Chapter 11

ISOMETRIES AND PATTERNS

What geometrician or arithmetician could fail to take pleasure in the symmetries, correspondences and principles of order observed in visible things? — Plotinus, The Enneads, II.9.16 [**AT:** Plotinus]

[Geometry is] the study of the properties of a space which are invariant under a given group of transformations. – F. Klein, Erlangen Program

Life forms illogical patterns. It is haphazard and full of beauties which I try to catch as they fly by, for who knows whether any of them will ever return? — Margot Fonteyn

Recall that in Chapter 1 we gave the following:

DEFINITION. An ***isometry*** is a transformation that preserves distances and angle measures.

In this chapter we will show (for the plane, spheres, and hyperbolic planes) that every isometry is the composition (product) of (not more than three) reflections, and we will determine all the different types of isometries. This finishes the study of reflections and rotations we started in Problem **5.4**. We will note the differences between the kinds of isometries that appear in the three geometries.

Then we will study patterns in these three spaces. Along the way we will look at some group theory through its origins, that is, geometrically.

This material originated primarily within the Art/Pattern Strand.

It would be good for the reader to start by reviewing Problem **5.4**. We will start the chapter with a further investigation of isometries and then with a discussion of definitions and terminology. We advise the reader to investigate this introductory material as concretely as possible, making drawings and/or moving about paper triangles.

PROBLEM 11.1 ISOMETRIES

Definitions:

A ***reflection through the line*** (***geodesic***) ***l*** *is an isometry* R_l *such that it fixes* only *those points that lie on l and, for each point P not on l, l is the perpendicular bisector of the geodesic segment joining P to* $R_l(P)$.

Reflections are closely related to our notion of geodesic (*intrinsically straight*). We say in Chapters 1, 2, 4, and 5 that the existence of (local) reflections through any geodesic is a major determining property of geodesic in the plane, spheres, cylinders, cones, and hyperbolic planes.

A ***directed angle*** *is an angle with one of its sides designated as the* ***initial side*** *(the other side is then the* ***terminal side****). It is usual to indicate the direction with an arrow, as in* Figure 11.1. *The usual convention is to say that angles with a counterclockwise direction are positive and those with a clockwise direction are negative. When we write* $\angle APB$ *for a directed angle, then we consider AP to be on the initial side*

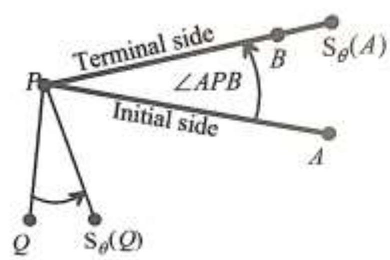


Figure 11.1 Directed angle and rotation about P through angle $\angle APB$

A ***rotation about P through the directed angle*** θ *is an isometry* S_θ *that leaves the point P fixed and is such that, for every* $Q \neq P$, $S_\theta(Q)$ *is on the same circle with center P that Q is on, and the angle* $\angle QPS_\theta(Q)$ *is congruent to* θ *and in the same direction*. See Figure 11.1.

a. Prove: *If P is a point on the plane, sphere, or hyperbolic plane and* $\angle APB$ *is any directed angle at P, then there is a rotation about P through the angle* $\theta = \angle APB$.

Refer to Problem **5.4** and Figure 5.13. Remember to show that the composition of these two reflections has the desired angle property. Remember to check that every point is rotated.

However, it is still not yet clear that anything we would want to call a translation exists on spheres or hyperbolic planes.

b. *Let m and n be two geodesics on the plane, a sphere, or a hyperbolic plane with a common perpendicular l. Look at the composition of the reflection* R_m *through m with the reflection* R_n *through n. Show why this composition* R_nR_m *could be called a translation of the surface along l. How far are points on l moved? What will happen to points not on l?* See Figure 11.2.

Remember that R_nR_m denotes: *First* reflect about *m* and *then* reflect about *n*. Let *Q* be an arbitrary point on *l* (but not on *m* or *n*). Investigate where *Q* is sent by R_m and then by R_nR_m. Then investigate what will happen to points not on *l* — note that they stay the same distance from *l*. Be sure to draw pictures.

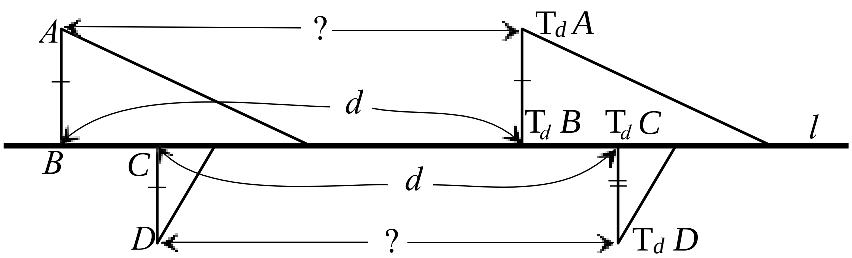


Figure 11.2 Translation of distance *d* along *l*

Pattern in Alhambra which has only translation symmetry

We can now formulate a definition that works on all surfaces:

A ***translation of distance d along the line*** (***geodesic***) ***l*** *is an isometry* T_d *that takes each point on l to a point on l at the distance* (*along l*) *of d and takes each point not on l to another point on the* same side *of l and at the same distance from l.* See Figure 11.2.

Note that on a sphere a translation along a great circle *l* is the same as a rotation about the poles of that great circle.

c. Prove: *If l is a geodesic on the plane, a sphere, or a hyperbolic plane and d is a distance, then there is a translation of distance d along l.*

Use part **b**.

In Figure 11.3 (which can be considered to be on either the plane, a sphere, or a hyperbolic plane), two congruent geometric figures, **F** and **G**, are given, but there is not a single reflection, or rotation, or translation that will take one onto the other.

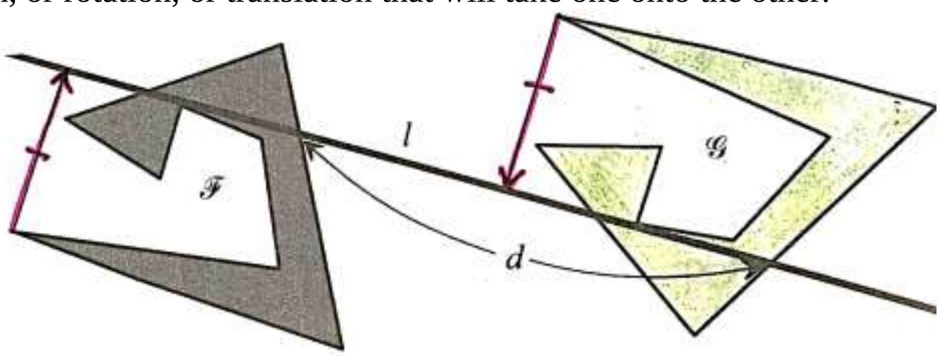


Figure 11.3 Glide reflection

However, it is clear that there is some composition of translations, rotations, and reflections that will take F onto G. In fact, the composition of a reflection through the line *l* and a translation along *l* will take F onto G. This isometry is called ***a glide reflection along l***.

A ***glide reflection*** (or just plain ***glide***) ***of distance d along the line*** (***geodesic***) ***l*** *is an isometry* G_d *that takes each point on l to a point on l at the distance* (*along l*) *of d and takes each point not on l to another point on the* other side *of l and at the same distance from l.*

d. *If l is a geodesic on the plane, sphere, or hyperbolic plane and if d is a distance, then there is a glide of distance d along l.*

Later in this chapter we will show that these are the only isometries of the plane and spheres. However, on the hyperbolic plane there is another isometry that is not a reflection, rotation, translation, or glide.

Asymptotic geodesics

On the hyperbolic plane there are some pairs of geodesics, called ***asymptotic geodesics***, that do not intersect and do not have a common perpendicular (and thus are not parallel transports). See Problem **8.4b**. For example, two radial geodesics in the annular hyperbolic plane are asymptotic. The composition of reflections about two asymptotic geodesics is defined to be a ***horolation***. See Figure 11.4, where

$$A' = R_m(A),\ A'' = R_n(R_m(A)),\ B'' = R_n(R_m(B)),\ C'' = R_n(R_m(C)).$$

Figure 11.4 Horolation

Note that every point moves along an annular line (not a geodesic) such as *a* in Figure 11.4. The annular lines could be called "circles of infinite radius". [*Do you see why? If you are given a circular arc in the plane how to you find its center?*] In much of the literature these annular lines, or circles of infinite radius, are called ***horocycles***. [In the past, some students have affectionately called them "horror-cycles."] Note that *on the plane circles of infinite radius are straight lines.*

For the horolation depicted in Figure 11.4 the two reflection lines are radial geodesics. This is not really the special case it looks to be: If *l* and *m* are any two asymptotic (but not radial) geodesics, then *l* must intersect a radial geodesic *r*, in fact, infinitely many radial geodesics. Reflect the whole hyperbolic plane through the bisector *b* of the angle between the end of l at which it is asymptotic to m and the end of r at which it is asymptotic to other radial geodesics. See Figure 11.5. The images of l and m under the reflection are now radial geodesics, $r = \mathrm{R}_b(l)$ and $\mathrm{R}_b(m)$.

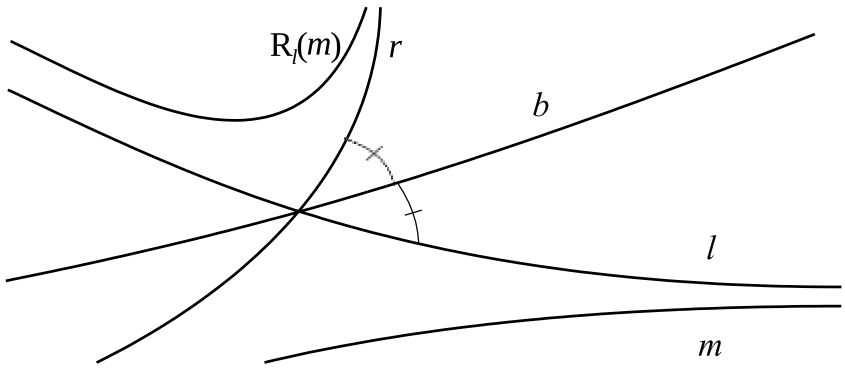


Figure 11.5 Asymptotic geodesics can be reflected to radial geodesics

A horolation is an isometry that is neither a rotation nor a translation but can be thought of as a rotation about the point at infinity where the two asymptotic lines converge. In the case of radial geodesics, a horolation will take each annulus to itself because the radial geodesics are perpendicular to the annuli. See Figure 11.4.

In Chapter 17 we will also see that a horolation corresponds to a translation parallel to the x-axis in the hyperbolic coordinate system introduced in Problem **5.2** and the upper half-plane model introduced in Problems **17.1** and **17.2**. In the context of the models of the hyperbolic plane introduced in Chapter 17 many authors use the terms “elliptic isometry”, “parabolic isometry”, and “hyperbolic isometry” to refer to what we are calling, “rotation”, “horolation”, and “translation.”

PROBLEM 11.2 THREE POINTS DETERMINE AN ISOMETRY

In order to analyze what all isometries are, we need the following very important property of isometries:

> *Prove the following: On the plane, spheres, or hyperbolic planes, if **f** and **g** are isometries and **A**, **B**, **C** are three non-collinear points, such that **f**(**A**) = **g**(**A**), **f**(**B**) = **g**(**B**), and **f**(**C**) = **g**(**C**), then **f** and **g** are the same isometry, that is, **f**(**X**) = **g**(**X**) for every point **X**.*

Let us see an example of how this works before you see why this property holds. In Figure 11.6, H_P represents the half-turn around P, R_m represents the reflection through line m, G_l represents a glide reflection along l, and

$\mathscr{F}_1 = \{A, B, C\}$, $\mathscr{F}_2 = \{A', B', C'\}$ and $\mathscr{F}_3 = \{A'', B'', C''\}$.

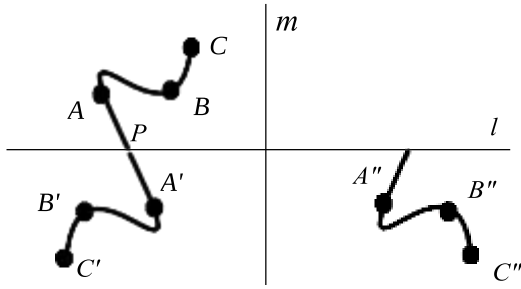


Figure 11.6 A glide equals a half-turn followed by a reflection

We can see that $H_P(\mathscr{F}_1) = \mathscr{F}_2$, $R_m(\mathscr{F}_2) = \mathscr{F}_3$, and $G_l(\mathscr{F}_1) = \mathscr{F}_3$. But then G_l and R_mH_p perform the same action on the three points. If we apply the above result, we can say that $G_l = R_mH_P$; that is,

$G_l(X) = R_mH_P(X)$, for all points X on the plane. You can see now the usefulness of proving Problem **11.2**.

If you have trouble getting started with this problem, then take a specific example of two congruent triangles ΔABC and $\Delta A'B'C'$ such as in Figure 11.7. Pick another point X and convince yourself as concretely as possible that there is only one location that X will be taken to by any isometry that takes A, B, C onto A', B', C'.

PROBLEM 11.3 CLASSIFICATION OF ISOMETRIES

a. *Prove that on the plane, spheres, or hyperbolic planes, every isometry is the composition of one, two, or three reflections.*

Look back at what we did in the proofs of SAS and ASA. Also use Problem **11.2**, which allows you to be more concrete when thinking about isometries because you only need to look at the effect of the isometries on any three non-collinear points that you pick. To get started on **11.3**, cut a triangle out of an index card and use it to draw two congruent triangles in different orientations on a sheet of paper. For example, see Figure 11.7. Can you move one triangle to the other by three (or fewer) reflections? You can use your cutout triangle for the intermediate steps.

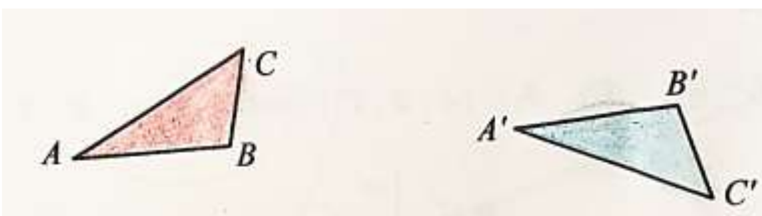


Figure 11.7 Can you make triangles coincide with three reflections?

We have already showed in Problems **5.4** and **11.1** that the product of two reflections is a rotation (if the two reflection lines intersect) and a translation (if the two reflection lines have a common perpendicular, that is, if the two lines are parallel transports). Thus, we can

b. *Prove that on the plane and spheres, every composition of two reflections is either the identity, a translation, or a rotation. What are the other possibilities on a hyperbolic plane? Why are you sure that these are the only isometries that are the composition of two reflections? What happens if you switch the order of the two reflections?*

In this part start with the two reflections as given. What are the different ways in which the two reflection lines can intersect? To fully answer this part on a hyperbolic plane, you will have to use the result (found informally in Chapters 5 and 8 and proved in Problem **17.3**) that *Two geodesics in a hyperbolic plane either intersect in a point, or have a common perpendicular, or are asymptotic*.

Your proof of part **b** or Problem **11.1** probably already shows that

THEOREM 11.3. *On the plane, spheres, and hyperbolic planes, a rotation is determined by two intersecting reflection lines (geodesics). The lines are determined only by their point of intersection and the angle between them; that is, any two lines with the same intersection point and the same angle between them will produce the same rotation.* See Figure 11.8.

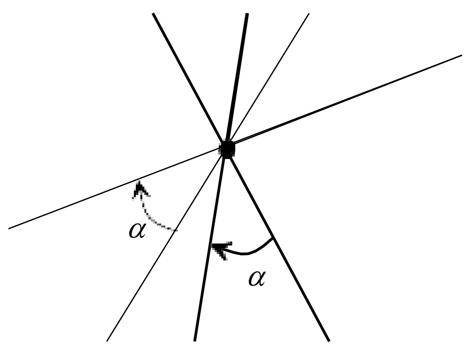


Figure 11.8 Two pairs of reflection lines, which determine the same rotation

c. *On the plane, spheres, and hyperbolic plane, the product of two rotations* (in general, about different points) *is a single rotation, translation, or horolation. Show how to determine geometrically which specific isometry you obtain, including the center and angle of any rotation.*

Use Theorem 11.3 and write the composition of the two rotations with each rotation the composition of two reflections in a way that the middle two reflections cancel out. Be careful to keep track of the directions of the rotations. See Figure 11.9.

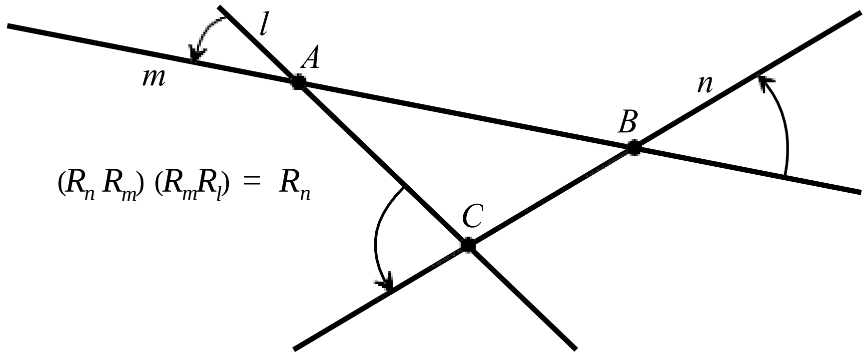


Figure 11.9 Composition of two rotations

d. *On the plane, spheres, and hyperbolic planes, every composition of three reflections is either a reflection or a glide reflection. How can you tell which one*?

Approach using Theorem 11.3: Write the composition of three reflections, R_l, R_m, R_n, as either $(R_lR_m)R_n$ or $R_l(R_mR_n)$. If the reflection lines, *l*, *m*, *n*, intersect, then use part **c** to replace the reflection lines within the parentheses with reflection lines that produce the same result. Try to produce the situation where the first (or last) reflection line is perpendicular to the other two. What is the situation if none of *l*, *m*, *n* intersect each other? This approach works well on the plane and spheres but is more difficult on a hyperbolic plane because the situation when all three lines do not intersect is more complicated.

Approach using Problem 11.2 and triangles: Let ΔABC be a triangle and let $\Delta A'B'C'$ be its image under the composition of the three reflections. Note that the two triangles cannot be directly congruent (see the discussion around Figure 6.5). Extend two corresponding sides (say *AB* and *A'B'*) of the triangles to lines (geodesics) *l* and *m*. Then there are three cases: The lines *l* and *m* intersect, or have a common perpendicular, or are asymptotic.

If parts **a**, **b**, and **d** are true, then

e. *Every isometry of the plane or a sphere or a hyperbolic plane is either a reflection, a translation, a rotation, a horolation, a glide reflection, or the identity.*

Notice that in our proofs of SAS and ASA and (probably) in your proof of part **b** we only need to use reflections about the perpendicular bisectors or segments joining two points. Because, on a sphere, we only need great circles to join two points and for perpendicular bisectors, we can restate part **b** on a sphere to read

Every isometry of a sphere is the composition of one, two, or three reflections through great circles.

Thus, if there were a reflection of the sphere that was not through a great circle, then that reflection (being an isometry) would also be composition of one, two, or three reflections through great circles.

You now have powerful tools to make a classification of discrete strip patterns on the plane, spheres, and hyperbolic planes and finite patterns on the plane and hyperbolic planes.

KLEIN'S ERLANGEN PROGRAM

Felix Klein (1849-1925)

In 1872, on the occasion of his appointment as professor in the University of Erlangen at the age of 23, Felix Klein (1849–1925) proposed that the view of symmetries should be radically extended. He proposed viewing geometry as "the study of the properties of a space which are invariant under a given group of transformations." The proposal is now universally known as the *Erlangen Program*.

Klein provided several examples of geometries and their associated transformations. In Euclidean plane geometry the transformations are all the isometries (reflections, rotations, translations, and glides) and including the similarity transformations (dilations), that are not isometries because they do not preserve distances but do preserve angles and thus take triangles to similar triangles (see Problem **13.4**).

Within the Erlangen Program, spherical geometry is the study of the isometries of the sphere (reflections, rotation/translations, and glides) and hyperbolic geometry is the study of its isometries (reflections, rotations, translations, glides, and horolations). In general, symmetry was used as an underlying principle and it was shown that different geometries could coexist because they dealt with different types of propositions and invariances related to different types of (symmetry) transformations. Topology, projective geometry, and other theories have all been placed within the classification of geometries in the Erlangen Program. The long-term effects of the Erlangen Program can be seen all over pure mathematics, and the idea of transformations and of synthesis using groups of symmetry is now standard also in physics. Klein's demonstration of the relationship of geometry to groups of transformations helped to provide impetus for the development of the abstract notion of a *group* by the end of 19th century. For more discussion of Klein's program, see [**HM**: Yaglom].

SYMMETRIES AND PATTERNS

Symmetries leave deep relationships invariant. Albert Einstein (1879-1955) showed that the relationship between space and time always stays the same, even as space contracts and time dilates. Emmy Noether (1882-1935) showed that the symmetries of general relativity – its invariance under transformations between reference frames – ensure that energy is always conserved. Paul Dirac (1902-1984), trying to make quantum mechanics compatible with the symmetry requirements of special relativity, found a minus sign in an equation suggesting that "antimatter" must exist to balance the books. Wolfgang Pauli (1900-1958), in an attempt to account for the energy that seemed to go missing during the disintegration of radioactive particles, speculated that perhaps the missing energy was carried away by some unknown, elusive particle. It was neutrino. (more https://www.quantamagazine.org/einstein-symmetry-and-the-future-of-physics-20190626/)

In Chapter 1 we talked about symmetries of the line. All of those symmetries can be seen as isometries of the plane except for similarity symmetry and 3-D rotation symmetry (through any angle not an integer multiple of 180°). Similarity symmetry changes lengths between points of the geometric figure and thus is not an isometry. Three-dimensional rotation symmetry is an isometry of 3-space, but it moves any plane off itself and thus cannot be an isometry of a plane (unless the angle of rotation is a multiple of 180°). The notion of symmetry grew out of the Art/Pattern Strand of history. Well before written history, symmetry and patterns were part of the human experience of weaving and decorating.

Which symmetries can you find in this border?

What is a symmetry of a geometric figure? A ***symmetry*** of a geometric figure is an isometry that takes the figure onto itself. For example, reflection through any median is a symmetry of an equilateral triangle. See Figure 11.10.

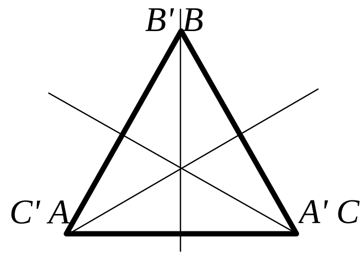


Figure 11.10 Reflection symmetries

It is easy to see that rotations through 1/3 and 2/3 of a revolution, $S_{1/3}$ and $S_{2/3}$, are also symmetries of the equilateral triangle. In addition, the identity, Id, is (trivially, by the definition) an isometry and, thus, is also a symmetry of the equilateral triangle. Therefore, the equilateral triangle has six symmetries: R_A, R_B, R_C, $S_{1/3}$, $S_{2/3}$, Id, where R_A, R_B, R_C, denote the reflections through the medians from *A*, *B*, *C*. These are the only symmetries of the equilateral triangle.

Now look at the geometric figure in Figure 11.11. It has exactly the same symmetries as an equilateral triangle. Though the two figures look very different, we say they are ***isomorphic patterns***.

A ***pattern*** is a figure *together with all its symmetries*; and we call the collection of all symmetries of a geometric figure its ***symmetry group***. We want to be able to denote that two different patterns have the same symmetries, as is the case with the pattern in Figure 11.11 and the equilateral triangle in Figure 11.10. We do this by saying that two patterns are ***isomorphic*** if they have the same symmetries. This should become clearer through more examples.

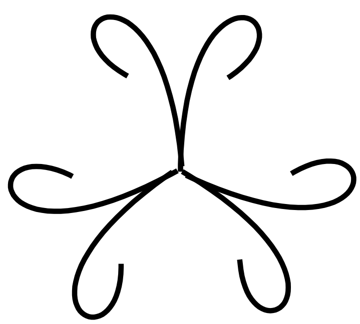


Figure 11.11 Same symmetries as the equilateral triangle

The letters S and N each have only half-turn and the identity as symmetries; thus, we say they are isomorphic patterns with symmetry group {Id, $S_{1/2}$}. Similarly, the letters A and M each have only vertical reflection and the identity as symmetries and thus are isomorphic patterns with symmetry group {Id, R}. Note that the letters S and A each have the same number of symmetries, but we do **not** call them isomorphic patterns because the symmetries are different symmetries. You should check your understanding by finding isomorphic patterns among the other letters of the alphabet in normal printing by hand.

To construct further examples, we can start with a geometric figure, often called a ***motif***, that has no symmetry (except the identity). For example, the geometric figures in

Figure 11.12 are possible motifs.

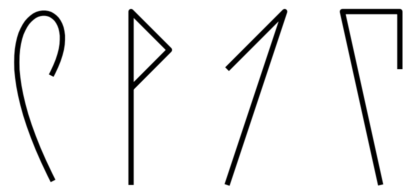


Figure 11.12 Motifs

To make examples of patterns with a specific isometry, we can start with any motif and then use the isometry and its inverse to make additional copies of the motif over and over again. In the process we obtain another geometric figure for which the initial isometry is a symmetry. Let us look at an example: If we start with the first motif in Figure 11.12 and the isometry is translation to the right through a distance *d*, then, using this isometry and its inverse (translation to the left through a distance *d*) and repeating them over and over, we obtain the pattern in Figure 11.13.

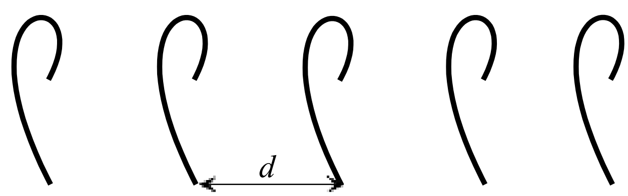


Figure 11.13 Pattern with translation symmetries

The symmetry group of the pattern in Figure 11.13 is

$$\{\text{Id (the identity)}, T_{nd} \text{ (where } n = \pm 1, \pm 2, \pm 3, \ldots)\}.$$

If the isometry is clockwise rotation through 1/3 of a revolution about the lower endpoint of the motif, then we obtain the pattern in Figure 11.14. This figure is a pattern with symmetries $\{\text{Id}, S_{1/3}, S_{2/3}\}$. Note that this pattern is not isomorphic with the equilateral triangle pattern.

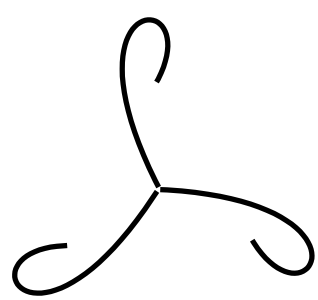


Figure 11.14 Rotation symmetry

If, in the constructions depicted in Figures 11.13 and 11.14, we replace the motif with any other motif (with no non-trivial symmetries), then we will get other patterns that are isomorphic to the original ones because the symmetries are the same.

We call the collection of all symmetries of a geometric figure its ***symmetry group***. If g, h are symmetries of a figure $\mathscr{F}$, then you can easily see that

> *The* ***composition*** *gh* (*first transform by h and then follow it by g*) *is also a symmetry.* For example, for an equilateral triangle the composition of the reflection R_A with the reflection R_B is a rotation, $S_{2/3}$. In symbols, $R_BR_A = S_{2/3}$. See Figure 11.15.
> *Composition of symmetries is* ***associative***: *That is, if h, g, k are symmetries of the same figure, then* $(hg)k = h(gk) \equiv hgk$.

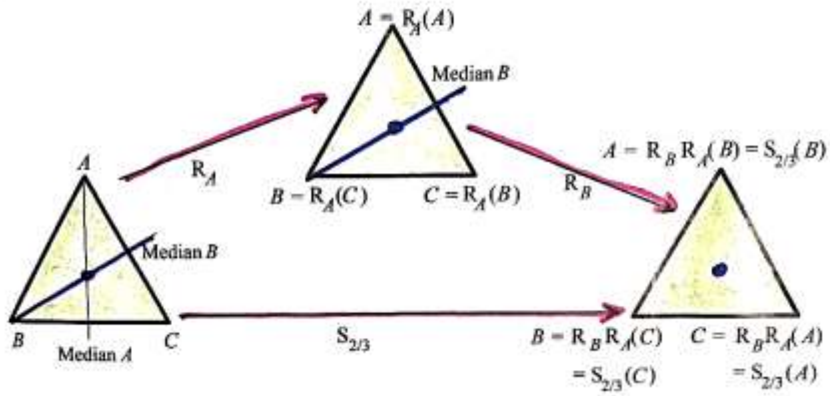


Figure 11.15 Composition of symmetries

> *The* ***identity*** *transformation,* Id (*the transformation that takes every point to itself*), *is a symmetry.*
> *For every symmetry g of a figure there is another symmetry f such that gf and fg are the identity — in this case we call f the* ***inverse*** *of g.* For example, the inverse of the rotation $S_{2/3}$ is the rotation $S_{1/3}$ and vice versa.
> In symbols, $S_{1/3}S_{2/3} = S_{2/3}S_{1/3} = \mathrm{Id}$.

Those readers who are familiar with abstract group theory will recognize the above as the axioms for an abstract group. We will discuss further connections with group theory at the end of this chapter. If two patterns are isomorphic, then their symmetry groups are isomorphic as abstract groups. The converse is often, but not always, true. For example, the symmetry groups of the letter A and the letter S are isomorphic (as abstract groups) to $\mathbf{Z}_2$, but they are not isomorphic as patterns because they have different symmetries.

A ***strip*** (or linear, or frieze) ***pattern*** is a pattern that has a translation symmetry, with all of its symmetries also symmetries of a given line. For example, the pattern in Figure 11.13 is a strip pattern as is the pattern in Figure 11.11. The strip pattern in Figure 11.16 has symmetry group: $\{\mathrm{Id}, R_l, T_{nd}, G_{nd} = R_lT_{nd}$ (where $n = 0, \pm1, \pm2, \pm3, \cdots)\}$.

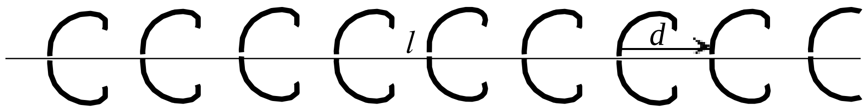


Figure 11.16 A strip pattern

You are now able to start to study properties about patterns and isometries.

PROBLEM 11.4 EXAMPLES OF PATTERNS

a. *Go through all the letters of the alphabet* (in normal printing by hand) *and decide which are isomorphic as patterns.*

b. *Find as many* (*non-isomorphic*) *patterns as you can that have only finitely many symmetries. List all the symmetries of each pattern you find.*

c. *Find as many* (*non-isomorphic*) *strip patterns as you can*. *List all the symmetries of each strip pattern you find.*

The purpose of this problem is to get you looking at and thinking about patterns. Examples of different strip patterns and many finite patterns can be found on buildings everywhere: houses of worship, courthouses, and most older buildings. Also look for other decorations on plates or on wallpaper edging.

In the next two problems we will determine if your list of strip patterns and finite patterns contains all possible strip and finite patterns.

Pattern examples from Alhambra

PROBLEM 11.5 CLASSIFICATION OF DISCRETE STRIP PATTERNS

A strip pattern is ***discrete*** if every translation symmetry of the strip pattern is a multiple of some shortest translation.

- **a.** *Prove there are only seven non-isomorphic strip patterns on the plane that are discrete.*
- **b.** *What are some non-discrete strip patterns*?
- **c.** *What happens with strip patterns on spheres and hyperbolic planes*?

Hint: Use Problem **11.3**.

PROBLEM 11.6 CLASSIFICATION OF FINITE PLANE PATTERNS

Look at all the finite patterns on the plane that you found in Problem **11.5b**. Do you notice that for each there is a point (not necessary on the figure) such that every symmetry of the pattern leaves the point fixed? See Figure 11.17.

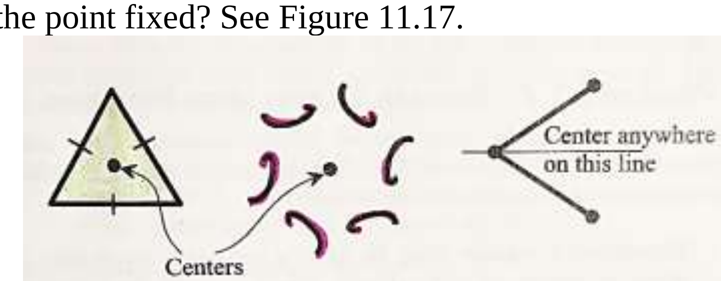


Figure 11.17 Finite patterns have centers

This leads to the following:

a. *Show that any pattern on the plane with only finitely many symmetries has a center. That is, there is a point in the plane* (*not necessarily on the figure*) *such that every symmetry of the pattern leaves the point fixed. Is this true on spheres and hyperbolic planes*?

This was first proved by Leonardo da Vinci (1452–1519, Italian), and you can prove it too! Hint: Start by looking at what happens if there is a translation or glide symmetry.

b. *Describe all the patterns on the plane and hyperbolic planes with only finitely many symmetries.*

Hint: Use part **a**. What rotations are possible if there are only finitely many symmetries?

c. *Describe all the patterns on the sphere that are finite and that have centers.*

Hint: If there is one center, then its antipodal point is necessarily also a center. *Why*?

If you take a cube with its vertices on a sphere and project from the center of the sphere the edges of the cube onto the sphere, then the result is a pattern on the sphere with only finitely many symmetries. This pattern does not fit with the patterns you found in part **c** because this pattern has no center (on the sphere). See Problems **11.7b** and **23.5** for more examples.

PROBLEM 11.7 REGULAR TILINGS WITH POLYGONS

We will now consider some special infinite patterns of the plane, spheres, and hyperbolic planes. These special patterns are called ***regular tilings*** (or ***regular tessellations***, or ***mosaics***).

DEFINITION. *A* ***regular tiling {n, k}*** *of a geometric space* (*the plane, a sphere, or a hyperbolic plane*) *is made by taking identical copies of a regular n-gon* (*a polygon* *with n edges*) *and using these n-gons to cover every point in the space so that there are no overlaps except each edge of one n-gon is also the edge of another n-gon and each vertex is a vertex of k n-gons.*

You probably know the three familiar regular tilings of the plane: {3, 6}, the regular tiling of the plane by triangles with six triangles coming together at a vertex; {4, 4}, the regular tiling of the plane by squares with four squares coming together at a vertex; and {6, 3}, the regular tiling of the plane by hexagons with three hexagons coming together at each vertex. There is only one way to tile a plane with regular hexagons; however, there are other ways to tile a plane with regular triangles and squares but only one for each that is a *regular* tiling.

Note that each regular tiling can also be thought of as a (infinite) pattern. The notation {n, k} is called the ***Schläfli symbol*** of the tiling, named after Ludwig Schläfli (1814–1895, Swiss). We now study the possible regular tilings.

a. *Show that,* {3, 6}, {4, 4}, {6, 3} *are the only Schläfli symbols that represent regular tilings of the plane.*

Focus on what happens at the vertices.

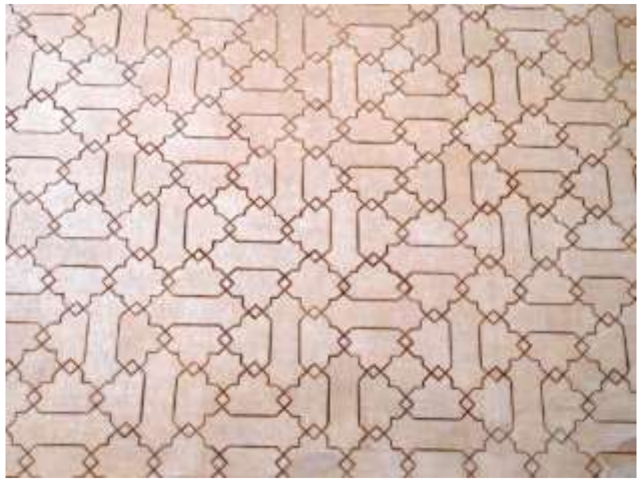
b. *Find all the regular tilings of a sphere.*

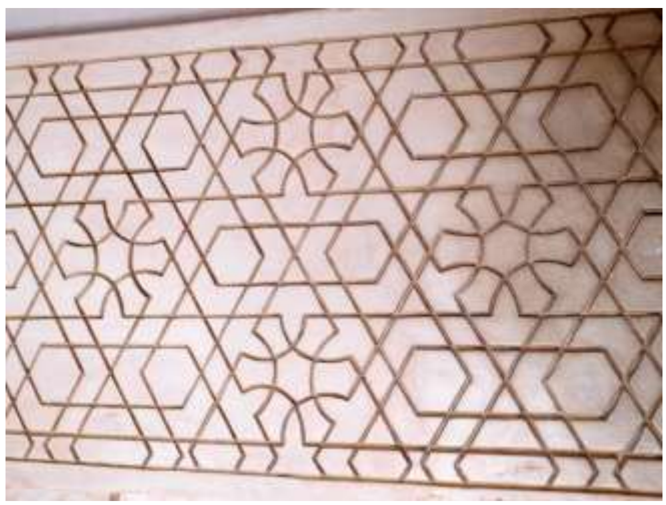
Again, focus on what happens at the vertices. Remember that angles of a regular n-gon on the sphere are larger than the angles of the corresponding regular n-gon on the plane (*Why*?); and use Problems **7.1** and **7.4**.

There are more finite patterns of the sphere besides those given in Problem **11.6c** and **11.7b**. See a soccer ball for an example and [**SG**: Montesinos] for the complete classification.

c. *Show that each of the Schläfli symbols* {n, k} *with both n and k greater than* 1 *represents a regular tiling of the plane* (part **a**), *or of a sphere* (part **b**), *or of a hyperbolic plane.*

Again, focus on what happens at the vertices.





More examples of tilings from Alhambra

OTHER PERIODIC (AND NON-PERIODIC) PATTERNS

There are numerous non-regular tilings and other patterns other than regular tilings. Besides the three that come from regular tilings, there are 14 more (non-isomorphic) infinite periodic patterns on the plane. ("Periodic" means that the pattern has a minimal translation symmetry.) See Figure 11.18. These 17 periodic patterns in the plane are often called ***wallpaper patterns***. See Escher's (Maurits Cornelius Escher, 1898–1972, Dutch) drawings for examples and [**SG:** Budden] for proofs and more examples. For a complete exposition on periodic patterns and tilings on the plane, see [**SG:** Grünbaum].

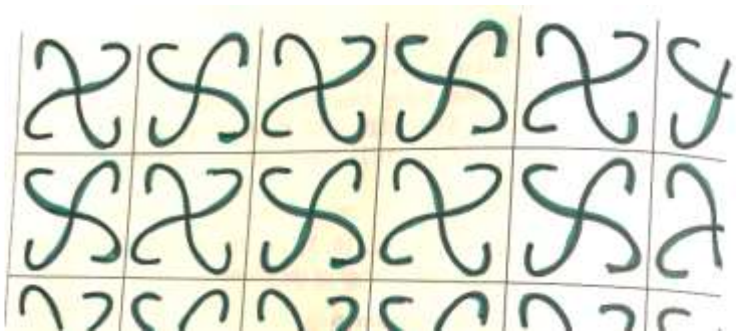


Figure 11.18 A repeating pattern that is not a regular tiling — the thin straight lines (forming squares) are lines of reflection symmetry, there is half-turn symmetry about corners of the squares, and there is 4-fold rotation symmetry about the center of each square.

There also exist non-periodic tilings of the plane; see Figure 11.19. Notice the 5-fold local rotation symmetry at the center — this will never repeat anywhere in the pattern and thus the pattern cannot be periodic.

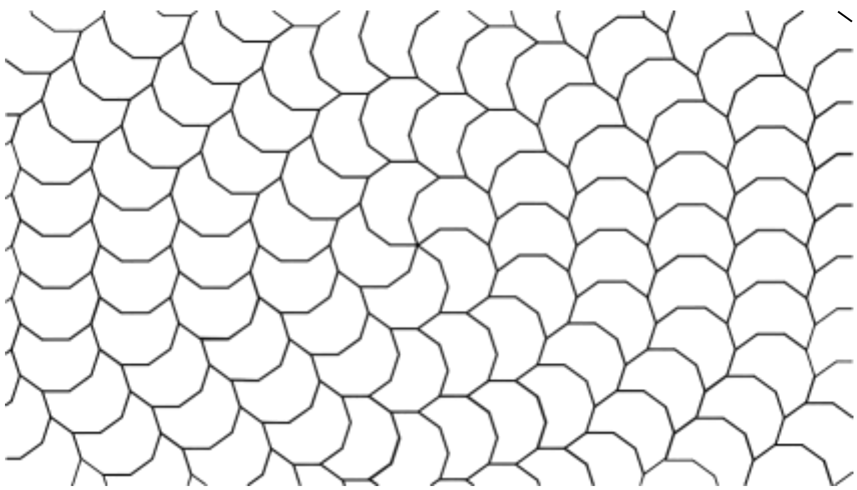


Figure 11.19 Non-periodic tiling of the plane

For more discussion of non-periodic tilings, see [**SG**: Senechal] and [**SG:** Grünbaum].

See [**SG**: Montesinos] for a discussion about other patterns and tilings on hyperbolic spaces.

There are 230 non-isomorphic repeating patterns in Euclidean 3-space. These are the patterns that appear in crystals. However, in protein crystals there are no reflection (through planes) or central (reflection through a point) symmetries — as a consequence there are only 65 repeating patterns without plane reflections or central symmetries in 3-space. The corresponding groups are call ***chiral groups***. See [**SG**: Giacovazzo] for more information on crystallographic groups.

GEOMETRIC MEANING OF ABSTRACT GROUP TERMINOLOGY

The collection of symmetries of a geometric figure with the operation of composition is an ***abstract group.*** We showed above how the usual axioms of a group are satisfied. For example, the pattern in Figure 11.20 has symmetry group $\{R_A, R_B, R_C, S_{1/3}, S_{2/3}, \mathrm{Id}\}$ that is isomorphic as an abstract group to what is usually called $\mathbf{D_3}$, the third dihedral group.

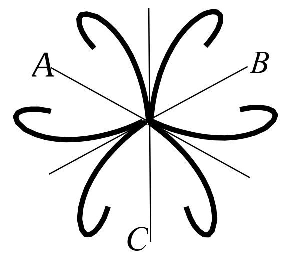


Figure 11.20 Isomorphic to $\mathbf{D_3}$

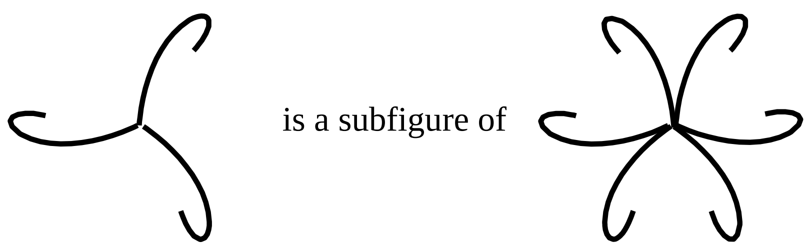


Figure 11.21 Subgroups

If a figure has a subfigure and if the symmetries of the subfigure are all also symmetries of the original figure, then the symmetry group of the subfigure is a ***subgroup*** of the symmetry group of the original figure. See Figure 11.21.

The symmetry group of the subfigure is $\{S_{1/3}, S_{2/3}, \mathrm{Id}\}$, which is isomorphic as an abstract group to $\mathbf{Z_3}$. This group is a subgroup of the symmetry group of the original figure, $\{S_{1/3}, S_{2/3}, R_A, R_B, R_C, \mathrm{Id}\}$, which is isomorphic to $\mathbf{D_3}$.

In $\mathbf{D_3}$ the two ***cosets*** of $\mathbf{Z_3}$, Id $\mathbf{Z_3}$ and $R_A\mathbf{Z_3} = R_B\mathbf{Z_3} = R_C\mathbf{Z_3}$, correspond to the two copies of the subfigure in Figure 11.22.

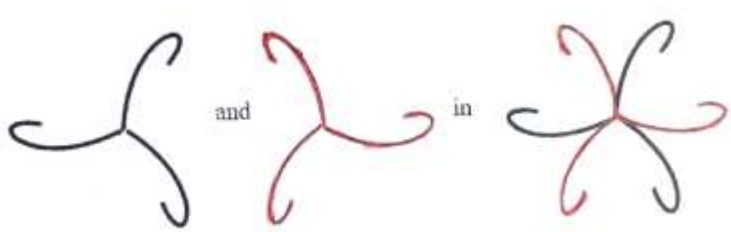


Figure 11.22 Two cosets of the subgroup in the group

In general, if G is the symmetry group of a figure and H is a subgroup that is the symmetry group of a subfigure in the figure, then the ***cosets*** of H correspond to the several congruent copies of the subfigure that exist within the larger figure.

The term *group* was coined (as *groupe* in French) by E. Galois (1811–1832) in 1830. The modern definition of a group is somewhat different from that of Galois, for whom the term denoted a subgroup of the group of permutations of the roots of a given polynomial. F. Klein and S. Lie (1842–1899) used the term *closed system* in their earliest writing on the subject of groups. *Group* appears in English in an article by Arthur Cayley, who wrote "A set of symbols, 1, α, β, ..., all of them different, and such that the product of any two of them (no matter what order), or the product of any one of them into itself, belongs to the set, is said to be a group." See [**SG**: Lyndon] for more discussion on groups.

To further read about symmetries we highly recommend two beautifully illustrated books: *The Symmetries of Things* by John H. Conway, Heidi Burgiel, and Chaim Goodman-Strauss (AKPeters, 2008) and *Creating Symmetry: The Artful Mathematics of Wallpaper Patterns* by Frank A. Farris (Princeton University Press, 2015). You can also check out *Beautiful Symmetry: A Coloring Book about Math* by Alex Berke (http://www.coloring-book.co/?pageNumber=0&pageName=cover)

Islamic artists mastered regular division of plane using, in particular, circles on triangular or square grids, because the circle – which has no beginning and no end and thus symbolizes infinity – was considered to be the most perfect geometric form. M. C. Escher (1898-1972) visited 14th century Alhambra palace (Granada, Spain) in 1922 and 1936. He became fascinated with geometry there and made many sketches. Alhambra had a great influence on his work. In October 2015 both authors had a chance to spend two days in Alhambra which was very memorable experience. We took many pictures, few of them are now in this book.