

Chapter 23

POLYHEDRA



... if four equilateral triangles are put together, three of their plane angles meet to form a single solid angle,... When four such angles have been formed the result is the simplest solid figure ...

The second figure is composed of ... eight equilateral triangles, which yield a single solid angle from four planes. The formation of six such solid angles completes the second figure.

The third figure ... has twelve solid angles, each bounded by five equilateral triangles, and twenty faces, each of which is an equilateral triangle.

... Six squares fitted together complete eight solid angles, each composed by three plane right angles. The figure of the resulting body is the cube ...

There is remained a fifth construction, which the god used for arranging the constellations on the whole heaven. — Plato, *Timaeus*, 54e–55c [AT: Plato]

DEFINITIONS AND TERMINOLOGY

[The text in the brackets applies to polyhedra on a 3-sphere or a hyperbolic 3-space.] A *tetrahedron*, $\Delta ABCD$, in 3-space [in a 3-sphere or a hyperbolic 3-space] is determined by any four points, A , B , C , D , called its *vertices*, such that all four points do not lie on the same plane [great 2-sphere, great hemisphere] and no three of the points lie on the same line [geodesic]. The *faces* of the tetrahedron are the four [small] triangles ΔABC , ΔBCD , ΔCDA , ΔDAB . The *edges* of the tetrahedron are the six line [geodesic] segments AB , AC , AD , BC , BD , CD . The *interior* of the tetrahedron is the [smallest] 3-dimensional region that it bounds.

Tetrahedra are to three dimensions as triangles are to two dimensions. Every polyhedron can be dissected into tetrahedra, but the proofs are considerably more difficult than the ones from Problem 7.5b, and in the discussion to Problem 7.5b there is a polyhedron that is impossible to dissect into tetrahedra without adding extra vertices. There

are numerous congruence theorems for tetrahedra, analogous to the congruence theorems for triangles. We say two tetrahedra are **congruent** if one can be transformed by an isometry of 3-space to coincide with the other. All of the problems below apply to tetrahedra in Euclidean 3-space or a 3-sphere or a hyperbolic 3-space.

The **dihedral angle**, $\angle AB$, at the edge AB is the angle formed at AB by $\triangle ABC$ and $\triangle ABD$. The dihedral angle is measured by intersecting it with a plane that is perpendicular to AB at a point between A and B . The **solid angle** at A , $\angle A$, is that portion of the interior of the tetrahedron “at” the vertex A . See Figure 23.1.

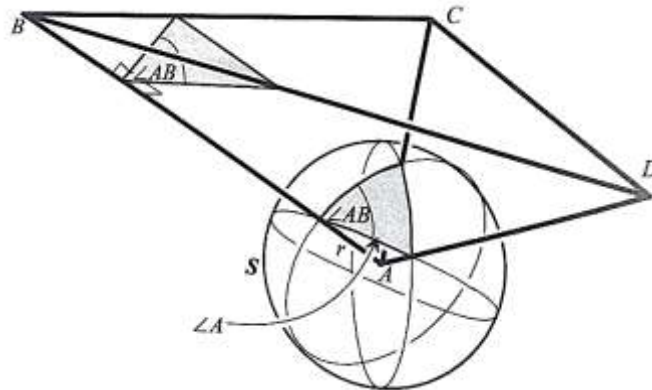


Figure 23.1 Dihedral and solid angles

You may find it helpful with these problems to construct some tetrahedra out of cardboard.

PROBLEM 23.1 MEASURE OF A SOLID ANGLE

The measure of the solid angle is defined as the ratio

$$m(\angle A) = [\lim_{r \rightarrow 0}] \text{area}\{(\text{interior of } \triangle ABCD) \cap \mathbf{S}\} / r^2,$$

where \mathbf{S} is any small 2-sphere with center at A whose radius, r , is smaller than the distance from A to each of the other vertices and to each of the edges and faces not containing A . Note that this definition is analogous to the definition of radian measure of an angle. *Do you see why?*

- a. Show that the measures of the solid and dihedral angles of a tetrahedron satisfy the following relationship:

$$m(\angle A) = m(\angle AB) + m(\angle AC) + m(\angle AD) - \pi.$$

- b. Show that two solid angles with the same measure are not necessarily congruent. We say the two solid angles are **congruent** if one can be transformed by an isometry to coincide with the other.

SUGGESTIONS

Solid angles, whether in Euclidean 3-space or a 3-sphere or a hyperbolic 3-space, are closely related to spherical triangles on a small sphere around the vertex. You can think of starting with a sphere, \mathbf{S} , and creating a solid angle by extending three sticks out from the

center of the sphere. If you connect the ends of these sticks, you will have a tetrahedron. The important thing to notice is how the sticks intersect the sphere. They will obviously intersect the sphere at three points, and you can draw in the great circle arcs connecting these points. Look at the planes in which the great circles lie. In this problem you need to figure out the relationships between the angles of the spherical triangle and the dihedral angles.

The formula given above for the definition of the solid angle uses the intersection of the interior of the solid angle with any small sphere S . This intersection is the small triangle that you just drew, and the area of the intersection is the area of the triangle. Because the measure of a solid angle is defined in terms of an area, it is possible for two solid angles to have the same measure without being congruent — they can have the same area without having the same shape.

What you are asked to prove here is the relationship between the measure of a solid angle and the measures of its dihedral angles. Because they are closely related to spherical triangles on the small sphere, you can use everything you know about small triangles on a sphere.

PROBLEM 23.2 EDGES AND FACE ANGLES

We will study congruence theorems for tetrahedra that can be thought of as the three-dimensional analogue of triangles. A tetrahedron has 4 vertices, 4 faces, and 6 edges and we can denote it by $\triangle ABCD$, where A, B, C, D are the vertices. Figure 23.2.

Show that if $\triangle ABCD$ and $\triangle A'B'C'D'$ are two tetrahedra such that
 $\angle BAC \cong \angle B'A'C', \angle CAD \cong \angle C'A'D', \angle BAD \cong \angle B'A'D',$
 $CA \cong C'A', BA \cong B'A', DA \cong D'A',$
then $\triangle ABCD \cong \triangle A'B'C'D'$.

Part of your proof must be to show that the solid angles $\angle A$ and $\angle A'$ are congruent and not merely that they have the same measure.

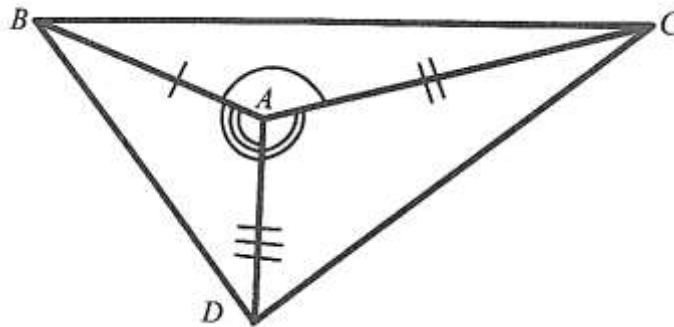


Figure 23.2 Edges and faces

SUGGESTIONS

If \mathbf{S} is a small sphere with center at A and radius r , then $\mathbf{S} \cap$ (interior of $\triangle ABCD$) is a spherical triangle whose sides have lengths $r \angle BAC$, $r \angle CAD$, $r \angle BAD$. In the last problem, you saw how solid angles are related to spherical triangles. This problem asks you to prove the congruence of tetrahedra based on certain angle and length measurements. (Note that the angles shown above are not the dihedral angles of the tetrahedron.) So, since you can use spherical triangles to relate solid and dihedral angle measurements, why not use them to prove tetrahedra congruencies? Use the hint given to see what measurements of the spherical triangle are defined by measurements of the tetrahedron. Then see if the measurements given do in fact show congruence and show why.

PROBLEM 23.3 EDGES AND DIHEDRAL ANGLES

Show that if

$$AB \cong A'B', \angle AB \cong \angle A'B', AC \cong A'C', \\ \angle AC \cong \angle A'C', AD \cong A'D', \angle AD \cong \angle A'D',$$

then $\triangle ABCD \cong \triangle A'B'C'D'$. See Figure 23.3

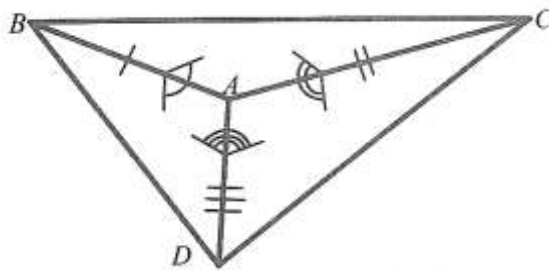


Figure 23.3 Edges and dihedral angles

This is very similar to the previous problem but uses different measurements — here we have the dihedral angles instead of the angles on the faces of tetrahedron. Look at this problem the same way you looked at the previous one — see how the measurements given relate to a spherical triangle, and then prove the congruence.

PROBLEM 23.4 OTHER TETRAHEDRAL CONGRUENCE THEOREMS

Make up your own congruence theorems! Find and prove at least two other sets of conditions that will imply congruence for tetrahedra; that is, make up and prove other theorems like those in Problems 23.2 and 23.3.

It is important to make sure your conditions are sufficient to prove that the solid angles are congruent, not just that they have the same measure.

Alexander Graham Bell (1847-1922) is best known for his invention of the first practical telephone and co-founding AT&T. Less known are his experiments in aerodynamics, particularly his obsession with tetrahedral kites. Some historic pictures can be seen <https://publicdomainreview.org/collections/alexander-graham-bells-tetrahedral-kites-1903-9/>.

PROBLEM 23.5 THE FIVE REGULAR POLYHEDRA

A *regular polygon* is a polygon lying in a plane or 2-sphere or hyperbolic plane such that all of its edges are congruent, and all of its angles are congruent. For example, on the plane a regular quadrilateral is a square. On a 2-sphere and a hyperbolic plane a regular quadrilateral is constructed as shown in Figure 23.4. See also Figure 18.16 for a regular octagon on a hyperbolic plane.

Note that half of a regular quadrilateral is a Khayyam quadrilateral (see Chapter 12). On 2-spheres and hyperbolic planes there are no similar polygons; for example, a regular quadrilateral (congruent sides and congruent angles) will have the same angles as another regular quadrilateral if and only if they have the same area. (Do you see why?)

A *polyhedron* in 3-space [or in a 3-sphere or in a hyperbolic 3-space] is *regular* if all of its edges are congruent, all of its face angles are congruent, all of its dihedral angles are congruent, and all of its solid angles are congruent. The faces of a polyhedron are assumed to be polygons that lie on a plane [a great 2-sphere, a great hemisphere].

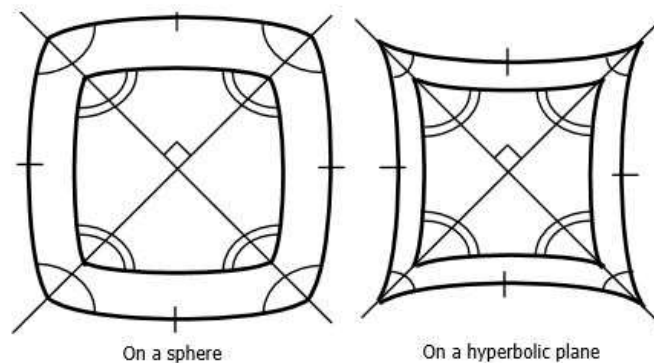


Figure 23.4 Regular quadrilaterals on a sphere and on a hyperbolic plane

Show that there are only five regular polyhedra. In Euclidean 3-space, to say “there are only five regular polyhedra” is to mean that any regular polyhedra is similar (same shape, but not necessarily the same size) to one of the five. It still makes sense on a 3-sphere and a hyperbolic 3-space to say that “there are only five regular polyhedra,” but you need to make clear what you mean by this phrase.

These polyhedra are often called the Platonic Solids and are described by Greek philosopher Plato (429–348 B.C.) as “forms of bodies which excel in beauty” (*Timaeus*, 53e [AT: Plato]), but there is considerable evidence that they were known well before Plato’s time. See T. L. Heath’s discussion in [AT: Euclid, Elements], Vol. 3, pp. 438–39, for evidence that the five regular solids were known by Greeks before the time of Plato.

There is a description of the discovery in Scotland of a complete set of the five regular polyhedra carefully carved out of stone by Neolithic persons some 4000 to 6000 years ago in [HI: Critchlow], pp. 148–49. The regular polyhedra are also the subject of the thirteenth (and last) book in [AT: Euclid, Elements].



Neolithic stone polyhedra

SUGGESTIONS

Your argument should be essentially the same whether you are considering 3-space, or a 3-sphere, or a hyperbolic 3-space. There are many widely different ways to do this problem. The following are some approaches that we suggest:

First Approach: Note that the faces of a regular polyhedron must be regular polygons. Then focus on the vertices of regular polyhedra. Show that if the faces are regular quadrilaterals or regular pentagons, then there must be precisely three faces intersecting at each vertex. Show that it is impossible for regular hexagons to intersect at a vertex to form the solid angle of a regular polyhedron. If the faces are regular (equilateral) triangles, then show that there are three possibilities at the vertices.

Second Approach: Refer to Problem 18.5. Each regular polyhedron can be considered to be projected out from its center onto a sphere and thus determine a cell division of the sphere. The Euler number of this spherical subdivision is $v - e + f = 2$, where v is the number of vertices, e is the number of edges, and f is the number of faces. Then

$$2e = nf \text{ and } 2e = kv. \text{ (Why?)}$$

Thus, deduce that,

$$e = \frac{2}{\frac{2}{k} + \frac{2}{n} - 1}$$

and remember that e must be a positive integer.

In both approaches you should then finish the problem by using earlier problems from this chapter to show that any two polyhedra constructed from the same polygons, with the same number intersecting at each vertex, must be congruent. This step is necessary because there are polyhedra that are not rigid (that is, there are polyhedra that can be

continuously moved into a non-congruent polyhedra without changing any of the faces or changing the number of faces coming together at each vertex). See Robert Connelly's "The Rigidity of Polyhedral Surfaces" (Mathematics Magazine, vol. 52, no. 5 (1979), pp. 275–83 <http://pi.math.cornell.edu/~web3040/Math-mag-rigidity.pdf>). The five regular polyhedra are usually named the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. (See Figure 23.5.) There is a duality (related to but not exactly the same as the duality in Chapter 20, Trigonometry and Duality) among regular polyhedra: If you pick the centers of the faces of a regular polyhedron, then these points are the vertices of a regular polyhedron, which is called the dual of the original polyhedron. You can see that the cube is dual to the octahedron (and vice versa), that the icosahedron is dual to the dodecahedron (and vice versa), and that the tetrahedron is dual to itself. (<https://www.georgehart.com/virtual-polyhedra/duality.html>)

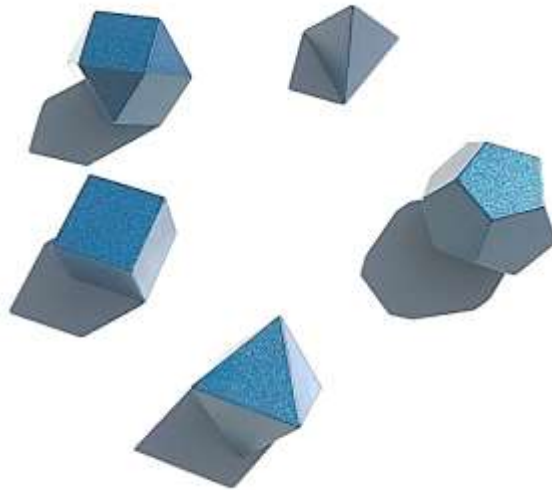


Figure 23.5 The five Platonic solids