Appendix B

Analysis from a Geometric Point of View

B.1. Smooth Functions

We start with the notion of field of view as in Chapter 2, Problems 2.1 and 2.2.

**Definition.** A function \( f \) from a neighborhood \( U \) of a point \( p \) in \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is **smooth** if the graph \( G \) of \( f \) is a smooth \( n \)-submanifold of \( \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m} \) and the projection of each tangent space of \( G \) to \( \mathbb{R}^n \) is one-to-one and onto. An \( n \)-submanifold \( M \) is a subset of an Euclidean space such that \( M \) is **infinitesimally \( n \)-spatial**; that is, for every point \( p \) in \( M \), there is an \( n \)-hyperplane \( T_p \) (called the **tangent space** at \( p \)) such that, for every tolerance \( \tau = (1/N) \), there is a radius \( \rho = (1/M) \), such that in any f.o.v. centered at \( p \) with radius less than \( \rho \), the projection of \( M \) onto \( T_p \) is one-to-one and onto and moves each point less than \( \tau \rho \) (we describe this by saying that if you zoom in on \( p \), then \( M \) and \( T_p \) become indistinguishable). The submanifold is said to be **smooth** if the zooming is uniform in the sense that (for each tolerance) the same \( \rho \) can be used for every point in some neighborhood of \( p \).

**Lemma.** The last sentence above is equivalent to saying that the tangent spaces vary continuously over \( M \).

The proof is essentially the same as Problems 2.2.e and 3.1.e.

**Definition.** If \( f \) is a smooth function from a neighborhood \( U \) in \( \mathbb{R}^n \) to \( \mathbb{R}^m \), then for each \( p \) in \( U \), the **differential**, \( df_p \), is the linear function from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) such that the tangent space \( T_p \) is the graph of the affine linear function \( t(q) = f(p) + df_p(q - p) \). In the terminology of Appendix A.1, we can more accurately say that \( df_p \) is a linear transformation from the tangent space \( (\mathbb{R}^n)_p \) to the tangent space \( (\mathbb{R}^m)_{f(p)} \).

**Theorem B.1.** A function, which maps a neighborhood \( U \) of \( p \) in \( \mathbb{R}^n \) to \( \mathbb{R}^m \), is smooth (in the above geometric sense) if and only if it is \( C^1 \) (in the sense of having for every point \( p \) in \( U \) a differential \( df_p \) that varies continuously with \( p \)).

The proof is essentially the same as the proofs of Problem 2.2.b,c,e and Problem 3.1.e.

B.2. Invariance of Domain

In the next section we will need the following result:

**Theorem B.2.** Any continuous function that maps an open subset of \( n \)-space one-to-one to \( n \)-space is open (that is, the image of every open set is open).

This result is commonly known as **Brouwer’s Invariance of Domain**. It was first proved in about 1910 by L.E.J. Brouwer. The proofs of this theorem involve the topological fields of dimension theory or homology theory, and all require a fair amount of machinery. There are proofs in any of the three books listed in the Bibliography in Section Tp. Topology. In the context of differentiable functions, there is an easier proof, which involves explicitly constructing a continuous inverse (see [An: Strichartz], the proof of Theorem 13.1.1.)
B.3. Inverse Function Theorem

**Theorem B.3.** If \( f \) is a smooth function from \( n \)-space to \( n \)-space such that, for the point \( p_0 = (y_0, f(y_0)) \) on the graph of \( f \), the tangent space \( T_p \) projects one-to-one onto the range, then there is a neighborhood \( U \) of \( x_0 = f(y_0) \) and a smooth function \( g \) from \( U \) to \( n \)-space such that \( f(g(x)) = x \), for every \( x \) in \( U \). Furthermore, \( g \) maps \( U \) one-to-one onto a neighborhood \( V \) of \( y_0 \) and \( g(f(y)) = y \), for every \( y \) in \( V \).

**Proof:** This proof uses the Invariance of Domain but is otherwise shorter and more geometric than the usual proofs in analysis. The only nontrivial things to show are (a) that \( f \) is one-to-one in a neighborhood of \( y_0 \), and (b) that \( f \) maps a neighborhood of \( y_0 \) onto a neighborhood of \( x_0 \).

(a) Suppose \( f \) is not one-to-one in a neighborhood of \( y_0 \). Then there is a sequence of point pairs \( \{a_n, b_n\} \) such that \( f(a_n) = f(b_n) \), for all \( n \), and both sequences \( \{a_n\} \) and \( \{b_n\} \) converge to \( y_0 \). Let \( l_n \) be the line segment joining \( a_n \) to \( b_n \). Applying the Mean Value Theorem for Space Curves (Problem 4.2.b), there is a point \( c_n \) on \( l_n \) between \( a_n \) and \( b_n \) such that a vector tangent to the graph of \( f \) \( l_n \) (and, therefore, tangent also to the graph of \( f \)) projects to a point on the range \( n \)-space. But then the tangent spaces to the graph of \( f \) cannot be varying continuously.

(b) The fact that \( f \) is onto a neighborhood follows from Invariance of Domain (B.2).

For an analytic proof of B.3, see [An: Strichartz], Theorem 13.1.2.

B.4. Implicit Function Theorem

**Theorem B.4.1.** Let \( F(x,y) \) be a smooth function defined in a neighborhood of \( (x_0, y_0) \) in \( \mathbb{R}^n \) and \( \mathbb{R}^m \) taking values in \( \mathbb{R}^n \), with \( F(x_0, y_0) = c \). Then, if the function \( f(y) = F(x_0, y) \) is such that, for the point \( p_0 = (x_0, y_0, f(y_0)) \) on the graph of \( f \), the tangent space \( T_p \) projects one-to-one onto the range, then there is a neighborhood \( U \) of \( x_0 \) and a smooth function \( h \) from \( U \) to \( \mathbb{R}^m \) such that \( h(x_0) = y_0 \) and \( F(x, h(x)) = c \) for every \( x \) in \( U \).

Note that the condition on the graph of \( f \) is equivalent to the analytic condition that \( F(x_0, y_0) \) is invertible, where \( F_x \) is the submatrix of \( dF \) corresponding to using only the partial derivatives with respect to \( y \).

**Proof:** We will describe three different proofs:

1. Define \( f(x, y) = (x, F(x, y)) \). Then it is easy to check that \( f \) satisfies the hypotheses of Theorem B.3. Thus, if there is a smooth inverse function \( g \) defined on a neighborhood of \( (x_0, F(x_0, y_0)) \), then there is a function \( h(x) \) such that \( g(x, c) = (x, h(x)) \). This \( h \) is the desired function.

2. It is possible to construct a direct geometric proof (using Invariance of Domain) along the same lines as the proof of Theorem B.3.

3. There is an analysis proof that explicitly constructs the function \( h \). (See [An: Strichartz], Theorem 13.1.1.)

**Theorem B.4.2.** Let \( F: \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a \( C^1 \) function, and suppose \( dF(x) \) has maximal rank \( n-m \) at every point on a level set \( M = \{x | F(x) = c\} \). Then \( M \) is a \( C^1 \) \( m \)-submanifold of \( \mathbb{R}^n \).

We can prove this as a corollary of B.4.1, (see [An: Strichartz], Theorem 13.2.2.) But there is a more geometric proof. First, change the hypotheses to geometric ones. That \( dF \) has maximal rank at \( p \) is equivalent to the tangent space \( T_{p, F(p)} \) of the graph of \( F \) projecting onto the range space. Now, if we take the inverse of \( c \) under this projection, we get a linear \( m \)-dimensional subspace of the tangent space. The projection of this \( m \)-subspace onto the domain is a tangent space of the level set \( M \). We have proved:

**Theorem B.4.3.** Let \( F: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n \) be a smooth function, such that, for every point \( p = (x, y, c) \) on the graph of the level set \( M = \{x | F(x, y) = c\} \), the tangent space \( T_p \) projects onto the range. Then \( M \) is a \( C^1 \) \( m \)-submanifold of \( \mathbb{R}^{n+m} \).