Chapter 7

Applications of Gaussian Curvature

In this chapter we will use the hard-won result from Chapter 6 to express Gaussian (intrinsic) curvature in local coordinates and to find several intrinsic descriptions of Gaussian curvature. Along the way, we will investigate the exponential map and finally come to some resolution concerning the tension between *shortest* and *straight*.

PROBLEM 7.1. Gaussian Curvature in Local Coordinates

In local coordinates **x**, the second fundamental form

$$\mathrm{II}(\mathbf{X},\mathbf{Y}) = \mathrm{II}(X^1\mathbf{x}_1+X^2\mathbf{x}_2,\ Y^1\mathbf{x}_1+Y^2\mathbf{x}_2)$$

can be written as:

$$\mathrm{II}\left(\begin{pmatrix} X^1 \\ X^2 \end{pmatrix},\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}\right) = \begin{pmatrix} X^1 & X^2 \end{pmatrix}\begin{pmatrix} \langle \mathbf{x}_{11},\mathbf{n}\rangle & \langle \mathbf{x}_{12},\mathbf{n}\rangle \\ \langle \mathbf{x}_{21},\mathbf{n}\rangle & \langle \mathbf{x}_{22},\mathbf{n}\rangle \end{pmatrix}\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}.$$

Now express the **unit** principal directions in these coordinates:

$$\mathbf{T}_1 = \begin{pmatrix} T_1^1 \\ T_1^2 \end{pmatrix} \text{ and } \mathbf{T}_2 = \begin{pmatrix} T_2^1 \\ T_2^2 \end{pmatrix}.$$

Since (from Problem **6.2**),

$$\nabla_{\mathbf{T}_1}\mathbf{n} = -\kappa_1\mathbf{T}_1 \text{ and } \nabla_{\mathbf{T}_2}\mathbf{n} = -\kappa_2\mathbf{T}_2,$$

we have

$$\mathrm{II}(\mathbf{T}_1,\mathbf{T}_1) = \kappa_1,\ \mathrm{II}(\mathbf{T}_2,\mathbf{T}_2) = \kappa_2,\text{ and } \mathrm{II}(\mathbf{T}_1,\mathbf{T}_2) = \mathrm{II}(\mathbf{T}_2,\mathbf{T}_1) = 0.$$

Thus, we can see that:

$$\begin{pmatrix} T_1^1 & T_1^2 \\ T_2^1 & T_2^2 \end{pmatrix}\begin{pmatrix} \langle \mathbf{x}_{11},\mathbf{n}\rangle & \langle \mathbf{x}_{12},\mathbf{n}\rangle \\ \langle \mathbf{x}_{21},\mathbf{n}\rangle & \langle \mathbf{x}_{22},\mathbf{n}\rangle \end{pmatrix}\begin{pmatrix} T_1^1 & T_2^1 \\ T_1^2 & T_2^2 \end{pmatrix} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$

Thus (using the result from matrix algebra that the determinant of a product is the product of the determinants):

$$K = \kappa_1\kappa_2 = \det\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} =$$

$$= \det\begin{pmatrix} T_1^1 & T_1^2 \\ T_2^1 & T_2^2 \end{pmatrix}\det\begin{pmatrix} \langle \mathbf{x}_{11},\mathbf{n}\rangle & \langle \mathbf{x}_{12},\mathbf{n}\rangle \\ \langle \mathbf{x}_{21},\mathbf{n}\rangle & \langle \mathbf{x}_{22},\mathbf{n}\rangle \end{pmatrix}\det\begin{pmatrix} T_1^1 & T_2^1 \\ T_1^2 & T_2^2 \end{pmatrix}.$$

But, also,

$$\langle \mathbf{T}_1, \mathbf{T}_1 \rangle = 1 = \langle \mathbf{T}_2, \mathbf{T}_2 \rangle \text{ and } \langle \mathbf{T}_1, \mathbf{T}_2 \rangle = 0 = \langle \mathbf{T}_2, \mathbf{T}_1 \rangle$$

and thus:

$$\begin{pmatrix} T_1^1 & T_1^2 \\ T_2^1 & T_2^2 \end{pmatrix} \begin{pmatrix} \langle \mathbf{x}_1, \mathbf{x}_1 \rangle & \langle \mathbf{x}_1, \mathbf{x}_2 \rangle \\ \langle \mathbf{x}_2, \mathbf{x}_1 \rangle & \langle \mathbf{x}_2, \mathbf{x}_2 \rangle \end{pmatrix} \begin{pmatrix} T_1^1 & T_2^1 \\ T_1^2 & T_2^2 \end{pmatrix} =$$
$$= \begin{pmatrix} T_1^1 & T_1^2 \\ T_2^1 & T_2^2 \end{pmatrix} (g_{ij}) \begin{pmatrix} T_1^1 & T_2^1 \\ T_1^2 & T_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

and

$$\det \begin{pmatrix} T_1^1 & T_1^2 \\ T_2^1 & T_2^2 \end{pmatrix} \det \begin{pmatrix} T_1^1 & T_2^1 \\ T_1^2 & T_2^2 \end{pmatrix} = (\det(g_{ij}))^{-1}.$$

Therefore, we conclude:

THEOREM. *For any local coordinates* $\mathbf{x}$ *the Gaussian curvature is given by*

$$K = \kappa_1 \kappa_2 = (\det(g_{ij}))^{-1} \det \begin{pmatrix} \langle \mathbf{x}_{11}, \mathbf{n} \rangle & \langle \mathbf{x}_{12}, \mathbf{n} \rangle \\ \langle \mathbf{x}_{21}, \mathbf{n} \rangle & \langle \mathbf{x}_{22}, \mathbf{n} \rangle \end{pmatrix}.$$

***a.** *If the surface has a Monge patch, show that it is the graph of a function* $(x,y,f(x,y))$ *such that the surface is tangent to the* (x,y)*-plane at* $(0,0,f(0,0))$

$$[\text{thus}, f(0,0) = 0 = f_x(0,0) = f_y(0,0)],$$

then show that

$$K = f_{xx} f_{yy} - (f_{xy})^2, \text{ at } p = (0,0,f(0,0)).$$

Warning: This formula does not hold, away from the point p, and this formula is also still extrinsic.

b. *If* $\mathbf{x}(u^1,u^2)$ *is any local coordinates with Riemannian metric matrix*

$$(g_{ij}) = \begin{pmatrix} h^2 & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$h(u^1,u^2) = |\mathbf{x}_1(u^1,u^2)| \text{ and } h^2 = \langle \mathbf{x}_1, \mathbf{x}_1 \rangle,$$

and such that the second coordinate curves $g(s) \equiv \mathbf{x}(u^1,s)$ *are geodesics (parametrized by arclength, then show that the Gaussian curvature at the point* $\mathbf{x}(a,b)$ *is given by*

$$K = -\frac{h_{22}(a,b)}{h(a,b)} = -\frac{1}{h(a,b)} \left(\frac{\partial}{\partial u^2} \left(\frac{\partial}{\partial u^2} h(a, u^2) \right) \right)_{u^2=b}$$

Note that h and its derivatives are intrinsic and thus K is intrinsic. (Note that geodesic rectangular coordinates and geodesic polar coordinates both satisfy the hypotheses of **7.1.b**, and so do the standard local coordinates on the strake.)

Outline of a proof of **7.1.b**:

1. Since

$$h = |\mathbf{x}_1| = \sqrt{\langle \mathbf{x}_1, \mathbf{x}_1 \rangle}\ ,$$

we can calculate that

$$h_2 = \frac{\langle \mathbf{x}_{21}, \mathbf{x}_1 \rangle}{\sqrt{\langle \mathbf{x}_1, \mathbf{x}_1 \rangle}}$$

and that

$$-\frac{h_{22}}{h} = -\frac{(\langle \mathbf{x}_{221}, \mathbf{x}_1 \rangle + \langle \mathbf{x}_{21}, \mathbf{x}_{21} \rangle)\langle \mathbf{x}_1, \mathbf{x}_1 \rangle - \langle \mathbf{x}_{21}, \mathbf{x}_1 \rangle^2}{\langle \mathbf{x}_1, \mathbf{x}_1 \rangle^2}.$$

2. Show that

$$\begin{aligned}\langle \mathbf{x}_{221}, \mathbf{x}_1 \rangle = \langle \mathbf{x}_{122}, \mathbf{x}_1 \rangle = \\ = \mathbf{x}_1 \langle \mathbf{x}_{22}, \mathbf{x}_1 \rangle - \langle \mathbf{x}_{22}, \mathbf{x}_{11} \rangle = 0 - \langle \mathbf{x}_{22}, \mathbf{x}_{11} \rangle,\end{aligned}$$

where the "0" results because $\mathbf{x}_{22}(a,b)$ is the curvature of the curve $\gamma(s) = \mathbf{x}(a,s)$ at $s=b$ (*Why?*) and thus is perpendicular to $\mathbf{x}_1$.

3. We can then calculate that:

$$\begin{aligned}-\frac{h_{22}}{h} &= \frac{\langle \mathbf{x}_{22}, \mathbf{x}_{11} \rangle}{\langle \mathbf{x}_1, \mathbf{x}_1 \rangle} - \frac{\langle \mathbf{x}_{21}, \mathbf{x}_{12} \rangle\langle \mathbf{x}_1, \mathbf{x}_1 \rangle - \langle \mathbf{x}_{21}, \mathbf{x}_1 \rangle^2}{\langle \mathbf{x}_1, \mathbf{x}_1 \rangle^2} = \\ &= \frac{\langle \mathbf{x}_{22}, \mathbf{x}_{11} \rangle}{\langle \mathbf{x}_1, \mathbf{x}_1 \rangle} - \frac{|\mathbf{x}_{21}|^2(1-\cos^2\theta)}{\langle \mathbf{x}_1, \mathbf{x}_1 \rangle},\end{aligned}$$

where θ is the angle from $\mathbf{x}_1$ to $\mathbf{x}_{21}$.

4. At the same time (explain the steps)

$$\begin{aligned}K &= (\det(g_{ij}))^{-1} \times \det\begin{pmatrix} \langle \mathbf{x}_{11}, \mathbf{n} \rangle & \langle \mathbf{x}_{12}, \mathbf{n} \rangle \\ \langle \mathbf{x}_{21}, \mathbf{n} \rangle & \langle \mathbf{x}_{22}, \mathbf{n} \rangle \end{pmatrix} = \\ &= \frac{\langle \mathbf{x}_{11}, \mathbf{n} \rangle\langle \mathbf{x}_{22}, \mathbf{n} \rangle - \langle \mathbf{x}_{12}, \mathbf{n} \rangle^2}{\langle \mathbf{x}_1, \mathbf{x}_1 \rangle} = \\ &= \frac{\langle \mathbf{x}_{22}, \mathbf{x}_{11} \rangle}{\langle \mathbf{x}_1, \mathbf{x}_1 \rangle} - \frac{|\mathbf{x}_{21}|^2 \cos^2\phi}{\langle \mathbf{x}_1, \mathbf{x}_1 \rangle},\end{aligned}$$

where, since $\mathbf{x}_{22}$ is in the same direction as the normal $\mathbf{n}$,

$$\langle \mathbf{x}_{11}, \mathbf{n} \rangle\langle \mathbf{x}_{22}, \mathbf{n} \rangle = \langle \mathbf{x}_{11}, \mathbf{x}_{22} \rangle,$$

and where ϕ is the angle from $\mathbf{x}_{21}$ to $\mathbf{n}$.

5. But $\mathbf{x}_{21}$ lies in the plane of $\mathbf{x}_1$ and $\mathbf{n}$. (*Why?*) Therefore,

$$\theta + \phi = (\text{the angle from } \mathbf{x}_1 \text{ to } \mathbf{n}) = \pi/2 \text{ and } \cos\phi = \sin\theta,$$

and the above expressions imply that the Gaussian curvature is given by

$$K = -\frac{h_{22}}{h}.$$

c. *For geodesic rectangular coordinates* $\mathbf{x}(u^1,u^2)$ *with base a geodesic (parametrized by arclength), show that the function*

$$f(t) = h(u^1,t) = |\mathbf{x}_1(u^1,t)|$$

satisfies for each u^1*:*

$$f(0) = 1 \text{ and } f'(0) = \langle \mathbf{x}_{12},\mathbf{x}_1\rangle = -\langle \mathbf{x}_2,\mathbf{x}_{11}\rangle = 0.$$

Thus, show that, for each u^1, $f(t)$ *has a local maximum at* $t = 0$ *when* $K > 0$ *and a local minimum at* $t = 0$ *when* $K < 0$.

[Hint: Use first semester calculus.]

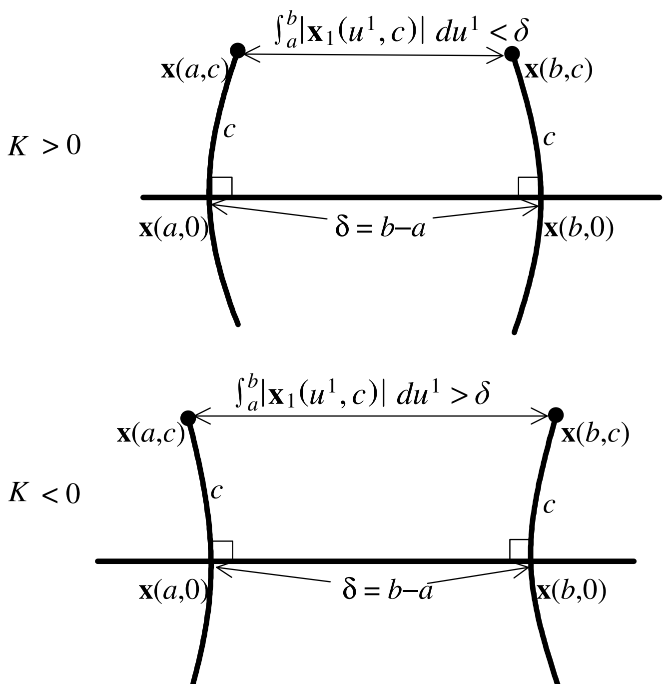

Figure 7.1. Rectangular geodesic coordinates with nonzero Gaussian curvature.

Thus, in a region in which the tangent plane is indistinguishable from the surface, h will appear to be constantly 1, and we will not be able to intrinsically determine K within such a region. However, at a distance c from the base geodesic, we have the pictures in Figure 7.1.

The results in Problem **7.1** were basically known to Gauss in 1827 and were used as the basis for his published proof [**DG**: Gauss] of the Theorema Egregium (Problem **6.4.d**).

**Problem 7.2. Curvature on Sphere, Strake, Catenoid*

a. *Check that the formula*

$$K = -\frac{h_{22}}{h}$$

from Problem **7.1.b** *on a sphere of radius R gives* $K = 1/R^2$. *Does the formula give zero curvature on cylinders and cones?*

[Hint: Use geodesic rectangular coordinates on the sphere, that is, the equator and the longitude must be parametrized by arclength, $u^1 = R\theta$ and $u^2 = R\phi$.]

b. *Calculate the Gaussian curvature for points on a strake.*

c. *Calculate the Gaussian curvature for points on a catenoid and helicoid.* (See Problem **6.6.e**.)

d. *On a sphere of radius R, show that the circumference of a circle that has intrinsic radius r can be expressed by*

$$\textit{Circumference} = 2\pi R \sin(r/R)$$

or, as a power series in $r^2/R^2 = r^2K$,

$$\textit{Circumference} = 2\pi r\,(1 - r^2K(1/6) + r^4K^2(\ldots)).$$

e. *On a sphere of radius R, show that the area of a circular disk that has intrinsic radius r is given by*

$$\begin{aligned} \textit{Area} &= 2\pi R^2 (1 - \cos(r/R)) = \\ &= \pi r^2 (1 - r^2K(1/12) + r^4K^2(\ldots)). \end{aligned}$$

In the next problem we will use geodesic polar coordinates to show that the power series in **7.2.d-e** hold also on any C^2 surface and thus can be used to find more intrinsic descriptions of Gaussian curvature.

Problem 7.3. Circles, Polar Coordinates, and Curvature

Let M be a C^2 surface. An ***intrinsic circle*** (or ***geodesic circle***) in M with radius a and center at p is the collection of all points in M that lie at a distance a along a geodesic from p. If $\mathbf{y}(\theta,r)$ ($\mathbf{y}(0,0)=p$) is geodesic polar coordinates around p, then the intrinsic circle with radius a is just the points of the form $\mathbf{y}(\theta,a)$.

If a is too large, then the circle may be distorted in various ways. For example, the intrinsic circles of radius πR on a sphere of radius R are just a point. However, for a small enough, the intrinsic circle will have a well defined area and circumference. Our goal is to find expressions for the area and circumference that are analogous to those in **7.2.d-e**.

To calculate arclength and area we first need an expression for the Riemannian metric in geodesic polar coordinates. In Problem **4.9** we showed that

$$g_{ij} = \begin{pmatrix} h^2 & 0 \\ 0 & 1 \end{pmatrix}, \text{ where } h(\theta,r) = \left|\mathbf{y}_1(\theta,r)\right|.$$

Now we (you) can

a. *Show that, for fixed* θ*, the third Taylor approximation for h is*

$$h(\theta,r) = r - \tfrac{K(p)r^3}{6} + R(\theta,r), \textit{ where } \lim_{r\to 0} \tfrac{R(\theta,r)}{r^3} = 0 \textit{ uniformly in } \theta.$$

[Hint: If you have forgotten about Taylor polynomials (remember, "polynomials," not "series"), then read about it in your favorite calculus text. If $f(r) \equiv h(\theta,r)$, then you can find $f(0)$ by looking directly at its definition, and $f'(0)$ you can calculate by zooming in on p sufficiently and expanding the two derivative in $f'(0)$ to their definitions. Problem **7.1.b** gives information about $f''(r)$, then take the limit. Find $f'''(0)$

by differentiating the result from **7.1.b** and then (carefully) taking the limit. That the remainder term over r^3 goes to zero, follows from the theory of Taylor polynomials, or can be checked directly and will submit to several applications of L'Hôpital's Rule.]

b. *If $C(r)$ is the circumference of a geodesic circle of (intrinsic) radius r with center at the point p, then show that*

$$C(r) = 2\pi r\,(1 - r^2 K_p(1/6)) + R_C(r),$$

where $\lim_{r\to 0}(1/r^3)R_C(r) = 0$ *and K_p is the Gaussian curvature of the surface at p. Thus, we conclude*

$$K_p = \lim_{r\to 0} 3\frac{2\pi r - C(r)}{\pi r^3}.$$

[Hint: Integrate (see Problem **4.5**) and use **7.1.a**.]

c. *If $A(r)$ is the area of a geodesic circle of (intrinsic) radius r with center at the point p, then show that*

$$A(r) = \pi\, r^2\,(1 - r^2 K_p(1/12)) + R_A(r),$$

where $\lim_{r\to 0}(1/r^4)R_A(r) = 0$. *Thus, we conclude*

$$K_p = \lim_{r\to 0} 12\frac{\pi r^2 - A(r)}{\pi r^4}.$$

[Hint: Integrate (see Problem **4.5**) and use **7.1.a**.]

Notice that the expressions in **7.3.b** and **7.3.c** are additional intrinsic descriptions of Gaussian curvature. In [**DG**: Spivak] these results are attributed to Diquet, Bertrand, and Puiseux in 1848.

PROBLEM 7.4. Exponential Map and Shortest Is Straight

We now return to the issue of the connections between shortest and straight that we first encountered in Problem **1.3**. Recall that we saw then that straight paths were not always the shortest distance between their endpoints and that on the cone with cone angle 450°, even locally (near the cone point) the shortest paths were not straight. First, we investigate all the geodesics which emanate from a single point p.

Let M be a C^2 surface and $T_p M$ be the tangent space (plane) at the point p in M. If $\mathbf{V} \in T_p M$ is a tangent vector, then there is a geodesic

$$\boldsymbol{\gamma}\colon [0,1] \longrightarrow M, \text{ with } \boldsymbol{\gamma}(0) = p,\ \ \boldsymbol{\gamma}'(0) = \mathbf{V},\ \text{ and }\ |\boldsymbol{\gamma}'(t)| = |\mathbf{V}| = \text{constant}.$$

We define the ***exponential*** of **V** to be

$$\exp(\mathbf{V}) = \exp_p(\mathbf{V}) = \boldsymbol{\gamma}(1).$$

The name "exponential" comes from the form it takes on Lie groups or spaces of matrices, see [**DG**: Spivak], Volume 1, Chapter 10. That the exponential is a C^2 map in some neighborhood of p follows from standard theorems about the solutions of differential equations varying smoothly with respect to their initial conditions (see [**DG**: Spivak], Volume 1, Chapter 5, for a detailed discussion).

a. *Show that, if* $\mathbf{U}(\theta)$ *is the unit vector in the direction* θ, *then the function defined by*

$$\mathbf{y}(\theta,r) = \exp(r\mathbf{U}(\theta))$$

is geodesic polar coordinates in some open neighborhood U_p of p. Thus, conclude that all the geodesics in U_p that pass through p are perpendicular to the level curves

$$\{ \exp_p(\mathbf{V}) \mid |\mathbf{V}| = \text{constant} \}.$$

[Hint: Use Problem **4.9**.]

b. *Show that any geodesic $\boldsymbol{\gamma}$ in U_p that joins p to p* is the shortest path joining p to p*.* (This was apparently first proved by J.H.C. Whitehead in 1932.)

Outline of a proof of **7.4.b**:

1. Assume that there is a piecewise smooth path $\boldsymbol{\alpha}$: $[0,b] \to U_p$ from p to p^* that is shorter than γ. Then, using geodesic polar coordinates $\mathbf{y}(\theta,r)$ we can write $\boldsymbol{\alpha}(t) = \mathbf{y}(\theta(t),r(t))$. Differentiate and show that, for $0 < a \le t \le b$,

$$|\boldsymbol{\alpha}'(t)| \ge |r'(t)|,$$

with equality if and only if $\theta'(t) = 0$.

2. Then integrate and show that

$$\int_a^b |\boldsymbol{\alpha}'(t)|dt \ \ge | \, r(a) - r(b) \, |,$$

with equality if and only if $r(t)$ is monotone and $\theta(t)$ is constant.

3. Take the limit as $a \to 0$ and conclude the desired result.

c. *Let $C(a)$ be a circle of radius a and center p. Let $\boldsymbol{\lambda}$ be a path that joins two points, p*, p**, on $C(a)$ and which is the union of two geodesic pieces, which form an angle at p of angle $\phi<\pi$ as in Figure 7.2. Show that, if a is sufficiently small, there is a path $\boldsymbol{\alpha}$ in the interior of $C(a)$ that has length less than $2a$ (= the length of $\boldsymbol{\lambda}$).*

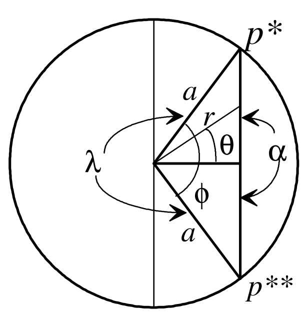

Figure 7.2. A geodesic angle is not shortest.

Outline of a proof of **7.4.c**:

1. Look at the path α marked in the figure with parametrization

$$\boldsymbol{\alpha}(\theta) = \mathbf{y}(\theta,r(\theta)),\ -\phi/2 \le \theta \le \phi/2$$

where

$$r(\theta) = a\frac{\cos\phi/2}{\cos\theta}.$$

Calculate $|\boldsymbol{\alpha}'(\theta)|$.

2. Look at the integral that expresses the length of α:

$$\int_{-\phi/2}^{\phi/2} |\boldsymbol{\alpha}'(\theta)| d\theta \ ,$$

and use the estimate in **7.3.a** when expanding the integrand.

3. Show that

$$\int_{-\phi/2}^{\phi/2} |\boldsymbol{\alpha}'(\theta)| = \int_{-\phi/2}^{\phi/2} \frac{a\cos\phi/2}{\cos^2\theta} \sqrt{1 + a(A(a,\theta)) + \frac{R(\theta,r(\theta))}{a^2}(B(a,\theta))}\, d\theta$$

where $A(a,\theta)$ and $B(a,\theta)$ are bounded for $-\phi/2 \le \theta \le \phi/2$ and $0 < a \le 1$. Thus, for sufficiently small a

$$\sqrt{1 + a(A(a,\theta)) + \frac{R(\theta,r(\theta))}{a^2}(B(a,\theta))} \le C < \frac{1}{\sin\phi/2},$$

and

$$\int_{-\phi/2}^{\phi/2} |\boldsymbol{\alpha}'(\theta)| \le \int_{-\phi/2}^{\phi/2} \frac{a\cos\phi/2}{\cos^2\theta} C d\theta = 2aC\sin\phi/2 < 2a.$$

For the main result of this problem we need a notion of *completeness*:

M is ***geodesically complete*** if every geodesic in M can be extended indefinitely. This is a direct interpretation of Euclid's first postulate which says: "Every straight line can be extended indefinitely."

Now we can prove the main result of this problem, which (together with **7.4.e**) is usually called the Hopf-Rinow-de Rham Theorem (proved in 1931):

d. *If M is **geodesically complete** then any two points can be joined by a geodesic that is the shortest path between them.*

Outline of a proof of **7.4.d**:

1. Let p, q be any two points in M with their distance $d(p,q) = b$. Let C be a circle of radius δ and center p so that $C \subset U_p$. There is a point p^* on C such that

$$d(p^*,q) \le d(x,q), \text{ for all } x \in C.$$

Now $p^* = \exp_p(\delta\mathbf{V})$, for some unit tangent vector $\mathbf{V} \in T_pM$.

$$\text{CLAIM: } \exp_p(b\mathbf{V}) = q;$$

this will show that the geodesic $\boldsymbol{\gamma}(t) = \exp_p(t\mathbf{V})$ is a geodesic of length b joining p to q.

2. The claim will be true (*Why?*) if

$$b \in A \equiv \{\ t \mid d(\boldsymbol{\gamma}(t),q) = b - t\ \}.$$

3. Since every curve from p to q must cross C, we have

$$d(p,q) = \min_{x\in C}[d(p,x) + d(x,q)] = \delta + d(p^*,q).$$

So $d(p^*,q) = b-\delta$ and $\delta \in A$.

4. Let t^* be the least upper bound of all t in A. Then $t^* \in A$. Suppose that $t^* < b$. Let C^* be the circle of radius δ^* around $\gamma(t^*)$, and let q^* be the point on C^* that is closest to q. (See Figure 7.3.)

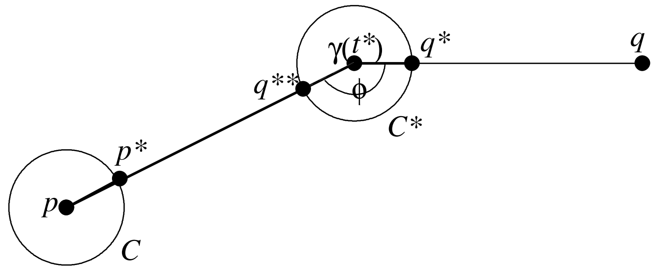

Figure 7.3. Finding shortest path from *p* to *q*.

5. Use **7.4.c** to argue that the angle ϕ in Figure 7.3 must be equal to π and thus t^* is not the least upper bound of elements in A.

There is another notion of completeness that is familiar from analysis:

M is ***Cauchy complete*** (or, simply, ***complete***) if every Cauchy sequence in M converges. A sequence $\{x_i\}$ is a Cauchy sequence if, for every integer m, there is an integer n such that

$$| x_i - x_j | < 1/m, \text{ whenever } i > n \text{ and } j > n.$$

Notice that Cauchy completeness is a local concept while geodesic completeness is a more global notion. Nevertheless, you can prove that:

***e.** *A surface M is Cauchy complete if and only if it is geodesically complete.*

[Hint: Work locally and use the results above, including that the exponential map is continuous.]

Problem 7.5. Surfaces with Constant Curvature

a. *Let M be a surface with constant Gaussian curvature K. Let $\mathbf{x}(u^1,u^2)$ be a geodesic rectangular coordinate chart with base curve a geodesic. Let $\mathbf{y}(\theta,r)$ be geodesic polar coordinates. Show that the Riemannian metric matrix is*

$$(g_{ij}) = \begin{pmatrix} h^2 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\textit{where, for } \mathbf{x}(u^1,u^2),\ h = h(u^1,u^2) = \begin{cases} \cos\sqrt{K}\,u^2, & \textit{if } K \ge 0 \\ \cosh\sqrt{|K|}\,u^2, & \textit{if } K \le 0 \end{cases},$$

$$\textit{and, for } \mathbf{y}(\theta,r),\ h = h(\theta,r) = \begin{cases} K^{-\frac{1}{2}}\sin\sqrt{K}\,r, & \textit{if } K \ge 0 \\ |K|^{-\frac{1}{2}}\sinh\sqrt{|K|}\,r, & \textit{if } K \le 0 \end{cases},$$

[Hint: Use Problems **4.9**, **7.1.b**, **7.1.c**, and **7.3.a** and the theory of second-order linear differential equations.]

b. *Prove that any two surfaces with the same constant Gaussian curvature are locally isometric.*

[Hint: Use geodesic rectangular coordinates (with the base curve a geodesic) on both surfaces and define a map that takes a point on the first surface to the point on the other surface with the same coordinates. With respect to these coordinates, show that the Riemannian metrics of the two surfaces are equal. The arclength of any curve γ on a surface is given by the integral

$$\int |\gamma'(t)|\, dt = \int \sqrt{\langle \gamma'(t), \gamma'(t) \rangle}\ dt\ .$$

Thus, if the Riemannian metrics are the same, then all lengths are the same on the two surfaces.]

c. *Show that on a surface of constant curvature there exist locally: rotations about any point through any angle, translations along any geodesic, and reflections across any geodesic.*

[Hint: Apply your argument in Part **b** to different local geodesic coordinates on the surface.]

Since the annular hyperbolic plane is constructed the same everywhere (as $\delta \to 0$), it is homogeneous (that is, intrinsically and geometrically every point has a neighborhood that is isometric to a neighborhood of any other point). Thus the Gaussian curvature is constant. In addition (if the construction is continued indefinitely), every geodesic can be continued indefinitely in both directions. Thus:

The annular hyperbolic plane has global translations, rotations, and reflections, which are isometries of the whole surface onto itself.

Problem 7.6. Ruled Surfaces and Ribbons

Now we are in a position to finish some details about ruled surfaces and the converse of the Ribbon Test, as was promised at the end of Chapter 3.

As explained at the end of Chapter 3, a ***regular ruled surface*** is a surface with a single coordinate patch of the form

$$\mathbf{x}(t,s) = \alpha(t) + s\mathbf{r}(t),$$

where $\alpha(t)$ is a smooth curve parametrized by arclength and, at each point of the curve, $\mathbf{r}(t)$ is a unit vector such that

1. $\mathbf{r}(t)$ is a differentiable function of t, and
2. each point $\alpha(t)$ is in the interior of an (extrinsically) straight segment in M that is parallel to $\mathbf{r}(t)$, and
3. the vectors, $\mathbf{x}_1(t,s) = \alpha'(t) + s\mathbf{r}'(t)$, $\mathbf{x}_2(t,s) = \mathbf{r}(t)$ form a basis for the tangent space.

a. *Show that a regular ruled surface is* ***developable*** (that is, *isometric to a region in the plane*) *if and only if*

$$[\mathbf{r}(t),\mathbf{r}'(t),\alpha'(t)] = 0, \text{ for all } t,$$

where $[\mathbf{r}(t),\mathbf{r}'(t),\alpha'(t)]$ *denotes the triple product, which by* **A.5.2** *in the Appendix A, is equal to* $\langle \mathbf{r}(t) \times \mathbf{r}'(t), \alpha'(t)\rangle$.

[Hint: Show that in this setting

$$K = 0 \Leftrightarrow \det\begin{pmatrix} \langle \mathbf{x}_{11}, \mathbf{n}\rangle & \langle \mathbf{x}_{12}, \mathbf{n}\rangle \\ \langle \mathbf{x}_{21}, \mathbf{n}\rangle & \langle \mathbf{x}_{22}, \mathbf{n}\rangle \end{pmatrix} = 0,$$

and note that (see **A.5.2**) $\langle \mathbf{V},\mathbf{n}\rangle = [\mathbf{V},\mathbf{x}_1,\mathbf{x}_2]$.]

b. *Let* $\alpha(t)$ *be a smooth curve parametrized by arclength on the surface M. If* α *has nonzero normal curvature* κ_n *at every point, then, for* $|s|$ *sufficiently small, show that*

$$\mathbf{x}(t,s) = \alpha(t) + s\frac{\mathbf{n}(\alpha(t))\times\mathbf{n}'(\alpha(t))}{|\mathbf{n}'(\alpha(t))|}$$

is a developable regular ruled surface, which is tangent to M along α.

[Hint: Check that it is regular and then developable, and then calculate the normal vector to the ruled surface along α. You may need some of the formulas in the *Appendix* **A.5**.]

c. *Show that on a smooth surface M, if α is a geodesic with nonzero normal curvature κ_n at each point, then a ribbon can be laid tangent along α.* (Remember that "laid tangent" means that the ribbon is tangent to the surface along its center line.)

The properties of ruled surfaces discussed in this problem were mostly worked out in the nineteenth century, but the applications to the Ribbon Test (and the Ribbon Test, itself) are, as far as I can tell, first published in this book and were apparently not known (or, at least, not widely known) before.

Problem 7.7. Curvature of the Hyperbolic Plane

What is the Gaussian curvature of the hyperbolic plane constructed out of annular strips? Calculate the Gaussian curvature in three different ways:

a. *using the extrinsic definition from Problem* **6.1** *and using the formulas from Problem* **6.2.f**.

[Hint: To use the extrinsic description you must first find a particular extrinsic embedding of a portion of the surface such as in Problem **3.1.f**.]

b. *using the intrinsic description in terms of local coordinates in Problem 7.1.*

[Hint:

$$K = -h_{22}/h, \text{ where } |\mathbf{x}_1| = h,\ |\mathbf{x}_2| = 1,\ \langle \mathbf{x}_1, \mathbf{x}_2 \rangle = 0.$$

Pick the local coordinates so that the coordinate curves $\mathbf{x}(u^1,b)$ follow the annular strips, and the coordinate curves $\mathbf{x}(a,u^2)$ are perpendicular to the annular strips.]

c. *using the intrinsic calculation from Problem* **5.7.d**.

d. *Discuss the differences between these three methods and how each affects your understanding.*