of this map relative to these bases. The entries of A_v are linear functions of $v \in V$, with values in $\mathbb{C}$. For some $v = v_0$, we know that $\varphi(\mathfrak{g})v_0 = V$ since V is assumed prehomogeneous. Thus the rank of A_{v_0} is $\dim V$, and some $(\dim V) \times (\dim V)$ minor of A_{v_0} has to be nonzero. If F denotes the vector-valued function on V whose value at v is the tuple of all $(\dim V) \times (\dim V)$ minors of A_v, then F is a vector-valued polynomial function on V whose value at v_0 is not zero. By Lemma 2.14 the set of v for which $F(v) \neq 0$ is connected, and it is certainly open and dense. Hence the subset Ω of $v \in V$ for which $\varphi(\mathfrak{g})v = V$ is open, dense, and connected.

If g is in G and $\varphi(\mathfrak{g})v = V$, then $\varphi(\mathfrak{g})\Phi(g)v = \Phi(g)\varphi(\mathrm{Ad}(g)^{-1}\mathfrak{g})v = \Phi(g)\varphi(\mathfrak{g})v = \Phi(g)V = V$, and it follows that Ω is carried to itself by $\Phi(G)$. Thus Ω is the union of disjoint orbits under $\Phi(G)$. For any

$v \in \Omega$, we have $\varphi(\mathfrak{g})v = V$, and hence the orbit $\Phi(G)(v)$ is open in V. Consequently Ω is exhibited as the disjoint union of open orbits, and the connectivity of Ω implies that there is just one orbit in Ω.

Proposition 10.2. Let G be the complexification of a compact connected group U, let V be a prehomogeneous vector space for G, and suppose that the G orbit of v_0 is open. If U_{v_0} denotes the subgroup of U fixing v_0, then $S(V)$ embeds in a natural one-one U equivariant fashion into $L^2(U/U_{v_0})$. In particular the multiplicity of any irreducible representation of U in $S(V)$ is bounded by the degree of the representation.

REMARK. For example, in the action of $GL(1, \mathbb{C})$ on $\mathbb{C}^1$, the group U is $U(1)$. Fix a nonzero member Z of $\mathbb{C}^1$. The action by $U(1)$ is $(e^{i\theta})Z = e^{i\theta}Z$, and the subgroup U_Z is trivial. The symmetric algebra $S(\mathbb{C}^1)$ consists of all polynomial expressions $p(Z)$, and the action is $(e^{i\theta})p(Z) = p(e^{i\theta}Z)$. The embedding of $S(\mathbb{C}^1)$ is into L^2 of the circle; if $e^{i\varphi}$ denotes a point on the circle, the embedding sends $p(Z)$ into the function $e^{i\varphi} \mapsto p(e^{i\varphi})$. The closure of the image is the subspace of members of L^2 that are boundary values of analytic functions on the unit disc.

PROOF. Let $P(V)$ be the space of all holomorphic polynomial functions from V into $\mathbb{C}$, and let $P^n(V)$ be the subspace of those functions that are homogeneous of degree n. The space $P^n(V)$ is the vector space dual of $S^n(V)$ by Corollary A.24, and the representation of G or U on $P^n(V)$ given by $g(p(v)) = p(g^{-1}v)$ is contragredient to the representation on $S^n(V)$. For each p in $P(V)$, define $\widetilde{p} : G \to \mathbb{C}$ by $\widetilde{p}(g) = p(gv_0)$; this is holomorphic, being the composition of the function $G \times v_0 \to V$ followed by p. The map $p \mapsto \widetilde{p}$ is one-one because the only holomorphic function vanishing on the open set Gv_0 is the 0 function. Restriction of holomorphic functions from G to U is one-one since, in a chart about the identity, the function on G can be reconstructed from the power series expansion of the function on U. In this way we obtain an embedding of $P(V)$ into $L^2(U/U_{v_0})$, and this embedding certainly respects the action by U.

To complete the proof, we pass from each $P^n(V)$ to its contragredient $S^n(V)$, and thereby embed each $S^n(V)$ into the contragredient of a finite-dimensional invariant subspace of $L^2(U/U_{v_0})$. Complex conjugation of functions carries invariant subspaces within L^2 to their contragredients, and in this way $S(V)$ is embedded into $L^2(U/U_{v_0})$. We may regard $L^2(U/U_{v_0})$ as a subspace of $L^2(U)$, and thus the bound on the multiplicities follows from the Peter–Weyl Theorem (Theorem 4.20).

2. Jacobson–Morozov Theorem

A member e of a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is said to be **nilpotent** if $\operatorname{ad} e$ is a nilpotent linear transformation. In this section we develop tools for working with nilpotent elements.

A triple (h, e, f) of nonzero elements in $\mathfrak{g}$ is called an $\mathfrak{sl}_2$ **triple** if the elements satisfy the bracket relations of (1.6): $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. In this case the span of the elements h, e, f is isomorphic with $\mathfrak{sl}(2, \mathbb{C})$. Theorem 1.67 shows that the complex-linear representation of this copy of $\mathfrak{sl}(2, \mathbb{C})$ on $\mathfrak{g}$ by ad is completely reducible, and Theorem 1.66 allows us to conclude that $\operatorname{ad} e$ is nilpotent on $\mathfrak{g}$; consequently the member e of the $\mathfrak{sl}_2$ triple (h, e, f) is a nilpotent element in $\mathfrak{g}$. The Jacobson–Morozov Theorem is a converse to this fact when $\mathfrak{g}$ is semisimple.

Theorem 10.3 (Jacobson–Morozov). If e is a nonzero nilpotent element in a complex semisimple Lie algebra $\mathfrak{g}$, then e can be included in an $\mathfrak{sl}_2$ triple (h, e, f). More specifically, there exists a nonzero h in $(\operatorname{ad} e)(\mathfrak{g})$ such that $[h, e] = 2e$, and, for any nonzero h in $(\operatorname{ad} e)(\mathfrak{g})$ with $[h, e] = 2e$, there exists a unique f in $\mathfrak{g}$ such that (h, e, f) is an $\mathfrak{sl}_2$ triple.

The proof will be preceded by a lemma.

Lemma 10.4. If V is a finite-dimensional complex vector space and if A and B are linear transformations from V to itself with A nilpotent and with $[A, [A, B]] = 0$, then AB is nilpotent.

PROOF. Put $C = [A, B]$. Then $[A, C] = 0$ by hypothesis, and it follows for every integer $n \geq 0$ that

$$[A, BC^n] = ABC^n - BC^nA = ABC^n - BAC^n = [A, B]C^n = C^{n+1}.$$

Consequently C^{n+1} is exhibited as a commutator, and it follows that C^p has trace 0 for every $p \geq 1$. Let us see that C is therefore nilpotent. Arguing by contradiction, suppose that C is not nilpotent, so that the number d of distinct nonzero roots of the characteristic polynomial of C is ≥ 1. Let $\lambda_1, \dots, \lambda_d$ be these distinct nonzero roots, and let $m_1, \dots, m_d$ be the multiplicities. The condition on the trace is that

$$\sum_{q=1}^{d} m_q \lambda_q^p = 0$$

for every $p \geq 1$. If we regard this condition for $1 \leq p \leq d$ as a homogeneous linear system with the m_q as unknowns, then the 0 solution is the only solution because the determinant of the coefficient matrix $\{\lambda_q^p\}_{p,q=1}^d$ is $\prod_{q=1}^d \lambda_q$ times a Vandermonde determinant and is therefore nonzero. Thus we have a contradiction, and we conclude that C is nilpotent.

Now let λ be any eigenvalue of AB, and let $v \neq 0$ be an eigenvector for λ. Since $[B, A]A = A[B, A]$ by hypothesis, we have

$$[B, A^n] = \sum_{j=0}^{n-1} A^j[B, A]A^{n-j-1} = \sum_{j=0}^{n-1} [B, A]A^{n-1} = n[B, A]A^{n-1}.$$

The transformation A is assumed nilpotent, and thus there exists an integer $r > 0$ such that $A^{r-1}v \neq 0$ and $A^r v = 0$. For this r,

$$\lambda A^{r-1}v = A^{r-1}ABv = A^r Bv = BA^r v - [B, A^r]v = 0 - r[B, A]A^{r-1}v,$$

and we see that $-\lambda/r$ is an eigenvalue of $[B, A]$. Since $[B, A]$ is nilpotent, we conclude that $\lambda = 0$. Therefore AB is nilpotent.

PROOF OF THEOREM 10.3. Let B be the Killing form. If $\mathfrak{n}$ denotes the kernel of $(\operatorname{ad} e)^2$, then every $z \in \mathfrak{n}$ has $0 = (\operatorname{ad} e)^2 z = [e, [e, z]]$ and therefore $0 = \operatorname{ad}[e, [e, z]] = [\operatorname{ad} e, [\operatorname{ad} e, \operatorname{ad} z]]$. Applying Lemma 10.4 with $A = \operatorname{ad} e$ and $B = \operatorname{ad} z$, we find that $(\operatorname{ad} e)(\operatorname{ad} z)$ is nilpotent. Hence $\operatorname{Tr}((\operatorname{ad} e)(\operatorname{ad} z)) = 0$ and

$$B(e, \mathfrak{n}) = 0. \tag{10.5}$$

The invariance of B implies that

$$B((\operatorname{ad} e)^2 x, y) = B(x, (\operatorname{ad} e)^2 y) \tag{10.6}$$

for all x and y in $\mathfrak{g}$. Taking y arbitrary in $\mathfrak{n}$ and using (10.6), we obtain $B((\operatorname{ad} e)^2\mathfrak{g}, \mathfrak{n}) = 0$. Therefore

$$(\operatorname{ad} e)^2\mathfrak{g} \subseteq \mathfrak{n}^\perp, \tag{10.7}$$

where $(\,\cdot\,)^\perp$ is as in §I.7. Taking y arbitrary in $((\operatorname{ad} e)^2\mathfrak{g})^\perp$ in (10.6) and using the nondegeneracy of B given in Theorem 1.45, we see that

$$((\operatorname{ad} e)^2\mathfrak{g})^\perp \subseteq \ker((\operatorname{ad} e)^2) = \mathfrak{n}.$$

Application of the operation $(\,\cdot\,)^\perp$ to both sides and use of (10.7) gives

$$(\operatorname{ad} e)^2\mathfrak{g} \subseteq \mathfrak{n}^\perp \subseteq ((\operatorname{ad} e)^2\mathfrak{g})^{\perp\perp}. \tag{10.8}$$

But Proposition 1.43 and the nondegeneracy of B combine to show that $V^{\perp\perp} = V$ for every subspace V of $\mathfrak{g}$, and therefore (10.8) yields

$$\mathfrak{n}^\perp = (\operatorname{ad} e)^2\mathfrak{g}. \tag{10.9}$$

From (10.5) and (10.9), it follows that $e = (\operatorname{ad} e)^2 x$ for some $x \in \mathfrak{g}$. If we put $h = -2[e, x]$, then h is a nonzero member of $(\operatorname{ad} e)\mathfrak{g}$ with $[h, e] = -2[[e, x], e] = 2[e, [e, x]] = 2e$. This proves the existence of h.

Next let h be any nonzero member of $(\operatorname{ad} e)\mathfrak{g}$ such that $[h, e] = 2e$. If $\mathfrak{k} = \ker(\operatorname{ad} e)$, then the equation $(\operatorname{ad} h)(\operatorname{ad} e) - (\operatorname{ad} e)(\operatorname{ad} h) = 2(\operatorname{ad} e)$ shows that $(\operatorname{ad} h)(\mathfrak{k}) \subseteq \mathfrak{k}$. Choose z such that $h = -[e, z]$. For A in $\operatorname{End}_{\mathbb{C}} \mathfrak{g}$, define $L(A)$, $R(A)$, and $\operatorname{ad} A$ to mean left by A, right by A, and $L(A) - R(A)$, respectively. Then we have

$$\begin{aligned}
(\operatorname{ad}(\operatorname{ad} e))(\operatorname{ad} z) &= [\operatorname{ad} e, \operatorname{ad} z] = \operatorname{ad}[e, z] = -\operatorname{ad} h\\
(\operatorname{ad}(\operatorname{ad} e))^2(\operatorname{ad} z) &= [\operatorname{ad} e, -\operatorname{ad} h] = \operatorname{ad}[e, -h] = 2\operatorname{ad} e\\
(\operatorname{ad}(\operatorname{ad} e))^3(\operatorname{ad} z) &= [\operatorname{ad} e, 2\operatorname{ad} e] = 0.
\end{aligned}$$

Imitating part of the proof of Lemma 5.17, we obtain, for every $n > 0$,

$$\begin{aligned}
(L(\operatorname{ad} e))^n(\operatorname{ad} z) &= (R(\operatorname{ad} e) + \operatorname{ad}(\operatorname{ad} e))^n(\operatorname{ad} z)\\
&= (R(\operatorname{ad} e))^n(\operatorname{ad} z) + n(R(\operatorname{ad} e))^{n-1}(\operatorname{ad}(\operatorname{ad} e))(\operatorname{ad} z)\\
&\quad + \tfrac{1}{2}n(n-1)(R(\operatorname{ad} e))^{n-2}(\operatorname{ad}(\operatorname{ad} e))^2(\operatorname{ad} z) + 0\\
&= (R(\operatorname{ad} e))^n(\operatorname{ad} z) - n(\operatorname{ad} h)(\operatorname{ad} e)^{n-1}\\
&\quad + n(n-1)(\operatorname{ad} e)^{n-1}.
\end{aligned}$$

Therefore

$$n(\operatorname{ad} h - (n-1))(\operatorname{ad} e)^{n-1} = (\operatorname{ad} z)(\operatorname{ad} e)^n - (\operatorname{ad} e)^n(\operatorname{ad} z).$$

This equation applied to u with $v = (\operatorname{ad} e)^{n-1}u$ shows that

$$n(\operatorname{ad} h - (n-1))v - (\operatorname{ad} z)(\operatorname{ad} e)v \quad \text{is in} \quad (\operatorname{ad} e)^n(\mathfrak{g}).$$

If v is in $\mathfrak{k}$ in addition, then $(\operatorname{ad} h)v$ is in $\mathfrak{k}$ and $(\operatorname{ad} z)(\operatorname{ad} e)v = 0$, so that

$$n(\operatorname{ad} h - (n-1))v \quad \text{is in} \quad \mathfrak{k}.$$

Thus $(\operatorname{ad} h - (n-1))$ carries $\mathfrak{k} \cap (\operatorname{ad} e)^{n-1}(\mathfrak{g})$ into $\mathfrak{k} \cap (\operatorname{ad} e)^{n}(\mathfrak{g})$. For some N, $(\operatorname{ad} e)^N(\mathfrak{g}) = 0$ since $\operatorname{ad} e$ is nilpotent. It follows that

$$\Big(\prod_{p=0}^{N-1} (\operatorname{ad} h - p)\Big)(\mathfrak{k}) = 0.$$

Consequently the eigenvalues of $\operatorname{ad} h$ on $\mathfrak{k}$ are all ≥ 0, and $(\operatorname{ad} h + 2)$ must be invertible on $\mathfrak{k}$. The element $[h, z] + 2z$ is in $\mathfrak{k}$ because

$$\begin{aligned}[e, [h, z]] + 2[e, z] &= -[h, [z, e]] - [z, [e, h]] + 2[e, z] \\ &= [h, h] + 2[z, e] + 2[e, z] = 0.\end{aligned}$$

Thus we can define $z' = (\operatorname{ad} h + 2)^{-1}([h, z] + 2z)$ as a member of $\mathfrak{k}$. Then we have $[h, z'] + 2z' = [h, z] + 2z$ and hence $[h, z' - z] = -2(z' - z)$. In other words the element $f = z' - z$ has $[h, f] = -2f$. Since z' is in $\mathfrak{k}$, we have $[e, f] = [e, z'] - [e, z] = h$, and f has the required properties.

Finally we are to show that f is unique. Thus suppose that f' has $[h, f'] = -2f'$ and $[e, f'] = h$. Theorem 1.67 shows that $\mathfrak{g}$ is fully reducible under the adjoint action of the span $\mathfrak{s}$ of h, e, f, and we may take $\mathfrak{s}$ itself to be one of the invariant subspaces. Write $f' = \sum f_i'$ according to this decomposition into invariant subspaces. From $-2\sum f_i' = -2f' = [h, f'] = \sum [h, f_i']$, we see that $[h, f_i'] = -2f_i'$ for all i. Also $h = [e, f'] = \sum [e, f_i']$ shows that $[e, f_i'] = 0$ for all components f_i' other than the one in $\mathfrak{s}$. If any f_i' outside $\mathfrak{s}$ is nonzero, then we obtain a contradiction to Theorem 1.66 since that theorem shows that $\operatorname{ad} e$ cannot annihilate any nonzero vector whose eigenvalue under $\operatorname{ad} h$ is -2. We conclude that $f_i' = 0$ except in the component $\mathfrak{s}$, and therefore we must have $f' = f$.

Theorem 10.10 (Malcev–Kostant). Let G be a complex semisimple group with Lie algebra $\mathfrak{g}$, and let (h_0, e_0, f_0) be an $\mathfrak{sl}_2$ triple in $\mathfrak{g}$. For each integer k, define $\mathfrak{g}^k = \{X \in \mathfrak{g} \mid [h_0, X] = kX\}$, and let G^0 be the analytic subgroup of G with Lie algebra $\mathfrak{g}^0$. Then the set Ω of all e in $\mathfrak{g}^2$ such that $\operatorname{ad} e$ carries $\mathfrak{g}^0$ onto $\mathfrak{g}^2$

(a) contains e_0,
(b) is open, dense, and connected in $\mathfrak{g}^2$,
(c) is a single orbit under G^0, and
(d) consists of all $e \in \mathfrak{g}^2$ that can be included in an $\mathfrak{sl}_2$ triple (h_0, e, f).

PROOF. Let $\mathfrak{s}$ be the span of $\{h_0, e_0, f_0\}$. Theorem 1.67 allows us to decompose $\mathfrak{g}$ into the direct sum of irreducible spaces V_i under $\operatorname{ad}\mathfrak{s}$, and

Theorem 1.66 describes the possibilities for the V_i. Since $\operatorname{ad} h_0$ carries each V_i into itself, we have $\mathfrak{g}^k = \bigoplus_i (V_i \cap \mathfrak{g}^k)$ for all k. From Theorem 1.66, $(\operatorname{ad} e_0)(V_i \cap \mathfrak{g}^0) = V_i \cap \mathfrak{g}^2$, and therefore $[e_0, \mathfrak{g}^0] = \mathfrak{g}^2$. This proves part (a).

Part (a) says that $\mathfrak{g}^2$ is a prehomogeneous vector space for G^0, and (b) and (c) then follow from Proposition 10.1.

Finally if $e \in \mathfrak{g}^2$ is in Ω, write $e = \operatorname{Ad}(g)e_0$ with $g \in G^0$, by (a) and (c). Then e is included in the $\mathfrak{sl}_2$ triple $(h_0, \operatorname{Ad}(g)e_0, \operatorname{Ad}(g) f_0)$. Conversely the argument that proves (a) shows that any e included in some $\mathfrak{sl}_2$ triple (h_0, e, f) lies in Ω. This proves (d).

Proposition 10.11. If $\mathfrak{g}$ is a complex reductive Lie algebra, then

(a) any abelian subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ for which the members of $\operatorname{ad}_{\mathfrak{g}} \mathfrak{s}$ are diagonable can be extended to a Cartan subalgebra and

(b) the element h of any $\mathfrak{sl}_2$ triple in $\mathfrak{g}$ lies in some Cartan subalgebra.

PROOF. Part (a) follows from Proposition 2.13, and part (b) is the special case of (a) in which $\mathfrak{s} = \mathbb{C}h$.

Proposition 10.12. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra, and let (h, e, f) be an $\mathfrak{sl}_2$ triple such that h lies in $\mathfrak{h}$. Then in a suitable system of positive roots, each simple root β has $\beta(h)$ equal to 0, 1, or 2.

PROOF. Theorems 1.67 and 1.66 show that the eigenvalues of $\operatorname{ad} h$ are integers, and hence $\alpha(h)$ is an integer for every root α. Consequently h lies in the real form $\mathfrak{h}_0$ of $\mathfrak{h}$ on which all roots are real, and we can take h to be the first member of an orthogonal basis of $\mathfrak{h}_0$ that defines a system of positive roots. Then $\alpha(h)$ is ≥ 0 for every simple root α, and we are to prove that $\alpha(h)$ cannot be ≥ 3.

Using Theorem 1.67, write $\mathfrak{g} = \bigoplus V_i$ with each V_i invariant and irreducible under the span of $\{h, e, f\}$. Suppose that α is a root with $\alpha(h) = n \geq 3$. Decompose a nonzero root vector X_α as $\sum X_i$ with X_i in V_i. From the equality $n \sum X_i = \alpha(h) X_\alpha = [h, X_\alpha] = \sum [h, X_i]$ and the invariance of V_i under $\operatorname{ad} h$, we see that $[h, X_i] = n X_i$ whenever $X_i \neq 0$. Since $n \geq 1$, Theorem 1.66 shows that $[f, X_i] \neq 0$ for any such i, and therefore $[f, X_\alpha] \neq 0$. Writing f as a sum of root vectors and possibly a member of $\mathfrak{h}$, we see in the same way that f is a sum of root vectors $X_{-\gamma}$ with $\gamma(h) = 2$. Since $[f, X_\alpha] \neq 0$, we must have $[X_{-\gamma}, X_\alpha] \neq 0$ for some γ with $\gamma(h) = 2$. Then $\beta = \alpha - \gamma$ is a root with $\beta(h) = n - 2 > 0$, and β must be positive. Since $\gamma(h) = 2 > 0$, γ is positive as well. Thus $\alpha = \beta + \gamma$ exhibits α as not being simple.

Corollary 10.13. Let $\mathfrak{g}$ be a complex reductive Lie algebra, and let $G = \operatorname{Int} \mathfrak{g}$. Up to the adjoint action of G on $\mathfrak{g}$, there are only finitely many elements h of $\mathfrak{g}$ that can be the first element of an $\mathfrak{sl}_2$ triple in $\mathfrak{g}$.

PROOF. All $\mathfrak{sl}_2$ triples lie in $[\mathfrak{g}, \mathfrak{g}]$, and thus we may assume $\mathfrak{g}$ is semisimple. If h is given, Proposition 10.11b produces a Cartan subalgebra $\mathfrak{h}$ containing h. With $\mathfrak{h}$ fixed, Proposition 10.12 shows, for a certain system of positive roots, that there are at most 3^l possibilities for h, where l is the rank. Any two Cartan subalgebras are conjugate via G, according to Theorem 2.15, and the number of distinct positive systems equals the order of the Weyl group. The corollary follows.

Proposition 10.14. Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\mathfrak{h}$ be a Cartan subalgebra. If $\mathfrak{s}$ is a subspace of $\mathfrak{h}$, then the centralizer $Z_{\mathfrak{g}}(\mathfrak{s})$ is the Levi subalgebra of some parabolic subalgebra of $\mathfrak{g}$, and hence it is reductive.

PROOF. Let Δ be the set of roots, and let $\mathfrak{h}_0$ be the real form of $\mathfrak{h}$ on which all roots are real. The centralizer of $\mathfrak{s}$ contains $\mathfrak{h}$, is therefore stable under $\operatorname{ad} \mathfrak{h}$, and consequently is a subspace of the form $\mathfrak{h} \oplus \bigoplus_{\gamma \in \Psi} \mathfrak{g}_\gamma$ for some subset Ψ of Δ, $\mathfrak{g}_\gamma$ being the root space for the root γ. If γ is in Ψ, then γ vanishes on $\mathfrak{s}$, and conversely. Hence $\Psi = \{\gamma \in \Delta \mid \gamma(\mathfrak{s}) = 0\}$. If bar denotes the conjugation of $\mathfrak{h}$ with respect to $\mathfrak{h}_0$, then each $\gamma \in \Psi$, being real on $\mathfrak{h}_0$, vanishes on $\bar{\mathfrak{s}}$. Hence each $\gamma \in \Psi$ vanishes on $\mathfrak{s} + \bar{\mathfrak{s}}$, which we write as $\mathfrak{t}$. Since $\mathfrak{t}$ is stable under bar, it is the complexification of the real form $\mathfrak{t}_0 = \mathfrak{t} \cap \mathfrak{h}_0$ of $\mathfrak{t}$. Thus $\Psi = \{\gamma \in \Delta \mid \gamma(\mathfrak{t}_0) = 0\}$. Let $\mathfrak{t}_0^\perp$ be the orthogonal complement of $\mathfrak{t}_0$ in $\mathfrak{h}_0$ relative to the Killing form, choose an orthogonal basis of $\mathfrak{h}_0$ consisting of an orthogonal basis of $\mathfrak{t}_0$ followed by an orthogonal basis of $\mathfrak{t}_0^\perp$, and let Π be the simple roots for the corresponding ordering. Define Π' to be the set of members of Π that vanish on $\mathfrak{t}_0$. If a positive root in Ψ is expanded in terms of simple roots, then each of the simple roots with nonzero coefficient must vanish on $\mathfrak{t}_0$ as a consequence of the choice of ordering; thus each simple root with nonzero coefficient is in Π'. Consequently $Z_{\mathfrak{g}}(\mathfrak{s})$ is the Levi subalgebra of the parabolic subalgebra corresponding to Π' in Proposition 5.90. The Levi subalgebra is reductive by Corollary 5.94c.

3. Vinberg's Theorem

A complex semisimple Lie algebra $\mathfrak{g}$ is said to be **graded** if vector subspaces $\mathfrak{g}^k$ are specified such that $\mathfrak{g} = \bigoplus_{k=-\infty}^{\infty} \mathfrak{g}^k$ and $[\mathfrak{g}^j, \mathfrak{g}^k] \subseteq \mathfrak{g}^{j+k}$ for all integers j and k. In other words, $\mathfrak{g}$ is to be graded as a vector space in the sense of (A.35), and the grading is to be consistent with the bracket structure. Since $\mathfrak{g}$ is by assumption finite dimensional, $\mathfrak{g}^k$ has to be 0 for all but finitely many k. The statement of Theorem 10.10 gives an example, showing how any $\mathfrak{sl}_2$ triple (h, e, f) leads to a grading; the indices in the grading are integers because Theorems 1.67 and 1.66 show that $\operatorname{ad} h$ acts diagonably with integer eigenvalues. Examples 3 and 4 of §1 arise from gradings associated with special parabolic subalgebras of $\mathfrak{g}$; more generally any parabolic subalgebra of $\mathfrak{g}$ leads to gradings as follows.

EXAMPLE. Gradings associated with a parabolic subalgebra. Fix a Cartan subalgebra $\mathfrak{h}$ and a choice Δ^+ of a system of positive roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let Π be the set of simple roots, and let $\mathfrak{n}$ be the sum of the root spaces for the members of Δ^+, so that $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$. Proposition 5.90 shows how to associate a parabolic subalgebra $\mathfrak{q}_{\Pi'}$ containing $\mathfrak{b}$ to each subset Π' of simple roots. Fix Π', associate a positive integer m_β to each member β of the complementary set $\Pi - \Pi'$, and let H be the member of $\mathfrak{h}$ such that

$$\beta(H) = \begin{cases} 0 & \text{if } \beta \text{ is in } \Pi' \\ m_\beta & \text{if } \beta \text{ is in } \Pi - \Pi'. \end{cases}$$

Then $\alpha(H)$ is an integer for every root α. Define

$$\mathfrak{g}^0 = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \Delta, \\ \alpha(H)=0}} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{g}^k = \bigoplus_{\substack{\alpha \in \Delta, \\ \alpha(H)=k}} \mathfrak{g}_\alpha \quad \text{for } k \neq 0.$$

Then $\mathfrak{g} = \bigoplus_k \mathfrak{g}^k$ exhibits $\mathfrak{g}$ as graded in such a way that $\mathfrak{q}_{\Pi'} = \bigoplus_{k \geq 0} \mathfrak{g}^k$, the Levi factor of $\mathfrak{q}_{\Pi'}$ is $\mathfrak{g}^0$, and the nilpotent radical of $\mathfrak{q}_{\Pi'}$ is $\bigoplus_{k>0} \mathfrak{g}^k$.

In fact, the next proposition shows that any grading $\mathfrak{g} = \bigoplus_k \mathfrak{g}^k$ of a complex semisimple Lie algebra $\mathfrak{g}$ arises as in the above example. First we prove a lemma.

Lemma 10.15. If $\mathfrak{g} = \bigoplus_k \mathfrak{g}^k$ is a graded complex semisimple Lie algebra, then there exists H in $\mathfrak{g}^0$ such that $\mathfrak{g}^k = \{X \in \mathfrak{g} \mid [H, X] = kX\}$ for all k.

PROOF. Define a member D of $\operatorname{End}_{\mathbb{C}} \mathfrak{g}$ to be multiplication by k on $\mathfrak{g}^k$. Direct computation shows that D is a derivation, and Proposition 1.121 produces an element H in $\mathfrak{g}$ such that $D(X) = [H, X]$ for all X in $\mathfrak{g}$. Since $[H, H] = 0$, H is in $\mathfrak{g}^0$.

Proposition 10.16. If $\mathfrak{g} = \bigoplus_k \mathfrak{g}^k$ is a graded complex semisimple Lie algebra, then there exist a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, a subset Π' of the set Π of simple roots, and a set $\{m_\beta \mid \beta \in \Pi - \Pi'\}$ of positive integers such that the grading arises from the parabolic subalgebra $\mathfrak{q}_{\Pi'}$ and the set $\{m_\beta\}$ of positive integers.

PROOF. Let H be as in Lemma 10.15. Proposition 10.11a with $\mathfrak{s} = \mathbb{C}H$ produces a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing H. The members X of $\mathfrak{g}$ that commute with H are exactly those with $D(X) = [H, X] = 0$ and hence are exactly those in $\mathfrak{g}^0$. In particular, $\mathfrak{h}$ is contained in $\mathfrak{g}^0$. The eigenvalues of $\operatorname{ad} H$ are integers, and thus H is in the real form $\mathfrak{h}_0$ of $\mathfrak{h}$ on which all the roots are real. Extend H to an orthogonal basis of $\mathfrak{h}_0$, and use this basis to define positivity of roots. Let Π be the set of simple roots, and let Π' be the subset on which $\beta(H) = 0$. For β in $\Pi - \Pi'$, define $m_\beta = \beta(H)$; since H comes first in the ordering, the nonzero integer m_β has to be positive. Then the given grading is the one associated to the parabolic subalgebra $\mathfrak{q}_{\Pi'}$ and the set of positive integers $\{m_\beta \mid \beta \in \Pi - \Pi'\}$.

Corollary 10.17. In any graded complex semisimple Lie algebra $\mathfrak{g} = \bigoplus_k \mathfrak{g}^k$, the subalgebra $\mathfrak{g}^0$ is reductive.

PROOF. Combine Proposition 10.16 and Corollary 5.94c.

Lemma 10.18. Let $\mathfrak{g} = \bigoplus_k \mathfrak{g}^k$ be a graded complex semisimple Lie algebra, and suppose that e is a nonzero element in $\mathfrak{g}^1$. Then there exist h in $\mathfrak{g}^0$ and f in $\mathfrak{g}^{-1}$ such that (h, e, f) is an $\mathfrak{sl}_2$ triple.

PROOF. Since $(\operatorname{ad} e)^j(\mathfrak{g}^k) \subseteq \mathfrak{g}^{j+k}$, e is nilpotent. Theorem 10.3 produces elements h' and f' in $\mathfrak{g}$ such that (h', e, f') is an $\mathfrak{sl}_2$ triple. Decompose h' and f' according to the grading as $h' = \sum h'_k$ and $f' = \sum f'_k$. From $2e = [h', e] = \sum [h'_k, e]$, we see that $[h'_0, e] = 2e$ and $[h'_k, e] = 0$ for $k \neq 0$. From $\sum [e, f'_k] = [e, f'] = h' = \sum h'_k$, we see that $[e, f'_{-1}] = h'_0$, hence that h'_0 is in $(\operatorname{ad} e)(\mathfrak{g})$. A second application of Theorem 10.3 shows that there exists f'' such that (h'_0, e, f'') is an $\mathfrak{sl}_2$ triple. Writing $f'' = \sum f''_k$, we obtain $[e, f''_{-1}] = h'_0$ and $[h'_0, f''_{-1}] = -2f''_{-1}$. Therefore (h'_0, e, f''_{-1}) is the required $\mathfrak{sl}_2$ triple.

In any grading $\mathfrak{g} = \bigoplus_k \mathfrak{g}^k$ of the complex semisimple Lie algebra $\mathfrak{g}$, $\operatorname{ad}\mathfrak{g}^0$ provides a complex-linear representation of $\mathfrak{g}^0$ on each $\mathfrak{g}^k$. Let G be a connected complex Lie group with Lie algebra $\mathfrak{g}$, for example $G = \operatorname{Int}\mathfrak{g}$, and let G^0 be the analytic subgroup of G with Lie algebra $\mathfrak{g}^0$. Then the adjoint action of G on $\mathfrak{g}$ yields a holomorphic representation of G^0 on each $\mathfrak{g}^k$.

Theorem 10.19 (Vinberg). Let G be a complex semisimple Lie group with a graded Lie algebra $\mathfrak{g} = \bigoplus_k \mathfrak{g}^k$, and let G^0 be the analytic subgroup of G with Lie algebra $\mathfrak{g}^0$. Then the adjoint action of G^0 on $\mathfrak{g}^1$ has only finitely many orbits. Hence one of them must be open.

REMARK. In other words the representation of G^0 on $\mathfrak{g}^1$ makes $\mathfrak{g}^1$ into a prehomogeneous vector space for G^0. This kind of prehomogeneous vector space is said to be of **parabolic type**.

PROOF. Once it is proved that there are only finitely many orbits, one of them must be open as a consequence of (8.18). To prove that there are only finitely many orbits, we shall associate noncanonically to each element e of $\mathfrak{g}^1$ a member of a certain finite set of data. Then we shall show that two elements that can be associated to the same member of the finite set are necessarily in the same orbit of G^0.

Let e be in $\mathfrak{g}^1$, and extend e by Lemma 10.18 to an $\mathfrak{sl}_2$ triple (h, e, f) with h in $\mathfrak{g}^0$ and f in $\mathfrak{g}^{-1}$. Write $\mathfrak{sl}_2$ for the copy of $\mathfrak{sl}(2,\mathbb{C})$ spanned by $\{h, e, f\}$. By Lemma 10.15, there exists an element H in $\mathfrak{g}^0$ such that, for every integer k, $[H, X] = kX$ for all X in $\mathfrak{g}^k$.

Among all abelian subalgebras of the centralizer $Z_{\mathfrak{g}^0}(\mathfrak{sl}_2)$ whose members T have $\operatorname{ad}T$ diagonable, let $\mathfrak{t}$ be a maximal one. The subalgebra $\widetilde{\mathfrak{t}} = \mathfrak{t} \oplus \mathbb{C}h$ of $\mathfrak{g}^0$ is abelian. The element H commutes with every member of $\widetilde{\mathfrak{t}}$ because $\widetilde{\mathfrak{t}} \subseteq \mathfrak{g}^0$, and hence so does $h-2H$. Also $[h-2H, e] = [h, e] - 2[H, e] = 2e - 2e = 0$ since $\operatorname{ad}H$ acts as the identity on $\mathfrak{g}^1$. Thus $h - 2H$ centralizes e and h. From Theorems 1.67 and 1.66 we know that any element of $\mathfrak{g}$ that centralizes e and h automatically centralizes $\mathfrak{sl}_2$. Thus $h - 2H$ is a member of $Z_{\mathfrak{g}^0}(\mathfrak{sl}_2)$ such that $\operatorname{ad}(h - 2H)$ is diagonable and $[h - 2H, X] = 0$ for all X in $\mathfrak{t}$. By maximality of $\mathfrak{t}$, $h - 2H$ is in $\mathfrak{t}$. Let us write

$$h = 2H + T_0 \qquad \text{with } T_0 \text{ in } \mathfrak{t}. \tag{10.20}$$

By Proposition 10.11a we can extend $\widetilde{\mathfrak{t}}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. From (10.20) we see that $[H, \mathfrak{h}] = 0$, and therefore $\mathfrak{h} \subseteq \mathfrak{g}^0$.

Let $\mathfrak{z} = Z_{\mathfrak{g}}(\mathfrak{t})$. By Proposition 10.14, $\mathfrak{z}$ is a Levi subalgebra of $\mathfrak{g}$, and the definition of $\mathfrak{t}$ implies that $\mathfrak{sl}_2 \subseteq \mathfrak{z}$. Let us see that the grading of $\mathfrak{g}$ induces a grading on $\mathfrak{z}$, i.e., that the subspaces $\mathfrak{z}^k = \mathfrak{z} \cap \mathfrak{g}^k$ have the property that $\mathfrak{z} = \bigoplus \mathfrak{z}^k$. If X is in $\mathfrak{z}$, decompose X according to the grading of $\mathfrak{g}$ as $X = \sum X_k$. For any T in $\mathfrak{t}$, we have $0 = [X, T] = \sum [X_k, T]$. Since T is in $\mathfrak{g}^0$, $[X_k, T]$ is in $\mathfrak{g}^k$, and thus $[X_k, T] = 0$ for all k. Hence each X_k is in $\mathfrak{z}$, and we conclude that $\mathfrak{z}$ is graded.

Since $\mathfrak{z}$ is a Levi subalgebra, Proposition 5.94c shows that $\mathfrak{z}$ is reductive. Using Corollary 1.56, write $\mathfrak{z}$ as the sum of its center and its commutator ideal, the latter being semisimple:

$$\mathfrak{z} = Z_{\mathfrak{z}} \oplus \mathfrak{s} \qquad \text{with } \mathfrak{s} = [\mathfrak{z}, \mathfrak{z}]. \tag{10.21}$$

We shall identify $Z_{\mathfrak{z}}$ as $\mathfrak{t}$. In fact, we know that $\mathfrak{h}$ is contained in $\mathfrak{z}$, and hence so is the subalgebra $\mathfrak{t}$. Since $\mathfrak{z}$ is defined as the centralizer of $\mathfrak{t}$, $\mathfrak{t}$ commutes with each member of $\mathfrak{z}$. Therefore $\mathfrak{t} \subseteq Z_{\mathfrak{z}}$. In the reverse direction let X be in $Z_{\mathfrak{z}}$. Then $[X, \mathfrak{h}] = 0$. Since $\mathfrak{h}$ satisfies $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ by definition of Cartan subalgebra, X must be in $\mathfrak{h}$. Therefore ad X is diagonable. We know that $\mathfrak{sl}_2$ is contained in $\mathfrak{z}$, and therefore $[X, \mathfrak{sl}_2] = 0$. Consequently X is in $Z_{\mathfrak{g}^0}(\mathfrak{sl}_2)$, and the maximality of $\mathfrak{t}$ shows that X is in $\mathfrak{t}$. Thus indeed $Z_{\mathfrak{z}} = \mathfrak{t}$.

Let us see that $\mathfrak{s}$ is graded, i.e., that the subspaces $\mathfrak{s}^k = \mathfrak{s} \cap \mathfrak{g}^k$ have the property that $\mathfrak{s} = \bigoplus \mathfrak{s}^k$. The subalgebra $\mathfrak{s}$ is generated by all $[\mathfrak{z}^j, \mathfrak{z}^k]$, and such a subspace is contained in $\mathfrak{g}^{j+k}$, hence in $\mathfrak{s}^{j+k}$. Thus every member of $\mathfrak{s}$ lies in $\bigoplus \mathfrak{s}^k$, and $\mathfrak{s}$ is graded. We can identify each $\mathfrak{s}^k$ a little better; since $\mathfrak{t}$ centralizes $\mathfrak{z}$, (10.20) yields

$$\mathfrak{s}^k = \mathfrak{s} \cap \mathfrak{g}^k = \{X \in \mathfrak{s} \mid [H, X] = kX\} = \{X \in \mathfrak{s} \mid [h, X] = 2kX\}$$

for all k.

The subalgebra $Z_{\mathfrak{z}}$ is graded, being completely contained in $\mathfrak{z}^0$. Hence (10.21) gives $\mathfrak{z}^k = (Z_{\mathfrak{z}})^k \oplus \mathfrak{s}^k$ for all k, and we conclude that $\mathfrak{s}^k = \mathfrak{z}^k$ for all $k \neq 0$. Thus e is in $\mathfrak{z}^1 = \mathfrak{s}^1$ and f is in $\mathfrak{z}^{-1} = \mathfrak{s}^{-1}$, and we see that the triple (h, e, f) lies in $\mathfrak{s}$. Let S^0 be the analytic subgroup of G with Lie algebra $\mathfrak{s}^0$. Since $\mathfrak{s}$ is semisimple and $\mathfrak{s}^0 = \{X \in \mathfrak{s} \mid [h, X] = 0\}$, Theorem 10.10 applies and shows that e lies in the unique open orbit of S^0 in $\mathfrak{s}^1$.

Let us now exhibit a finite set of data in the above construction. The grading of $\mathfrak{g}$ was fixed throughout, and the other gradings were derived from it. Starting from e, we worked with the tuple $(e, h, \mathfrak{t}, \mathfrak{h}, \mathfrak{z}, \mathfrak{s})$, and then we located e in the open orbit of S^0 in $\mathfrak{s}^1$. If we had started with e', let us write $(e', h', \mathfrak{t}', \mathfrak{h}', \mathfrak{z}', \mathfrak{s}')$ for the tuple we would have obtained. Before comparing our two tuples, we introduce a normalization. The Lie algebra $\mathfrak{g}^0$ is reductive, and $\mathfrak{h}$ and $\mathfrak{h}'$ are Cartan subalgebras of it. By Theorem 2.15 we can find $g \in G^0$ such that $\operatorname{Ad}(g)\mathfrak{h}' = \mathfrak{h}$. We replace $(e', h', \mathfrak{t}', \mathfrak{h}', \mathfrak{z}', \mathfrak{s}')$ by $\operatorname{Ad}(g)$ of the tuple, namely

$$(e'', h'', \mathfrak{t}'', \mathfrak{h}, \mathfrak{z}'', \mathfrak{s}'') = (\operatorname{Ad}(g)e', \operatorname{Ad}(g)h', \operatorname{Ad}(g)\mathfrak{t}', \mathfrak{h}, \operatorname{Ad}(g)\mathfrak{z}', \operatorname{Ad}(g)\mathfrak{s}'),$$

and then we readily check that if we had started with $\operatorname{Ad}(g)e'$, we could have arrived at this tuple through our choices. Since e' and e'' are in the same G^0 orbit, we may compare e with e'' rather than e with e'. That is our normalization: we insist on the same $\mathfrak{h}$ in every case.

Once $\mathfrak{h}$ is fixed, $\mathfrak{z}$ is the Levi subalgebra of a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$, h is an element of $\mathfrak{h}$ that is constrained by Proposition 10.12 to lie in a finite set, $\mathfrak{t}$ is the center of $\mathfrak{z}$, and $\mathfrak{s}$ is the commutator subalgebra of $\mathfrak{z}$. Our data set consists of all pairs

$$(\text{Levi subalgebra containing } \mathfrak{h}, \text{ element } h \text{ in } \mathfrak{h} \text{ as in Proposition 10.12}).$$

The number of Borel subalgebras containing $\mathfrak{h}$ equals the order of a Weyl group, and the number of parabolic subalgebras containing a given Borel subalgebra is finite; therefore the number of Levi subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$ is finite. Consequently our data set is finite.

What we have seen is that any e, possibly after an initial application of some member of $\operatorname{Ad}(G^0)$, leads to a member of this finite set. Suppose that e and e'' lead to the same member of the set. Then $\mathfrak{s} = \mathfrak{s}''$, e lies in the unique open orbit of S^0 on $\mathfrak{s}^1$, and e'' lies in that same orbit. Since $S^0 \subseteq G^0$, e and e'' lie in the same orbit under G^0. This completes the proof.

Corollary 10.22. Let G be a complex semisimple Lie group with a graded Lie algebra $\mathfrak{g} = \bigoplus_j \mathfrak{g}^j$, and let G^0 be the analytic subgroup of G with Lie algebra $\mathfrak{g}^0$. Then the adjoint action of G^0 on any $\mathfrak{g}^k$, with $k \neq 0$, has only finitely many orbits. Hence one of them must be open.

PROOF. Let H be as in Lemma 10.15, and let Φ be the automorphism of $\mathfrak{g}$ given by $\Phi = \operatorname{Ad}(\exp 2\pi i H/k)$. The subalgebra $\mathfrak{s}$ fixed by Φ is $\bigoplus_{jk} \mathfrak{g}^{jk}$, and thus $\mathfrak{s}$ is graded with $\mathfrak{s}^0 = \mathfrak{g}^0$ and $\mathfrak{s}^1 = \mathfrak{g}^k$. Extend $\mathbb{C}H$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that lies within $\mathfrak{g}^0$. Then $\mathfrak{s}$ contains $\mathfrak{h}$, and we find that $\mathfrak{s} = \mathfrak{h} \oplus \bigoplus_{\gamma\in\Psi} \mathfrak{g}_\gamma$, where Ψ is the set of roots γ for which $\gamma(H)$ is a multiple of k. The set Ψ is closed under $\gamma \mapsto -\gamma$, and that is all that is needed for the proof of Corollary 5.94cto show that $\mathfrak{s}$ is reductive with its center contained in $\mathfrak{s}^0 = \mathfrak{g}^0$. Replacing $\mathfrak{s}$ by $[\mathfrak{s}, \mathfrak{s}]$ and applying Theorem 10.19, we obtain the corollary.

Examples 3 and 4 in §1 are cases of Theorem 10.19 that contain the overlay of a real form of the underlying complex group. This additional structure can be imposed in complete generality. The grading of the complex semisimple Lie algebra $\mathfrak{g}$ leads, via an element H as in Lemma 10.15, to a parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. It does not completely specify a Cartan subalgebra or a system of positive roots, only the 1-dimensional subspace $\mathbb{C}H$ of a Cartan subalgebra and the positivity of the roots that are positive on H, namely those that contribute to $\mathfrak{u}$. Let us therefore extend $\mathbb{C}H$ to a Cartan subalgebra by means of Proposition 10.11a and then introduce a system of positive roots that takes H first in the ordering. To this much information we can associate a Dynkin diagram for $\mathfrak{g}$. This diagram we make into an abstract Vogan diagram by imposing zero 2-element orbits and by painting the simple roots that contribute to $\mathfrak{u}$. Theorem 6.88 says that this abstract Vogan diagram arises from a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ and a Cartan involution θ of $\mathfrak{g}_0$. Changing the meaning of G, let us write G for the analytic group corresponding to $\mathfrak{g}_0$ and $G^{\mathbb{C}}$ for its complexification with Lie algebra $\mathfrak{g}$. Let K be the maximal compact subgroup of G corresponding to θ. Since the Vogan diagram has zero 2-element orbits, we have $\operatorname{rank} G = \operatorname{rank} K$. The closure of $\exp(i\mathbb{R}H)$ is a torus in K and its centralizer L is a connected compact group whose Lie algebra is the real form $\mathfrak{l}_0 = \mathfrak{g}^0 \cap \mathfrak{g}_0$ of $\mathfrak{l} = \mathfrak{g}^0$. The complexification $L^{\mathbb{C}}$ of L has Lie algebra $\mathfrak{l}$. Then Theorem 10.19 says that $L^{\mathbb{C}}$ acts on $\mathfrak{g}^1$ with an open orbit. In the definition of prehomogeneous space in §1, the complex group is therefore $L^{\mathbb{C}}$, and the vector space is $V = \mathfrak{g}^1$. The compact form of $L^{\mathbb{C}}$, which was called U in the definition of prehomogeneous space, is the group L.

We will be especially interested in the special case in which the parabolic subalgebra is maximal parabolic. This is the case in which $\Pi - \Pi'$ consists of just one root, say β. If $m_\beta = k$, then the indexing for the grading uses only the integers in $k\mathbb{Z}$; so we may as well normalize matters by making $m_\beta = 1$. If the complexified Cartan decomposition is written as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, then $\mathfrak{k} = \bigoplus_{j \text{ even}} \mathfrak{g}^j$ and $\mathfrak{p} = \bigoplus_{j \text{ odd}} \mathfrak{g}^j$. Instances of this situation arise in Examples 3 and 4 in §1. Example 3 covers all cases in which the underlying group is simple and the unique noncompact simple root occurs just once in the largest root. The instances of SO with $m \geq 2$ in Example 4 are some classical cases in which the underlying group is simple and the unique noncompact simple root occurs twice in the largest root.

4. Analysis of Symmetric Tensors

Using notation as at the end of §3, let us examine Examples 3 and 4 of §1 from the point of view of decomposing $S(\mathfrak{g}^1)$ under the adjoint action of L or $L^{\mathbb{C}}$.

We begin with the instance of Example 3 in which $G = SU(m, n)$. This example is discussed at length in §VII.9. The notation will be less cumbersome if we work instead with $G = U(m, n)$ and $G^{\mathbb{C}} = GL(m + n, \mathbb{C})$. Here $K = U(m) \times U(n)$, and $K^{\mathbb{C}} = GL(m, \mathbb{C}) \times GL(n, \mathbb{C})$. We can write members of $\mathfrak{g} = \mathfrak{gl}(m + n, \mathbb{C})$ in blocks of sizes m and n as $\begin{pmatrix} * & * \\ * & * \end{pmatrix}$. In the complexified Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{k}$ consists of all the matrices $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, and $\mathfrak{p}$ consists of all the matrices $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$. We are interested in the action of K or $K^{\mathbb{C}}$ on $\mathfrak{p}^+$, which consists of all the matrices $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. Then Ad of a member $\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ of $K^{\mathbb{C}}$ on the $\mathfrak{p}^+$ matrix $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ is the $\mathfrak{p}^+$ matrix $\begin{pmatrix} 0 & k_1 x k_2^{-1} \\ 0 & 0 \end{pmatrix}$. Thus we can identify $\mathfrak{p}^+$ with the space $M_{mn}(\mathbb{C})$ of m-by-n matrices, and K or $K^{\mathbb{C}}$ is acting by $(k_1, k_2)(x) = k_1 x k_2^{-1}$. On the Lie algebra level, $\mathfrak{k} = \mathfrak{gl}(m, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C})$ is acting on $\mathfrak{p}^+$ by $(X_1, X_2)(x) = X_1 x - x X_2$. We use the direct sum of the diagonal subalgebras as Cartan subalgebra, and the positive roots are the $e_i - e_j$ with $i < j$. We are interested in the decomposition of $S(M_{mn}(\mathbb{C}))$ under $K = U(m) \times U(n)$, and the result is as follows.

Theorem 10.23. Let $r = \min(m, n)$. In the action of $U(m) \times U(n)$ on $S(M_{mn}(\mathbb{C}))$, the irreducible representations that occur are exactly the outer tensor products $\tau_\lambda^m \widehat{\otimes} (\tau_\lambda^n)^c$, where λ is any nonnegative highest weight of depth $\leq r$, and the multiplicities are all 1. Here τ^m and τ^n refer to irreducible representations of $U(m)$ and $U(n)$, respectively, and $(\,\cdot\,)^c$ indicates contragredient.

REMARK. Let $m \leq n$ for definiteness, so that $r = m$; the argument for $m > n$ is similar. If $\lambda = (a_1, \dots, a_m)$, then τ_λ^m has highest weight $(a_1, \dots, a_m)$, and $(\tau_\lambda^n)^c$ has *lowest* weight $(-a_1, \dots, -a_m, 0, \dots, 0)$. The highest weight of $(\tau_\lambda^n)^c$ is therefore $(0, \dots, 0, -a_m, \dots, -a_1)$.

FIRST PART OF THE ARGUMENT. Let us prove that the indicated irreducible representations actually occur. It is more convenient to work with the space $P(M_{mn}(\mathbb{C}))$ of polynomials with action $(k_1, k_2)(p)(x) = p(k_1^{-1} x k_2)$ than to work with the space of symmetric tensors; we take contragredients, one degree at a time, to get the decomposition of $S(M_{mn}(\mathbb{C}))$.

Let $P^d(M_{mn}(\mathbb{C}))$ be the subspace of polynomials homogeneous of degree d. Since the representation of $K^{\mathbb{C}}$ on each $P^d(M_{mn}(\mathbb{C}))$ is holomorphic, $\mathfrak{k}$ acts by

$$((X_1, X_2)p)(x) = \tfrac{d}{dt}\, p((\exp tX_1)^{-1}x(\exp tX_2))|_{t=0}. \tag{10.24}$$

We are to show that each $((-a_m, \ldots, -a_1), (a_1, \ldots, a_m, 0, \ldots, 0))$ occurs as a highest weight.

For $1 \le l \le m$, let $x^\# = x^\#(l)$ be the l-by-l submatrix of x obtained by using rows $m - l + 1$ through m and columns 1 through l, and let $d_l(x) = \det(x^\#)$. Suppose that k_1 and k_2 are upper triangular. Let $k_1^\#$ be the lower right l-by-l block of k_1, and let $k_2^\#$ be the upper left l-by-l block of k_2. A little computation shows that $d_l(k_1^{-1}xk_2) = \det(k_1^{\#-1}x^\# k_2^\#) = (\det k_1^\#)^{-1} d_l(x)(\det k_2^\#)$, and it follows that d_l is a nonzero highest weight vector with weight $-\sum_{i=m-l+1}^{m} e_i + \sum_{j=m+1}^{m+l} e_j$. From formula (10.24) we see that a product of powers of highest weight vectors is a highest weight vector and the weights are additive. If $a_1 \ge \cdots \ge a_m \ge 0$, then $d_1^{a_1-a_2} d_2^{a_2-a_3} \cdots d_{m-1}^{a_{m-1}-a_m} d_m^{a_m}$ is a highest weight vector with the required highest weight.

SECOND PART OF THE ARGUMENT. We give a heuristic proof that the multiplicities are 1 and that the only highest weights are the ones mentioned; the heuristic proof can be made rigorous without difficulty, but we will omit here the steps needed for that purpose.

There is one rigorous part. The linear functions $x \mapsto x_{ij}$ on $\mathfrak{p}^+$ with $i \le m < j$ form a basis for $P^1(M_{mn}(\mathbb{C}))$, and (10.24) shows that such a function is a weight vector with weight $-e_i + e_j$. Since linear combinations of products of such functions yield all polynomials, we can conclude that the only weights are sums of the expressions $-e_i + e_j$. That is, all the weights are of the form $((b_1, \ldots, b_m), (c_1, \ldots, c_n))$ with all $b_i \le 0$ and all $c_j \ge 0$. In particular, this is true of the highest weight of any irreducible constituent.

For the heuristic part, we use the choice $\gamma_j = e_j - e_{m+j}$ for $1 \le j \le m$ in Example 3 of §1. Then $e = \sum_{j=1}^{m} E_{j,m+j}$ is a member of $\mathfrak{p}^+$ in the unique open orbit under $K^{\mathbb{C}}$. If we write $(m+n)$-by-$(m+n)$ matrices in block form with blocks of sizes m, m, and $n - m$, then $e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where 1 is the m-by-m identity matrix. The members of $K^{\mathbb{C}}$ are anything invertible of the form $\begin{pmatrix} z & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$. Let $(K^{\mathbb{C}})_e$ be the subgroup of $K^{\mathbb{C}}$ fixing

e; direct computation shows that $(K^{\mathbb{C}})_e$ consists of all invertible matrices $\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & c & d \end{pmatrix}$. We can identify $K^{\mathbb{C}}/(K^{\mathbb{C}})_e$ with the open subset $\mathrm{Ad}(K^{\mathbb{C}})e$ of $\mathfrak{p}^+$. We are interested in identifying the action of $K^{\mathbb{C}}$ on the restrictions of the holomorphic polynomials to this set, and we only make the space of functions bigger if we consider *all* holomorphic functions on $K^{\mathbb{C}}/(K^{\mathbb{C}})_e$. The result is something like an induced representation except that only holomorphic functions are allowed. We introduce the notation "holo-ind" for this ill-defined construction, which we might call "holomorphic induction." We seek to understand $\text{holo-ind}_{(K^{\mathbb{C}})_e}^{K^{\mathbb{C}}} 1$. If we write $(K^{\mathbb{C}})_{\mathrm{int}}$ for the intermediate group consisting of all invertible matrices $\begin{pmatrix} z & 0 & 0 \\ 0 & a & 0 \\ 0 & c & d \end{pmatrix}$, then the formal computation, which we explain in a moment, is

$$\begin{aligned}\text{holo-ind}_{(K^{\mathbb{C}})_e}^{K^{\mathbb{C}}} 1 &= \text{holo-ind}_{(K^{\mathbb{C}})_{\mathrm{int}}}^{K^{\mathbb{C}}} \big(\text{holo-ind}_{(K^{\mathbb{C}})_e}^{(K^{\mathbb{C}})_{\mathrm{int}}} 1\big) \\ &= \text{holo-ind}_{(K^{\mathbb{C}})_{\mathrm{int}}}^{K^{\mathbb{C}}} \Big(\bigoplus_\lambda (\tau_\lambda^m)^c \widehat{\otimes} \tau_\lambda^m \widehat{\otimes} 1\Big) \\ &= \bigoplus_\lambda (\tau_\lambda^m)^c \widehat{\otimes} \big(\text{holo-ind}_{(K^{\mathbb{C}})_{\mathrm{int}}}^{K^{\mathbb{C}}} (\tau_\lambda^m \widehat{\otimes} 1)\big) \\ &= \bigoplus_\lambda (\tau_\lambda^m)^c \widehat{\otimes} \tau_\lambda^n.\end{aligned}$$

The symbol $\bigoplus$ here admits an interpretation as an orthogonal sum of Hilbert spaces, but let us not belabor the point. What deserves attention is the formal reasoning behind each line: The first line is holomorphic induction in stages, and the second line is the usual result for induction when a group H is embedded diagonally in $H \times H$. The embedding here is of the a as the diagonal subgroup of pairs (z, a); the inner representation does not depend on the variables c and d. The parameter λ varies over all highest weights of depth $\le m$. The third line uses commutativity of $\bigoplus$ and holomorphic induction, and again the innermost representation does not depend on the variables c and d. The fourth line is the crux of the matter and follows from the Borel–Weil Theorem, which is discussed briefly in the Historical Notes. The highest weight λ of τ_λ^m has to be nonnegative, as we saw above, and we obtain the desired upper bound for the multiplicities.

Let us state without proof a generalization of Theorem 10.24 that handles all instances of Example 3 of §1.

Theorem 10.25 (Schmid). If G/K is Hermitian and if a good ordering is used to define positivity of roots, introduce $\{\gamma_1, \ldots, \gamma_s\}$ as follows: γ_1

is the largest positive noncompact root, and, inductively, γ_j is the largest positive noncompact strongly orthogonal to all of $\gamma_1, \dots, \gamma_{j-1}$. Then the highest weights of the representations of $K^{\mathbb{C}}$ that occur in $S(\mathfrak{p}^+)$ are exactly all expressions $\sum_{j=1}^{s} a_j\gamma_j$ with all $a_j \in \mathbb{Z}$ and with $a_1 \geq \cdots \geq a_s \geq 0$. Moreover, all these representations occur in $S(\mathfrak{p}^+)$ with multiplicity 1.

Lemma 7.143 shows that s in Theorem 10.25 is the real rank of G. For this s, the theorem says that an s-parameter family of representations of $K^{\mathbb{C}}$ handles the analysis of $S(\mathfrak{p}^+)$.

Now let us turn to Example 4 in §1. The two classes of groups behave similarly, and we concentrate on $G = SO(2m, 2n)_0$. A look at the roots shows that $\mathfrak{l}_0 = \mathbb{R} \oplus \mathfrak{su}(m) \oplus \mathfrak{so}(2n)$, and one readily checks that $L \cong U(m) \times SO(2n)$. The noncompact positive roots, namely all $e_i \pm e_j$ with $i \leq m < j$, are the weights occurring in $\mathfrak{u} \cap \mathfrak{p}$. The various e_i's are the weights of the standard representation of $U(m)$, and the $\pm e_j$'s are the weights of the standard representation of $SO(2n)$. As a result we can check that $\mathfrak{u} \cap \mathfrak{p} \cong M_{m,2n}(\mathbb{C})$ and that the action of L on $P(\mathfrak{u} \cap \mathfrak{p})$ corresponds to the action on $P(M_{m,2n}(\mathbb{C}))$ by $U(m)$ on the left and $SO(2n)$ on the right. Hence the action of L on $S(\mathfrak{u} \cap \mathfrak{p})$ corresponds to the action on $S(M_{m,2n}(\mathbb{C}))$ by $U(m)$ on the left and $SO(2n)$ on the right. This is the natural restriction of the action of $U(m) \times U(2n)$ on $S(M_{m,2n}(\mathbb{C}))$, which is addressed in Theorem 10.23. According to that theorem, the irreducible constituents are all $\tau_\lambda^m \widehat{\otimes} (\tau_\lambda^{2n})^c$ for λ nonnegative of depth $\leq m$, and the multiplicities are all 1. Since $m \leq n$, the restriction of τ_λ^{2n} to $SO(2n)$ is given by Littlewood's result stated as Theorem 9.75; from the theorem we see that only the first m entries of the n-tuple highest weight of an irreducible constituent can be nonzero. Moreover, the resulting reducible representation of $SO(2n)$ is its own contragredient, and hence the restriction of $(\tau_\lambda^{2n})^c$ is the same as the restriction of τ_λ^{2n}. This much argument proves Theorem 10.26 below for $SO(2m, 2n)_0$, and a similar argument handles $SO(2m, 2n+1)_0$.

Theorem 10.26 (Greenleaf). For G equal to either of the groups $SO(2m, 2n)_0$ or $SO(2m, 2n+1)_0$ with $m \leq n$, every highest weight of L in the adjoint action on $S(\mathfrak{u} \cap \mathfrak{p})$ is in the span of $e_1, \dots, e_{2m}$.

Thus the number of parameters of irreducible representations of L appearing in $S(\mathfrak{u} \cap \mathfrak{p})$ is bounded above by the real rank $2m$ of G. (The multiplicities may be greater than 1, however.) Of course, the number of parameters for all the irreducible representations of L is the (complex) rank

$m + n$ of G, and hence only very special representations of L can occur in $S(\mathfrak{u} \cap \mathfrak{p})$ when m is much less than n.

Theorem 9.75 is explicit enough so that one can say more about the decomposition. The groups $SO(2, 2n)_0$ and $SO(2, 2n+1)_0$ are handled by Theorem 10.25. Here is a precise result about $SO(4, 2n)_0$. To avoid becoming too cumbersome, the statement takes liberties with the notion of representation, allowing a countable sum of irreducible representations, with no topology, to be considered as a representation.

Theorem 10.27 (Gross–Wallach). For $SO(4, 2n)_0$ with $n \geq 2$, the 1-dimensional representation τ of L with highest weight $2e_1 + 2e_2$ occurs in $S^4(\mathfrak{u} \cap \mathfrak{p})$ and has the property that the adjoint representation of L on $S(\mathfrak{u} \cap \mathfrak{p})$ decomposes as the tensor product of $1 \oplus \tau \oplus \tau^2 \oplus \tau^3 \oplus \cdots$ with a multiplicity-free representation σ whose irreducible constituents have highest weights described as follows: Let (a, b, k, d) be any integer 4-tuple satisfying

$$a \geq b \geq 0, \quad 0 \leq k \leq [a/2], \quad \max(0, b - 2k) \leq d \leq \min(b, a - 2k).$$

Then the corresponding highest weight for $n \geq 3$ is $ae_1 + be_2 + ce_3 + de_4$, where $c = a + b - 2k - d$. For $n = 2$, the same parameters are to be used, but the 4-tuple yields two highest weights $ae_1 + be_2 + ce_3 \pm de_4$ if $d \neq 0$.

PROOF. As we observed before the statement of Theorem 10.26, we are to decompose, for each integer pair (a, b) with $a \geq b \geq 0$, the representation of $U(2) \times U(2n)$ with highest weight $ae_1 + be_2 + ae_3 + be_4$ under the subgroup $U(2) \times SO(2n)$. We use Theorem 9.75 for this purpose. The expression μ in that theorem takes values of the form $2ke_3 + 2le_4$ with $k \geq l \geq 0$, $2k \leq a$, and $2l \leq b$. The contributions from $\mu = 2ke_3$ will be part of σ, and the other contributions will have $k \geq l \geq 1$. Writing σ both for the representation and for the space on which it acts and comparing the analysis that is to be done for (k, l) with that for $(k-1, l-1)$, we see that $S^m(\mathfrak{u} \cap \mathfrak{p}) \cong (\sigma \cap S^m(\mathfrak{u} \cap \mathfrak{p})) \oplus (\tau \otimes S^{m-4}(\mathfrak{u} \cap \mathfrak{p}))$ for $m \geq 4$. The tensor product relation follows, and we are left with analyzing σ.

With a and b fixed, we now want to work with $\lambda = ae_3 + be_4$ and $\mu = 2ke_3$, where $0 \leq k \leq [a/2]$. Consider the possibilities for an expression $\nu = ce_3 + de_4$ that is to contribute a Littlewood–Richardson coefficient $c^{\lambda}_{\mu\nu}$; ν is at least to have $c \geq d \geq 0$, $c \leq a$, and $d \leq b$. The diagram that arises in the statement of Theorem 9.74 has two rows. The first row consists of $2k$ 0's followed by $a - 2k$ x's, and the second row has b x's. The number of x's must match $c + d$, and thus $c + d = a + b - 2k$. The pattern of ν consists of

c 1's and d 2's, and only 1's can be used for the x's in the first row because of (a) and (c) in Theorem 9.74. Also the substitution of 1's and 2's for the x's in the second row must result in 1's followed by 2's because of (a) in that theorem. This fact already means that the diagram can be completed in at most one way, and we see as a result that σ is multiplicity free. The count of 1's and 2's is that we must have $c - (a - 2k)$ 1's and d 2's in the second row. Condition (b) in the theorem says that no column in the completed diagram can have a 1 above a 1; this means that the number of 1's in the second row, which is $c-(a-2k)$, must be $\leq 2k$. This condition simplifies to $c \leq a$ and is already satisfied. Finally condition (c) in the theorem says that the number of 2's in the appropriate listing, when all 2's have been listed, must not exceed the number of 1's to that point, and this means that $d \leq a - 2k$. The complete list of constraints is therefore

$$c + d = a + b - 2k,\ \ 0 \leq c \leq a,\ \ 0 \leq d \leq b,\ \ c \geq a - 2k,\ \ d \leq a - 2k.$$

Define c by $c = a + b - 2k - d$. The condition $c \geq a - 2k$ is equivalent with $d \leq b$, and $c \geq a - 2k$ forces $c \geq 0$. Thus the condition $0 \leq d \leq \min(b, a - 2k)$ incorporates all the inequalities except $c \leq a$. From the definition of c, this is equivalent with $d \geq b - 2k$. The theorem follows.

Apart from Examples 3 and 4 in §1, what can be said in some generality? We give just one result of this kind. It allows the induced representation in Proposition 10.2 to be analyzed in stages using three compact symmetric spaces.

Proposition 10.28. Suppose that the grading of the complex semisimple $\mathfrak{g}$ is built from a maximal parabolic subalgebra, and suppose that (h, e, f) is an $\mathfrak{sl}_2$ triple with $h \in \mathfrak{l}$, $e \in \mathfrak{g}^1$, and $f \in \mathfrak{g}^{-1}$ such that $\bar{h} = -h$ and $\bar{e} = f$, where bar is the conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$. Define $\mathbf{c} = \mathrm{Ad}(\exp \frac{1}{4}\pi i(e + f))$. This is an element of $\mathrm{Int}\,\mathfrak{g}$ of order dividing 8. Then the set of $X \in \mathfrak{l}_0$ with $[X, e] = 0$ equals the subalgebra of $\mathfrak{l}_0$ fixed by $\mathbf{c}$.

SKETCH OF PROOF. If $X \in \mathfrak{l}_0$ has $[X, e] = 0$, then $[X, f] = [X, \bar{e}] = 0$ and $\mathbf{c}(X) = X$. Conversely if X is in $\mathfrak{l}$ and $\mathbf{c}(X) = X$, let H be as in Lemma 10.15. The Lie algebra $\mathfrak{s} = \mathrm{span}\{H, h, e, f\}$ is reductive with center $\mathbb{C}(H - \frac{1}{2}h)$. Decompose $\mathfrak{g}$ into irreducibles V_i under $\mathrm{ad}\,\mathfrak{s}$, and write $X = \sum X_i$ accordingly. Then $[H, X_i] = 0$ and $\mathbf{c}(X_i) = X_i$ for all i. Also $\mathrm{ad}(H - \frac{1}{2}h)$ is scalar on each V_i, and hence X_i is a weight vector for $\mathrm{ad}\,h$. An easy check shows that $\mathbf{c}$ cannot fix a nonzero weight vector in V_i unless

$\dim V_i = 1$; in this case, $[e, X_i] = 0$. Summing on i gives $[e, X] = 0$. The result follows.

5. Problems

1. Let G be $SO(n, \mathbb{C})$ with the nonzero scalar matrices adjoined. Prove that the standard n-dimensional representation of G yields a prehomogeneous vector space for G.
2. Prove that the usual representation of $GL(2n, \mathbb{C})$ on $\bigwedge^2 \mathbb{C}^{2n}$ makes $\bigwedge^2 \mathbb{C}^{2n}$ into a prehomogeneous vector space for $GL(2n, \mathbb{C})$. Prove that the corresponding statement is false for $\bigwedge^3 \mathbb{C}^n$ if n is large enough.
3. Fix a complex semisimple group G. Prove that, up to isomorphism, there can be only finitely many representations of G that yield prehomogeneous vector spaces.
4. Let $\mathfrak{g}$ be a complex reductive Lie algebra, and let $G = \operatorname{Int} \mathfrak{g}$. Starting from Corollary 10.13, prove that, up to the adjoint action of G, there are only finitely many nilpotent elements in $\mathfrak{g}$.
5. Let the grading $\mathfrak{g} = \bigoplus_k \mathfrak{g}^k$ of the complex semisimple Lie algebra be associated to a maximal parabolic subalgebra, and suppose that $\mathfrak{g}^1 \neq 0$. Prove that the representation of $\mathfrak{g}^0$ on $\mathfrak{g}^1$ is irreducible.
6. State and prove a converse result to Problem 5.

Problems 7–9 develop and apply a sufficient condition for recognizing the open orbit in a prehomogeneous vector space of parabolic type.

7. Let $\mathfrak{g} = \bigoplus_k \mathfrak{g}^k$ be a graded complex semisimple Lie algebra, let $G = \operatorname{Int} \mathfrak{g}$, and let G^0 be the analytic subgroup of G with Lie algebra $\mathfrak{g}^0$. Suppose that $e \neq 0$ is in $\mathfrak{g}^1$, and suppose that e can be included in an $\mathfrak{sl}_2$ triple (h, e, f) such that h is a multiple of the element H given in Lemma 10.15. Prove that the G^0 orbit of e is open in $\mathfrak{g}^1$.
8. For the group $Sp(2, 2)$, let the simple roots be as in (2.50), and take $e_2 - e_3$ to be the only simple root that is noncompact. In the notation at the end of §3, $\mathfrak{u} \cap \mathfrak{p}$ is then spanned by root vectors E_α for α equal to $e_1 \pm e_3$, $e_1 \pm e_4$, $e_2 \pm e_3$, and $e_2 \pm e_4$. Prove for all nonzero constants a and b that the orbit under $L^{\mathbb{C}}$ of $e = aE_{e_1+e_3} + bE_{e_2-e_3}$ is open in $\mathfrak{u} \cap \mathfrak{p}$.
9. In the notation of Example 4 of §1 and the end of §3, the vector space $\mathfrak{u} \cap \mathfrak{p}$ was shown in §1 to be prehomogeneous for the subgroup $L^{\mathbb{C}}$ of $SO(2m, 2n)_0$ when $m \leq n$, but Vinberg's Theorem says that $\mathfrak{u} \cap \mathfrak{p}$ is prehomogeneous without this restriction. By mixing the definitions in Example 4 of §1 and in Problem 8 and by using Problem 7, obtain an explicit formula for an element e in the open orbit under the weaker restriction $m \leq 2n$.