Chapter 24

3-MANIFOLDS — THE SHAPE OF SPACE

Jeff Weeks, Reuben Hersh, and David Henderson at Fall 2002 MAA Seaway Section meeting

> ... if we look at the extreme points of the sky, all the visual rays appear equal to us, and if diametrically opposed stars describe a great circle, one is setting while the other is rising. If the universe, instead of being spherical, were a cone or a cylinder, or a pyramid or any other solid, it would not produce this effect on earth: one of its parts would appear larger, another smaller, and the distances from earth to heaven would appear unequal. — Theon of Smyrna (~70–~135, Greek), [AT: Theon]

> It will be shown that a multiply extended quantity [three-dimensional manifold] is susceptible of various metric relations, so that Space constitutes only a special case of a triply extended quantity. From this however it is a necessary consequence that the theorems of geometry cannot be deduced from general notions of quantity, but that those properties that distinguish Space from other conceivable triply extended quantities can only be inferred from experience. — G. F. B. Riemann (1826–1866, German) On the Hypotheses Which Lie at the Foundations of Geometry, translated in [DG: Spivak], Vol. II, p. 135.

Now we come to where we live. We live in a physical 3-dimensional space, that is locally like Euclidean 3-space. The fundamental question we will investigate in this chapter is, how can we tell what the shape of our Universe is? This is the same question both Theon of Smyrna and Riemann were attempting to answer in the above quotes. The question about the shape of the Universe is the question about two geometries: local (primarily described by its curvature) and global (topology describing general global properties of the shape of Universe as of a continuous object). The shape of the Universe is related to the theory of general relativity, describing curved spacetime.

SPACE AS AN ORIENTED GEOMETRIC 3-MANIFOLD

A geometric 3-manifold is a space in which each point in the space has a neighborhood that is isometric with a neighborhood of either Euclidean 3-space, a 3-sphere, or a hyperbolic 3-space. (The notion of a 3-manifold was introduced in the work by Riemann quoted above.) We say "globally a geometric 3-manifold" in the same sense in which we say that globally Earth is a sphere (and spherical geometry is the appropriate geometry for intercontinental airplane flights) even though it is clear almost anywhere on the earth that locally there are many hills and valleys that make Earth not locally isometric to a sphere. However, the highest point on Earth (Mount Everest) is 8.85 km above sea level and the lowest point on the floor of the ocean (the Mariana Trench) is 10.99 km below sea level — the difference is only about 0.3% of the 6368 km radius of Earth (variations in the radius are of the same magnitude).

It was predicted by Einstein's general theory of relativity that our physical Universe is affected by any mass (especially large masses like our sun and other stars) [Albert Einstein (1879–1955)]. This effect is fairly accurately illustrated by imagining a 2-dimensional universe that is the surface of a flat rubber sheet. If you place steel balls on this rubber sheet, the balls will locally make dents or dimples in the sheet and thus will locally distort the flat Euclidean geometry. Einstein's prediction has been confirmed in two ways, as follows:

1. The orbits of the planets Mercury, Jupiter, and Saturn are (quite accurately) ellipses, and the major axes of these elliptical orbits change directions (precess). Classical Newtonian mechanics (based on Euclidean geometry) predicts that in a century the precession will be Mercury: 1.48°, Jupiter: 1.20°, Saturn: 0.77°, measured in degrees of an arc. Astronomers noticed that the observed amount of precession agreed accurately with these values for Saturn and Jupiter; however, for Mercury (the closest planet to the Sun) the observed precession is 1.60°, which is 0.12° more than is predicted by Newtonian/Euclidean methods. But if one does the computations based on the curvature of spacetime near the sun that is predicted by Einstein, the calculations agree accurately with the observed precession. For a mathematician's description of this calculation (which uses formulas from differential geometry), see [**DG**: Morgan], Chapter 7.

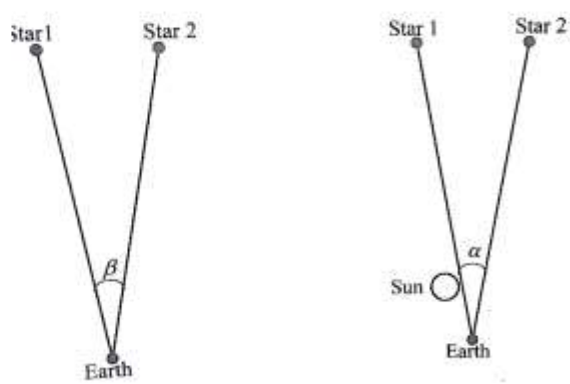

Figure 24.1 Observing local non-Euclidean geometry in the Universe

2. In 1919 British astronomers, led by Arthur Eddington (1882– 1944), measured the angle subtended by two stars from Earth: once when the Sun was not near the path of the light from the stars to Earth and once when the path of light from one of the stars went very close to the Sun. See Figure 24.1. In order to be able to see the star when its light passes close to the Sun, they had to make the second observation during a total eclipse of the Sun. They observed that the angle (α) measured with the Sun near was smaller than the angle (β) measured when the Sun was not near. The difference between the two measured angles was exactly what Einstein's theory predicted. Some accounts of this experiment talk about the Sun "bending" the light's rays but it is more accurate to say that light's worldline (the path that object traces in 4-dimensional spacetime) is a geodesic in spacetime and that the Sun distorts the geometry (curvature) of the spacetime. (Newtonian physics also predicts that the light bends but with a different bending angle).

But most of our physical Universe is empty space with only scattered planets, stars, galaxies, and the "dimples" that the stars make in space is only a very small local effect near the star. The vast empty space appears to locally (on a medium scale much larger than the scale of the distortions near stars) be a geometry of the local symmetries of Euclidean 3-space. In particular, our physical space is observed to be locally (on a medium scale) the same in all directions with isometric rotations, reflections, and translations as in Euclidean 3-space.

PROBLEM 24.1 COULD OUR UNIVERSE BE NON-EUCLIDEAN?

Carl Friedrich Gauss (1777–1855, German) found the formula for the sum of the angles of geodesic triangle on *any* surface. His important discovery was that the curvature is one of the quantities that can be determined just by making measurements on the surface or we now can also say in space.

a. *Could we show that the Universe is non-Euclidean by measuring the angles of a large triangle in our solar system? How accurately would we have to measure the angles?*

Note that, if the Universe would be a 3-sphere or a hyperbolic 3-space, the radius R of the Universe would have to be at least as large as the diameter of our galaxy, which is about 10^{18} km. In the foreseeable future, the largest triangle whose angles we could measure has area less than the area of our solar system, which is about 8×10^{19} km^2. Use the formulas you found in Problems **7.1** and **7.2**.

You will find that in order to determine the geometry of space, we have to look further than our solar system.

b. *If the stars were distributed uniformly in space* (it is not actually true, the stars are clumped in galaxies), *how could you tell by looking at stars at different distances whether space was locally Euclidean, spherical, or hyperbolic?*

If you have trouble envisioning this, then start with the analogous problem for a 2-dimensional bug on a plane, sphere, or hyperbolic plane. What would this bug observe? Assume that you can tell how far away each star is — this is something that astronomers know how to do.

There are certain types of stars (called by astronomers, ***standard candles***) that have a fixed known amount of brightness. These include the so-called Type Ia Supernovae.

c. *Suppose we can see several of these standard candles and can determine accurately their distances from Earth. How could we tell from this information whether the Universe is Euclidean, spherical, or hyperbolic?*

Hint: The apparent brightness of a shining object in Euclidean space is inversely proportional to the square of the distance to the object.

d. *It is impractical to measure the excess of triangles in our solar system by only taking measurements of angles within our solar system. What observations of only distant stars and galaxies would tell us that the Universe is not Euclidean 3-space?*

If you have trouble conceptualizing a 3-sphere or hyperbolic 3-space, then you can do this problem, first, for a very small bug on a 2-sphere or hyperbolic plane who can see distant points (stars) but who is restricted to staying inside its "solar system," which is so small that any triangle in it has excess (or defect) too small to measure. Always think intrinsically! You can assume, generally, that light will travel along geodesics, so think about looking at various objects and the relationships you would expect to find. For example, if the Universe were a 3-sphere and you could see all the way around the Universe (the distance of a great circle), how would you know that the Universe is spherical? Why? What if we could see halfway around the Universe? Or a quarter of the way around? Think of looking at stars at these distances.

All the ways discussed in parts **a–d** have been tried by astronomers. Astronomers attempted to find at a far (but approximately known) distance in the Universe a structure whose size is known (based on physical assumptions). If such a structure is found, then it is possible to measure the angle subtended by this structure from Earth. If the geometry of the Universe is Euclidean, then the measure of the observed angle is predicted by the Law of Cosines. If the observed angle is larger than the prediction, then the Universe would be spherical and if it is smaller, then the Universe would be hyperbolic. (*Do you see why?*)

In spring 2000, a group of astronomers announced (see editorial and article in *Nature*, volume **404**, April 27, 2000, p. 955-959) that they have observed such a structure in the cosmic background radiation and that their observations are "consistent with a flat, Euclidean Universe"; however, they also wrote that the precise details of their observations are inconclusive.

In June 2001 the United States' National Aeronautics and Space Administration (NASA) launched the Wilkinson Microwave Anisotropy Probe (WMAP) to make measurements of the cosmic background radiation. From 2009 to 2013 the European Space Agency operated the Planck satellite – the space observatory that substantially improved

WMAP observations. As the result it was determined that the observable universe is *flat* (no curvature) within very precise tolerances. (Cosmologists distinguish between the observable universe and the global universe, see discussion at the end of this chapter.)

PROBLEM 24.2 EUCLIDEAN 3-MANIFOLDS

We now consider the 3-dimensional analogue of the flat torus. Consider a cube in Euclidean 3-space with the opposite faces glued through a translation in the plane that is midway between the opposite faces. See Figure 24.2. In this figure, we have drawn a closed straight path that starts from A on the bottom right edge and then hits the middle of the front face at B. It continues from the middle of the back face and finishes at the middle of the top left edge at a point that is glued to A.

a. *Show that the cube with opposite faces glued by a translation through the plane midway between is a Euclidean 3-manifold. That is, check that a neighborhood of each point is isometric to a neighborhood in Euclidean 3-space.* This Euclidean 3-manifold is called ***the 3-torus***.

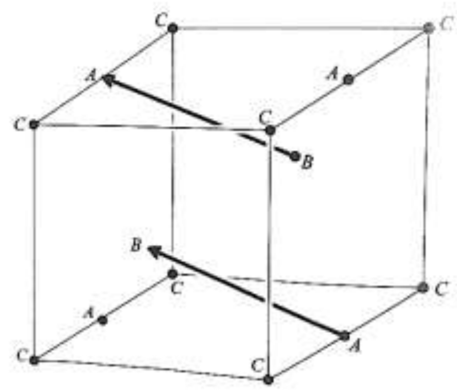

Figure 24.2 A closed geodesic path on 3-torus

Look separately at points (such as A) that are in the middle of edges, points (such as B) that are the middle of faces, and points that are vertices (such as C). Next glue the vertical faces of the cube the same way but glue the top and bottom faces by a quarter turn. In this case we get the closed geodesic depicted in Figure 24.3.

b. *Show that you obtain a Euclidean 3-manifold from the cube with vertically opposite faces glued by a translation through the plane midway between, and the top and bottom faces glued by a quarter-turn rotation.* This Euclidean 3-manifold is called the ***quarter-turn manifold***.

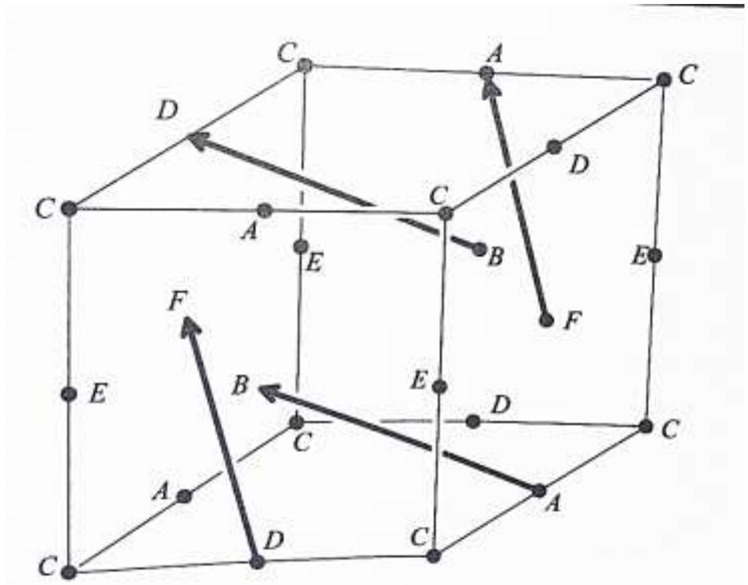

Figure 24.3 A closed geodesic in the quarter-turn manifold

c. *Draw a picture similar to Figures 24.2 and 24.3, for the **half-turn manifold**, which is the same as the quarter-turn manifold except that it is obtained by gluing the top and bottom faces with a half-turn.*

In Problem **18.1** we represented the flat torus in two different ways — one starting with a rectangle or square and the other starting from a hexagon. The above discussion of the 3-torus corresponds to the construction of the flat torus from a square. Now we want to look at what happens if we use an analogue of the hexagon construction.

Consider a hexagonal prism as in Figure 24.4. We will make gluings on the vertical sides by gluing each vertical face with its opposite in such a way that each horizontal cross-section (which are all hexagons) has the same gluings as the hexagonal flat torus (see Problem **18.2b** and Figure 18.3). The top and bottom face we glue in one of three ways. If we glue the top and bottom face through a reflection in the halfway plane, then we obtain the ***hexagonal 3-torus***. If we glue the top and bottom faces with a one-sixth rotation, then we obtain the ***one-sixth-turn manifold***. If we glue the top and bottom with a one-third rotation, then we will get the ***one-third-turn manifold***.

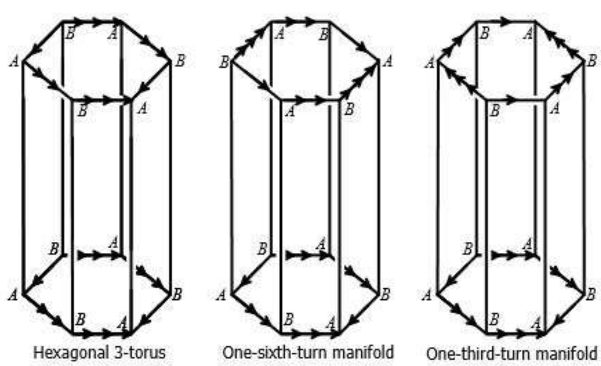

Figure 24.4 Hexagonal 3-manifolds

d. *Show that the hexagonal 3-torus, the one-sixth-turn manifold, and the one-third-turn manifold are Euclidean 3-manifolds and that the hexagonal 3-torus is homeomorphic to the 3-torus. What happens if we consider the two-thirds-turn manifold and the three-sixths-turn manifold and the five-sixths-turn manifold?*

It can be shown that

> **THEOREM.** *There are exactly ten Euclidean 3-manifolds up to homeomorphism. Of these, four are non-orientable and six are orientable. Five of the six orientable Euclidean manifolds are the 3-torus, the quarter-turn manifold, the half-turn manifold, the one-sixth-turn manifold, and the one-third-turn manifold.*

See **[DG**: Weeks], chapter 18, for a discussion of this theorem. For more detail and a proof, see [**DG**: Thurston], Section 4.3.

PROBLEM 24.3 DODECAHEDRAL 3-MANIFOLDS

Spherical and hyperbolic 3-manifolds are more complicated than Euclidean 3-manifolds. In fact, no one knows what all the hyperbolic 3-manifolds are. We will only look at a few examples in order to get an idea of how to construct spherical and hyperbolic 3-manifolds in this problem and the next.

There are two examples that can be obtained by making gluings of the faces of a dodecahedron (see Problem **23.5**). It will be best for this problem for you to have a model of the dodecahedron that you can look at and touch. We want to glue the opposite faces of the dodecahedron. Looking at your dodecahedron (or Figure 24.5), you should see that the opposite faces are not lined up but rather are rotated one-tenth of a full turn from each other. Thus, there are three possibilities for gluings: We can glue with a one-tenth rotation, or a three-tenths rotation, or a five-tenths (= one-half) rotation. When making the rotations it is important always to rotate in the same direction (say clockwise as you are facing the face from the outside). *You should check with your model that this is the same as rotating clockwise while facing the opposite face.*

Figure 24.5 Dodecahedron and its shadow

a. *When you glue the opposite faces of a dodecahedron with a one-tenth clockwise rotation, how many edges are glued together? What if you use a three-tenths rotation? Or a one-half rotation?*

The best way to do this counting is to take a model and mark one edge with tape. Then for each of the two pentagon faces on which the edge lies, the gluing glues the edge to another edge on the other side of the dodecahedron — mark those edges also. Continue with those marked edges until you have marked all the edges that are glued together.

Your answers to part **a** should be 2, 3, 5 (*not* in that order!). Thus, for these manifolds to be geometric manifolds, we must use dodecahedrons with dihedral angles of 180°, 120°, and 72° in either Euclidean space, a sphere, or a hyperbolic 3-space. But before we go further, we must figure out the size of the dihedral angles of the dodecahedron in Euclidean space, which from our model seems to be close to (if not equal to) 120°.

b. *Calculate the size ϕ of the dihedral angle of the (regular) dodecahedron in Euclidean 3-space.*

Imagine a small sphere with center at one of the vertices of the dodecahedron. This sphere will intersect the dodecahedron in a spherical equilateral triangle. The angles of this triangle are the dihedral angles — *Do you see why?* This triangle is called the ***link of a vertex*** in the dodecahedron. Determine the lengths of the sides of this triangle and then use the Law of Cosines (Problem **20.2**).

Now imagine a very small dodecahedron in 3-sphere. Its dihedral angles will be very close to the Euclidean angle ϕ (*Why is this the case?)*. If you now imagine the dodecahedron growing in the 3-sphere, its dihedral angles will grow from ϕ. If you start with a very small dodecahedron in a hyperbolic space, then its dihedral angles will start very close to ϕ and then decrease as the dodecahedron grows.

c. *Show that the manifold from part a with three edges being glued together is a spherical 3-manifold if $\phi < 120°$, or a Euclidean 3-manifold if $\phi = 120°$, or a hyperbolic 3-manifold if $\phi > 120°$.* This geometric 3-manifold is called the ***Poincaré dodecahedral space*** in honor of Henri Poincaré (1854–1912, French), who first described (not using the dodecahedron) a space homeomorphic to this geometric 3-manifold.

Show that each vertex of the dodecahedron is glued to three other vertices and that the four solid angles fit together to form a complete solid angle in the model (either Euclidean 3-space, 3-sphere, or hyperbolic 3-space).

d. *Can the dihedral angles of a dodecahedron in a 3-sphere grow enough to be* 180°? *What does such a dodecahedron look like*? *Is the manifold with two edges being glued together a spherical* 3-*manifold*? This spherical 3-manifold is called the ***projective* 3*-space*** or $\mathbf{RP}^3$.

e. *Can the dihedral angles of a dodecahedron in a hyperbolic 3-space shrink enough to be equal to 72°? If so, the dodecahedral manifold with five edges being glued together is a hyperbolic 3-manifold.*
This hyperbolic 3-manifold is called the ***Seifert-Weber dodecahedral space***, after H. Seifert (1907–1996) and C. Weber, who first described both dodecahedral spaces in a 1933 article, H. Seifert and C. Weber, "Die beiden Dodekaederräume," Mathematische Zeitschrift, vol. 37 (1933), no. 2, p. 237.

Imagine that the dodecahedron grows until its vertices are at infinity (thus on the bounding plane in the upper half space model). Use the fact that angles are preserved in the upper-half-space model and look at the three great hemispheres that are determined by the three faces coming together at a vertex. Remember also to check that the solid angles at the vertices of the dodecahedron fit together to form a complete solid angle.

PROBLEM 24.4 SOME OTHER GEOMETRIC 3-MANIFOLDS

We now look at three more examples of geometric 3-manifolds.

a. *Start with a tetrahedron and glue the faces as indicated in Figure 24.6. Does this gluing produce a manifold? Can the tetrahedron be put in a 3-sphere or hyperbolic 3-space so that the gluings produce a geometric 3-manifold?* We call this the ***tetrahedral space***.

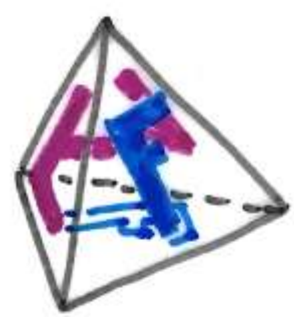

Figure 24.6 Tetrahedral space

b. *Start with a cube and glue each face to the opposite face with a one-quarter-turn rotation. Does this gluing produce a manifold? Can the cube be put in a 3-sphere or hyperbolic 3-space so that the gluings produce a geometric 3-manifold?* This is called the ***quaternionic manifold*** because its symmetries can be expressed in the quaternions (a four-dimensional version of the complex numbers with three imaginary axes and one real axis).

Again, investigate how many edges are glued together and what happens near the vertices.

c. *Start with an octahedron and glue each face to the opposite face with a one-sixth-turn rotation. Does this gluing produce a manifold? Can the octahedron be put in a* 3*-sphere or hyperbolic* 3*-space so that the gluings produce a geometric* 3*-manifold?* This is called the ***octahedral space***.

Once more, investigate how many edges are glued together and what happens near the vertices. You can also use your knowledge of solid angles from Chapter 23.

Cosmologists distinguish between the observable universe and the global universe. The observable universe consists of the part of the universe that can be observed by light reaching Earth within the age of the universe. It is thought as a ball centered at Earth of estimated radius 46.5 billion light-years (4.40×10^{26} m). This is not the age of the universe. The universe is measured to be 13.8 billion years old. The size of the observable universe is larger because the space has expanded and therefore cause light to travel longer.

The question about the shape of the universe actually has three parts:

1. Is it finite or infinite?
2. Is it flat (no curvature), open (negative curvature), or closed (positive curvature)?
3. Is the universe a simply connected space or it is multiply connected?

The exact shape is still a matter of debate in physical cosmology, but experimental data from various independent sources confirm that the universe is flat. Theorists have been trying to construct a formal mathematical model of the shape of the universe. The model most theorists currently use is the Friedmann–Lemaître–Robertson–Walker model. However, even if the observational data best fit with the conclusion that the shape of the global universe is infinite and flat, there are still possible other shapes that are consistent with those data, such Poincaré dodecahedral space and the Sokolov–Starobinskii space.

(see: https://en.wikipedia.org/wiki/Shape_of_the_universe)

If you are interested to study more about manifolds, topologist Levin Nicolaescu has a nice, clearly written, online book *Lectures on the Geometry of Manifolds* (bit.ly/2MSM4Im).