IDEAS FOR MORE GEOMETRIC EXPLORATIONS

Tell me and I forget. Teach me and I remember. Involve me and I learn. – Benjamin Franklin

SOME RESOURCES

These suggestions how to experience more geometry are gathered from David's notes and various online sources after the 3rd edition (as available October 2019 but not all guaranteed to be there always; search for new ones, too). More can be found in the Bibliography which is the same as it was in the 3rd edition.

- Many ideas to explore are in *Geometry and Imagination* by John Conway, Peter Doyle, Jane Gilman, and Bill Thurston: https://math.dartmouth.edu/~doyle/docs/gi/gi.pdf
- Explore geometry through paper folding, including origami. Some stunning geometric origami images: https://www.thisiscolossal.com/2019/10/origami-by-arseni-kazhamiakin/.
- Also see *Patty Paper Geometry* by Michael Serra (Playing It Smart, 2011)
- Explore scutoid – the shape discovered by biologists: https://www.livescience.com/63207-scutoid-new-shape-nature.html 3-D files and a video explaining this shape: https://www.thingiverse.com/thing:3024272
- The oldest (103 in 2019!) still working mathematician Richard Guy has put online a book about triangle which he calls *The Triangle "movie"* https://arxiv.org/pdf/1910.03379.pdf

- Another online book *Advanced Plane Geometry: some generalization in geometry* by Patrick D. Barry, 2019 http://www.logicpress.ie/2019-1/index.html
- Interesting geometry can be found in *Geometry Snacks* and *More Geometry Snacks* by Ed Southhall and Vincent Pantaloni (Tarquin, 2018; 2019)
- To learn more about history of geometry you can read *Geometry by Its History*, by Alexander Osterman and Gerhard Wanner (Springer, 2012); *Geometry Through History*, by Meighan I. Dillon (Springer, 2018); *5000 Years of Geometry*, by Christoph J. Scriba and Peter Schreiber (Springer, 2015), and *Proof!: How the World Became Geometrical*, by Amir Alexander (Scientific American / Farrar, Straus and Giroux, 2019)
- Tools for exploring hyperbolic geometry http://www.uwosh.edu/faculty_staff/szydliks/noneucl.shtml
- GeoGebra https://www.geogebra.org/geometry?lang=en

CURVAHEDRA

Curvahedra (https://curvahedra.com/) is a construction puzzle system consisting of sets of identical cut pieces which link together without the aid of glue or tape. They allow to build a variety of 3D structures with positive, negative and zero curvature, including complex three dimensional tilings. *Curvahedra* is the brainchild of British Mathematician and University of Arkansas Math Professor Edmund Harriss. It has successfully been used in a variety of educational and recreational settings. Edmund Harriss tells that he designed Curvahedra mainly in order to create non-flat objects using a laser cutter. The individual flat pieces can be connected together to create loops, thus controlling the curvature of the region surrounded by the loop. As a basic example you can take a piece with 5 symmetric branches. The angle between the arms is therefore 72 degrees, connecting three of these pieces together thus creates an equilateral triangle with three angles of 72 degrees. If you continue to connect these triangles, the result is a sphere. In contrast if you make triangles from pieces with 7 branches, the result is an organic looking hyperbolic plane with saddles everywhere. The resulting system can be used to play and explore many ideas in symmetry, curvature and geometry such as the three related minimal surfaces two discovered by Schwartz (the D and P surfaces) and the Gyroid discovered by Alan Schoen. See following six pictures (provided by Edmund Harriss).

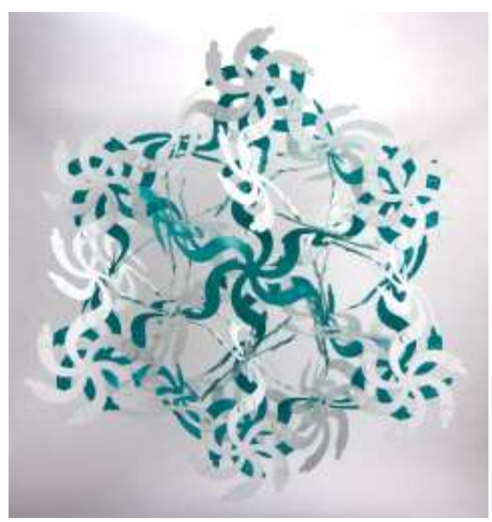
{3,7} Hyperbolic Plane

Sphere/ Icosahedron

{3,6} Torus with 12 5 fold centres for positive curvature and 6 4-sides loops to give the negative curvature.

{4,6} Structure related to Gyroid

{4,6} Structure related to Schwartz D surface



{4,6} Structure related to Schwartz P surface

EXPLORING POLYHEDRA

The following activities were designed by David Henderson for elementary school students. However, some of them can be of interest for any age.

Tetrahedron

Imagine the tetrahedron inside a sphere with its vertices on the surface of the sphere. Draw on the sphere the great circle segments that join these vertices. This produces tiling of the sphere. Draw this on an actual sphere.

What is the sum of the angles at each vertex of a tetrahedron?

Describe and demonstrate the different symmetries of a tetrahedron. [There are ½ turn, 1/3-turn, and reflection symmetries.]

Make a model of tetrahedron. Put the tetrahedron under a bright light so that it casts a shadow that is a triangle. Then orient the tetrahedron so that it casts a shadow that is a square!

Build a tetrahedral kite (directions can be found on-line, for example https://www.youtube.com/watch?v=noxjBfKtySQ)

Draw designs (with or without color) on the faces

so that all the symmetries are preserved

so that some symmetries are preserved, others not

so that there are no symmetries

You can find more about Alexander Graham Bell and his passion for the tetrahedron online, for example, https://en.wikipedia.org/wiki/Tetrahedral_kite.


Two of numerous A. G. Bell's tetrahedral kites

Octahedron

Imagine the octahedron inside the sphere with its vertices on the surface of a sphere – one vertex at the North Pole and the opposite vertex on the South Pole and the other four vertices evenly distributed around the equator. Draw the great circle segments on the sphere that join these vertices. This produces another tiling of the sphere. Draw this on an actual sphere.

What is the sum of the angles at each vertex of an octahedron?

Describe and demonstrate the symmetries of an octahedron. [There are ½-turn, 1/3-turn, ¼-turn, and two types of reflection symmetries.]

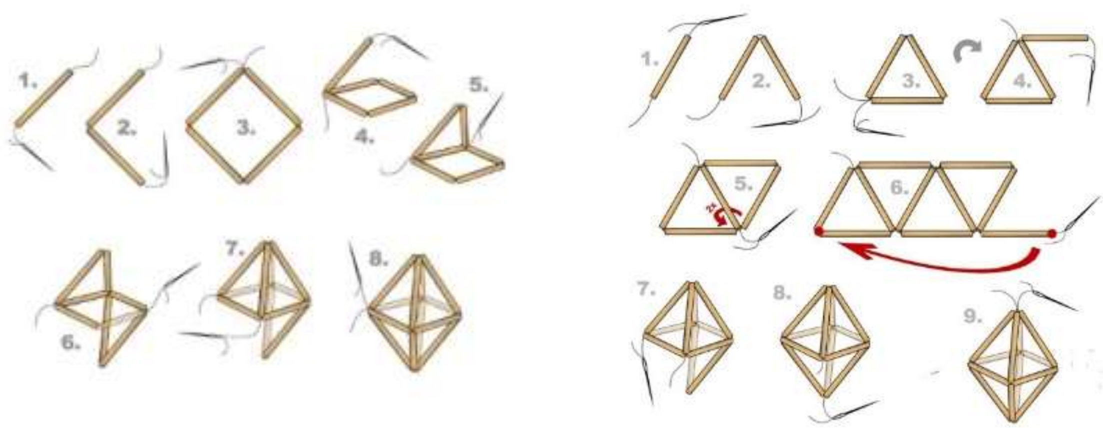
Make a model of octahedron. Put the octahedron under a bright light so that it casts a shadow that is a square. Then orient the octahedron so that it casts a shadow that is a regular hexagon. What do you observe?

Draw designs (with or without color) on the faces
- so that all the symmetries are preserved
- so that some symmetries are preserved, others not
- so that there are no symmetries

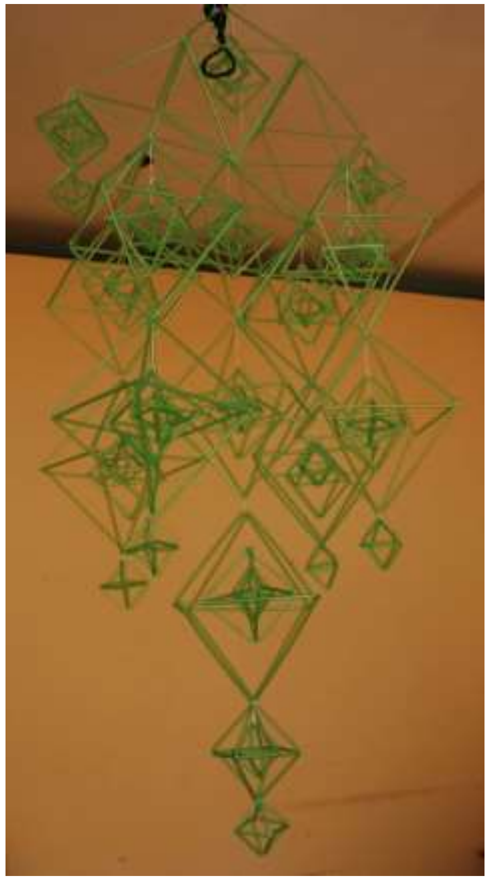
Show that the octahedron is an anti-prism with triangular bases.

Icosahedron

Make a hanging mobile from icosahedrons in various sizes.



Two ways to make straw icosahedron



This hanging mobile is made of drinking straws in 3 sizes – full size and cutting them in halves and quarters

Build one anti-prism with pentagonal faces and two pyramids with pentagon bases. Show how these can fit together to form an icosahedron.

The icosahedron has 1/5-turn, 1/3-turn and ½-turn rotation symmetries and also lots of reflection symmetries. Look for these symmetries and describe what you see.

What is the sum of the angles at each vertex of an icosahedron?

Draw designs (with or without color) on the faces
- so that all the symmetries are preserved
- so that some symmetries are preserved, others not
- so that there are no symmetries

Cube

Imagine the cube inside a sphere with its vertices on the surface of the sphere. Draw the great circle segments on the sphere that join these vertices. This produces another tiling of the sphere. Draw this on an actual sphere.

What is the sum of the angles at each vertex of a cube?

Describe symmetries of the cube. Are they different from the symmetries of an octahedron?

Put the cube under a bright light so that it casts a shadow that is a square. Then orient the cube so that it casts a shadow that is a regular hexagon. Describe what you observe.

Build a cubical kite.

Make a seamless origami cube (https://www.youtube.com/watch?v=i5QtqYWS5-I)

Draw designs (with or without color) on the faces:

so that all the symmetries are preserved
so that some symmetries are preserved, others not
so that there are no symmetries

Dodecahedron

Describe what symmetries you can find in the dodecahedron. You should be able to find all the same kinds of symmetries as you found on the icosahedron.

What are the differences between where you found the symmetries on the dodecahedron and where you found them on the icosahedron?

You can find many online resources how to build polyhedra (for example, https://www.ics.uci.edu/~eppstein/junkyard/polymodel.html). After regular polyhedra are made they can be turned into stellated ones by building a pyramid on each face.

Sport Balls as Tilings of a Sphere

Collect as many different balls as you can: baseball, softball, basketball, volleyball, tennis ball, soccer ball, and others that you can find that have seams or are divided by lines. In each case describe the tiling of the sphere that is indicated by the seams. What are the symmetries? What is the shape of the tiles? How many tiles? Do any have the same symmetries and the same number of symmetries as one of the Platonic Solids?

- Baseball
- Softball
- Basketball
- Volleyball
- Tennis ball
- Soccer balls (there are several different soccer ball designs with different symmetries, see https://archive.bridgesmathart.org/2015/bridges2015-151.pdf)

Why do you think the tilings and symmetries of these sports balls are different from each other? What do you think there is in how the sport is played for them to be different?

For further explorations of tilings and polyhedra

These are some on-line sources (there is no guarantee that these links will work forever!).

- Straw polyhedral http://www.math.nmsu.edu/~breakingaway/Lessons/straw/straw.html
- How to make straw geodesic dome: https://babbledabbledo.com/stem-kids-straw-geodesic-dome/
- How to make polyhedral kaleidoscope https://www.instructables.com/id/Easy-to-Make-Polyhedral-Kaleidoscopes/
- Historic polyhedra collection https://publicdomainreview.org/collections/max-bruckners-collection-of-polyhedral-models-1900/
- Interesting collaboration between a mathematician and an architect designing polyhedral that can collapse https://www.gathering4gardner.org/g4g12gift/Banchoff_Thomas-Hinge_Elastegrity.pdf
- Dissection tiling http://www.ics.uci.edu/~eppstein/junkyard/distile/
- Nakamura designs http://www.k4.dion.ne.jp/~mnaka/db_main.html
- Interlocked pavements http://www.geckostone.com/pavers.html
- Official website of Escher http://www.mcescher.com/
- Great article about tesselatins http://plus.maths.org/content/secrets-bathroom-floor
- About aperiodic tilings http://plus.maths.org/content/os/issue16/features/penrose/index
- Suggestions to exploree islamic tilings: https://docs.google.com/document/d/1-bYKI44sGxGZAj0U1OetiWoWrq3XAhTkgQLjshK82Pg/edit
- Alhambra tiling explorations using GeoGebra: https://www.geogebra.org/m/accseyfs
- Spherical tesselations http://www.cromp.com/pages/tess1.html
- How to tile plane using pentagons and other shapes http://plus.maths.org/content/os/issue45/features/kaplan/index
- Ron Resch origami tessellations https://vimeo.com/36122966 – this is a very nice story how simple exploration led to wonderful discoveries

3-D Symmetry

(David's unfinished notes, could be useful for prospective teachers)

The assumption here (which needs to be empirically checked) is that introducing "congruence" and "symmetry" in 3-D will be more powerful (and maybe easier) than in 2-D. The idea is to provide multiple experiences with reflections and rotations in 3-space so as to lead to the eventual goal of understanding them as transformations (functions), and to the goal of understanding 3-D reflection and rotation symmetries. The full goal is likely not to be accomplished immediately and may progress over some years.

A. 3-D Reflections – Mirror images

1. Interlocking Cubes: Have a large selection of Unifix® (or similar) interlocking cubes. The reason to use these (instead of something like MagFormers®) is to provide a more three-dimensional experience.

a. Play with the cubes seeing what you can build.

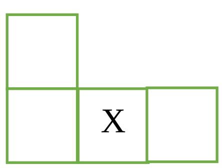
b. Make as many different shapes as you can with just 3 cubes – ignoring color. What do you find? Do you think anyone can find more? Why?

c. Make as many different shapes as you can with just 4 cubes – ignoring color. What do you find? Do you think anyone can find more? Why?

d. Make as many different 3-D shapes can you make with 5 cubes? -- These are called 3-D pentaminos.

e. When do you say that two 3-D pentaminos are the same shape?

- This will hopefully lead to class discussion.
- Be sure that the class considers the case of two pentaminos that are mirror-images of each other but cannot be flipped. For example, consider two pentaminos which, looking down from above, look like:



with four cubes sitting on the table and fifth cube attached to the cube marked with an "X" – on one above and on the other below.
- Ask students who see them as "the same" to say why. Ask students who see them as "different shapes" to say why.
- Look at them in a mirror. What do you notice?
- Hopefully, the students will say something like: "They are mirror images" – "If I look at one in a mirror then it looks like the other" – if not lead them in this direction.

2. Mirror (Reflected) Image: (all mirrors are considered to be <u>flat</u>)

a. Look at your left hand in a mirror and compare it to you right hand (not in the mirror). Can your left hand be "flipped" onto you right hand?

b. Consider your right shoe and your left shoe and look at them both in a mirror and not in a mirror. What do you notice?

c. Next to a vertical mirror, build of the floor (or table) a building with blocks and observe the image in the mirror.

d. On the other (back) side of the mirror build another block building that looks the same as the image in the mirror.

e. Keep both buildings but remove the mirror and draw a line on the floor (or table) where the mirror was. What do you observe?

f. **(for partners)** Start again with only the line left and imagine that there is an "invisible mirror" on that line. One person starts placing blocks (one at a time) on one side of the invisible mirror. After the first person places each block, the second person places a block of the other side of the invisible mirror so that it is where the image in the invisible mirror would be if there were a real mirror. When finished with the buildings, check by placing the mirror

The goal of these two activities is to arrive at experiences of two objects being:

- *directly congruent* = same size and shape and orientation = one can be rigidly (no stretching, breaking, and so forth) moved one onto the other
- *congruent* = same size and shape = one can be rigidly moved onto the other **OR** onto its mirror image.

It is desired that these notions first come from the students in their own words before the introducing the academic "congruent".

3. 3-D Mirror Symmetry:

a. Repeat the above building activities (with a real and invisible mirror) but this time design a building so that it is directly congruent to its mirror image. Such a building is said to have reflection (or mirror) symmetry.

b. Find examples of objects in your classroom and school that have reflection symmetry.

1. Note that most animal bodies are (very close to having) reflection symmetry.
2. Look at plants – do you see reflection symmetry?

c. What do you notice about objects with reflection symmetry?

- Guide the students' discussions to observing that object with reflection symmetry can be split into congruent halves with an invisible mirror separating the halves.
- The "invisible mirror" separating the two halves is called the "reflection plane" or "plane of reflection".

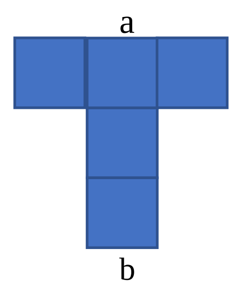
d. Which 3-D pentaminos have reflection symmetry? Do any have more than one reflection plane? Do any have only one plane of reflection? Do any have NO planes of reflection?

e. Create objects with MagFormers® that have 3-D reflection symmetry. Describe the planes of reflection. Is there a MagFormer object with only ONE plane of reflection?

f. Create an object with MagFormers that has NO reflection symmetry?

B. 3-D Rotation Symmetry

1. 3-D Pentaminos: Best with partners.



a. Start with the T-shaped 3-D pentamino:
Hold the pentamino between your two index fingers in the middle of the cubes at a and b. Now, keeping your fingers steady and the pentamino between them, have your partner turn (rotate) the pentamino. Note that for some rotations the pentamino looks in the same position it started – if a third person closed their eyes they would not be able to tell if it had been moved. [photo of this is needed]

- What do you notice?
  - A ½-turn leaves the pentamino the same.

[We say that this 3-D pentamino has ½-turn symmetry about the axis which is the line between the index fingers.]

b. Explore the other 3-D pentaminos. Which have ½-turn symmetry? Which do not?

c. Are there 3-D pentaminos with ¼-turn symmetry? Which?

d. Are there 3-D pentaminos with no rotation symmetries?

e. Are there 3-D pentaminos with more than one axis of rotation symmetry?

3. Using MagFormers.

- **a.** Consider a cube. Show that it has ¼-turn, 1/3-turn, 1/2-turn symmetry depending on what the axis is. Demonstrate.
- **b.** Consider a rectangular prism (box) with all three of the dimensions (length, width, height) different. What rotation symmetries does it have? [Only ½-turn.]
- **c.** Can you make an object with MagFormers which has rotation symmetry but NO reflection symmetry?
- **d.** **Local symmetry.** Consider a rectangular prism that is not a cube. There is no 1/3-turn rotation symmetry of the whole shape. But, if you imagine a small sphere centered at one of the prism's vertices, then the part of the prism inside of the sphere has 1/3-turn rotation symmetry. This is called *local 1/3-turn rotation symmetry at the vertex*.
- **e.** Make other shapes and investigate their local and global rotation symmetries.

4. Using blocks.

- **a.** **With partner.** On the floor or a table, mark a point and imagine a vertical line standing at this point – call this the axis. Partners sit on opposite sides of the table: One places a block near the axis and on her/his side of the axis, and then the other places a block on her/his side in such a way that the two blocks together have ½-turn symmetry about the axis. Continue is this way taking turns building a block "city" with ½-turn symmetry.
- **b.** **With 4-person group.** Do the same process except now there are 4 persons placing blocks in turn with the goal of ¼-turn symmetry.

C. 2-D Reflection and Rotation Symmetries.

1. 2-D Reflection Symmetry (also called "Line Symmetry")

- **a.** Place a small mirror (or Mira™) perpendicular to the paper along middle box line in each of the examples. What do you see?
- **b.** Trace the examples onto patty paper and fold to show that the traced images fall onto each other.
- **c.** Make your own: Fold the patty paper to form a line and then draw any figure on one side of the fold line and trace the figure onto the other side. Unfold. What do you notice?
- **d.** **(with a partner)** Start with a regular sheet of paper and draw a straight line down the middle. The first person **slowly** draws a figure on his/her side of the line – at

the same time the second person draws on the other side of the line a figure that is the reflected image of the figure that the first person draws. If available, try this with graph-paper.)

This activity can also be done by (instead of drawing) the first person arranging chips, stones, or some such in a pattern on one side of a line (on table or floor) and the second person arranging copies of the same objects on the other side of the line so that they form a reflective image.

2. 2-D reflection symmetry related 3-D symmetries.

Consider a figure draw on patty paper with 2-D reflection symmetry about a line draw on the patty paper. Now consider this patty paper in 3-space and image a mirror in space which is perpendicular to the patty paper and intersecting along the line. What do you notice about the 3-D symmetry of the figure? Consider

- 3-D reflection in the mirror
- 3-D ½-turn rotation with axis the line on the patty paper

3. 2-D ½-turn rotation symmetry

Start with examples of half-turn from Activity 1:

a. Trace the examples onto patty paper, hold a pencil point on the center, and turn the traced image until it covers the original figure.

b. Use patty paper to make figures with half-turn symmetry: Fold the patty paper in half and then fold in half again so that the fold lines divide the patty paper into four roughly equal quarters. Draw any figure in one of these. Mark with pencil a small “x” on the diagonally opposite quarter. Now, fold the patty paper back up along the two fold lines so that the “x” is on the outside. Then erase the “x” and trace the figure onto this side. Unfold. Check that this is a half-turn.cc

c. Fold another sheet of patty paper into quarters as described in **b**. While folded cut a curved edge in the folded patty paper. Unfold. Look for half-turn images.

d. **(with a partner)** Start with a regular sheet of paper and draw a dot in the middle of the paper. One person starts at the dot and **slowly** draws a figure on his/her side of the dot – at the same time the other person draws on the other side of the dot so that the second figure is a half-turn image of the first figure. (Try with graph-paper.)

This activity can also be done by (instead of drawing) the first person arranging chips, stones, or some-such in a pattern on one side of a point (on table or floor) and the second person arranging copies of the same objects on the other side of the point so that they form a half-turn image.