Chapter 19

GEOMETRIC SOLUTIONS OF QUADRATIC AND CUBIC EQUATIONS

Whoever thinks algebra is a trick in obtaining unknowns has thought it in vain. No attention should be paid to the fact that algebra and geometry are different in appearance. Algebras (jabbre and maqabeleh) are geometric facts which are proved by Propositions Five and Six of Book Two of [Euclid's] Elements.— Omar Khayyam, a paper [**AT:** Khayyam 1963]

In this chapter we will see how the results from Chapter 13 were used historically to solve equations. Quadratic equations were solved by "completing the square" — a real square. These results in turn lead to conic sections and cube roots and culminate in the beautiful general method from Omar Khayyam that can be used to find all the real roots of cubic equations. Along the way we shall see clearly some of the ancestral forms of our modern Cartesian coordinates and analytic geometry. We will point out several inaccuracies and misconceptions that have crept into the modern historical accounts of these matters. We urge you not to look at this only for its historical interest but also for the meaning it has in our present-day understanding of mathematics. This path is not through a dead museum or petrified forest; it passes through ideas that are very much alive and have something to say to our modern technological, increasingly numerical, world.

PROBLEM 19.1 QUADRATIC EQUATIONS

Finding square roots is the simplest case of solving quadratic equations. If you look in some history of mathematics books (e.g., [**HI**: Joseph] and [**HI**: Eves]), you will find that quadratic equations were solved extensively by the Babylonians, Chinese, Indians, and Greeks. However, the earliest known general discussion of quadratic equations took place between 800 and 1100 A.D. in the Muslim Empire. Best known are Mohammed Ibn Musa al'Khowarizmi (who lived in Baghdad from 780 to 850 and from whose name we get our word "algorithm") and Omar Khayyam (1048–1131), the Persian geometer who is mostly known in the West for his philosophical poetry *The Rubaiyat*). Both wrote books whose titles contain the phrase *Al-jabr w'al mugabalah* (from which we get our word "algebra"), al'Khowarizmi in about 820 and Khayyam in about 1100. (See [**HI**: Katz], Section 7.2.1.) An English translation of both books is available in many libraries, if you can figure out whose name it is catalogued under (see [**AT:** al'Khowarizmi] and [**AT:** Khayyam 1931]). (We already met Khayyam in Chapter 12.)

In these books you find geometric and numerical solutions to quadratic equations and geometric proofs of these solutions. But you will notice quickly that there is not one general quadratic equation as we are used to it: $ax^2 + bx + c = 0$. Rather, because the use of negative coefficients and negative roots was avoided, these books list six types of quadratic equations (we follow Khayyam's lead and set the coefficient of x^2 equal to 1):

1. $bx = c$, which has root $x = c/b$,
2. $x^2 = bx$, which has root $x = 0$ and $x = b$,
3. $x^2 = c$, which has root $x = \sqrt{c}$,
4. $x^2 + bx = c$, with root $x = \sqrt{\left(\frac{b}{2}\right)^2 + c} - b/2$,
5. $x^2 + c = bx$, with roots $x = \frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}$, if $c < (b/2)^2$, and
6. $x^2 = bx + c$, with root $= \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + c}$

Here b and c are always positive numbers or a geometric length (b) and area (c).

a. *Show that these are the only types. Why is $x^2 + bx + c = 0$ not included? Explain why b must be a length but c must be an area.*

The avoidance of negative numbers was widespread until a few hundred years ago. In the 16th century, European mathematicians called the negative numbers that appeared as roots of equations "numeri fictici" — fictitious numbers (see [**AT:** Cardano, page 11]). In 1759 Francis Masères (1731–1824, English), mathematician and a fellow at Cambridge University and a member of the Royal Society, wrote in his *Dissertation on the Use of the Negative Sign in Algebra*,

> ... [negative roots] serve only, as far as I am able to judge, to puzzle the whole doctrine of equations, and to render obscure and mysterious things that are in their own nature exceeding plain and simple.... It were to be wished therefore that negative roots had never been admitted into algebra or were again discarded from it: for if this were done, there is good reason to imagine, the objections which many leaned and ingenious men now make to algebraic computations, as being obscure and perplexed with almost unintelligible notions, would be thereby removed; it being certain that Algebra, or universal arithmetic, is, in its own nature, a science no less simple, clear, and capable of demonstration, than geometry.

More recently in 1831, Augustus De Morgan (1806–1871, English), the first professor of mathematics at University College, London, and a founder of the London Mathematical Society, wrote in his *On the Study and Difficulties of Mathematics*,

> *The imaginary expression $\sqrt{-a}$ and the negative $-b$ have this resemblance, that either of them occurring as solution of a problem indicates some inconsistency or absurdity. As far as real meaning is concerned, both are equally imaginary, since $0 - a$ is as inconceivable as $\sqrt{-a}$.*

Why did these mathematicians avoid negative numbers and why did they say what they said? To get a feeling for why, think about the meaning of 2×3 as two 3's and 3×2 as three 2's and then try to find a meaning for $3 \times (-2)$ and $-2 \times (+3)$. Also consider the quotation at the beginning of this chapter from Omar Khayyam about algebra and geometry. Some historians have quoted this passage but have left out all the words appearing after "proved." In our opinion, this omission changes the meaning of the passage. Euclid's propositions that are mentioned by Khayyam are the basic ingredients of Euclid's proof of the square root construction and form a basis for the construction of conic sections — see Problem **19.2**, below. Geometric justification when there are negative coefficients is at the least very cumbersome, if not impossible. (If you doubt this, try to modify some of the geometric justifications below.)

b. *Find geometrically the algebraic equations that express all the positive roots of each of the six types. Fill in the details in the following sketch of Khayyam's methods for Types 3–6.*

For the geometric justification of Type 3 and the finding of square roots, Khayyam refers to Euclid's construction of the square root in Proposition II 14, which we discussed in Problem **13.1**.

For Type 4, Khayyam gives as geometric justification the illustration shown in Figure 19.1.

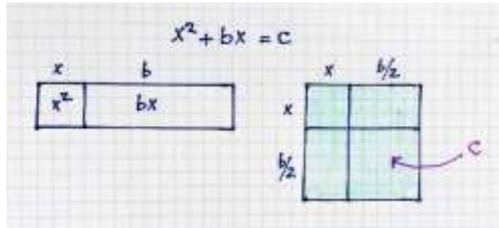

Figure 19.1 Type 4

Thus, by "completing the square" on $x + b/2$, we have

$$(x + b/2)^2 = c + (b/2)^2.$$

Thus, we have $x = \sqrt{c + (\frac{b}{2})^2} - b/2$. Note the similarity between this and Baudhayana's construction of the square root (see Chapter 13).

For Type 5, Khayyam first assumes $x < (b/2)$ and draws the equation as Figure 19.2.

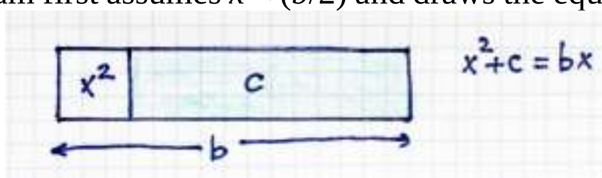

Figure 19.2 Type 5, $x < (b/2)$

Note (see Figure 19.3) that the square on $b/2$ is $(b/2 - x)^2 + c$.

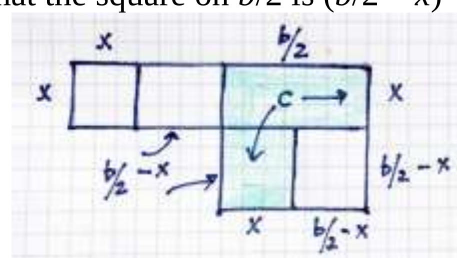

Figure 19.3 Type 5, $x < (b/2)$

This leads to $x = b/2 - \sqrt{(b/2)^2 - c}$. Note that if $c > (b/2)^2$, this geometric solution is impossible. When $x > (b/2)$, Khayyam uses the drawings like shown in Figure 19.4.

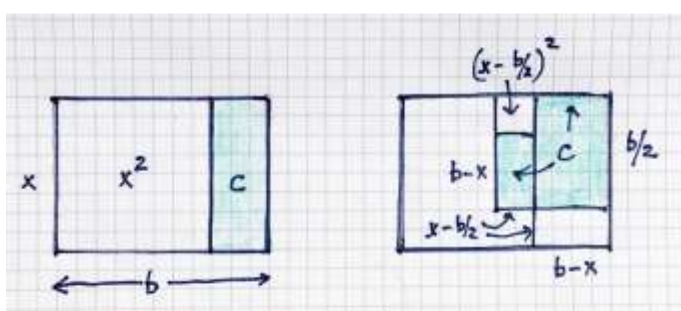

Figure 19.4 Type 5, $x > (b/2)$ $b - x$

For solutions of Type 6, Khayyam uses the drawing like in Figure 19.5.

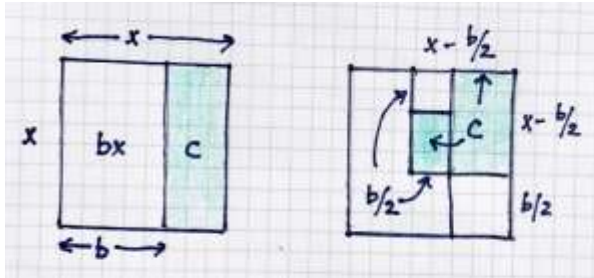

Figure 19.5 Type 6

***Do the above solutions find the negative roots*?** The answer is clearly "no" if you mean, "Did al'Khowarizmi and Khayyam mention negative roots?" But let us not be too hasty. Suppose $-r$ (r positive) is the negative root of $x^2 + bx = c$. Then $(-r)^2 + b(-r) = c$ or $r^2 = br + c$. Thus, r is a positive root of $x^2 = bx + c$! The absolute value of the negative root of $x^2 + bx = c$ is the positive root of $x^2 = bx + c$ and vice versa. Also, the absolute values of the negative roots of $x^2 + bx + c = 0$ are the positive roots of $x^2 + c = bx$. So, in this sense, *Yes, the above geometric solutions do find all the real roots of all quadratic equations*. Thus, it is misleading to state, as most historical accounts do, that the geometric methods failed to find the negative roots. The users of these methods did not find negative roots because they did not conceive of them. However, the methods can be used directly to find all the positive and negative roots of all quadratics.

c. *Use Khayyam's methods to find all roots of the following equations:*

$$x^2 + 2x = 2,\ x^2 = 2x + 2,\ x^2 + 3x + 1 = 0.$$

PROBLEM 19.2 CONIC SECTIONS AND CUBE ROOTS

The Greeks (for example, Archytas of Tarentum, 428–347 B.C., who was a Pythagorean in southern Italy, and Hippocrates of Chios in Asia Minor, 5th century B.C.) noticed that, if $a/c = c/d = d/b$, then $(a/c)^2 = (c/d)(d/b) = (c/b)$ and, thus, $c^3 = a^2b$. (For more historical discussion, see [**HI**: van der Waerden], page 150, and [**HI**: Katz], Chapter 2.) Now setting $a = 1$, we see that we can find the cube root of b if we can find c and d such that $c^2 = d$ and $d^2 = bc$. If we think of c and d as being variables and b as a constant, then we see these equations as the equations of two parabolas with perpendicular axes and the same vertex. The Greeks also saw it this way, but first they had to develop the concept of a parabola! The first general construction of conic sections was done by Menaechmus (a member of Plato's Academy) in the 4th century B.C.

To the Greeks, and later Khayyam, if AB is a line segment, then *the parabola with vertex B and parameter AB* is the curve P such that, if C is on P, then the rectangle $BDCE$ (see Figure 19.6) has the property that $(BE)^2 = BD \cdot AB$. Because in Cartesian coordinates the coordinates of C are (BE, BD), this last equation becomes a familiar equation for a parabola.

Points of the parabola may be constructed by using the construction for the square root given in Chapter 13. E is the intersection of the semicircle on AD with the line perpendicular to *AB* at *B*. (The construction can also be done by finding *D'* such that *AB* = *DD'*. The semicircle on *BD'* then intersects *P* at *C*.) We encourage you to try this construction yourself; it is very easy to do if you use a compass and graph paper.

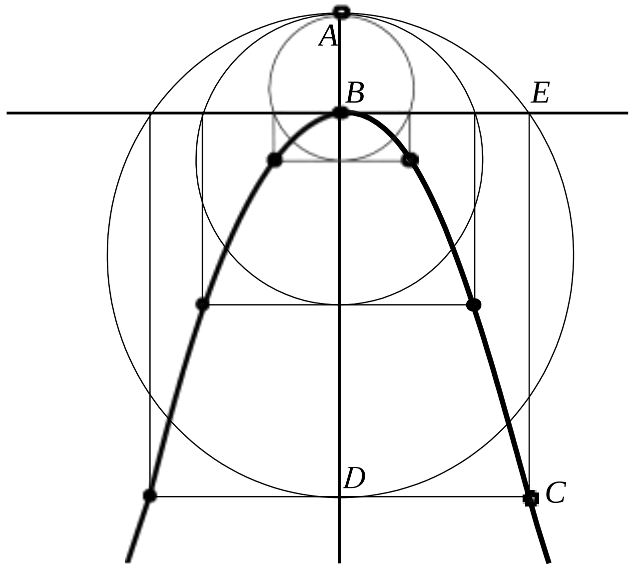

Figure 19.6 Construction of parabola

Now we can find the cube root. Let *b* be a positive number or length and let *AB* = *b* and construct *C* so that *CB* is perpendicular to *AB* and such that *CB* = 1. See Figure 19.7. Construct a parabola with vertex *B* and parameter *AB* and construct another parabola with vertex *B* and parameter *CB*. Let *E* be the intersection of the two parabolas. Draw the rectangle *BGEF*. Then

$$(EF)^2 = BF \cdot AB \text{ and } (GE)^2 = GB \cdot CB.$$

But, setting $c = GE = BF$ and $d = GB = EF$, we have $d^2 = cb$ and $c^2 = d$. Thus $c^3 = b$.
If you use a fine graph paper, it is easy to get three-digit accuracy in this construction.

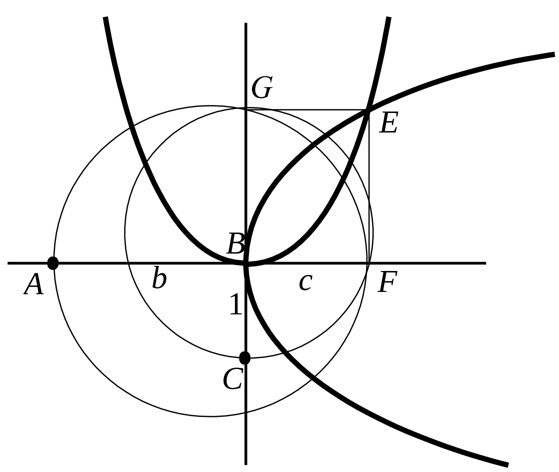

Figure 19.7 Finding cube roots geometrically

The Greeks did a thorough study of conic sections and their properties, culminating in Apollonius' (c. 260–170 B.C., Greek) book *Conics*, which appeared around 200 B.C. You can read this book in English translation (see [**AT:** Apollonius]).

a. *Use the above geometric methods with a fine graph paper to find the cube root of* 10.

To find roots of cubic equations, we shall also need to know *the* (*rectangular*) *hyperbola with vertex B and parameter AB*. This is the curve such that, if E is on the curve and $ACED$ is the determined rectangle (see Figure 19.8), then $(EC)^2 = BC \cdot AC$.

The point E can be determined using the construction from Chapter 13. Let F be the bisector of AB. Then the circle with center F and radius FC will intersect at D the line perpendicular to AB at A. From the drawing it is clear how these circles also construct the other branch of the hyperbola (with vertex A).

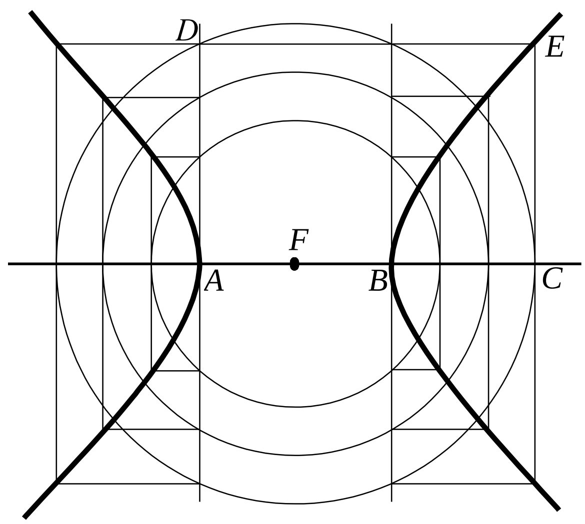

Figure 19.8 Construction of hyperbola

b. *Use the above method with graph paper to construct the graph of the hyperbola with parameter 5. What is an algebraic equation that represents this hyperbola?*

Notice how these descriptions and constructions of the parabola and hyperbola look very much as if they were done in Cartesian coordinates. *The ancestral forms of Cartesian coordinates and analytic geometry are evident here.* They are also evident in the solutions of cubic equations in the next section. The ideas of Cartesian coordinates did not appear to Rene Descartes (1596–1650, French) out of nowhere. The underlying concepts were developing in Greek and Muslim mathematics. One of the apparent reasons that full development did not occur until Descartes is that, as we have seen, negative numbers were not accepted. The full use of negative numbers is essential for the realization of Cartesian coordinates. However, even Descartes seems to have avoided negatives as much as possible when he was studying curves — he would start with a curve (constructed by some geometric or mechanical procedure) and then choose axes so that the important parts of the curve had both coordinates positive. However, it is not true (as asserted in some history of mathematics book) that Descartes always used x and y to stand for positive values. For example, in Book II of *Geometrie* [**AT:** Descartes] he describes the construction of a locus generated by a point C and defines on page 60 $y = CB$, where B is a given point, and derives an equation satisfied by y and other variables; and, in the same paragraph on page 63, he continues, "If y is zero or less than nothing in this equation ..."

PROBLEM 19.3 SOLVING CUBIC EQUATIONS GEOMETRICALLY

In his *Al-jabr wa'l muqabalah,* Omar Khayyam also gave geometric solutions to cubic equations. You will see that his methods are sufficient to find geometrically all real (positive or negative) roots of cubic equations; however; in the first chapter Khayyam says (see [**AT:** Khayyam 1931], p. 49),

> When, however, the object of the problem is an absolute number, neither we, nor any of those who are concerned with algebra, have been able to prove this equation — perhaps others who follow us will be able to fill the gap — except when it contains only the three first degrees, namely, the number, the thing and the square.

By "absolute number," Khayyam is referring to what we call algebraic solutions, as opposed to geometric ones. This quotation suggests, contrary to what many historical accounts say, that Khayyam expected that algebraic solutions would be found.

Khayyam found 19 types of cubic equations (when expressed with only positive coefficients). (See [**AT:** Khayyam 1931], p. 51.) Of these 19, 5 reduce to quadratic equations (for example, $x^3 + ax = bx$ reduces to $x^2 + ax = b$). The remaining 14 types Khayyam solved by using conic sections. His methods find all the positive roots of each type, although he failed to mention some of the roots in a few cases, and, of course, he ignored the negative roots. Instead of going through his 14 types, let us look how a simple reduction will reduce them to only four types in addition to types already solved, such as $x^3 = b$. Then we will look at Khayyam's solutions to these four types.

In the cubic $y^3 + py^2 + gy + r = 0$ (where p, g, r, are positive, negative, or zero), set $y = x - (p/3)$. Try it! The resulting equation in x will have the form $x^3 + sx + t = 0$, (where s and t are positive, negative, or zero). If we rearrange this equation so that all the coefficients are positive, we get the following four types:

$$(1)\ x^3 + ax = b,\quad (2)\ x^3 + b = ax,$$

$$(3)\ x^3 = ax + b, \text{ and } (4)\ x^3 + ax + b = 0,$$

where a and b are positive, in addition to the types previously solved.

a. *Show that in order to find all the roots of all cubic equations we need only have a method that finds the roots of Types* 1, 2, *and* 3.

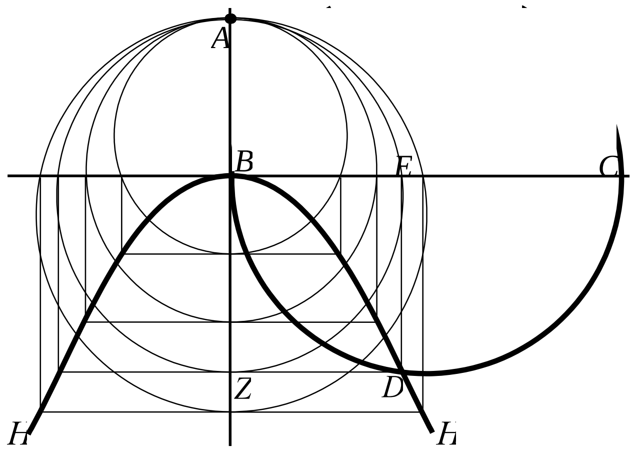

Figure 19.9 Type 1 cubic

KHAYYAM'S SOLUTION FOR TYPE 1: $x^3 + ax = b$

> A cube and sides are equal to a number. Let the line AB (see Figure 19.9) be the side of a square equal to the given number of roots [that is, $(AB)^2 = a$, the coefficient]. Construct a solid whose base is equal to the square on AB, equal in volume to the given number [b]. The construction has been shown previously. Let BC be the height of the solid. [That is, $BC \cdot (AB)^2 = b$.] Let BC be perpendicular to AB ... . Construct a parabola whose vertex is the point B ... and parameter AB. Then the position of the conic HBD will be tangent to BC. Describe on BC a semicircle. It necessarily intersects the conic. Let the point of intersection be D; drop from D, whose position is known, two perpendiculars DZ and DE on BZ and BC. Both the position and magnitude of these lines are known.

The root is *EB*. Khayyam's proof (using a more modern, compact notation) is as follows: From the properties of the parabola (Problem **19.2**) and circle (Problem **15.1**), we have

$$(DZ)^2 = (EB)^2 = BZ \cdot AB \text{ and } (ED)^2 = (BZ)^2 = EC \cdot EB,$$

thus

$$EB \cdot (BZ)^2 = (EB)^2 \cdot EC = BZ \cdot AB \cdot EC$$

and, therefore,

$$AB \cdot EC = EB \cdot BZ,$$

and

$$(EB)^3 = EB \cdot (BZ \cdot AB) = (AB \cdot EC) \cdot AB = (AB)^2 \cdot EC.$$

So

$$(EB)^3 + a(EB) = (AB)^2 \cdot EC + (AB)^2 \cdot (EB) = (AB)^2 \cdot CB = b.$$

Thus, *EB* is a root of $x^3 + ax = b$. Because $x^2 + ax$ increases as x increases, there can be only this one root.

KHAYYAM'S SOLUTIONS FOR TYPES 2 AND 3: $x^3 + b = ax$ AND $x^3 = ax + b$

Khayyam treated these equations separately; but, by allowing negative horizontal lengths, we can combine his two solutions into one solution of $x^3 \pm b = ax$. Let AB be perpendicular to BC and as before let $(AB)^2 = a$ and $(AB)^2 \cdot BC = b$. Place BC to the left if the sign in front of b is negative (Type 3) and place BC to the right if the sign in front of b is positive (Type 2). Construct a parabola with vertex B and parameter AB. Construct both branches of the hyperbola with vertices B and C and parameter BC. See Figure 19.10.

Each intersection of the hyperbola and the parabola (except for B) gives a root of the cubic. Suppose they meet at D. Then drop perpendiculars DE and DZ. The root is BE (negative if to the left and positive if to the right). Again, if you use fine graph paper, it is possible to get three-digit accuracy here. We leave it for you, the reader, to provide the proof, which is very similar to Type 1.

b. *Verify that Khayyam's method described above works for Types 2 and 3. Can you see from your verification why the extraneous root given by B appears?*

c. *Use Khayyam's method to find all roots of the cubic $x^3 = 15x + 4$. Use fine graph paper and try for three-place accuracy.*

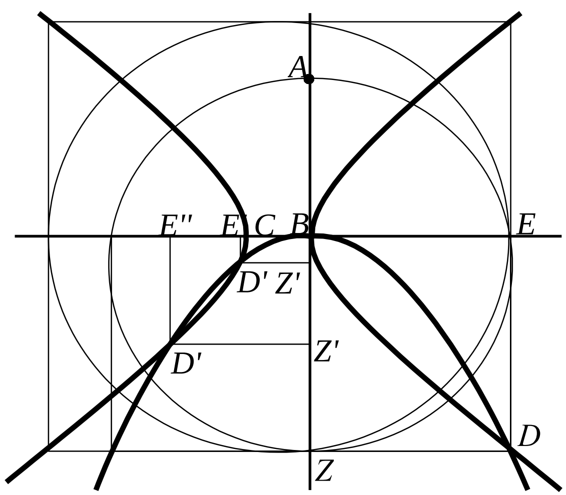

Figure 19.10 Type 2 and type 3 cubics

PROBLEM 19.4 ALGEBRAIC SOLUTION OF CUBICS

A little more history: Most historical accounts assert correctly that Khayyam did not find the negative roots of cubics. However, they are misleading in that they all fail to mention that his methods are fully sufficient to find the negative roots, as we have seen above. This contrasts with the common assertion (see, for example, [**EM**: Davis & Hersh]) that Girolamo Cardano (1501–1578, Italian) was the first to publish the general solution of cubic equations. In fact, as we shall see, Cardano himself admitted that his methods are insufficient to find the real roots of many cubics.

Cardano published his algebraic solutions in the book *Artis Magnae* (The Great Art) in 1545. For a readable English translation and historical summary, see [**AT:** Cardano]. Cardano used only positive coefficients and thus divided the cubic equations into the same 13 types (excluding $x^3 = c$ and equations reducible to quadratics) used earlier by Khayyam. Cardano also used geometry to prove his solutions for each type. As we did above, we can make a substitution to reduce these to the same types as above:

$$(1)\ x^3 + ax = b,\ (2)\ x^3 + b = ax,$$
$$(3)\ x^3 = ax + b, \text{ and } (4)\ x^3 + ax + b = 0.$$

If we allow ourselves the convenience of using negative numbers and lengths, then we can reduce these to one type: $x^3 + ax + b = 0$, where now we allow a and b to be either negative or positive.

The main "trick" that Cardano used was to assume that there is a solution of

$$x^3 + ax + b = 0 \text{ of the form } x = t^{1/3} + u^{1/3}.$$

Plugging this into the cubic, we get

$$(t^{1/3} + u^{1/3})^3 + a(t^{1/3} + u^{1/3}) + b = 0.$$

If you expand and simplify this, you get to

$$t + u + b + (3t^{1/3}u^{1/3} + a)\,(t^{1/3} + u^{1/3}) = 0.$$

(Cardano did this expansion and simplification geometrically by imagining a cube with sides $t^{1/3} + u^{1/3}$.) Thus $x = t^{1/3} + u^{1/3}$ is a root if

$$t + u = -b \text{ and } t^{1/3}u^{1/3} = -(a/3).$$

Solving, we find that t and u are the roots of the quadratic equation $z^2 + bz - (a/3)^3 = 0$, which Cardano solved geometrically (and so can you, Problem **19.1**) to get

$$t = -\frac{b}{2} + \sqrt{(b/2)^2 + (a/3)^3} \quad \text{and} \quad u = -\frac{b}{2} - \sqrt{(b/2)^2 + (a/3)^3}.$$

Thus, the cubic has roots $x = t^{1/3} \pm u^{1/3}$, where t and u are as above.

This is Cardano's cubic formula. But a strange thing happened. Cardano noticed that the cubic $x^3 = 15x + 4$ has a positive real root 4. However, if $a = -15$ and $b = -4$, and if we put these values into his cubic formula, we get that the roots of $x^3 = 15x + 4$ are

$$x = \left(2 + \sqrt{-121}\right)^{1/3} \pm \left(2 - \sqrt{-121}\right)^{1/3}.$$

But these are the sum of two complex numbers even though you have shown in Problem **19.3** that all three roots are real! How can this expression yield 4?

In Cardano's time there was no theory of complex numbers, and so he reasonably concluded that his method would not work for this equation, even though he did investigate expressions such as $\sqrt{-121}$. Cardano writes ([**AT:** Cardano, p. 103]),

> When the cube of one-third the coefficient of x is greater than the square of one-half the constant of the equation ... then the solution of this can be found by the aliza problem which is discussed in the book of geometrical problems.

It is not clear what book he is referring to, but the "aliza problem" presumably refers to the mathematician known as al'Hazen, Abu Ali al'Hasan ibu al'Haitam (965–1039), who was born in Persia and worked in Egypt and whose works were known in Europe in Cardano's time. Al'Hazen had used intersecting conics to solve specific cubic equations and the problem of describing the image seen in a spherical mirror — this latter problem in some books is called "Alhazen's problem."

We know today that each complex number has three cube roots and so the formula

$$x = \left(2 + \sqrt{-121}\right)^{1/3} \pm \left(2 - \sqrt{-121}\right)^{1/3}$$

is ambiguous. In fact, some choices for the two cube roots give roots of the cubic and some do not. (Experiment with $x^3 = 15x + 4$.) Faced with Cardano's formula and equations such as $x^3 = 15x + 4$, Cardano and other mathematicians of the time started exploring the possible meanings of these complex numbers and thus started the theory of complex numbers.

a. *Solve the cubic $x^3 = 15x + 4$ using Cardano's formula and your knowledge of complex numbers.*

Remember that we showed that $x = t^{1/3} + u^{1/3}$ is a root of the equation if $t + u = -b$ and $t^{1/3} u^{1/3} = -(a/3)$.

b. *Solve $x^3 = 15x + 4$ by dividing through by $x - 4$ and then solving the resulting quadratic.*

c. *Compare your answers and methods of solution from* Problems **19.3c**, **19.4a**, *and* **19.4b**.

William B. Brenson is giving his take on Cardano's geometric solution of the cubic equation, see:
https://www.maa.org/press/periodicals/convergence/solving-the-cubic-with-cardano-introduction

What Does This All Point To?

What does the experience of this chapter point to? It points to different things for each of us. We conclude that it is worthwhile to pay attention to the meaning in mathematics. Often in our haste to get to the modern, powerful analytic tools, we ignore and trod upon the meanings and images that are there. Sometimes it is hard even to get a glimpse that some meaning is missing. One way to get this glimpse and find meaning is to listen to and follow questions of "What does it mean?" that come up in ourselves, in our friends, and in our students. We must listen creatively because we and others often do not know how to express precisely what is bothering us.

Another way to find meaning is to read the mathematics of old and keep asking, “Why did they do that?” or “Why didn’t they do this?” Why did the early algebraists (up until at least 1600 and much later, we think) insist on geometric proofs? We have suggested some reasons above. Today, we normally pass over geometric proofs in favor of analytic ones based on the 150-year-old notion of Cauchy sequences and the Axiom of Completeness. However, for most students and, we think, most mathematicians, our intuitive understanding of the real numbers is based on the geometric real line. As an example, think about multiplication: What does $a \times b$ mean? Compare the geometric images of $a \times b$ with the multiplication of two infinite, non-repeating, decimal fractions. What is $\pi \times \sqrt{2}$?

There is another reason why a geometric solution may be more meaningful: Sometimes we need a geometric result instead of a numerical one. For example, David and a friend were building a small house using wood. The roof of the house consisted of 12 isosceles triangles that together formed a 12-sided cone (or pyramid). It was necessary for them to determine the angle between two adjacent triangles in the roof so they could appropriately cut the log rafters. David immediately started to calculate the angle using (numerical) trigonometry and algebra. But then he ran into a problem. He had only a slide rule with three-place accuracy for finding square roots and values of trigonometric functions. At one point in the calculation he had to subtract two numbers that differed only in the third place (for example, 5.68 – 5.65); thus, his result had little accuracy. As he started to figure out a different computational procedure that would avoid the subtraction, he suddenly realized *he didn’t want a number, he wanted a physical angle*. In fact, a numerical angle would be essentially useless — imagine taking two rough boards and putting them at a given numerical angle apart using only an ordinary protractor! What he needed was the physical angle, full size. So, David and his friend constructed the angle on the floor of the house using a rope as a compass. This geometric solution had the following advantages over a numerical solution:

- The geometric solution resulted in the desired physical angle, while the numerical solution resulted in a number.
- The geometric solution was quicker than the numerical solution.
- The geometric solution was immediately understood and trusted by David’s friend (and fellow builder), who had almost no mathematical training, while the numerical solution was beyond the friend’s understanding because it involved trigonometry (such as the Law of Cosines).
- And, because the construction was done full size, the solution automatically had the degree of accuracy appropriate for the application.

Meaning is important in mathematics, and geometry is an important source of that meaning.