Chapter 5

Area, Parallel Transport and Intrinsic Curvature

In Chapters 3 and 4 we were developing extrinsic descriptions of the intrinsic curvature of a curve on a surface. In this chapter we will develop intrinsic descriptions of both the ***intrinsic curvature of a curve*** on a surface and the ***intrinsic curvature of a surface***. This intrinsic curvature will be developed by investigating the relationships between surface area, normal curvature, and ***parallel transport***, a notion of local parallelism that is definable on all surfaces. We first develop these connections on a sphere that has known curvature and then use the results on the sphere to motivate the discussion on general surfaces. As an important part of this development, we introduce the notion of ***holonomy***, which also has many other applications in modern differential geometry and engineering.

PROBLEM 5.1. The Area of a Triangle on a Sphere

DEFINITION: On any surface we will call a triangle a ***geodesic triangle*** if its three sides are geodesic segments.

a. *Let Δ be a geodesic triangle on a sphere. Show that the formula*

$$\text{Area}\ (\Delta) = A/4\pi[\ \textstyle\sum \beta_i - \pi\ ] = A/4\pi[\ 2\pi - \sum \alpha_i\ ]$$

holds, where A is the area of the sphere, β_i are the interior angles, and α_i are the exterior angles of the triangle in radians. The quantity $(\Sigma\beta_i)-\pi$ is called the ***excess***.

We offer the following hint as a way to approach this problem: Find the area of a biangle (lune) with angle θ. (A ***biangle*** or ***lune*** is one of the connected regions between two great circles.) Notice that the great circles that contain the sides of the triangle divide the sphere into overlapping biangles. Focus on the biangles determined by either the interior angles or the exterior angles.

This is one of the problems that you almost certainly must do on an actual sphere. There are too many things to see, and the drawings we make on paper distort lines and angles too much. The best way to start is to make a small triangle on a sphere, and extend the lines of the triangle to complete great circles. Then look at the results. You will find an identical triangle on the other side of the sphere, and you can see several lunes extending out from the triangles. The key to this problem is to put everything in terms of areas that you know. *Find the areas of the lunes*, as noted above. After that, it is simply a matter of adding everything up properly.

We know that the area of the whole sphere is $4\pi R^2$, where R is the (extrinsic) radius of the sphere. With this additional information we can rewrite the formula of Problem **5.1.a**:

$$\text{Area}\ (\Delta) = [\ \textstyle\sum \beta_i - \pi\ ]\ R^2 = [\ 2\pi - \sum \alpha_i\ ]\ R^2.$$

b. *Use* **5.1.a** *to show that for a triangle in the plane the sum of the interior angles is π and the sum of the exterior angles is 2π.*

[Hint: The triangle in the plane is determined by three points. Imagine a large sphere resting on these three points. Then the three points determine a geodesic triangle of the sphere. What happens if you keep the three points fixed but allow the radius of the sphere goes to infinity?]

c. *Conclude that a sphere is not locally isometric with a plane.*

Introducing Parallel Transport

Imagine that you are walking along a straight line or geodesic, carrying a horizontal rod that makes a fixed angle with the line you are walking on. If you walk along the line maintaining the direction of the rod relative to the line constant, then you are performing a ***parallel transport*** of that "direction" along the path. (See Figure 5.1.)

To express the parallel transport idea, it is common terminology to say that:

- ***r′*** is a parallel transport of ***r*** along ***l***;
- ***r*** is a parallel transport of ***r′*** along ***l***;
- ***r*** and ***r′*** are parallel transports along ***l***;
- ***r*** can be parallel transported along ***l*** to ***r′***; or,
- ***r′*** can be parallel transported along ***l*** to ***r***.

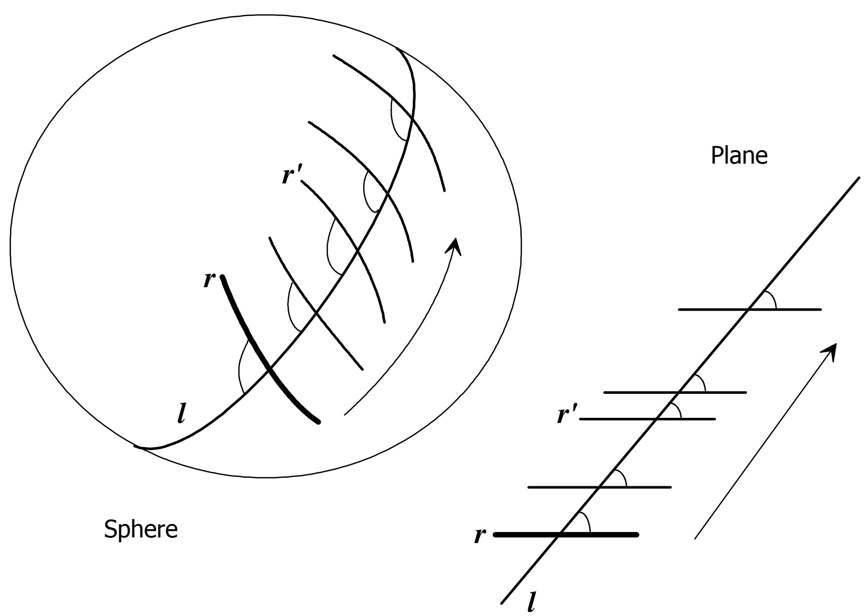

Figure 5.1. Parallel transport.

Parallel transport has become an important notion in Differential Geometry, Physics, and Mechanics. One important aspect of Differential Geometry is the study of properties of spaces (surfaces) from an intrinsic point of view. As we have seen, in general, it is not possible to have a global notion of direction from which we are able to determine when a direction (or vector) at one point is the same as a direction (or vector) at another point. However, we can say that they have the same direction *with respect to* a geodesic *g* if they are parallel transports of each other along *g*. Parallel transport can be extended to arbitrary curves as we shall discuss in Problem **5.4**. With this notion it is possible to talk about how a particular vector quantity changes intrinsically along a curve (covariant differentiation). In general, covariant differentiation is useful in the areas of physics and mechanics. In physics, the notion of parallel transport is central to some of the theories that have been put forward as possible candidates for a "Unified Field Theory," a hoped for, but as yet unrealized, theory that would unify all known physical laws about the forces of nature.

The Holonomy of a Small Geodesic Triangle

Let us imagine that we parallel transport a vector along a piecewise geodesic curve (such as a geodesic triangle) in a surface *M*. Since we do not want the parallel transported vector to change direction intrinsically as we parallel transport along the curve, we must change the angle between the transported vector and the velocity vector in order to undo the effect of the change in the direction of the curve. Thus the angle between the transported vector and the velocity vector changes at every vertex by the amount of the exterior angle. (See Figure 5.2.)

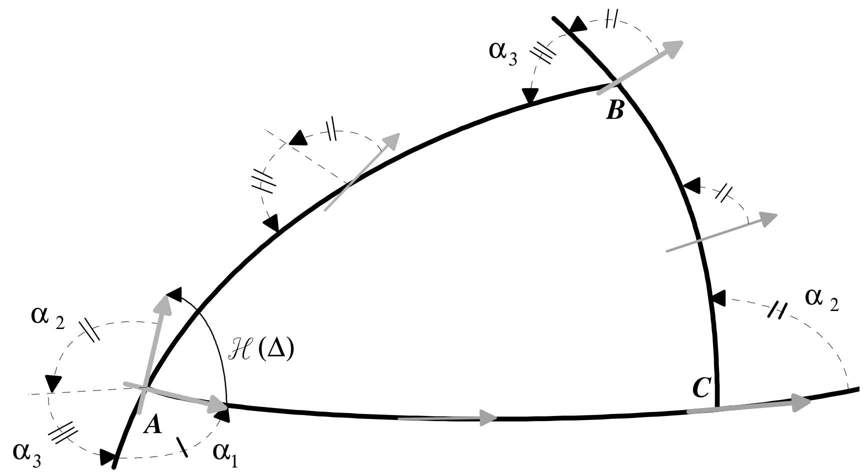

Figure 5.2. Holonomy of a geodesic triangle.

Definition: The ***holonomy*[†] *of a small*** (that is, contained in an open hemisphere) ***geodesic triangle***, $\mathcal{H}(\Delta)$, is defined as follows:

If you parallel transport a vector (a directed geodesic segment) counterclockwise around the three sides of a small triangle, then the ***holonomy*** *of the triangle is the smallest angle measured counterclockwise from the original position of the vector to its final position.* (See Figure 5.2.)

Later we will show how to extend this definition to apply to all triangles and polygons. By studying the above picture the reader should be able to see that:

The holonomy of a small geodesic triangle is equal to 2π *minus the sum of the exterior angles.*

Let β_1, β_2 and β_3 be the interior angles of the triangle and α_1, α_2, α_3, the exterior angles, then algebraically the statement above can be written as:

$$\mathcal{H}(\Delta) = 2\pi - (\alpha_1 + \alpha_2 + \alpha_3) = (\beta_1 + \beta_2 + \beta_3) - \pi.$$

The quantity $(\beta_1 + \beta_2 + \beta_3) - \pi$ is traditionally called the ***excess*** of the triangle.

Note one consequence of this formula :

The holonomy does not depend on either the vertex or the vector we start with.

[†] Some books define holonomy as [(our holonomy) − 2π]. I chose not to use this definition, because it would result in the holonomy of a planar triangle being equal to -2π. I think it is more natural for the holonomy of a planar triangle to be zero.

We can write the result from Problem **5.1.a** in the following form:

For a small geodesic triangle on the sphere:

$$\mathcal{H}(\Delta) = 2\pi - (\alpha_1 + \alpha_2 + \alpha_3) = A(\Delta)\, 4\pi/A = A(\Delta)\, R^{-2}.$$

The formula

$$2\pi - (\alpha_1 + \alpha_2 + \alpha_3) = A(\Delta)\, R^{-2}$$

is called the ***Gauss-Bonnet Formula*** (for triangles). The quantity R^{-2} is traditionally called the ***Gaussian curvature***, or ***intrinsic curvature***, or just plain ***curvature*** of the sphere. Can you see how this result gives the bug an intrinsic way of determining the extrinsic quantity R and the curvature

$$R^{-2} = \mathcal{H}(\Delta)/A(\Delta)\ ?$$

As soon as we have defined the notion of holonomy on a surface then we can define the intrinsic curvature, $K(p)$, at a point on the surface to be

$$K(p) = \lim_{n\to\infty} \mathcal{H}(R_n) / A(R_n),$$

where $\{R_n\}$ is a sequence of small (geodesic) triangles that converge to p. (See Problem **5.6**.) In Problem **6.4** we show that this intrinsic curvature is the same on C^2 surfaces as the usually defined Gaussian curvature.

The Gauss-Bonnet Formula not only holds on the sphere for small triangles but can be extended to any *small* (contained in an open hemisphere) *simple* (non intersecting) *polygon* (a closed curve consisting of a finite number of geodesic segments) on a sphere. We will need the result for small simple polygons on the sphere in order to make the transition from the sphere to arbitrary surfaces.

We shall prove the Gauss-Bonnet Formula for polygons on the sphere by first dividing the polygon into triangles as in the following:

Problem 5.2. Dissection of Polygons into Triangles

Prove that every small simple polygon on a sphere and plane can be dissected into small triangles without adding extra vertices and that large polygons on a sphere can be dissected into small triangles by adding a single vertex in the interior.

Consider this problem on both the plane and sphere. The difficulty in this problem lies in devising a method that works for all simple polygons, including very general or complex ones, like in Figure 5.3.

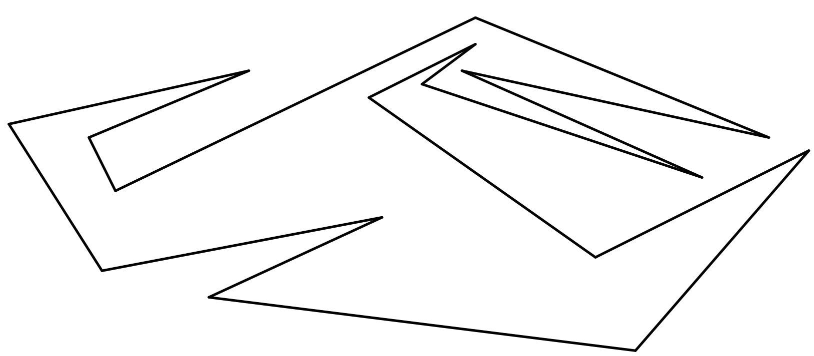

Figure 5.3. General simple polygon.

You may be tempted to try to connect nearby vertices to create triangles, but how do we know that this is always possible? How do you know that in any polygon there is even one pair of vertices that can be joined in the interior? The polygon may be so complex that parts of it get in the way of what you are trying to connect. You might start by giving a convincing argument that there is at least one pair of vertices that can be joined by a segment in the interior of the polygon. In order to see that there is something to prove here, look at Figure 5.4, which shows a polyhedron in 3-space with **no** pair of vertices that can be joined in the interior. The polyhedron consists of eight triangular faces and six vertices. Each vertex is joined by an edge to four of the other vertices and the straight line segment joining it to the fifth vertex lies in the exterior of the polyhedron. Therefore it is impossible to dissect this polyhedron into tetrahedra without adding extra vertices.

This example and some history of the problem are discussed in [**P**: Eves], page 211. Use Computer Exercise 5.2 to display images of this polyhedron, which may then be viewed from various perspectives.

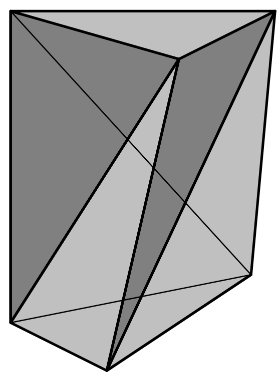

Figure 5.4. Cannot be dissected into tetrahedra without adding vertices.

Note that there is at least one convex vertex (a vertex with interior angle less than π) on every polygon (in fact it is not too hard to see that there must be at least three such vertices). For this and other problems, pick any (straight) line in the exterior of the polygon and parallel transport it towards the polygon until it first touches the polygon. It is easy to see that the line must now be intersecting the polygon at a convex vertex.

PROBLEM 5.3. Gauss-Bonnet for Polygons on a Sphere

DEFINITION: The holonomy of a small simple (geodesic) polygon, $\mathcal{H}(\Gamma)$, is defined as follows:

If you parallel transport a vector (a directed geodesic segment) counterclockwise around the sides of a small simple polygon, then the holonomy of the polygon is the smallest angle measured counterclockwise from the original position of the vector to its final position.

If you walk around a polygon with the interior of the polygon on the left, the exterior angle at a vertex is the change in the direction at that vertex. This change is positive if you turn counterclockwise and negative if you turn clockwise. (See Figure 5.5.)

Show that for Γ, a small simple polygon on a sphere,

$$\mathcal{H}(\Gamma) = A(\Gamma)\, 4\pi/A = A(\Gamma)\, R^{-2} = 2\pi - \Sigma\, \alpha_i\,,$$

where $\Sigma\alpha_i$ is the sum of the exterior angles of the polygon.

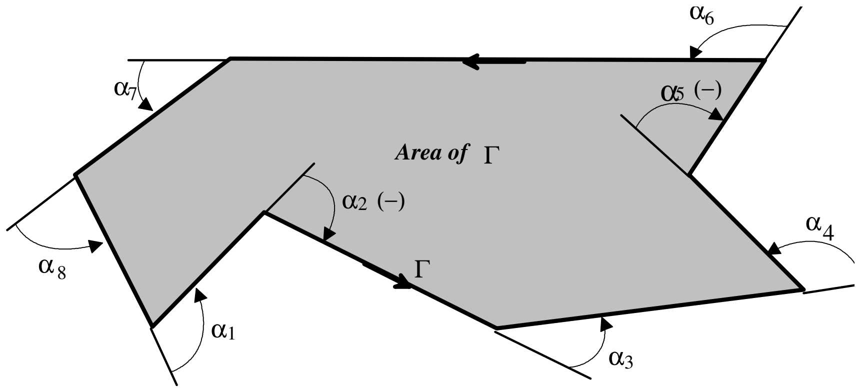

Figure 5.5. Exterior angles as used to determine holonomy.

Outline of a proof: Divide the polygon into small triangles. It is possible to do so by constructing geodesic segments in the interior of the polygon without adding any new vertices (see Problem **5.2**). Do this problem in two steps, and in each of these steps define the holonomy by parallel transporting a vector **V** around Γ, as in Figure 5.2.

First, as you go around the curve keep track of the angle between the parallel transported vector and the tangent vector to the curve and check directly that

$$\mathcal{H}(\Gamma) = 2\pi - \Sigma\, \alpha_i\,.$$

Second, by removing the small triangles one at a time, show that the holonomy of the polygon is the sum of the holonomies of the triangles.

PROBLEM 5.4. Parallel Fields and Intrinsic Curvature

When we parallel transport a vector along a piecewise geodesic curve γ in a surface M, then the angle between the transported vector and the velocity vector changes at every vertex by the amount of the exterior angle. (See Figure 5.6.)

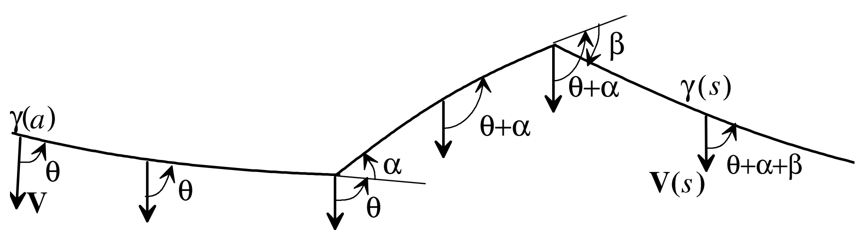

Figure 5.6. Parallel vector field along a piecewise geodesic curve.

Since we do not want the parallel transported vector to change direction as we parallel transport along the curve, we must change the angle between the transported vector and the velocity vector in order to undo the effect of the change in the direction of the velocity vector. If **V** is a vector tangent to the surface at $\gamma(a)$ such that θ is the angle from **V** to the velocity vector $\gamma'(a)$, then we can define

$$\mathbf{V}(s) \equiv P(\gamma,\gamma(a),\gamma(s))\mathbf{V},$$

the ***parallel transport of* V *along* γ *to the point*** $\gamma(s)$, to be the vector (with the same length as **V**) tangent to the surface at $\gamma(s)$ such that the angle from $\mathbf{V}(s)$ to $\gamma'(s)$ is equal to θ plus the sum of the exterior angles of γ between $\gamma(a)$ and $\gamma(s)$. (See Figure 5.6.) If we let s vary over all of γ, then the vectors, $\mathbf{V}(s) = P(\gamma,\gamma(a),\gamma(s))\mathbf{V}$, are called a ***parallel vector field*** along γ. We say that $\mathbf{W}(s)$ is simply a ***vector field*** along γ, if $\mathbf{W}(s)$ is a tangent vector in $T_{\gamma(s)}M$, the tangent space at $\gamma(s)$, for each s.

a. *Show that the following statements are equivalent for a vector field* $\mathbf{V}(s)$ *on a piecewise geodesic curve* γ *parametrized by arc-length*:

i. $\mathbf{V}(s)$ *is a parallel vector field along* γ *as defined above.*

ii. *The derivative* $\frac{d}{ds}\mathbf{V}(s)$ *is perpendicular to the tangent plane at* $\gamma(s)$.

[Hint: The only direction we have to compare the vector $\mathbf{V}(s)$ with is the velocity vector $\gamma'(s)$. So start by looking at $\langle \mathbf{V}(s), \gamma'(s) \rangle$ and differentiating it, noting that the derivative $\frac{d}{ds}\mathbf{V}(s)|_{s=a}$ is the same as the directional derivative $(\gamma'(a))\mathbf{V}(s)$. Use the fact that on a geodesic the angle between the transported vector and the velocity vector is constant. At the corners of a piecewise geodesic curve you can use one-sided derivatives:

$$\frac{d}{ds+}\mathbf{V}(s) = \lim_{h\to 0^+} \frac{1}{h}[\mathbf{V}(s+h) - \mathbf{V}(s)]$$

and

$$\frac{d}{ds-}\mathbf{V}(s) = \lim_{h\to 0^-} \frac{1}{h}[\mathbf{V}(s+h) - \mathbf{V}(s)] \quad .]$$

This motivates an ***extrinsic definition of parallel transport*** for arbitrary curves:

Definition: $\mathbf{V}(s)$ is a ***parallel vector field*** along the curve γ in the surface M if at each point $\gamma(s)$ the derivative $\frac{d}{ds}\mathbf{V}(s)$ is perpendicular to the tangent space $T_{\gamma(s)}M$.

In Chapter 2 we showed that if $\gamma(s)$ is a curve (parametrized by arclength) on the plane, then the magnitude of the curvature $|\kappa(s)|$ is equal to the magnitude of the rate of change of the angle between $\gamma'(s)$ and the horizontal. This is possible on the plane because the horizontal gives a global notion of parallel, and this is precisely what is impossible on more general surfaces. But if we have a parallel vector field $\mathbf{V}(s)$ along γ on a surface, then we might expect that the magnitude of the intrinsic curvature is equal to the rate of change of the angle between $\mathbf{V}(s)$ and $\gamma'(s)$. In fact, we can prove:

b. *If* $\mathbf{V}(s)$ *is a smooth vector field with constant length along the smooth* C^2 *curve* γ *on the surface M, then the following statements are equivalent:*

i. *The derivative* $\frac{d}{ds}\mathbf{V}(s)$ *is perpendicular to the tangent plane at* $\gamma(s)$.

ii. *If* $\theta(s)$ *is the counterclockwise angle from* $\mathbf{V}(s)$ *to the (unit) tangent vector* $\gamma'(s)$, *then*

$$\frac{d}{ds}\theta(s) = \kappa_g(s),$$

where $\kappa_g(s) = \pm|\boldsymbol{\kappa}_g(s)|$ *is the (scalar) geodesic curvature of* γ, *positive if* γ *is turning counterclockwise at* $\gamma(s)$. (See Figure 5.7.)

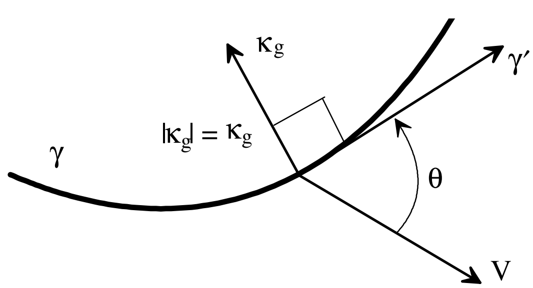

Figure 5.7. Positive geodesic curvature.

[Hint: Differentiate

$$\langle \mathbf{V}(s), \gamma'(s) \rangle = |\mathrm{V}(s)| \cos\theta$$

and note that

$$\gamma''(s) = \kappa(s) = \kappa_g(s) + \kappa_{\mathbf{n}}(s).]$$

We have at the beginning of this section an intrinsic definition of parallel transport for piecewise geodesics curves. We do not yet have an intrinsic definition of geodesic curvature or parallel transport on arbitrary piecewise smooth curves, this deficiency will be corrected in Problems **8.1** and **8.4**. What we do have now is the connection between parallel transport and geodesic curvature given in **5.4b** and we can give an intrinsic description of parallel transport using approximating piecewise geodesics:

c. *Let* γ *be a* C^2 *smooth curve that is parametrized by arclength s and defined for* $0 \le s \le b$. *Let* $\{\gamma_i(s)\}$ *be the sequence of piecewise geodesics curves defined on* $[0,b]$ *which are constant speed on each geodesic segment and the vertices of* γ_i *are*:

$$\gamma_i(0)=\gamma(0),\ \gamma_i(b/2^i)=\gamma(b/2^i),\ \gamma_i(2b/2^i)=\gamma(2b/2^i),\ \gamma_i(3b/2^i)=\gamma(3b/2^i),\ \ldots,$$
$$\gamma_i((2^i-1)b/2^i)=\gamma(((2^i-1)b/2^i),\ \gamma_i(b)=\gamma(b).$$

Show that the parallel transport of $\mathbf{V}$ *along* γ *to* $p = \gamma(b)$ *is the limit (if the limit exists) of the tangent vectors* $\mathbf{V}_i$ *at p which are parallel transports of* $\mathbf{V}$ *along* γ_i. (See Figure 5.8.)

[Hint: Follow these steps:

1. Define $\mathbf{V}'(t) \equiv \lim \mathbf{V}_i'(t)$ and argue that it is perpendicular to the tangent plane $T_{\gamma(t)}M$. You may need to use the fact that the tangent planes vary continuously.
2. Now, use Part **b.ii** to show that there is a unique vector field satisfying this property.]

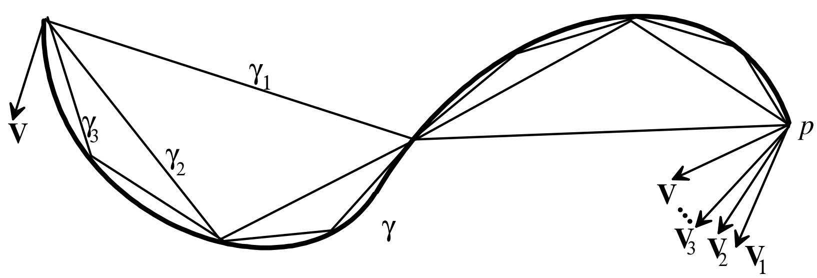

Figure 5.8. Defining parallel transport intrinsically on a smooth curve.

Since we can describe parallel transport along piecewise geodesics intrinsically, **5.4c** gives an ***intrinsic way of defining parallel transport along any smooth curve.***

Now we can give an ***intrinsic definition of intrinsic*** (***geodesic***) ***curvature*** as:

Definition: If $\mathbf{V}(s)$ is a parallel vector field along the curve γ, parametrized by arclength, in the surface M and if, at each point $\gamma(s)$, $\theta(s)$ is the counterclockwise angle from $\mathbf{V}(s)$ to $\gamma'(s)$, then

$$\kappa_g(s) = \tfrac{d}{ds}\theta(s).$$

We will return to this in Problems **8.1** and **8.4**.

d. *Derive the following corollaries:*

i. *Show that the smooth curve γ is a geodesic if and only if $\gamma'(s)$ is a parallel vector field along γ.*

ii. *For a piecewise smooth closed curve γ, the holonomy defined in terms of parallel transport is equal to $\mathcal{H}(\gamma) = n2\pi - [\, \Sigma\, \alpha_i + \int_\gamma \kappa_g(s)\, ds\,]$.*

The $n2\pi$ creates an unavoidable ambiguity unless we know more about the curve. For example, if the curve can be shrunk continuously (homotopied) to an arbitrarily small simple (does not intersect itself) curve, then [since the shrunk curve is indistinguishable from a planar curve (with zero homology) and the homology must varying continuous] we see that:

Theorem 5.4a. *The holonomy of a piecewise smooth closed curve that can be shrunk to a small simple curve is $\mathcal{H}(\gamma) = 2\pi - [\, \Sigma\, \alpha_i + \int_\gamma \kappa_g\,]$.*

We can now extend the Gauss-Bonnet formula (Problem **5.3**) on the sphere to piecewise smooth curves. Let $\{\gamma_i\}$ be the sequence of piecewise geodesic closed curves that is defined in Part **c**, then $\lim_{i\to\infty} \mathcal{H}(\gamma_i) = \mathcal{H}(\gamma)$ and thus we conclude (using Problem **5.3**):

Theorem 5.4b. *For any simple closed piecewise smooth curve γ on the sphere,*

$$\mathcal{H}(\gamma) = 2\pi - [\Sigma\, \alpha_i + \textstyle\int_\gamma \kappa_g] = A(\gamma)\, R^{-2},$$

where $A(\gamma)$ is the area on the left hand side of γ.

Problem 5.5. Holonomy on Surfaces

Find the holonomy of the following regions. Make note of which holonomies are positive, which are negative, and which are zero.

a. *Find the holonomy of a geodesic triangle on a cylinder or cone such that the area of the triangle is finite and does not contain the cone point.*

[Hint: Note that you can cut the cone or cylinder and flatten it to a plane.]

b. *Find the holonomy of an intrinsic circle with center at the cone point on a cone with cone angle α. Note when the holonomy is positive and when negative.*

c. *Find the holonomy of the two regions marked A and B on the torus in Figure 5.9.*

[Hint: The curves α, δ, and γ are geodesics. (*Why?*) The curve β has its extrinsic curvatures tangent to the surface, and therefore, the extrinsic and intrinsic curvatures coincide.]

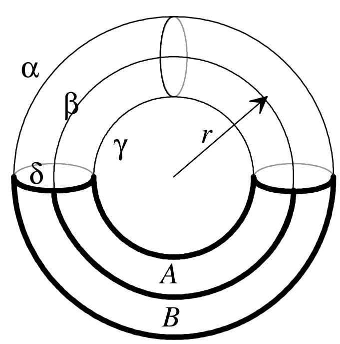

Figure 5.9. Holonomy on a torus.

d. *Find the holonomy, on a strake, of the region bounded by four coordinate curves:*

$$\mathbf{x}(\theta,t),\ \mathbf{x}(t,r),\ \mathbf{x}(\theta+\Delta,t),\ \mathbf{x}(t,r+\Delta).$$

[Hint: Note that the straight horizontal coordinate curves are geodesics and that the helical coordinate curves have their extrinsic curvatures equal to their intrinsic curvatures. Your answer should be negative.]

We can say that regions on a surface have ***negative, zero, or positive curvature*** depending on whether small polygons in the region have holonomy that is negative, zero, or positive. Computer Exercise 5.5 uses the intrinsic formula for intrinsic (Gaussian) curvature developed in Problem **7.1** to allow you display a surface with different colors marking the regions of positive, negative, and zero curvature.

PROBLEM 5.6. Holonomy Explains Foucault's Pendulum

We now give a physical example of holonomy at work. Jean Foucault, about 1851, built a pendulum consisting of a heavy iron ball on a wire 200 feet long. Foucault observed that, as time passed, the pendulum rotated, eventually returning to its original position after a period of

$$T = \frac{24}{\sin\phi},\quad \phi = \text{latitude},$$

the latitude on which the pendulum sits. Most science museums and physics departments have an example of such a pendulum which has come to be known as ***Foucault's Pendulum*** (or the ***Foucault pendulum***).

Imagine a sphere S with the same axis as the earth, which always stays with the earth but does not rotate on its axis. Because of the rotation of the earth, the pendulum moves around this stationary sphere along a latitude circle. Since the wire is very long compared to the small swing of the ball, we can consider the ball to be moving along a tangent line to the sphere. If we pick consistently one of the two possible directions along this line, then the swing plane of Foucault's Pendulum determines a unit tangent vector to the sphere. Over time this defines a vector field along the latitude circle. Since the pendulum is being moved very slowly around the sphere, the consequent centripetal force on the pendulum is negligible. Thus, the only significant force on the pendulum is due to gravity, which operates in a direction that is in the swing plane of the pendulum and perpendicular to the surface.

For the following problems, it may be helpful to know that the radius of the earth is approximately 6360 km.

a. *Argue that the vector field determined by the swing plane of Foucault's Pendulum is parallel along the latitude circle. Thus, after 24 hours (one revolution), the swing plane of Foucault's Pendulum rotates by an amount of the holonomy of the latitude circle minus* 2π.

[Hint: Look at Figure 5.2.]

b. *Calculate the holonomy of the latitude circle with angle* ϕ, *measured from the Equator.*

c. *What should be the period of rotation of Foucault's Pendulum?*

d. *What would happen to Foucault's Pendulum at the North Pole? on the Equator?*

e. *Use the above to calculate the area of the Earth's surface above the latitude* ϕ.

PROBLEM 5.7. Intrinsic Curvature of a Surface

DEFINITION: ***Intrinsic curvature*** $K(p)$ at a point p on any smooth surface can be defined intrinsically as:

$$K(p) = \lim_{n\to\infty} \mathcal{H}(R_n) / A(R_n),$$

where $\{R_n\}$ is a sequence of small (geodesic) polygons that converge to p.

It will follow from Problem **6.4** that this definition does not depend on which sequence of polygons is chosen.

a. *What is the intrinsic curvature at a point on a cylinder? What is the intrinsic curvature at points on a cone?*

b. *What is the intrinsic curvature at points on a sphere?*

***c.** *What is the intrinsic curvature at a point* $\mathbf{x}(a,b)$ *on a strake?*

[Hint: Use Problems **4.5.d** and **5.5.d**. Instead of the closed form expression for the area you may find it easier to leave it (at least partially) in its integral form and then evaluate the limit by using L'Hôpital's Rule.]

d. *What is the intrinsic curvature at a point on the annular hyperbolic plane?*

[Hints: Note that since the surface is constructed the same everywhere (as $\delta \to 0$), it is ***homogeneous*** (that is, intrinsically and geometrically, every point has a neighborhood that is isometric to a neighborhood of any other point). Thus, the intrinsic curvature is constant. There are (at least) two solution methods:

1. Using 4.5e and 5.4d. Consider the region indicated in Figure 5.10. Find areas by using **4.5e**, and find holonomy by using **5.4d**. Note that the geodesic curvature of the annular curves are $1/r$ (*Why?*).

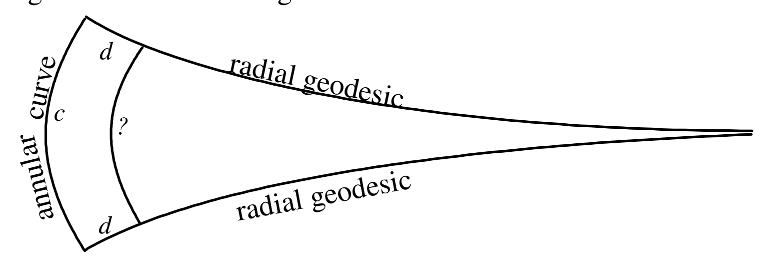

Figure 5.10. Region on the annular hyperbolic plane.

2. Using annulus approximations. (See Figure 5.11.)

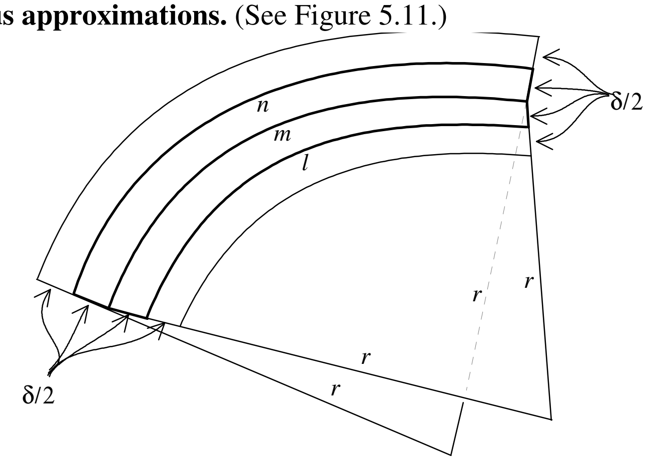

Figure 5.11. Region on annulus approximation of hyperbolic plane.

Pick a region that crosses the circular edge between two strips and then let $\delta \to 0$. Note that the inner bounding arc l and the outer bounding arc n on this region both have radii equal to $r + (\delta/2)$. In calculating the holonomy the exterior angles add up to 2π, and thus, the holonomy is determined by the geodesic curvatures on the two bounding arcs. But the bottom arc l is shorter and contributes positively to the holonomy (*Why?*), and the upper arc n is longer and contributes negatively to the holonomy (*Why?*). Therefore, the holonomy (and intrinsic curvature) is negative.]