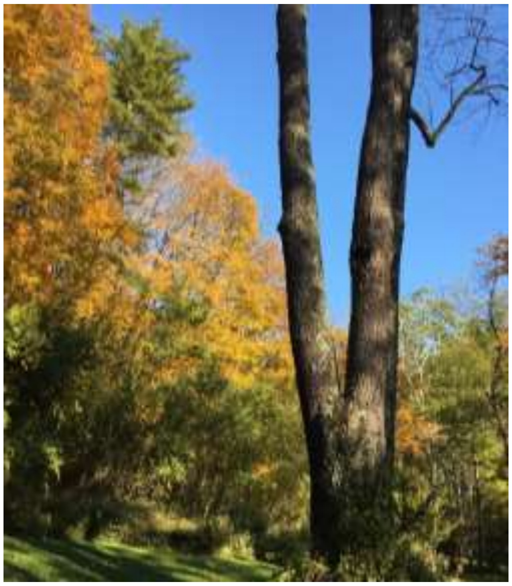
Chapter 3

WHAT IS AN ANGLE?



A (plane) **angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line. — Euclid, Elements, Definition 8

In this chapter you will be thinking about angles. In Problem **3.1** we will investigate various notions and definitions of angles and what it means for them to be considered to be the same (*congruent*). In Problem **3.2** we will prove the important *Vertical Angle Theorem* (VAT). It is not necessary to do these parts in order — you may find it easier to do Problem **3.2** before Problem **3.1** because it may help you think about angles. In a sense, you should be working on Problems **3.1** and **3.2** at the same time because they are so closely intertwined. This provides a valuable opportunity to apply and reflect on what you have learned about straightness in Chapters 1 and 2. This will also be helpful in the further study of straightness in Chapters 4 and 5; but, if you wish, you may study this chapter after Chapters 4 and 5.

PROBLEM 3.1 WHAT IS AN ANGLE?

Give some possible definitions of the term "*angle*." *Do all of these definitions apply to the plane as well as to spheres*? *What are the advantages and disadvantages of each*? *For each definition, what does it mean for two angles to be congruent*? *How can we check*?

SUGGESTIONS

Etymologically, "angle" comes through Old English, Old French, Old German, Latin, and Greek words for "hook." Textbooks usually give some variant of the definition: *An angle is the union of two rays* (*or segments*) *with a common endpoint*.

If we start with two straight line segments with a common endpoint and then add squiggly parts onto the ends of each one, would we say that the angle has changed as a result? Likewise, look at the angle formed at the lower-left-hand corner of this piece of paper. Even first grade students will recognize this as an example of an angle. Now, tear off the corner (at least in your imagination). Is the angle still there, on the piece you tore off? Now tear away more of the sides of the angles, being careful not to tear through the corner. The angle is still there at the corner, isn't it? See Figure 3.1.

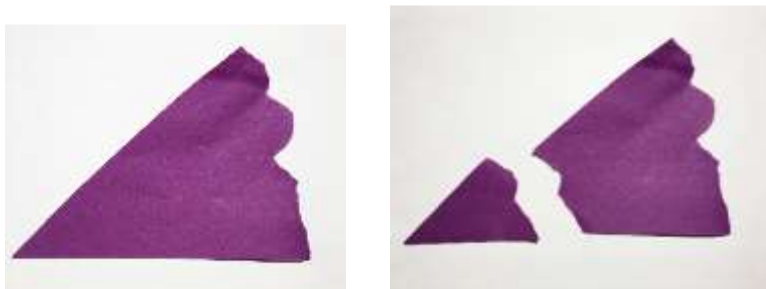


Figure 3.1 Where is the angle?

What part of the angle determines how large the angle is, or if it is an angle at all? What is the angle? Seems it cannot be merely a union of two rays. Here is one of the many cases where children seem to know more than we do. Paying attention to these insights, can we get better definitions of "angle"? Do not expect to find one formal definition that is completely satisfactory; it seems likely that no formal definition can capture all aspects of our experience of what an angle is.

There are at least three different perspectives from which we can define "angle," as follows:

- a **dynamic** notion of an angle – angle as movement;
- angle as a measure; and,
- angle as **geometric** shape.

A ***dynamic*** notion of angle involves an action: a rotation, a turning point, or a change in direction between two lines. Angle as ***measure*** may be thought of as the length of a circular arcs or the ratio between areas of circular sectors. Thought of as a ***geometric shape,*** an angle may be seen as the delineation of space by two intersecting lines. Each of these perspectives carries its methods for checking angle congruency. You can check the congruency of two dynamic angles by verifying that the actions involved in creating or replicating them are the same. If you feel that an angle is a measure, then you must verify that both angles have the same measure. If you describe angles as geometric shapes, then you describe how one angle can be made to coincide with the other using isometries. Which of the above definitions has the most meaning for you? Are there any other useful ways of describing angles?

Note that we sometimes talk about ***directed angles***, or angles with direction. When considered as directed angles, we say that the angles α and β in Figure 3.2 are not the same but have equal magnitude and opposite directions (or sense). Note the similarity to the relationship between line segments and vectors.

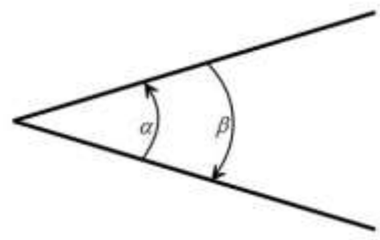


Figure 3.2 Directed angles

PROBLEM 3.2 VERTICAL ANGLE THEOREM (VAT)

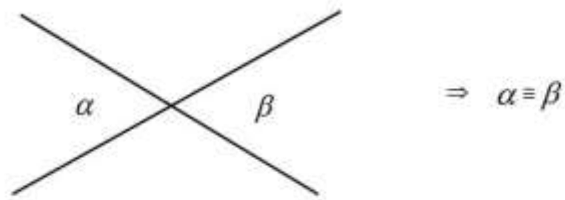


Figure 3.3 VAT

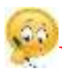
> *Prove: Opposite angles formed by two intersecting straight lines are congruent.* [Note: Angles such as α and β are called ***vertical angles***.] *What properties of straight lines and/or the plane are you using in your proof? Does your proof also work on a sphere? Why? Which definitions from Problem* **3.2** *are you using in your proof?*

Show how you would "move" α to make it coincide with β. We do not have in mind a formal two-column proof that used to be in American high school geometry. Mathematicians in actual practice usually use "proof" to mean "a convincing communication that answers — Why?" This is the notion of proof we ask you to use. There are three features of a proof:

- It must communicate (the words and drawings need to clearly express what it is that you want to say — and they must be understandable to your reader and/or listener.)
- It must be convincing (to yourself, to your fellow students, and to your teacher; preferably it should be convincing to someone who was originally skeptical).
- It must answer — Why? (Why is it true? What does it mean? Where did it come from?)

The goal is understanding. Without understanding we will never be fully satisfied. With understanding we want to expand that understanding and to communicate it to others.

Symmetries were an important element of your solutions for Problems **1.1** and **2.1**. They will be very useful for this problem as well. It is perfectly valid to think about measuring angles in this problem, but proofs utilizing line symmetries are generally simpler. It often helps to think of the vertical angles as whole geometric figures. Also, keep in mind that there are many different ways of looking at angles, so there are many ways of proving the vertical angle theorem. Make sure that your notions of angle and angle congruency in Problem **3.1** are consistent with your proofs in Problem **3.2**, and vice versa. Any of the definitions from Problem **3.1** can, separately or together, help you prove the Vertical Angle Theorem.

**You should pause and not read further until you have expressed your own thinking and ideas about Problems 3.1 and 3.2.**

HINTS FOR THREE DIFFERENT PROOFS

In the following section, we will give hints for three different proofs of the Vertical Angle Theorem. Note that a particular notion of angle is assumed in each proof. Pick one of the proofs or find your own different proof that is consistent with a notion of angle and angle congruence that is most meaningful to you.

1st proof:

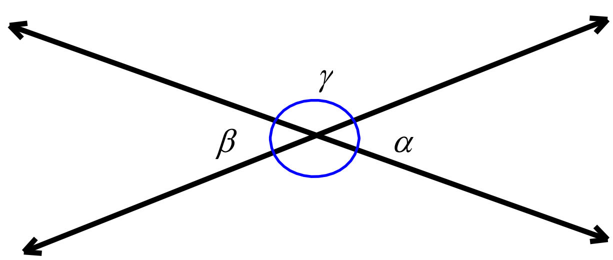


Figure 3.4 VAT using angle as measure

Each line creates a 180° angle. Thus, $\alpha + \gamma = \beta + \gamma$. See Figure 3.4. Therefore, we can conclude that $\alpha \cong \beta$. But why is this so? Is it always true that if we subtract a given angle from two 180° angles then the remaining angles are congruent? See Figure 3.5.

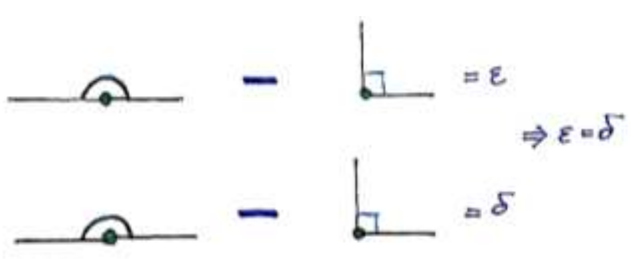


Figure 3.5 Subtracting angles and measures

Numerically, it does not make any difference how we subtract an angle, but geometrically it makes a big difference. Behold Figure 3.6! Here, ε really cannot be considered the same as δ. Thus, measure does not completely express what we *see* in the geometry of this situation. If you wish to salvage this notion of angle as measure, then you must explain *why* it is that in this proof of the Vertical Angle Theorem γ can be subtracted from both sides of the equation $\alpha + \gamma = \beta + \gamma$.

Figure 3.6 δ is not the same as ε

2nd proof: Consider two overlapping lines and choose any point on them. Rotate one of the lines, maintaining the point of intersection and making sure that the other line remains fixed as in Figure 3.7.

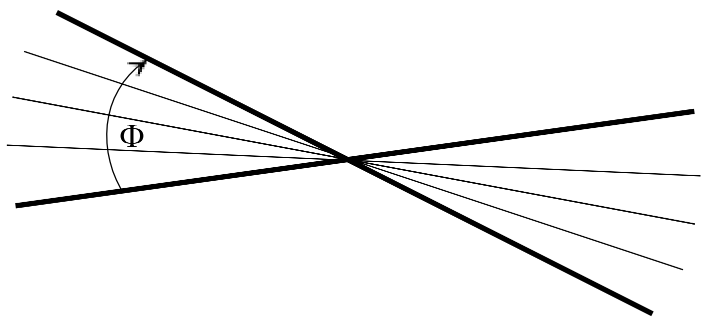


Figure 3.7 VAT using angle as rotation

What happens? What notion of angle and angle congruency is at work here?

3rd proof: What symmetries will take α onto β? See Figure 3.3 or 3.4. Use the properties of straight lines you investigated in Chapters 1 and 2.