Chapter 3

Extrinsic Descriptions of Intrinsic Curvature

When we view a curve on a smooth surface, then we can talk about its ***intrinsic curvature*** (sometimes called ***geodesic curvature***) with respect to the surface. As in Chapter 2, we want the curvature to be the rate of change of the tangent direction (with respect to arc length), except now we want to look at those changes that are intrinsic to the surface (that is, the changes that a 2-dimensional bug on the surface would be able to detect by perceiving only what is on the surface). The notions from Problem **2.3** can be used, but we have to be careful. For example, the machinery of ordinary linear algebra does not directly allow us to compare tangent vectors at two different points on the surface. This is because the vectors tangent to the surface do not, in general, form a vector space, since the difference of two vectors tangent to a surface at different points is not necessarily also a vector tangent to the surface. *Check this out on small portions of a cylinder and sphere.* (See discussion before Problem **2.2**.) We don't know how to talk about vectors at two different points on a sphere as being the "same". North/South/East/West terminology does not work on the sphere because these directions depend on the choice of pole and because at the north pole every direction is south.

On a surface, a curve with no intrinsic curvature is said to be ***intrinsically straight*** (with respect to that surface) or is called a ***geodesic***. These are the curves that a 2-dimensional bug would experience as straight. Along a geodesic, the tangent direction is not changing intrinsically, but of course, in general, it is changing extrinsically. So, we want to be able to find a way of talking about two tangent directions being equal intrinsically (along a curve) when they are not necessarily equal extrinsically — this we will do in Chapter 5.

In this chapter we give extrinsic descriptions of intrinsic curvature. These descriptions make sense to us viewing the surface extrinsically but are of no use to the 2-dimensional bug. Nor will they be useful to us in our experience as intrinsic "bugs" in our 3-dimensional physical universe where we have no extrinsic experience. We will remedy this situation later in Chapter 5 where we will finally arrive at intrinsic descriptions.

PROBLEM 3.1. Smooth Surfaces and Tangent Planes

An (***extrinsic***) ***smooth surface***, M, is a geometric figure in $\mathbf{R}^n$ that is uniformly infinitesimally planar (or flat). We say that a surface is *infinitesimally planar* at the point p in M if, when you zoom in on p, then M will become indistinguishable from a plane, T_pM, called the ***tangent plane at*** p; that is, for every tolerance $\tau > 0$, there is a δ, such that in any f.o.v. centered at p of radius $< \delta$, it is the case that the projection of M onto T_pM is one-to-one and moves points less than $\tau\delta$ in the f.o.v. [Remember that the tolerance is a percentage such that two points are indistinguishable (in the f.o.v.) if their distance apart is less than the tolerance times the diameter of the f.o.v. (See Problem **2.1**.)] The surface is uniformly infinitesimally planar if there is some neighborhood of p such that (for each tolerance τ) the same δ can be used for each point in the neighborhood. Computer Exercise 3.1 may be used to view a surface given parametrically and its tangent plane at a specified point. (See Problem **4.1** for further discussion of the tangent plane.)

***a**[†]*Show that a surface is smooth if and only if it is infinitesimally planar and the tangent planes, $T_p M$, vary continuously with respect to p.* (That is, for every tolerance τ there is a field of view with center p and radius ρ, such that, if p and x are both on the surface in the field of view, then in the field of view each point on T_p is within $\tau\rho$ of the tangent plane T_x.)

[Hint: Use the ideas from Problem **2.2.e**.]

b. *Let M be a surface with a coordinate patch* $\mathbf{x}(x,y)$. *If there exist continuous partial derivatives of* $\mathbf{x}(x,y)$ *that are linearly independent, then M is a smooth surface. Give an example to show that the converse is false.*

[Hint: Use Part **a**.]

c. *Show that cylinders, spheres, and strakes are smooth surfaces and that a cone is a smooth surface except at the cone point.* Argue analutically or geometrically.

d. *If the function f is nonzero, then show that a surface of revolution*

$$\mathbf{x}(\theta, x) = (x, f(x) \cos \theta, f(x) \sin \theta)$$

is a smooth surface if and only if the function f is smooth.

[Hint: Use Part **b** and Problem **2.2**.]

***e.** *Show that the graph of* $g(x, y)$ *is a smooth surface with every tangent plane projecting one-to-one onto the* (x, y)*-plane if and only if the function g has partial derivatives which exist everywhere and are continuous* (that is, g is C^1).

[Hint: Use the ideas from Problem **2.2**.]

We can now prove the following extension of Problem **1.9**:

Corollary. *If a smooth surface is the graph of a function, then the function is continuously differentiable.*

f. *Show that each point on the annular hyperbolic plane has a neighborhood that can be isometrically embedded into 3-space as a smooth surface.*

[Hint: See Problem **1.8** for the description of the annular hyperbolic plane made from annular strips. First, argue that each point on the annular hyperbolic plane is like any other point. Second, start with one of the annular strips and complete it to a full annulus in a plane. Third, construct a surface of revolution by attaching to the inside edge of this annulus other annular strips as described in the construction of the annular hyperbolic plane. Fourth, imagine the width of the annular strips, δ, shrinking to zero. (See Figure 3.1.)]

It is a theorem (see [**Tx:** Henderson/Taimina, Chapter 5] for references and a historical discussion) that there does not exist a C^2 (second derivatives exist and are continuous) isometric embedding of the whole hyperbolic plane into 3-space. However, the annular model can be extended indefinitely and provides (as $\delta \to 0$) an isometric embedding into 3-space. For a small section of the surface it is possible to have the embedding analytic (see Problem **3.1**), but it is possible to see in the model that as you take larger and larger sections, eventually "ruffles" will form, which cause the surface not to be analytic at some points and thus not to be C^2. If you play with a paper model you can see this phenomenon.

[†] Problems and Sections marked with an asterisk (*) are not essential later in this book.

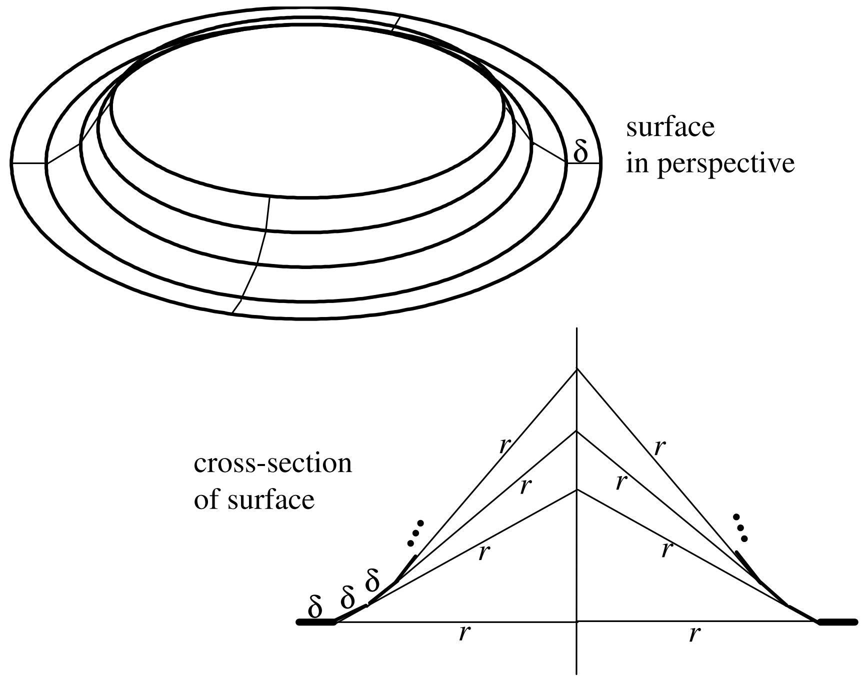

Figure 3.1. Hyperbolic surface of revolution.

The ***normal space*** $N_p M$ at a point p in the smooth surface M is the union of all lines which are perpendicular to $T_p M$ at p. In 3-space the normal space is a line called the ***normal line***.

PROBLEM 3.2. Extrinsic Curvature - Geodesics on Spheres

We now will start to explore curves on smooth surfaces and investigate the relationship between their (extrinsic) curvature vectors and the normal vectors and tangent planes of the surface. You may find two computer exercises helpful. Computer Exercise 3.2a allows you to display a surface and a curve in that surface. Computer Exercise 3.2b allows you to display extrinsic curvature vectors of this curve.

a. *Show that in every case the extrinsic curvature of any geodesic on a cylinder or cone is pointing in a direction that is normal to the surface.*

b. *Convince yourself that for any geodesic on the sphere the extrinsic curvature (or normal vector) must point towards (or directly away from) the center of the sphere.*

[Hint: If the extrinsic curvature were not so aligned, what would be the intrinsic experience of moving along the curve?]

c. *Prove that no other curve is a geodesic on the sphere except for an arc of a great circle.*

[Hint: Since great circles are precisely those curves on the sphere that are the intersection of the sphere with a plane that passes through the center of the sphere, you may find it helpful to use Problem **2.6.a-b**, especially **2.6a** which describes geometrically how the osculating plane changes.]

PROBLEM 3.3. Intrinsic Curvature: Curves on Spheres

a. *How much of what you said about curvature in Problem* **2.3** *will hold for the intrinsic curvature of curves on a cylinder or on a cone?*

[Hint: Remember that some of what we did in Problem **2.3** used Euclidean geometry; and the theorems of Euclidean geometry do not, in general, hold on nonplanar surfaces. However, note that locally the cylinder and cone (away from the cone point) are intrinsically the same as (locally isometric to) a local portion of a Euclidean plane.]

b. *What is the intrinsic curvature of the latitude circle that forms an angle of* α *with the equator* (see Figure 3.2)*? Which latitudes have no intrinsic curvature?*

[Hint: Remember that a sphere is not locally isometric to the plane, but we can look at the cone that is tangent to the sphere along the latitude. Argue that a 2-dimensional bug will experience the intrinsic curvature of the latitude on the sphere as the same curvature as the intrinsic curvature of the latitude on the cone.]

Note that the latitude circle in Figure 3.2 has four different centers: The ***extrinsic center*** (or ***center of extrinsic curvature***) is the point **c** in the plane of the latitude at the center of the circle. The ***intrinsic center*** is the point **b**, which is the center of the circle with respect to the surface of the sphere—it is the center of the circle from the point of view of a 2-dimensional bug on the surface. Then there is the point **a**, which is the intersection of all the planes tangent to the sphere along the latitude circle. The point **a** can be called the ***center of intrinsic curvature*** and s is the ***radius of intrinsic curvature***. Do you see why this name makes sense? What happens to the curve of tangency when you open up the cone? The point **d** is the center of the sphere and is the ***center of normal curvature*** for the latitude circle.

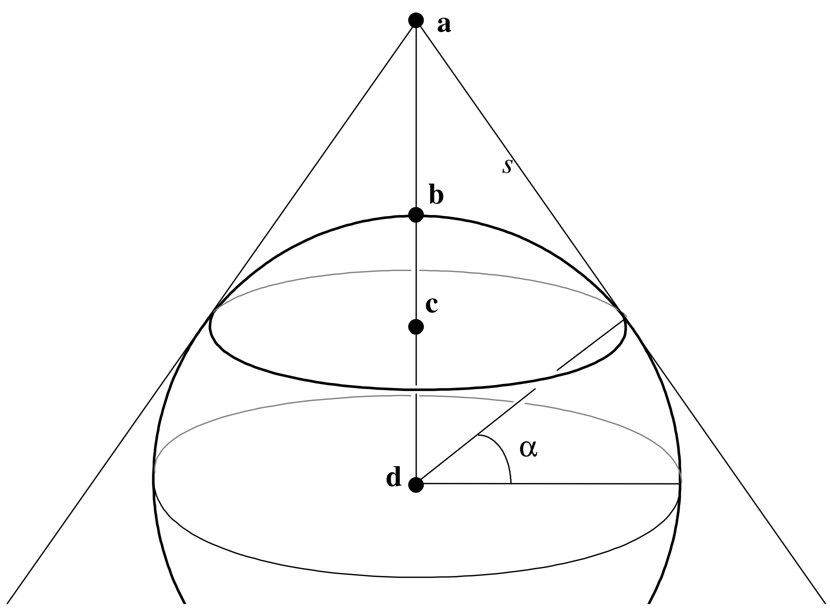

Figure 3.2. Circle on sphere with four centers.

c. *Show that, for all curves on cylinders, cones, and spheres that are extrinsic circles, the intrinsic curvature is the (orthogonal) projection onto the tangent plane of the extrinsic curvature vector.*

d. *Consider the extrinsic curvature vector* κ *at point p of a latitude circle on a sphere of radius R. Show that the projection of* κ *onto the tangent plane has length* $1/s$*. Show that the projection of* κ *onto the normal line to the sphere at p has length equal to* $1/R$*. How do you make sense out of the fact that these projections have the same length no matter which point on the sphere?* (See Figure 3.3.)

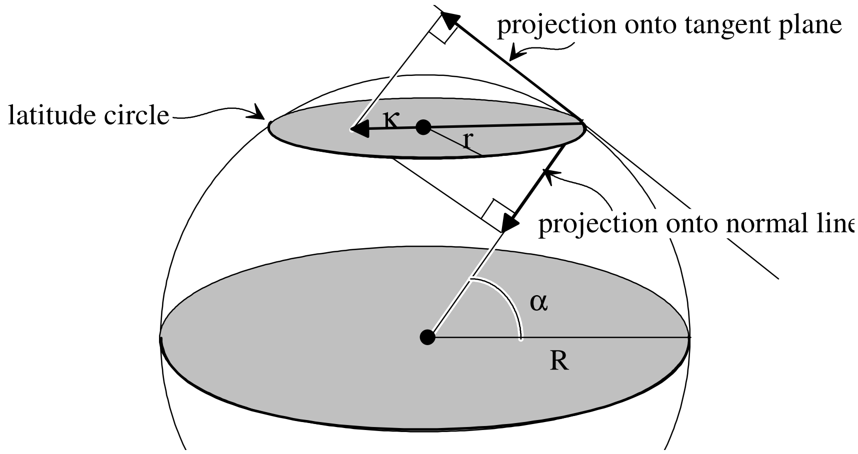

Figure 3.3. The three curvatures on a sphere.

Intrinsic (Geodesic) Curvature

In Problems **3.2-3.3** we saw that a curve γ on a surface S could be called a geodesic on S if and only if the curvature of γ is entirely due to the curvature of the surface S and not due to the intrinsic curving of the curve within the surface. We are, in effect, defining intrinsic curvature as the curvature observed by the bug. On the cone and cylinder we can define this precisely by flattening the cone or cylinder and then finding the curvature in the plane. We now wish to find a more formal definition of intrinsic curvature that will work on all surfaces. For now all that we can do is to find a formal *extrinsic* definition of the intrinsic curvature by defining the intrinsic curvature to be the projection of the extrinsic curvature onto the tangent plane. We eventually want to get intrinsic definitions because when we look at our own experiences in our three-dimensional universe (ignoring time for now) we have no access to extrinsic descriptions. But these intrinsic definitions will have to wait until later chapters.

We want to subtract out the component of the curvature of the curve that is perpendicular (normal) to the surface because this curvature is the curvature due to the surface. Thus we need to express this normal curvature.

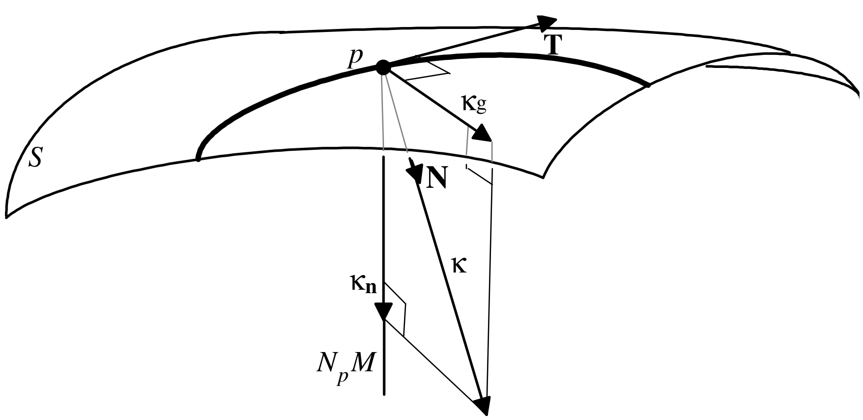

Figure 3.4. The three curvatures of a curve in a smooth surface.

This discussion leads to the following definitions. If γ is a smooth curve on the smooth surface S in $\mathbf{R}^n$, then let $\mathbf{T}$ and $\boldsymbol{\kappa}$ be the unit tangent vector and curvature vector to the curve at a point p on the curve. Define the ***intrinsic curvature** (or **geodesic curvature**)* of γ in S to be

$$\boldsymbol{\kappa}_g = [\text{projection of } \boldsymbol{\kappa} \text{ onto the tangent plane } T_pS\,]$$

and define the ***normal curvature*** of γ in S to be

$$\boldsymbol{\kappa}_n = [\text{projection of } \boldsymbol{\kappa} \text{ onto the normal space } N_pS\,].$$

(See Figure 3.4.) This same picture except for a latitude circle on a sphere is given in Figure 3.5. We see from Problem **3.3d** that on a sphere the normal curvature is constant and, in particular does not depend on the curve. In Problem **4.7** we will show that for arbitrary curves on smooth surfaces the normal curvature also does not depend on the curve but only on the tangent vector of the curve.

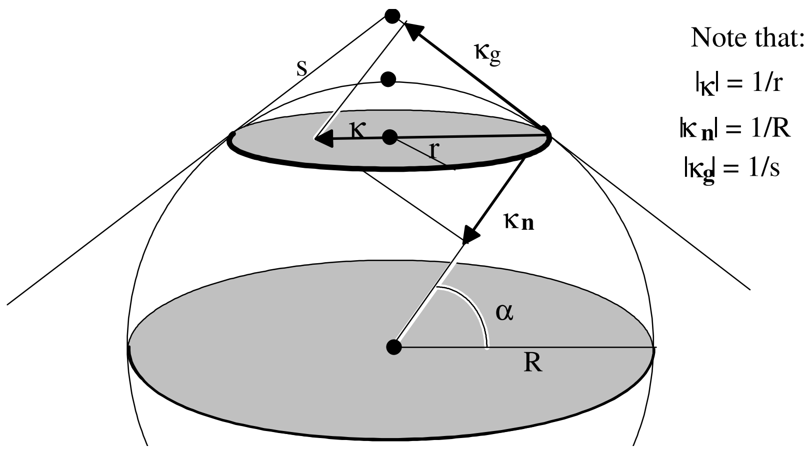

Figure 3.5. The three curvatures of a latitude circle on a sphere.

Computer Exercise 3.3 allows you to display a curve on a surface and the extrinsic, normal, and intrinsic curvature vectors of the curve at a specified point.

The directions of the three different curvatures, $\boldsymbol{\kappa}$, $\boldsymbol{\kappa}_n$, $\boldsymbol{\kappa}_g$, give rise to three different normals: the extrinsic normal to the curve, the normal to the surface, and the intrinsic normal to the curve in the surface.

The curvature $\boldsymbol{\kappa}$ is the extrinsic curvature of the curve as a curve in space without reference to any surface containing it. However, the normal and intrinsic curvatures depend on the surface that one is considering. If the curve lies on two different surfaces then, in general, the normal and intrinsic curvatures will be different. Note that the extrinsic curvature is the vector sum of the intrinsic curvature and the normal curvature:

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}_n + \boldsymbol{\kappa}_g\,.$$

The curve γ is called a ***geodesic*** if and only if $\boldsymbol{\kappa}_g = 0$ at *every* point. Note that a curve is a geodesic if and only if its extrinsic normal lies in the normal space N_pS at every point p. If $\boldsymbol{\kappa}_g = 0$ at an isolated point, then we do not say it is a geodesic at that point. It is a geodesic at a point only if it has zero intrinsic curvature on a whole interval containing the point.

PROBLEM 3.4. Geodesics on Surfaces — the Ribbon Test

a. *Show that if a curve γ on a surface is extrinsically a straight line, then it is a geodesic on the surface. Find as many examples of this as you can.*

b. ***Ribbon Test:*** *Consider a smooth embedding of a ribbon in* $\mathbf{R}^n$ (that is, the ribbon is a smooth surface) *such that the embedding is an isometry* (that is, distances measured along the ribbon do not change — no stretching).

i. *Argue that when the ribbon is laid tangent on a plane the center line is (intrinsically and extrinsically) straight.*

ii. *Argue intrinsically that the center line of the ribbon is a geodesic whenever it is isometrically embedded into* $\mathbf{R}^3$*. What does this say about the extrinsic curvature vector* $\boldsymbol{\kappa}$*?*

iii. *Use* **ii.** *to show extrinsically that a thin ribbon "laid tangent" on a smooth surface will always follow a geodesic. Here "laid tangent" means that the ribbon is tangent to the surface along its center line.*

iv. *Use this ribbon test to find some geodesics on some physical smooth surfaces around your room.*

c. *Show* (using the discussion above) *that, on a surface of revolution*

$$(r(z)\cos\theta,\ r(z)\sin\theta,\ z),$$

the curves with constant θ *are geodesics on the surface. Which generating circles* (z = constant) *are geodesics?*

d. *What geodesics can you find on the annular hyperbolic plane?*

[Hint: Use Parts **a** and **b** above and intrinsic symmetry.]

Note that a ribbon cannot be physically laid tangent through a saddle point (where a curve on the surface that is perpendicular to the ribbon curves in the direction of the outward pointing normal). But if the ribbon is narrow with respect to the curvature of the surface, then it can be laid tangent on the surface to a good approximation. To be precise with the ribbon test, we must imagine a ribbon that can pass through the surface so that it can be precisely tangent to the surface at every point along the centerline.

Ruled Surfaces and the Converse of the Ribbon Test

It is natural now to investigate whether the converse of the Ribbon Test holds: That is, if γ is a geodesic on a surface M, can a ribbon always be "laid tangent" along γ? To answer this question we must first study a special type of surfaces: ***ruled surfaces***.

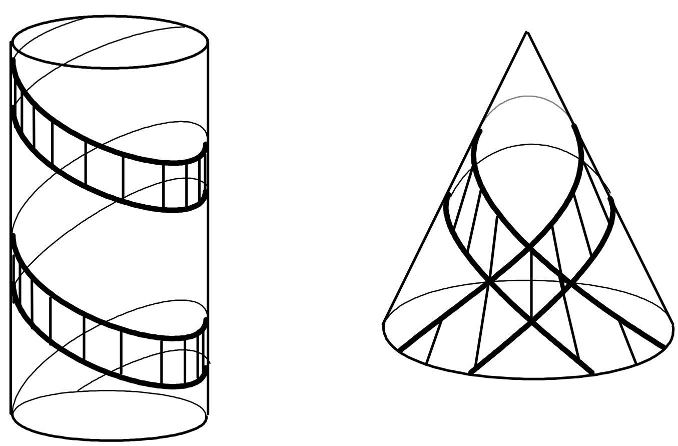

Figure 3.6. Rulings (segments with zero extrinsic curvature) along a ribbon.

Take a paper ribbon and bend it into many different smooth surfaces (no creases) in such a way that the center line is not extrinsically straight. Notice that at every point along the center line of the ribbon there is a direction along which the ribbon is (extrinsically) straight. It is important to notice that these directions of zero (extrinsic) curvature are not necessarily perpendicular to the center line and that these directions vary as you move along the center line. For example, lay the ribbon on a cylinder and on a cone and see which are the directions with zero extrinsic curvature (Figure 3.6).

A smooth surface M is called a ***regular ruled surface*** if on M there is a smooth curve $t \to \alpha(t)$ (parametrized by arclength) and at each point of the curve a unit vector $\mathbf{r}(t)$ such that

1. $\mathbf{r}(t)$ is a differentiable function of t,
2. each point $\alpha(t)$ is in the interior of an (extrinsically) straight segment in M that is parallel to $\mathbf{r}(t)$,
3. there is a (global) coordinate patch for M which can be expressed in the form: $\mathbf{x}(t,s) = \alpha(t) + s\mathbf{r}(t)$, and
4. the vectors, $\mathbf{x}_1(t,s) = \alpha'(t) + s\mathbf{r}'(t)$, $\mathbf{x}_2(t,s) = \mathbf{r}(t)$ form a basis for the tangent space.

The curve α is called the ***directrix*** of the surface, and the extrinsically straight segments are called the ***rulings*** of the surface.

Computer Exercise 3.4 allows you to display ruled surfaces.

Examples of ruled surfaces include cones (away from the cone point), cylinders, strakes, and helicoids.

In Problem **7.6** we will study ruled surfaces and show the following:

On a smooth surface M, if α is a geodesic with nonzero normal curvature at each point, then a ribbon can be laid tangent along α.

Note that if a geodesic has zero normal curvature at every point, then (since the extrinsic curvature is equal to the normal curvature) the geodesic is actually extrinsically straight. For such geodesics we do not need the Ribbon Test. The reader can check that the center line of the helicoid is such an extrinsically straight geodesic along which you cannot lay tangent a ribbon. (See Problem **1.6**.)