CHAPTER XVII.

THETA RELATIONS ASSOCIATED WITH CERTAIN GROUPS OF CHARACTERISTICS.

294. FOR the theta relations now to be considered*, the theory of the groups of characteristics upon which they are founded, is a necessary preliminary. This theory is therefore developed at some length. When the contrary is not expressly stated the characteristics considered in this chapter are half-integer characteristics†; a characteristic

$$\tfrac{1}{2}q = \tfrac{1}{2}\begin{pmatrix} q_1', & q_2', & \dots, & q_p' \\ q_1, & q_2, & \dots, & q_p \end{pmatrix}$$

is denoted by a single capital letter, say Q. The characteristic of which all the elements are zero is denoted simply by 0. If R denote another characteristic of half-integers, the symbol $Q+R$ denotes the characteristic, $S=\tfrac{1}{2}s$,

* The present chapter follows the papers of Frobenius, *Crelle*, LXXXIX. (1880), p. 185, *Crelle*, XCVI. (1884), p. 81. The case of characteristics consisting of n-th parts of integers is considered by Braunmühl, *Math. Annal.* XXXVII. (1890), p. 61 (and *Math. Annal.* XXXII. (1888), where the case $n=3$ is under consideration).

To the literature dealing with theta relations the following references may be given: Prym, *Untersuchungen über die Riemann'sche Thetaformel* (Leipzig, 1882); Prym u. Krazer, *Acta Math.* III. (1883); Krazer, *Math. Annal.* XXII. (1883); Prym u. Krazer, *Neue Grundlagen einer Theorie der allgemeinen Thetafunctionen* (Leipzig, 1892), where the method, explained in the previous chapter, of multiplying together the theta series, is fundamental: Noether, *Math. Annal.* XIV. (1879), XVI. (1880), where groups of half-integer characteristics are considered, the former paper dealing with the case $p=4$, the latter with any value of p; Caspary, *Crelle*, XCIV. (1883), XCVI. (1884), XCVII. (1884); Stahl, *Crelle*, LXXXVIII. (1879); Poincaré, Liouville, 1895; beside the books of Weber and Schottky, for the case $p=3$, already referred to (§§ 247, 199), and the book of Krause for the case $p=2$, referred to § 199, to which a bibliography is appended. References to the literature of the theory of the transformation of theta functions are given in chapter XX. In the papers of Schottky, in *Crelle*, CII. and onwards, and the papers of Frobenius, in *Crelle*, XCVII. and onwards, and in Humbert and Wirtinger (*loc. cit.* Ex. iv. p. 340), will be found many results of interest, directed to much larger generalizations; the reader may consult Weierstrass, *Berlin. Monatsber.*, Dec. 1869, and *Crelle*, LXXXIX. (1880), and subsequent chapters of the present volume.

† References are given throughout, in footnotes, to the case where the characteristics are n-th parts of integers. In these footnotes a capital letter, Q, denotes a characteristic whose elements are of the form q'_i/n, or of the form q_i/n, q_i', q_i being integers, which in the 'reduced' case are positive (or zero) and less than n. The abbreviations of the text are then immediately extended to this case, n replacing 2.

whose elements s_i', s_i are given by $s_i'=q_i'+r_i'$, $s_i=q_i+r_i$. The characteristic, $\frac{1}{2}t$, wherein $t_i'\equiv s_i'$, $t_i\equiv s_i$ (mod. 2) and each of $t_1', \dots, t_p$ is either 0 or 1, is denoted by QR. Unless the contrary is stated it is intended in any characteristic, $\frac{1}{2}q$, that each of the elements q_i', q_i is either 0 or 1. If $\frac{1}{2}q$, $\frac{1}{2}r$, $\frac{1}{2}k$ be any characteristics, we use the following abbreviations

$$|Q| = qq' = q_1q_1' + \dots\dots + q_pq_p', \quad |Q, R| = qr' - q'r = \sum_{i=1}^{p} (q_ir_i' - q_i'r_i),$$

$$|Q, R, K| = |R, K| + |K, Q| + |Q, R|, \quad \binom{Q}{R} = e^{\pi i q' r} = e^{\pi i (q_1' r_1 + \dots + q'_p r_p)};$$

further we say that two characteristics are congruent when their elements differ only by integers, and use for this relation the sign $\equiv$. In this sense the sum of two characteristics is congruent to their difference. And we say that two characteristics Q, R are syzygetic or azygetic according as $|Q, R| \equiv 0$ or $\equiv 1$ (mod. 2), and that three characteristics P, Q, R are syzygetic or azygetic according as $|P, Q, R| \equiv 0$ or $\equiv 1$ (mod. 2).

Ex. Prove that the $2p+1$ characteristics arising in § 202 associated with the half periods u^{a, c_1}, u^{a, a_1}, u^{a, c_2}, ..., u^{a, a_p}, $u^{a, c}$ are azygetic in pairs. Further that if any four of these characteristics, A, B, C, D, be replaced by the four, BCD, CAD, ABD, ABC, the statement remains true; and deduce that every two of the characteristics 1, 2, ..., 7 of § 205 are azygetic.

295. A preliminary lemma of which frequent application will be made may be given at once. Let $a_{1,1}, \dots, a_{1,n}, \dots, a_{r,1}, \dots, a_{r,n}$ be integers, such that the r linear forms

$$U_i = a_{i,1}x_1 + \dots\dots + a_{i,n}x_n, \qquad (i = 1, 2, \dots, r),$$

are linearly independent (mod. 2) for indeterminate values of $x_1, \dots, x_n$; then if $\alpha_1, \dots, \alpha_r$ be arbitrary integers, the r congruences

$$U_1 \equiv \alpha_1, \dots, U_r \equiv \alpha_r, \text{ (mod. 2)},$$

have 2^{n-r} sets of solutions* in which each of $x_1, \dots, x_n$ is either 0 or 1. For consider the sum

$$\frac{1}{2^r} \sum_{x_1, \dots, x_n} [1 + e^{\pi i (U_1 - \alpha_1)}] \dots [1 + e^{\pi i (U_r - \alpha_r)}],$$

wherein the 2^n terms are obtained by ascribing to $x_1, \dots, x_n$ every one of the possible sets of values in which each of $x_1, \dots, x_n$ is either 0 or 1. A term in which $x_1, \dots, x_n$ have a set of values which constitutes a solution of the proposed congruences, has the value unity. A term in which $x_1, \dots, x_n$ do not constitute such a solution will vanish; for one at least of its factors will vanish. Hence the sum of this series gives the desired number of sets of

* When the forms $U_1, \dots, U_r$ are linearly independent mod. m, the number of incongruent sets of solutions is m^{n-r}. In working with modulus m we use $\omega = e^{\frac{2i\pi}{m}}$, instead of $e^{i\pi}$; and instead of a factor $1 + e^{\pi i (U_1 - \alpha_1)}$ we use a factor $1 + \mu + \mu^2 + \dots + \mu^{n-1}$, where $\mu = \omega^{U_1 - \alpha_1}$.

solutions of the congruences. Now the general term of the series is typified by such a term as

$$\frac{1}{2^r}\sum_x e^{\pi i(U_1-a_1)+\pi i(U_2-a_2)+\dots+\pi i(U_\mu-a_\mu)},$$

where μ may be 0, or 1, or ..., or p; and this is equal to

$$\frac{1}{2^r}e^{-\pi i(a_1+\dots+a_\mu)}\sum_x e^{\pi i(c_1x_1+\dots+c_nx_n)},$$

where

$$c_i = a_{1,i} + \dots\dots + a_{\mu,i}, \qquad (i = 1, 2, \dots, n),$$

and, therefore, equal to

$$\frac{1}{2^r}e^{-\pi i(a_1+\dots+a_\mu)}(1+e^{\pi ic_1})(1+e^{\pi ic_2})\dots(1+e^{\pi ic_n});$$

now, when $\mu > 0$, one at least of the quantities $c_1, \dots, c_n$ must be $\equiv 1$ (mod. 2), since otherwise the sum of the forms $U_1, \dots, U_\mu$ is $\equiv 0$ (mod. 2), contrary to the hypothesis that the r forms $U_1, \dots, U_r$ are independent (mod. 2); hence the only terms of the summations which do not vanish are those arising for $\mu = 0$, and the sum of the series is

$$\frac{1}{2^r}\sum_x . 1,$$

or 2^{n-r}.

Ex. i. If, of all 2^{2p} half-integer characteristics, $\frac{1}{2}q$, the number of even characteristics be denoted by g, and h be the number of odd characteristics, prove by the method here followed that $g-h$, which is equal to $\Sigma e^{\pi iqq'}$, is equal to 2^p. This equation, with $g+h=2^{2p}$, determine the known numbers* $g=2^{p-1}(2^p+1)$, $h=2^{p-1}(2^p-1)$.

Ex. ii. If $\frac{1}{2}a$ denote any half-integer characteristic other than zero, and $\frac{1}{2}q$ become in turn all the 2^{2p} characteristics, the sum $\Sigma e^{\pi i|A,\,Q|}=\Sigma e^{\pi i(aq'-a'q)}$ vanishes. For it is equal to

$$(1+e^{\pi ia_1})(1+e^{\pi ia_2})\dots\dots(1+e^{-\pi ia_1'})\dots\dots(1+e^{-\pi ia'_p}),$$

and if $\frac{1}{2}a$ be other than zero, one at least of these factors vanishes. On the other hand it is obvious that $\Sigma e^{\pi i|0,\,Q|}=2^{2p}$.

We may deduce the result from the lemma of the text. For by what is there proved there are 2^{2p-1} characteristics for which $|A, Q|\equiv 0$ (mod. 2) and an equal number for which $|A, Q|\equiv 1$.

296. We proceed now to obtain a group of characteristics which are such that every two are syzygetic.

Let P_1 be any characteristic other than zero; it can be taken in $2^{2p}-1$ ways.

Let P_2 be any characteristic other than zero and other than P_1, such that

$$|P_1, P_2| \equiv 0 \text{ (mod. 2)};$$

* Among the n^{2p} incongruent characteristics which are n-th parts of integers, there are $n^{p-1}(n^p+n-1)$ for which $|Q|\equiv 0$ (mod. n), and $n^{p-1}(n^p-1)$ for which $|Q|\equiv r$ (mod. n), when r is not divisible by n.

by the previous lemma (§ 295), P_2 can be taken in $2^{2p-1}-2$ ways; also by the definition, if P_1P_2 be the reduced sum* of P_1, P_2,

$$|P_1, P_1P_2| = |P_1, P_1| + |P_1, P_2| \equiv 0 \text{ (mod. 2)}.$$

Let P_3 be any characteristic, other than one of the four $0, P_1, P_2, P_1P_2$, such that the two congruences are satisfied

$$|P_3, P_1| \equiv 0,\ |P_3, P_2| \equiv 0, \text{ (mod. 2)};$$

then P_3 can be chosen in $2^{2p-2}-2^2$ ways; also, by the definition,

$$|P_3, P_1P_2| = |P_3, P_1| + |P_3, P_2| \equiv 0, \text{ (mod. 2)},$$

and

$$|P_3, P_3P_1| \equiv 0, \text{ etc.}$$

Let P_4 be any characteristic, other than the 2^3 characteristics

$$0, P_1, P_2, P_3, P_1P_2, P_2P_3, P_3P_1, P_1P_2P_3,$$

which is such that

$$|P_4, P_1| \equiv 0,\ |P_4, P_2| \equiv 0,\ |P_4, P_3| \equiv 0, \text{ (mod. 2)};$$

then P_4 can be chosen in $2^{2p-3}-2^3$ ways, and we have

$$|P_2P_3, P_4| = |P_2, P_4| + |P_3, P_4| \equiv 0, \text{ (mod. 2), etc.},$$

and

$$|P_1P_2P_3, P_4| = |P_1, P_4| + |P_2, P_4| + |P_3, P_4| \equiv 0, \text{ (mod. 2)}.$$

Proceeding thus we shall obtain a group of 2^r characteristics,

$$0, P_1, P_2, \ldots, P_1P_2, \ldots, P_1P_2P_3, \ldots,$$

formed by the sums of r fundamental characteristics, and such that every two are syzygetic. The r-th of the fundamental characteristics can be chosen in $2^{2p-r+1}-2^{r-1}=2^{r-1}(2^{2p-2r+2}-1)$ ways; thus we may suppose r as great as p, but not greater. Such a group will be denoted by a single letter, (P); the r fundamental characteristics, $P_1, P_2, P_3, \ldots$, may be called the *basis* of the group. We have shewn that they can be chosen in

$$(2^{2p}-1)(2^{2p-1}-2)(2^{2p-2}-2^2)\ldots(2^{2p-r+1}-2^{r-1})/\underline{|r},$$

or

$$(2^{2p}-1)(2^{2p-2}-1)(2^{2p-4}-1)\ldots(2^{2p-2r+2}-1)\,2^{\frac{1}{2}r(r-1)}/\underline{|r}$$

ways. But all these ways will not give a different group; any r linearly independent characteristics of the group may be regarded as forming a basis of the group. For instance instead of the basis

$$P_1, P_2, \ldots, P_r$$

we may take, as basis,

$$P_1P_2, P_2, \ldots, P_r,$$

wherein P_1P_2 is taken instead of P_1; then P_1 will arise by the combination

* So that the elements of P_1P_2 are each either 0 or $\frac{1}{2}$.

of P_1P_2 and P_2. Hence, the number of ways in which, for a given group, a basis of r characteristics, $P_1', \ldots, P_r'$, may be selected is

$$(2^r-1)(2^r-2)\ldots(2^r-2^{r-1})/\underline{|r},$$

for the first of them, P_1', may be chosen, other than 0, in 2^r-1 ways; then P_2', other than 0 and P_1', in 2^r-2 ways; then P_3' may be chosen, other than 0, P_1', P_2', $P_1'P_2'$, in 2^r-2^2 ways, and so on, and the order in which they are selected is immaterial.

Hence on the whole the number of different groups, of the form

$$0, P_1, P_2, \ldots, P_1P_2, \ldots, P_1P_2P_3, \ldots$$

of 2^r characteristics, in which every two characteristics of the group are syzygetic*, is

$$\frac{(2^{2p}-1)(2^{2p-2}-1)\ldots\ldots(2^{2p-2r+2}-1)}{(2^r-1)(2^{r-1}-1)\ldots\ldots(2-1)}.$$

Such a group may be called a Göpel group of 2^r characteristics. The name is often limited to the case when $r=p$, such groups having been considered by Göpel for the case $p=2$ (cf. § 221, Ex. i.).

297. We now form a set of 2^r characteristics by adding an arbitrary characteristic A to each of the characteristics of the group (P) just obtained; let P, Q, R be three characteristics of the group, and A', A'', A''', the three corresponding characteristics of the resulting set; then

$$|A', A'', A'''| = |AP, AQ, AR| \equiv |P, Q, R| \equiv |Q, R| + |R, P| + |P, Q|, \text{(mod. 2)},$$

as is immediately verifiable from the definition of the symbols; thus the resulting set is such that every three of its characteristics are syzygetic, that is, satisfy the condition

$$|A', A'', A'''| \equiv 0, \text{ (mod. 2)};$$

this set is not a group, in the sense so far employed; we may choose $r+1$ fundamental characteristics A, A_1, ..., A_r, respectively equal to A, AP_1, AP_2, ..., AP_r, and these will be said to constitute the basis of the system; but the 2^r characteristics of the system are formed from them by taking only combinations which involve an *odd* number of the characteristics of the basis. The characteristics of the basis are not necessarily independent; there may, for instance, exist the relation $A + AP_1 \equiv AP_2$, or $A \equiv P_1P_2$. But there can be no relation connecting an *even* number of the characteristics of the basis; for such a relation would involve a relation connecting the set, $P_1, P_2, \ldots, P_r$, of the group before considered, and such a relation was expressly excluded. Hence it follows that there is at most one relation connecting an odd number

* When the characteristics are n-th parts of integers, the number of such syzygetic groups is $(n^{2p}-1)\ldots(n^{2p-2r+2}-1)$ divided by $(n^r-1)\ldots(n-1)$.

of the characteristics of the basis; for two such relations added together would give a relation connecting an even number.

Conversely if $A, A_1, \ldots, A_r$ be any $r+1$ characteristics, whereof no even number are connected by a relation, such that every three of them satisfy the relation

$$|A', A'', A'''| \equiv 0, \text{ (mod. 2)},$$

we can, taking $P_\alpha \equiv A_\alpha A$, obtain r independent characteristics $P_1, \ldots, P_r$, of which every two are syzygetic, and hence, can form such a group (P) of 2^r pairwise syzygetic characteristics as previously discussed. The aggregate of the combinations of an odd number of the characteristics $A, A_1, \ldots, A_r$ may be called a Göpel system* of characteristics. It is such that there exists no relation connecting an even number of the characteristics which compose the system, and every three of the 2^r characteristics of the system satisfy the conditions

$$|A', A'', A'''| \equiv 0, \text{ (mod. 2)}.$$

We shall denote the Göpel system by (AP).

To pass from a definite group, (P), of 2^r pairwise syzygetic characteristics to a Göpel system, the characteristic A may be taken to be any one of the 2^{2p} characteristics. But if it be taken to be any one of the characteristics of the group (P), we shall obtain, for the Göpel system, only the group (P); and more generally, if P denote in turn every one of the characteristics of the group (P), and A be any assigned characteristic, each of the 2^r characteristics AP leads, from the group (P), to the same Göpel system. Hence, from a given group (P) we obtain only 2^{2p-r} Göpel systems. Hence the number of Göpel systems is equal to

$$2^{2p-r}\frac{(2^{2p}-1)(2^{2p-2}-1)\ldots(2^{2p-2r+2}-1)}{(2^r-1)(2^{r-1}-1)\ldots(2-1)}.$$

We shall say that two characteristics, whose difference is a characteristic of the group (P), are congruent, mod. (P). Thus there exist only 2^{2p-r} characteristics which are incongruent to one another, mod. (P).

It is to be noticed that the 2^{2p-r} Göpel systems derived from a given group (P) have no characteristic in common; for if P_1, P_2 denote characteristics of the group, and A_1, A_2 denote two values of the characteristic A, a congruence $A_1P_1 \equiv A_2P_2$ would give $A_2 \equiv A_1P_1P_2$, which is contrary to the hypothesis that A_1 and A_2 are incongruent, mod. (P). Thus the Göpel systems derivable from a given group (P) constitute a division of the 2^{2p} possible characteristics into 2^{2p-r} systems, each of 2^r characteristics. We can however divide the 2^{2p} characteristics into 2^{2p-r} systems based upon any group (Q) of 2^r characteristics; it is not necessary that the characteristics of the group (Q) be syzygetic in pairs.

* By Frobenius, the name Göpel system is limited to the case when $r=p$.

Ex. For $p=2$, $r=2$, the number of groups (P) given by the formula is 15. And the number of Göpel systems derivable from each is 4. We have shewn in Example iv., § 289, Chap. XV., how to form the 15 groups, and shewn how to form the systems belonging to each one. The condition that two characteristics P, Q be syzygetic is equivalent to $|PQ| \equiv |P|+|Q|$ (mod. 2), or in words, two characteristics are syzygetic when their sum is even or odd according as they themselves are of the same or of different character. It is immediately seen that the 15 groups given in § 289, Ex. iv., satisfy this condition. The four systems derivable from any group were stated to consist of one system in which all the characteristics are even and of three systems in which two are even and two odd. We proceed to a generalization of this result.

298. Of the 2^{2p-r} Göpel systems derivable from one group (P), there is a certain definite number of systems consisting wholly of odd characteristics, and a certain number consisting wholly of even characteristics*. We shall prove in fact that when $p > r$ there are $2^{\sigma-1}(2^\sigma+1)$ of the systems which consist wholly of even characteristics, σ being $p-r$; these may then be described as even systems; and there are $2^{\sigma-1}(2^\sigma-1)$ systems which may be described as odd systems, consisting wholly of odd characteristics. When $p=r$, there is one even system, and no odd system. In every one of the $2^{2\sigma}(2^r-1)$ Göpel systems in which all the characteristics are not of the same character, there are as many odd characteristics as even characteristics.

For, if $P_1, \dots, P_r$ be the basis of the group (P), a characteristic A which is such that the characteristics $A, AP_1, \dots, AP_r$ are all either even or odd, must satisfy the congruences

$$|XP_1| \equiv |XP_2| \equiv \dots\dots \equiv |X|, \text{ (mod. 2)}$$

which are equivalent to

$$|X, P_i| \equiv |P_i|, \qquad (i=1, 2, \dots, r),$$

as is immediately obvious. Since, when $|X, P_1| \equiv |P_1|$, and $|X, P_2| \equiv |P_2|$,

$$\begin{aligned} |X, P_1P_2| \equiv |X, P_1| + |X, P_2| &\equiv |X, P_1| + |X, P_2| + |P_1, P_2| \\ &\equiv |P_1| + |P_2| + |P_1, P_2| \equiv |P_1P_2|, \end{aligned}$$

etc., it follows that these r congruences are sufficient, as well as necessary. These congruences have (§ 295) 2^{2p-r} solutions. If A be any solution, each of the 2^r characteristics forming the Göpel system (AP) is also a solution; for it follows immediately from the definition, if P, Q denote any two characteristics of the group, that

$$\begin{aligned} |APQ| &\equiv |A| + |P| + |Q| + |A, P| + |A, Q| + |P, Q| \\ &\equiv |A| + 2|P| + 2|Q| + |P, Q| \\ &\equiv |A|, \end{aligned}$$

because $|P, Q| \equiv 0$. Hence the 2^{2p-r} solutions of the congruences consist of

* This result holds for characteristics which are n-th parts of integers, provided the group (P) consist of characteristics in which either the upper line, or the lower line, of elements, are zeros.

$2^{2p-r}/2^r = 2^{2p-2r}$ characteristics A, and the characteristics derivable therefrom by addition of the characteristics, other than 0, of the group (P); namely they consist of the characteristics constituting 2^{2p-2r} Göpel systems, these systems being all derived from the group (P). In a notation already introduced, the congruences have 2^{2p-2r} solutions which are incongruent (mod. (P)).

Ex. If S be any characteristic which is syzygetic with every characteristic of the group (P), without necessarily belonging to that group, prove that the 2^{2p-2r} characteristics SA are incongruent (mod. P), and constitute a set precisely like the set formed by the characteristics A.

299. Put now $\sigma = p - r$, and consider, of the $2^{2\sigma}$ Göpel systems just derived, each consisting wholly either of odd or of even characteristics, how many there are which consist wholly of odd characteristics and how many which consist wholly of even characteristics. Let h be the number of odd systems, and g the number of even systems. Then we have, beside the equation

$$g + h = 2^{2\sigma},$$

also

$$g - h = 2^{-2r} \sum_R e^{\pi i|R|} \left[1 + e^{\pi i|R, P_1| - \pi i|P_1|} \right] \dots \left[1 + e^{\pi i|R, P_r| - \pi i|P_r|} \right],$$

wherein $P_1, \dots, P_r$ are the basis of the group (P), and R is in turn every one of the 2^{2p} possible characteristics. For, noticing that the congruence $|RP| \equiv |R|$ is the same as $|R, P| \equiv |P|$, it is evident that the element of the summation on the right-hand side has a zero factor when R is a characteristic for which all of $R, RP_1, \dots, RP_r$ are not of the same character, either even or odd, and that it is equal to $2^{-r}e^{\pi i|R|}$ when these characteristics are all of the same character; while, corresponding to any value of R, say $R = A$, for which all of $R, RP_1, \dots, RP_r$ are of the same character, there arise, on the right hand, 2^r values of R, the elements of the Göpel set (AP), for which the same is true.

Now if we multiply out the right-hand side we obtain

$$2^{2r}(g - h) = \sum_R e^{\pi i|R|} + \sum_{P_1, P_2, \dots} \left[\sum_R e^{\pi i|R| + \pi i|R, P_1| + \dots + |R, P_\mu|} \right] e^{-\pi i|P_1| - \dots - \pi i|P_\mu|},$$

wherein $\sum\limits_{P_1, P_2, \dots}$ denotes a summation extending to every set of μ of the characteristics $P_1, \dots, P_\mu$, and μ is to have every value from 1 to r; but we have, since $P_1, P_2, \dots$, are syzygetic in pairs,

$$|R| + |R, P_1| + \dots\dots + |R, P_\mu| \equiv |RP_1 \dots P_\mu| + |P_1| + \dots\dots + |P_\mu|,$$

and therefore

$$\sum_R e^{\pi i|R| + \pi i|R, P_1| + \dots + \pi i|R, P_\mu| - \pi i|P_1| - \dots - \pi i|P_\mu|} = \sum_R e^{\pi i|RP_1 \dots P_\mu|} = \sum_S e^{\pi i|S|},$$

where $S, = RP_1 \dots P_\mu$, will, as R becomes all 2^{2p} characteristics in turn, also become all characteristics in turn; also $\sum\limits_R e^{\pi i|R|} = \sum\limits_S e^{\pi i|S|}$ is immediately seen to be 2^p; it is in fact the difference between the whole number of even and odd characteristics contained in the 2^{2p} characteristics. Hence

$$2^{2r}(g-h) = 2^p\left[1 + r + \frac{r(r-1)}{2!} + \ldots\ldots + 1\right] = 2^p\left[(1+x)^r\right]_{x=1} = 2^{p+r},$$

and therefore $g - h = 2^{p-r} = 2^\sigma$.

This equation, with $g + h = 2^{2\sigma}$, when $\sigma > 0$, determines $g = 2^{\sigma-1}(2^\sigma + 1)$ and $h = 2^{\sigma-1}(2^\sigma - 1)$, and when $\sigma = 0$ determines $g = 1$, $h = 0$.

These results will be compared with the numbers $2^{p-1}(2^p+1)$, $2^{p-1}(2^p-1)$, of the even and odd characteristics, which make up the 2^{2p} possible characteristics.

If P_i denote every characteristic of the group (P) in turn, and P_m denote one characteristic of the bases $P_1, \ldots, P_r$, and R be such a characteristic that the 2^r characteristics RP_i are not all of the same character, at least one of the r quantities $|R, P_m| + |P_m|$ is $\equiv 1$ (mod. 2), and therefore the product

$$\prod_{m=1}^{r}\{1 + e^{\pi i|P_m| + \pi i|R, P_m|}\}$$

is zero. But, in virtue of the congruences,

$$|P_iP_j| \equiv |P_i| + |P_j|, \quad |R, P_i| + |R, P_j| \equiv |R, P_iP_j|,$$

this product is equal to

$$\sum_{i=1}^{2^r} e^{\pi i|P_i| + \pi i|R, P_i|}, \text{ or } e^{-\pi i|R|}\sum_{i=1}^{2^r} e^{\pi i|RP_i|}.$$

Now $e^{\pi i|RP_i|}$ is 1 or -1 according as RP_i is an even or odd characteristic. Hence the system of 2^r characteristics RP_i contains as many odd as even characteristics, and therefore 2^{r-1} of each, unless all its characteristics be of the same character.

300. The $2^{2\sigma}$ Göpel systems thus obtained, each of which consists wholly of characteristics having the same character, either even or odd, have a further analogy with the 2^{2p} single characteristics. We have shewn (§ 202, Chap. XI.) that the 2^{2p} characteristics can all be formed as sums of not more than p of $2p+1$ fundamental characteristics, whose sum is the zero characteristic; we proceed to shew that from the $2^{2\sigma}$ Göpel systems we can choose $2\sigma + 1$ fundamental systems having a similar property for these $2^{2\sigma}$ systems.

Let the $s = 2^{2\sigma}$ Göpel systems be represented by

$$(A_1P), \ldots, (A_sP),$$

the first of them, in a previous notation, consisting of A_1 and all characteristics which are congruent to A_1 for the modulus (P), and similarly with the others. Then we prove that it is possible, from $A_1, \ldots, A_s$ to choose $2\sigma + 1$ characteristics, which we may denote by $A_1, \ldots, A_{2\sigma+1}$, such that every three of them, say A', A'', A''', satisfy the condition

$$|A', A'', A'''| \equiv 1, \text{(mod. 2)};$$

but it is necessary to notice that, if P be any characteristic of the group (P),

$$|A'P, A'', A'''|, \equiv |A', A'', A'''| + |P, A''| + |P, A'''|,$$

is $\equiv |A', A'', A'''|$; for $|P, A''|, \equiv |P|$, is also $\equiv |P, A'''|$; hence, if B', B'', B''' be any three characteristics chosen respectively from the systems $(A'P)$, $(A''P)$, $(A'''P)$, the condition $|A', A'', A'''| \equiv 1$ will involve also $|B', B'', B'''| \equiv 1$; hence we may state our theorem by saying that it is possible, from the $2^{2\sigma}$ Göpel systems, to choose $2\sigma + 1$ *systems*, whereof every three are azygetic.

Before proving the theorem it is convenient to prove a lemma; if B be any characteristic not contained in the group (P), in other words not $\equiv 0$ (mod. (P)), and R become in turn all the $2^{2\sigma}$ characteristics $A_1, \ldots, A_s$, then*

$$\sum_R e^{\pi i |R, B|} = 0.$$

For let a characteristic be chosen to satisfy the $r + 1$ congruences

$$|X, B| \equiv 1, \; |X, P_1| \equiv 0, \ldots, |X, P_r| \equiv 0, \text{(mod. 2)},$$

and, corresponding to any characteristic R which is one of $A_1, \ldots, A_s$, and therefore satisfies the r congruences $|R, P_i| \equiv |P_i|$, take a characteristic $S = RX$; then

$$|S, B| - |R, B| \equiv |X, B| \equiv 1, \text{and } |S, P_i| = |RX, P_i| \equiv |R, P_i| + |X, P_i| \equiv |P_i|,$$

because $|X, P_i| \equiv 0$; hence the characteristics $A_1, \ldots, A_s$ can be divided into pairs, such as R and S, which satisfy the equation $e^{\pi i |S, B|} = -e^{\pi i |R, B|}$. This proves† that $\sum_R e^{\pi i |R, B|} = 0$.

We now prove the theorem enunciated. Let the characteristic A_1 be chosen arbitrarily from the s characteristics $A_1, \ldots, A_s$; this is possible in $2^{2\sigma}$ ways. Let A_2 be chosen, also from among $A_1, \ldots, A_s$, other than A_1; this is possible in $2^{2\sigma} - 1$ ways. Then A_3 must be one of the characteristics $A_1, \ldots, A_s$, other than A_1, A_2, and‡ must satisfy the congruence $|A_1, A_2, X| \equiv 1$. The number of characteristics satisfying these conditions is equal to

$$\tfrac{1}{2} \sum_R [1 - e^{\pi i |A_1, A_2, R|}],$$

* We have proved an analogous particular proposition, that if B be not the zero characteristic, and R be in turn all the 2^{2p} characteristics, $\sum_R e^{\pi i |R, B|} = 0$ (§ 295, Ex. ii.).

† If R be all the 2^{2p} characteristics in turn, $\sum_R e^{\pi i |0, R|} = 2^{2p}$. If P be one of the group (P), and R be one of $A_1, \ldots, A_s$, so that $|R, P| \equiv |P|$, we have $\sum_R e^{\pi i |P, R|} = e^{\pi i |P|} 2^{2\sigma}$.

‡ We do not exclude the possibility $A_3 \equiv A_1 A_2$. Since $|A_1, A_2, A_1 A_2| \equiv |A_1, A_2|$, it is a possibility only if $|A_1, A_2| \equiv 1$.

wherein R is in turn equal to all the characteristics $A_1, \ldots, A_s$. For a term of this series, in which R satisfies the conditions for A_3, is equal to unity*, while for other values of R the terms vanish. Now, since $|A_1, A_2, R| \equiv |R, A_1A_2| + |A_1, A_2|$, the series is equal to

$$2^{2\sigma-1} - \tfrac{1}{2}e^{\pi i|A_1, A_2|}\sum_R e^{\pi i|R, A_1A_2|};$$

the characteristic A_1A_2 cannot be one of the group (P), for if $A_1A_2 = P$, then $A_2 = A_1P$, which is contrary to the hypothesis that $A_1, \ldots, A_s$ are incongruent for the modulus (P); hence by the lemma just proved the sum of the series is $2^{2\sigma-1}$, and A_3 can be chosen in $2^{2\sigma-1}$ ways.

We consider next in how many ways A_4 can be chosen; it must be one of $A_1, \ldots, A_s$ other than A_1, A_2, A_3 and must satisfy the congruences

$$|A_1, A_2, X| \equiv 1, \quad |A_1, A_3, X| \equiv 1,$$

which, in virtue of the congruence $|A_1, A_2, A_3| \equiv 1$, and the identity

$$|A_2, A_3, X| + |A_3, A_1, X| + |A_1, A_2, X| \equiv |A_1, A_2, A_3|,$$

involve also $|A_2, A_3, X| \equiv 1$. The number of characteristics which satisfy these conditions is equal to

$$2^{-2}\sum_R (1 - e^{\pi i|A_1, A_2, R|})(1 - e^{\pi i|A_1, A_3, R|})$$

or

$$2^{2\sigma-2} - 2^{-2}\sum_R e^{\pi i|A_1, A_2, R|} - 2^{-2}\sum_R e^{\pi i|A_1, A_3, R|} + 2^{-2}\sum_R e^{\pi i|A_1, A_2, R| + \pi i|A_1, A_3, R|},$$

where R is in turn equal to every one of $A_1, \ldots, A_s$; hence, in virtue of the lemma proved, using the equations,

$$|A_1, A_2, R| \equiv |A_1, A_2| + |R, A_1A_2|,$$

$$|A_1, A_2, R| + |A_1, A_3, R| \equiv |A_1, A_2| + |A_1, A_3| + |A_2A_3, R|,$$

the number of solutions obtained is $2^{2\sigma-2}$. But we have

$$|A_1A_2A_3, A_1, A_2| \equiv |A_1, A_2| + |A_1A_2A_3, A_1A_2| \equiv |A_1, A_2| + |A_3, A_1A_2| \equiv |A_1, A_2, A_3| \equiv 1,$$

so that $A_1A_2A_3$ also satisfies the conditions.

Now it is to be noticed that, for an odd number of characteristics $B_1, \ldots, B_{2k+1}$, the condition that every three be azygetic excludes the possibility of the existence of any relation connecting an even number of these characteristics, and for an even number of characteristics $B_1, \ldots, B_{2k}$, the condition that every three be azygetic excludes the possibility of the existence of any relation connecting an even number except the relation $B_1B_2 \ldots B_{2k} \equiv 0$. For, B being any one of $B_1, \ldots, B_{2k+1}$ other than $B_1, \ldots, B_{2m}$, we have, as is easy to verify,

$$|B_1B_2 \ldots B_{2m-1}, B_{2m}, B| \equiv |B_1, B_{2m}, B| + |B_2, B_{2m}, B| + \ldots + |B_{2m-1}, B_{2m}, B|,$$

* It is immediately seen that $|A, B, B| \equiv 0$.

so that the left hand is $\equiv 1$; therefore, as $|B_{2m}, B_{2m}, B| \equiv 0$, we cannot have $B_{2m} = B_1 B_2 \dots B_{2m-1}$. This holds for all values of m not greater than k, and proves the statement.

Hence, $2\sigma + 1$ being greater than 4, we cannot have $A_4 = A_1 A_2 A_3$, for we are determining an odd number, $2\sigma + 1$, of characteristics. On the whole, then, A_4 can be chosen in $2^{2\sigma-2} - 1$ ways.

To find the number of ways in which A_5 can be chosen we consider the congruences

$$|A_1, A_2, X| \equiv 1, \quad |A_1, A_3, X| \equiv 1, \quad |A_1, A_4, X| \equiv 1,$$

which include such congruences as $|A_2, A_3, X| \equiv 1$, $|A_2, A_4, X| \equiv 1$, etc. The characteristic A_5 must be one of $A_1, \dots, A_s$, other than A_1, A_2, A_3, A_4; the condition that A_5 be not the sum of any three of A_1, A_2, A_3, A_4 is included in these conditions. The number of ways in which A_5 can be chosen is therefore

$$2^{-3} \sum_R \left(1 - e^{\pi i |A_1, A_2, R|}\right)\left(1 - e^{\pi i |A_1, A_3, R|}\right)\left(1 - e^{\pi i |A_1, A_4, R|}\right),$$

where R is in turn equal to every one of $A_1, \dots, A_s$; making use of the fact that $A_1 A_2 A_3 A_4$ is not $\equiv 0$, we find the number of ways to be $2^{2\sigma-3}$.

Proceeding in this way, we find that a characteristic A_{2m+1} can be chosen in a number of ways equal to the sum of a series of the form

$$2^{-(2m-1)} \sum_R \left[1 - e^{\pi i |A_1, A_2, R|}\right]\left[1 - e^{\pi i |A_1, A_3, R|}\right] \dots \left[1 - e^{\pi i |A_1, A_{2m}, R|}\right],$$

and therefore in $2^{2\sigma-(2m-1)}$ ways, and that a characteristic A_{2m} can be chosen in $2^{2\sigma-(2m-2)} - 1$ ways, the value $A_{2m} = A_1 A_2 \dots A_{2m-1}$ being excluded. In particular $A_{2\sigma}$ can be chosen in $2^2 - 1$ ways, and $A_{2\sigma+1}$ in 2 ways.

To the $2\sigma + 1$ characteristics thus determined it is convenient* to add the characteristic $A_{2\sigma+2} = A_1 A_2 \dots A_{2\sigma+1}$; if A_i, A_j be any two of $A_1, \dots, A_{2\sigma+1}$ we have

$$|A_{2\sigma+2}, A_i, A_j| \equiv |A_i, A_j, A_1| + \dots\dots + |A_i, A_j, A_{2\sigma+1}| \equiv 1,$$

the expressions $|A_i, A_j, A_i|$, $|A_i, A_j, A_j|$ being both zero. We have then the result: *From the* $2^{2\sigma}$ *characteristics* $A_1, \dots, A_s$ *it is possible to choose a set* $A_1, \dots, A_{2\sigma+2}$, *such that every three of them satisfy the condition*

$$|A', A'', A'''| \equiv 1,$$

in

$$\frac{2^{2\sigma}(2^{2\sigma}-1)\, 2^{2\sigma-1}(2^{2\sigma-2}-1) \dots (2^2-1)\, 2}{\lfloor 2\sigma+2} = \frac{2^{2\sigma+\sigma^2}(2^{2\sigma}-1)(2^{2\sigma-2}-1) \dots (2^2-1)}{\lfloor 2\sigma+2}$$

ways; there exists no relation connecting an even number of the characteristics $A_1, \dots, A_{2\sigma+2}$ *except the prescribed condition that their sum is zero;* since the sum of two relations each connecting an odd number is a relation connecting

* In the particular case of § 202, Chap. XI., $A_{2\sigma+2}$ is zero.

an even number, there can be at most* only one independent relation connecting an odd number of the characteristics $A_1, \ldots, A_{2\sigma+2}$. And, as before remarked, to every one of the characteristics $A_1, \ldots, A_{2\sigma+2}$ is associated a Göpel system of 2^r characteristics.

301. The $2^{2\sigma}$ systems $(A_1P), \ldots, (A_sP)$, which have been considered, were obtained by limiting our attention to one group (P) of 2^r pairwise syzygetic characteristics. We are now to limit our attention still further to the sets $A_1, \ldots, A_{2\sigma+2}$ just obtained satisfying the condition that every three are azygetic.

If to any one of the characteristics $A_1, \ldots, A_{2\sigma+2}$, say A_k, we add the characteristic X, the conditions that the resulting characteristic may still be a characteristic of the set $A_1, \ldots, A_s$, are (§ 298) the r congruences $|XA_k, P_i| \equiv |P_i|$, in which $i = 1, \ldots, r$; in virtue of the conditions $|A_k, P_i| \equiv |P_i|$, these are equivalent to the r congruences $|X, P_i| \equiv 0$, which are independent of k; these latter congruences have 2^{2p-r} solutions, but from any solution we can obtain 2^r others by adding to it all the characteristics of the group (P). There are therefore $2^{2p-2r} = 2^{2\sigma}$ congruences X, incongruent with respect to the modulus (P), each of which†, added to the set $A_1, \ldots, A_{2\sigma+2}$, will give rise to a set $A_1', \ldots, A'_{2\sigma+2}$, also belonging to $A_1, \ldots, A_s$. Further $|A_i', A_j', A_k'| \equiv |XA_i, XA_j, XA_k| \equiv |A_i, A_j, A_k| \equiv 1$; and any relation connecting an even number of the characteristics $A_1', \ldots, A'_{2\sigma+2}$ gives a relation connecting the corresponding characteristics of $A_1, \ldots, A_{2\sigma+2}$. Thus the $2^{2\sigma}$ sets derivable from $A_1, \ldots, A_{2\sigma+2}$ have the same properties as the set $A_1, \ldots, A_{2\sigma+2}$.

Hence all the sets $A_1, \ldots, A_{2\sigma+2}$ can be derived from

$$\frac{2^{\sigma^2}(2^{2\sigma}-1)(2^{2\sigma-2}-1)\ldots(2^2-1)}{\lfloor 2\sigma+2}$$

root sets by adding any one of the $2^{2\sigma}$ characteristics X to each characteristic of the *root set*.

302. Fixing attention upon one of these root sets, and selecting arbitrarily $2\sigma+1$ of its characteristics, which shall be those denoted by $A_1, \ldots, A_{2\sigma+1}$, we proceed to shew that of the $2^{2\sigma}$ characteristics X, there is just one such that the characteristics $XA_1, \ldots, XA_{2\sigma+1}$, derived from $A_1, \ldots, A_{2\sigma+1}$, have all the same character, either even or odd. The conditions for this are

$$|XA_1| \equiv |XA_2| \equiv \ldots\ldots \equiv |XA_{2\sigma+1}|,$$

* If the characteristic of which all the elements, except the i-th element of the first line, are zero, be denoted by E_i', and E_i denote the characteristic in which all the elements are zero except the i-th element of the second line, every possible characteristic is clearly a linear aggregate of $E_1', \ldots, E_p', E_1, \ldots, E_p$. Thus when σ has its greatest value, $=p$, there is certainly one relation, at least, connecting any $2\sigma+1$ characteristics.

† It is only in case all the characteristics of the group (P) are even that the values of X can be the characteristics $A_1, \ldots, A_s$.

which are equivalent to the 2σ congruences

$$|X, A_1A_i| \equiv |A_1| + |A_i|, \quad (i = 2, 3, \ldots, (2\sigma+1));$$

if X be a solution of these congruences, and P be any characteristic of the group (P), we have

$$|XP, A_1A_i| \equiv |X, A_1A_i| + |P, A_1| + |P, A_i| \equiv |A_1| + |A_i| + 2|P|,$$

so that XP is also a solution; since the other congruences satisfied by X were in number r, and similarly, associated with any solution, there were 2^r other solutions congruent to one another in regard to the group (P), it follows that the total number of characteristics X satisfying all the conditions is $2^{2p-r-2\sigma-r} = 1$. Thus, as stated, from any $2\sigma+1$ characteristics, $A_1, \ldots, A_{2\sigma+1}$, of a root set, we can derive one set of $2\sigma+1$ characteristics $\bar{A}_1, \ldots, \bar{A}_{2\sigma+1}$, which are all of the same character, their values being of the form $\bar{A}_i = XA_i$.

Starting from the same root set, and selecting, in place of $A_1, \ldots, A_{2\sigma+1}$, another set of $2\sigma+1$ characteristics, say $A_2, \ldots, A_{2\sigma+2}$, we can similarly derive a set of the form

$$X'A_2, \ldots, X'A_{2\sigma+2},$$

consisting of $2\sigma+1$ characteristics of the same character. The question arises whether this can be the same set as $\bar{A}_1, \ldots, \bar{A}_{2\sigma+1}$. The answer is in the negative. For if the set $X'A_2, \ldots, X'A_{2\sigma+2}$ be in some order the same as the set $XA_1, \ldots, XA_{2\sigma+1}$, or the set $XX'A_2, \ldots, XX'A_{2\sigma+2}$ the same as the set $A_1, \ldots, A_{2\sigma+1}$, it follows by addition that $XX'A_1 \equiv A_{2\sigma+2}$ or $XX' \equiv A_1A_{2\sigma+2}$. Thence the set $A_1A_2A_{2\sigma+2}, A_1A_3A_{2\sigma+2}, \ldots, A_1A_{2\sigma+1}A_{2\sigma+2}, A_1$ is the same as $A_1, A_2, \ldots, A_{2\sigma+1}$, or we have 2σ equations of the form $A_1A_iA_{2\sigma+2} \equiv A_j$, in which $i = 2, \ldots, 2\sigma+1$, $j = 2, \ldots, 2\sigma+1$. Since there is no relation connecting an even number of the characteristics $A_1, \ldots, A_{2\sigma+2}$ except the one expressing that their sum is 0, these equations are impossible*.

Similarly the question may arise whether such a set as $\bar{A}_1, \ldots, \bar{A}_{2\sigma+1}$, of $2\sigma+1$ characteristics of the same character, azygetic in threes, subject to no relation connecting an even number, and incongruent for modulus (P), can arise from two different root sets. The answer is again in the negative. For if $A_1, \ldots, A_{2\sigma+1}$, and $B_1, \ldots, B_{2\sigma+1}$ be two sets taken from different root sets, the $2\sigma+1$ conditions $XA_i \equiv X'B_i$, for $i = 1, \ldots, 2\sigma+1$, to which by addition may be added $XA_{2\sigma+2} \equiv X'B_{2\sigma+2}$, shew that the set $B_1, \ldots, B_{2\sigma+2}$ is derivable from the set $A_1, \ldots, A_{2\sigma+2}$ by addition of the characteristic XX' to every constituent. This is contrary to the definition of root sets. Conversely if $A_1', \ldots, A'_{2\sigma+2}$ be any one of the $2^{2\sigma}$ sets which are derivable from the root set $A_1, \ldots, A_{2\sigma+2}$ by equations of the form $A_i' \equiv ZA_i$, the set of $2\sigma+1$

* To the sets $\bar{A}_1, \ldots, \bar{A}_{2\sigma+1}$ and $X'A_2, \ldots, X'A_{2\sigma+2}$ we may adjoin respectively their respective sums. The two sets of $2\sigma+2$ characteristics thus obtained are not necessarily the same. When σ is odd they cannot be the same, as will appear below (§ 303).

characteristics of the same character, say $\bar{A}_1', \dots, \bar{A}'_{2\sigma+1}$, which are derivable from $A_1', \dots, A'_{2\sigma+1}$ by equations of the form $\bar{A}_i' = X'A_i'$, will also be derived from $A_1, \dots, A_{2\sigma+1}$ by the equations $\bar{A}_i' = XA_i$, in which $X = X'Z$.

On the whole then it follows that there are

$$\frac{2^{\sigma^2}(2^{2\sigma}-1)(2^{2\sigma-2}-1)\dots(2^2-1)}{\lfloor 2\sigma+1}$$

different sets, $\bar{A}_1, \dots, \bar{A}_{2\sigma+1}$, of $2\sigma+1$ characteristics of the same character, azygetic in threes, subject to no relation connecting an even number, and incongruent for the modulus (P).

Of the characteristics $\bar{A}_1, \dots, \bar{A}_{2\sigma+1}$ there can be formed

$$(2\sigma+1, 1) + (2\sigma+1, 3) + \dots + (2\sigma+1, 2\sigma+1) = 2^{2\sigma}$$

combinations*, each consisting of an odd number; and, since there is no relation connecting an even number of $\bar{A}_1, \dots, \bar{A}_{2\sigma+1}$, no two of these combinations can be equal. These combinations all belong to the characteristics $A_1, \dots, A_s$, satisfying the r congruences $|X, P_i| \equiv |P_i|$; for

$$|\bar{A}_1\bar{A}_2\dots\bar{A}_{2k-1}, P_i| \equiv |\bar{A}_1, P_i| + \dots + |\bar{A}_{2k-1}, P_i| \equiv |P_i|.$$

And no two of them are congruent in regard to the modulus (P); for a relation of the form

$$\bar{A}_1 \dots \bar{A}_{2k-1} \equiv \bar{A}_m\bar{A}_{m+1}\dots\bar{A}_{m+2\mu}P,$$

wherein P is a characteristic of the group (P), would lead to a relation of the form $\bar{A}_{2\rho} = \bar{A}_1\bar{A}_2\dots\bar{A}_{2\rho-1}P$, and thence give $|\bar{A}_1\dots\bar{A}_{2\rho-1}P, \bar{A}_{2\rho}, \bar{A}_{2\rho+1}| \equiv 0$, whereas

$$\begin{aligned}|\bar{A}_1\dots\bar{A}_{2\rho-1}P, \bar{A}_{2\rho}, \bar{A}_{2\rho+1}| &\equiv |\bar{A}_1\dots\bar{A}_{2\rho-1}, \bar{A}_{2\rho}, \bar{A}_{2\rho+1}| + |\bar{A}_{2\rho}, P| + |\bar{A}_{2\rho+1}, P| \\ &\equiv |\bar{A}_1\dots\bar{A}_{2\rho-1}, \bar{A}_{2\rho}, \bar{A}_{2\rho+1}| \\ &\equiv |\bar{A}_1, \bar{A}_{2\rho}, \bar{A}_{2\rho+1}| + \dots + |\bar{A}_{2\rho-1}, \bar{A}_{2\rho}, \bar{A}_{2\rho+1}| \equiv 1.\end{aligned}$$

Thus the $2^{2\sigma}$ combinations, each consisting of an odd number of the characteristics $\bar{A}_1, \dots, \bar{A}_{2\sigma+1}$, are in fact the characteristics $A_1, \dots, A_s$. We† call the set $\bar{A}_1, \dots, \bar{A}_{2\sigma+1}$ a *fundamental set.* We may associate therewith the characteristic $\bar{A}_{2\sigma+2} = \bar{A}_1\dots\bar{A}_{2\sigma+1}$, which is azygetic with every two of the set $\bar{A}_1, \dots, \bar{A}_{2\sigma+1}$; the case in which it has the same character as these will appear in the next article. And it should be remarked that the argument establishes, for the $2^{2\sigma}$ Göpel systems $(A_1P), \dots, (A_sP)$, the existence of fundamental sets, $(\bar{A}_1P), \dots, (\bar{A}_{2\sigma+1}P)$, which are Göpel systems, by the odd combinations of the constituents of which, the constituents of the systems $(A_1P), \dots, (A_sP)$ can be represented.

* Where (n, k) denotes $n(n-1)\dots(n-k+1)/k!$

† By Frobenius the term Fundamental Set is applied to any $2\sigma+2$ characteristics (incongruent mod. (P)) of which every three are azygetic.

303. The characteristics $\overline{A}_1, \ldots, \overline{A}_{2\sigma+1}$ have been derived to have the same character. We proceed to shew now, in conclusion, that this character is the same for every one of the possible fundamental sets, and depends only on σ. Let $\left(\frac{\sigma}{4}\right)$ be the usual sign which is $+1$ or -1 according as σ is a quadratic residue of 4 or not, in other words, $\left(\frac{\sigma}{4}\right)=1$ when σ is $\equiv 1$ or $\equiv 0$ (mod. 4), and $\left(\frac{\sigma}{4}\right)=-1$ when σ is $\equiv 2$ or $\equiv 3$ (mod. 4); then the character of the sets $\overline{A}_1, \ldots, \overline{A}_{2\sigma+1}$ is $\left(\frac{\sigma}{4}\right)$, that is, $\overline{A}_1, \ldots, \overline{A}_{2\sigma+1}$ are even when $\left(\frac{\sigma}{4}\right)=+1$ and are otherwise odd, and the character of the sum $\overline{A}_{2\sigma+2}=\overline{A}_1 \ldots \overline{A}_{2\sigma+1}$ is $e^{\pi i\sigma}\left(\frac{\sigma}{4}\right)$. Or, we may say

when $\sigma \equiv 1$ (mod. 4), $\overline{A}_1, \ldots, \overline{A}_{2\sigma+1}$ are even, $\overline{A}_{2\sigma+2}$ is odd;

when $\sigma \equiv 0$, $\overline{A}_1, \ldots, \overline{A}_{2\sigma+1}$ are even, $\overline{A}_{2\sigma+2}$ is even,

when $\sigma \equiv 2$ (mod. 4), $\overline{A}_1, \ldots, \overline{A}_{2\sigma+1}$ are odd, $\overline{A}_{2\sigma+2}$ is odd;

when $\sigma \equiv 3$, $\overline{A}_1, \ldots, \overline{A}_{2\sigma+1}$ are odd, $\overline{A}_{2\sigma+2}$ is even.

For if $\overline{A}_1, \ldots, \overline{A}_{2\sigma+1}$ be all of character ϵ we have

$$|\overline{A}_1\overline{A}_2 \ldots \overline{A}_{2k+1}| \equiv |\overline{A}_1| + \ldots + |\overline{A}_{2k+1}| + \Sigma\, |\overline{A}_i, \overline{A}_j|,$$

where $\overline{A}_i, \overline{A}_j$ consist of every pair from $\overline{A}_1, \ldots, \overline{A}_{2k+1}$; also

$$(2k-1)\, \Sigma\, |\overline{A}_i, \overline{A}_j| = \Sigma\, |\overline{A}_i, \overline{A}_j, \overline{A}_h|,$$

where $\overline{A}_i, \overline{A}_j, \overline{A}_h$ consist of every triad from $\overline{A}_1, \ldots, \overline{A}_{2k+1}$; hence, since $|\overline{A}_i, \overline{A}_j, \overline{A}_h| \equiv 1$, and, as is easily seen, $n(n-1)(n-2)/3!$ is even or odd according as n is of the form $4m+1$ or $4m+3$, it follows that $\Sigma\, |\overline{A}_i, \overline{A}_j|$ is even or odd according as $2k+1$ is of the form $4m+1$ or $4m+3$; therefore $\overline{A}_1\overline{A}_2 \ldots \overline{A}_{2k+1}$ has the character ϵ or $-\epsilon$ according as $2k+1 \equiv 1$ or $\equiv 3$ (mod. 4). Thus the number of combinations of an odd number from $\overline{A}_1, \ldots, \overline{A}_{2\sigma+1}$ which have the character ϵ is

$$\begin{aligned}(2\sigma+1, 1)&+(2\sigma+1, 5)+(2\sigma+1, 9)+\ldots \\ &= \tfrac{1}{4}\left\{(1+x)^{2\sigma+1}-(1-x)^{2\sigma+1}+i(1-ix)^{2\sigma+1}-i(1+ix)^{2\sigma+1}\right\}_{x=1} \\ &= 2^{2\sigma-1}+2^{\sigma-\frac{1}{2}}\sin\frac{2\sigma+1}{4}\pi;\end{aligned}$$

this number is $2^{2\sigma-1}+2^{\sigma-1}$ when $\sigma \equiv 0$ or $\sigma \equiv 1$ (mod. 4); otherwise it is $2^{2\sigma-1}-2^{\sigma-1}$; now we have shewn (§ 298) that the characteristics $A_1, \ldots, A_s$ contain respectively $2^{2\sigma-1}+2^{\sigma-1}$, $2^{2\sigma-1}-2^{\sigma-1}$ even and odd characteristics, and (§ 302) that every one of $A_1, \ldots, A_s$ can be formed as an odd combination from $\overline{A}_1, \ldots, \overline{A}_{2\sigma+1}$; hence $\epsilon=+1$ when $\sigma \equiv 0$ or $\sigma \equiv 1$ (mod. 4), and otherwise $\epsilon=-1$; this agrees with the statement made. Further, by the same argument $\bar{A}_1\bar{A}_2\ldots\bar{A}_{2\sigma+1}$ has the character ϵ or $-\epsilon$ according as $2\sigma+1\equiv 1$ or $\equiv 3$ (mod. 4); and this leads to the statement made for $\bar{A}_{2\sigma+2}$.

The reader will find it convenient to remember that the combinations, from the fundamental set $\bar{A}_1, \ldots, \bar{A}_{2\sigma+1}$, consisting of 1, 5, 9, 13, ... of them, are all of the same character, and the combinations consisting of 3, 7, 11, ... are all of the opposite character.

Ex. If $A_1, \ldots, A_{2p+1}$ be half-integer characteristics azygetic in pairs, and S be the sum of the odd ones of these, prove that a characteristic formed by adding S to a sum of any $p+r$ characteristics of these is even when $r\equiv 0$ or $\equiv 1$ (mod. 4), and odd when $r\equiv 2$ or $\equiv 3$ (mod. 4). (Stahl, *Crelle*, LXXXVIII. (1879), p. 273.)

304. It is desirable now to frame a connected statement of the results thus obtained. It is possible, in

$$(2^{2p}-1)(2^{2p-2}-1)\ldots(2^{2p-2r+2}-1)/(2^r-1)(2^{r-1}-1)\ldots(2-1)$$

ways, to form a group,

$$0,\ P_1,\ P_2,\ \ldots,\ P_1P_2,\ \ldots,\ P_1P_2P_3,\ \ldots$$

of 2^r characteristics, consisting of the combinations of r independent characteristics $P_1, \ldots, P_r$, such that every two characteristics P, P' of the group are syzygetic, that is, satisfy the congruence $|P, P'|\equiv 0$, (mod. 2). Such a group is denoted by (P), and two characteristics whose difference is a characteristic of the group are said to be congruent for the modulus (P).

From such a group (P), by adding the same characteristic A to each constituent, we form a system, which we call a Göpel system, consisting of the combinations of an odd number of $r+1$ characteristics $A, AP_1, \ldots, AP_r$, among an even number of which there exists no relation; this system is such that every three of its constituents, say L, M, N, satisfy the congruence $|L, M, N|\equiv 0$, or, as we say, are syzygetic. Such a Göpel system is represented by (AP).

It is shewn that by taking 2^{2p-r} different values of A and retaining the same group (P), we can thus divide the 2^{2p} possible characteristics into 2^{2p-r} Göpel systems. Among these 2^{2p-r} Göpel systems there are 2^{2p-2r} systems of which all the elements have the same character. Putting $2p-2r=2\sigma$ we shew further that $2^{\sigma-1}(2^\sigma+1)$ of these Göpel systems consist wholly of even characteristics, and that $2^{\sigma-1}(2^\sigma-1)$ of them consist wholly of odd characteristics. Putting $s=2^{2\sigma}$ we denote the $2^{2\sigma}$ Göpel systems which have a distinct character by $(A_1P), \ldots, (A_sP)$; and, still retaining the same group (P), we proceed to consider how to represent these $2^{2\sigma}$ systems by means of $2\sigma+1$ fundamental systems.

It appears then that from the characteristics $A_1, \ldots, A_s$ we can choose $2\sigma+1$ characteristics $\bar{A}_1, \ldots, \bar{A}_{2\sigma+1}$ in

$$2^{\sigma^2}(2^{2\sigma}-1)(2^{2\sigma-2}-1)\ldots(2^2-1)/\underline{|2\sigma+1}$$

ways, such that every three of them are azygetic, and all have the same character; this character is not at our disposal but is that of $\left(\frac{\sigma}{4}\right)$; the sum of $\bar{A}_1, \ldots, \bar{A}_{2\sigma+1}$, denoted by $\bar{A}_{2\sigma+2}$, has the character $e^{\pi i\sigma}\left(\frac{\sigma}{4}\right)$. Then all the combinations of 1, 5, 9, ... of $\bar{A}_1, \ldots, \bar{A}_{2\sigma+1}$ have the character $\left(\frac{\sigma}{4}\right)$, and all the combinations of 3, 7, 11, ... have the opposite character. These combinations in their aggregate are the characteristics $A_1, \ldots, A_s$. The characteristics $\bar{A}_1, \ldots, \bar{A}_{2\sigma+1}$ are, like $A_1, \ldots, A_s$, incongruent for the modulus (P). To each of them, say $\bar{A}_i$, corresponds a Göpel system $(\bar{A}_i P)$, to any constituent of which statements may be applied analogous to those made for $\bar{A}_i$ itself.

The characteristic $\bar{A}_{2\sigma+2}$ is such that every three of the set $\bar{A}_1, \ldots, \bar{A}_{2\sigma+2}$ are azygetic. This set is in fact derived, as one of $2\sigma+2$ such, from a set of $2\sigma+2$ characteristics, here called a root set, which satisfies the condition that every three of its constituents are azygetic without satisfying the condition that $2\sigma+1$ of them are of the same character. There are

$$2^{\sigma^2}(2^{2\sigma}-1)\ldots(2^2-1)/\underline{|2\sigma+2}$$

such root sets. It is not possible, from any root set, to obtain another by adding the same characteristic to each constituent of the former set.

The root sets are not the most general possible sets of $2\sigma+2$ characteristics of which every three are azygetic. Of such sets there are

$$2^{\sigma^2+2\sigma}(2^{2\sigma}-1)\ldots(2^2-1)/\underline{|2\sigma+2},$$

but they break up into batches of $2^{2\sigma}$, each derivable from a root set by the addition of a proper characteristic to all the constituents of the root set.

305. As examples of the foregoing theory we consider now the cases $\sigma=0$, $\sigma=1$, $\sigma=2$, $\sigma=p$. When $\sigma=0$, the number of Göpel groups of 2^p pairwise syzygetic characteristics is

$$(2^p+1)(2^{p-1}+1)\ldots\ldots(2+1);$$

from any such group we can, by the addition of the same characteristic to each of its constituents obtain one Göpel system consisting wholly of characteristics of the same *even* character. These results have already been obtained in case $p=2$ (§ 289, Ex. iv.), and, as in that particular case, the 2^p-1 other systems obtainable from the Göpel group by the addition of the same characteristic to each constituent, contain as many odd characteristics as even characteristics.

When $\sigma=1$, we can, from any Göpel group of 2^{p-1} pairwise syzygetic characteristics, obtain 4 Göpel systems, three of them consisting of 2^{p-1} even characteristics and one of 2^{p-1} odd characteristics. The characteristics of the latter (odd) system are obtainable as the sums of three characteristics taken one from each of the three even systems.

When $\sigma=2$, the number of fundamental sets $\bar{A}_1, \ldots, \bar{A}_5$ is

$$\frac{2^4(2^4-1)(2^2-1)}{\underline{|5}}=6;$$

each of them has the character $\left(\frac{\sigma}{4}\right)$, or is odd, and their sum, $\bar{A}_6$, is odd. Among the $2^{2\sigma}=16$ characteristics $A_1, \ldots, A_s$ there are $2^{2\sigma-1}-2^{\sigma-1}$ or 6 odd characteristics; these clearly consist of the characteristics $\bar{A}_1, \ldots, \bar{A}_6$; the six fundamental sets are obtained by neglecting each of $\bar{A}_1, \ldots, \bar{A}_6$ in turn. Among the characteristics $A_1, \ldots, A_s$ there are 10 even characteristics, obtainable by combining $\bar{A}_1, \ldots, \bar{A}_s$ in threes. And, to each of the characteristics $A_1, \ldots, A_s$ corresponds a Göpel system of $2^r=2^{p-\sigma}=2^{p-2}$ characteristics, for the constituents of which similar statements may be made.

Of the cases for which $\sigma=2$, the case $p=2$ is the simplest. After what has been said in Chap. XI., and elsewhere, we can leave that case aside here. For $p=3$ the Göpel systems consist of two characteristics; adopting, for instance, as the group (P), the pair $\frac{1}{2}\begin{pmatrix}000\\000\end{pmatrix}$, $\frac{1}{2}\begin{pmatrix}000\\100\end{pmatrix}$, the condition for the characteristics $A_1, \ldots, A_s$, namely $|X, P_1| \equiv |P_1|$, reduces to the condition that the first element of the upper row of the characteristic symbol of X shall be zero; hence the 16 characteristics $A_1, \ldots, A_s$ may be taken to be $\frac{1}{2}\begin{pmatrix}0 & a_1' & a_2'\\0 & a_1 & a_2\end{pmatrix}$, where $\frac{1}{2}\begin{pmatrix}a_1' & a_2'\\a_1 & a_2\end{pmatrix}$ represents in turn all the characteristic symbols for $p=2$.

Taking next the case $\sigma=3$, there are $s=2^{2\sigma}=64$ Göpel systems, (AP), each consisting wholly either of odd characteristics or of even characteristics, there being $2^{\sigma-1}(2^\sigma-1)$, $=28$, odd systems, and 36 even systems. From the representatives, $A_1, \ldots, A_s$, of these systems, which are incongruent mod. (P), we can choose a fundamental set of 7 characteristics $\bar{A}_1, \ldots, \bar{A}_7$ in

$$\frac{2^9(2^6-1)(2^4-1)(2^2-1)}{\underline{|7}}, \; =288,$$

ways; $\bar{A}_1, \ldots, \bar{A}_7$ will be odd, and their sum, $\bar{A}_8$, will be even; for $\left(\frac{\sigma}{4}\right)=(\frac{3}{4})=-1$, $e^{\pi i\sigma}\left(\frac{\sigma}{4}\right)=1$. The set $\bar{A}_1, \ldots, \bar{A}_7, \bar{A}_8$ is, in accordance with the theory, derived from one of $288/(2\sigma+2)$, $=36$, root sets $A_1, \ldots, A_8$ (§ 301), by equations of the form $\bar{A}_i=XA_i$, in which X is so chosen that $\bar{A}_1, \ldots, \bar{A}_7$ are of the same character; from this root set we can similarly derive 8 fundamental sets of seven odd characteristics, according as it is A_8 or is one of $A_1, \ldots, A_7$ which is left aside. Now the fact is, that, in whichever of the eight ways we pass from the root set to the seven fundamental odd characteristics, the sum of these seven fundamental characteristics is the same. We see this immediately in an indirect way. Let $\bar{A}_1, \ldots, \bar{A}_7$ be a fundamental set of odd characteristics derived from the root set $A_1, \ldots, A_8$ by the equations $\bar{A}_i=XA_i$; putting $\bar{A}_8=\bar{A}_1 \ldots \bar{A}_7$, consider the set $\bar{A}_8, \bar{A}_8\bar{A}_1\bar{A}_2, \ldots, \bar{A}_8\bar{A}_1\bar{A}_7, \bar{A}_1$, derived from $\bar{A}_1, \ldots, \bar{A}_8$ by adding $\bar{A}_8\bar{A}_1$ to each; in the first place it consists of one even characteristic, $\bar{A}_8$, and seven odd characteristics; for

$$|\bar{A}_8\bar{A}_1\bar{A}_i| \equiv |\bar{A}_8|+|\bar{A}_1|+|\bar{A}_i|+|\bar{A}_8, \bar{A}_1, \bar{A}_i| \equiv |\bar{A}_8, \bar{A}_1, \bar{A}_i| \equiv 1, \text{ (mod. 2)},$$

because $\bar{A}_1, \ldots, \bar{A}_8$ are azygetic in threes; in the next place

$$|\bar{A}_8, \bar{A}_1, \bar{A}_8\bar{A}_1\bar{A}_i| \equiv |\bar{A}_8, \bar{A}_1, \bar{A}_i| \equiv 1,$$

so that every three of its constituents are azygetic. Hence the characteristics $\bar{A}_8\bar{A}_1\bar{A}_2, \ldots, \bar{A}_8\bar{A}_1\bar{A}_7, \bar{A}_1$, which, as easy to see, are not congruent to $\bar{A}_1, \ldots, \bar{A}_7$ mod. (P), form, equally with $\bar{A}_1, \ldots, \bar{A}_7$, a fundamental set, whose sum is likewise $\bar{A}_8$; they are derived from $A_1, \ldots, A_8$ by adding $\bar{A}_8\bar{A}_1X$ to each of these. There are clearly six other such fundamental sets, derived from $A_1, \ldots, A_8$ by adding respectively $\bar{A}_8\bar{A}_2X, \ldots, \bar{A}_8\bar{A}_7X$. Hence to each of the 36 root sets there corresponds a certain even characteristic and to each of these even characteristics there correspond 8 fundamental sets. We can now shew further that the even characteristics, thus associated each with one of the 36 root sets, are in fact the 36 possible* even characteristics of the set $A_1, \ldots, A_8$. This again we shew indirectly by shewing how to form the remaining $7 . 36$ fundamental systems from the system $\bar{A}_1, \ldots, \bar{A}_7$. The seven characteristics $\bar{A}_8\bar{A}_2\bar{A}_3, \bar{A}_8\bar{A}_3\bar{A}_1, \bar{A}_8\bar{A}_1\bar{A}_2, \bar{A}_4, \bar{A}_5, \bar{A}_6, \bar{A}_7$, are in fact incongruent mod. (P), they are all odd, have for sum $\bar{A}_1\bar{A}_2\bar{A}_3$, which is even, and are azygetic in threes; for $\bar{A}_8\bar{A}_2\bar{A}_3$ is a combination of five of $\bar{A}_1, \ldots, \bar{A}_7$, and

$$|\bar{A}_4, \bar{A}_5, \bar{A}_8\bar{A}_2\bar{A}_3| \equiv |\bar{A}_8, \bar{A}_4, \bar{A}_5| + |\bar{A}_2, \bar{A}_4, \bar{A}_5| + |\bar{A}_3, \bar{A}_4, \bar{A}_5| \equiv 1, \quad |\bar{A}_4, \bar{A}_5, \bar{A}_6| \equiv 1,$$

$$|\bar{A}_4, \bar{A}_8\bar{A}_1\bar{A}_2, \bar{A}_8\bar{A}_1\bar{A}_3| \equiv |\bar{A}_8\bar{A}_1\bar{A}_4, \bar{A}_2, \bar{A}_3| \equiv 1, \; |\bar{A}_8\bar{A}_2\bar{A}_3, \bar{A}_8\bar{A}_3\bar{A}_1, \bar{A}_8\bar{A}_1\bar{A}_2| \equiv |\bar{A}_1, \bar{A}_2, \bar{A}_3| \equiv 1,$$

(the modulus in each case being 2); hence these seven characteristics form a fundamental system. There are 35 sets of three characteristics, such as $\bar{A}_1, \bar{A}_2, \bar{A}_3$, derivable from the seven $\bar{A}_1, \ldots, \bar{A}_7$; each of these corresponds to such a fundamental system as that just explained; and each of these fundamental systems is associated with seven other fundamental systems, derived from it by the process whereby the set $\bar{A}_i, \bar{A}_i\bar{A}_8\bar{A}_2, \ldots, \bar{A}_i\bar{A}_8\bar{A}_7$ is derived from $\bar{A}_1, \ldots, \bar{A}_7$.

When $\sigma = p$, a Göpel system consists of one characteristic only; we can, in

$$2^{p^2}(2^{2p}-1)(2^{2p-2}-1)\ldots\ldots(2^2-1)/\underline{|2p+1}$$

ways, determine a set of $2p+1$ characteristics, all of character $\left(\frac{p}{4}\right)$, of which every three are azygetic; their sum will be of character $e^{\pi i p}\left(\frac{p}{4}\right)$; all the possible 2^{2p} characteristics can be represented as combinations of an odd number of these.

306. We pass now to some applications of the foregoing theory to the theta functions. The results obtained are based upon the consideration of the theta function of the second order defined by

$$\phi(u, a; \tfrac{1}{2}q) = \vartheta(u+a; \tfrac{1}{2}q)\,\vartheta(u-a; \tfrac{1}{2}q),$$

where $\frac{1}{2}q$ is a half-integer characteristic; as theta function of the second order this function has zero characteristic; the addition of any integers to the elements of the characteristic $\frac{1}{2}q$ does not affect the value of the function. By means of the formulae (§ 190, Chap. X.),

$$\vartheta(u+a; \tfrac{1}{2}q+N) = e^{\pi i N q'}\,\vartheta(u+a; \tfrac{1}{2}q),$$

$$\vartheta(u+\tfrac{1}{2}\Omega_k; \tfrac{1}{2}q) = e^{\lambda(u;\, \frac{1}{2}k) - \frac{1}{4}\pi i k' q}\,\vartheta(u; \tfrac{1}{2}k+\tfrac{1}{2}q),$$

wherein N denotes a row of integers and $\lambda(u; s) = H_s(u+\frac{1}{2}\Omega_s) - \pi i s s'$, we immediately find

$$\phi(u+\tfrac{1}{2}\Omega_k, a; \tfrac{1}{2}q) = e^{2\lambda(u;\, \frac{1}{2}k)}\begin{pmatrix}K\\Q\end{pmatrix}\phi(u, a; \tfrac{1}{2}kq),$$

where $\frac{1}{2}kq$ denotes the sum of the characteristics $\frac{1}{2}k, \frac{1}{2}q$; to save the repetition of the $\frac{1}{2}$, this equation will in future be written in the form (cf. § 294)

$$\phi(u+\Omega_K, a; Q) = e^{2\lambda(u;\, K)}\begin{pmatrix}K\\Q\end{pmatrix}\phi(u, a; KQ);$$

when the contrary is not stated capital letters will denote half-integer characteristics, and KQ will denote the *reduced* sum of the characteristics K, Q, having for each of its elements either 0 or $\frac{1}{2}$.

* Thus, when $p=3=\sigma$, the result quoted in § 205, Chap. XI., is justified.

We shall be concerned with groups of 2^r pairwise syzygetic characteristics, such as have been called Göpel groups, and denoted by (P); corresponding to the r characteristics $P_1, \ldots, P_r$ from which such a group is formed, we introduce r fourth roots of unity, denoted by $\epsilon_1, \ldots, \epsilon_r$, which are such that

$$\epsilon_1^2 = e^{\pi i |P_1|}, \ldots, \epsilon_r^2 = e^{\pi i |P_r|};$$

the signs of these symbols are, at starting, arbitrary, but are to be the same throughout unless the contrary be stated. Since the characteristics of the group (P) satisfy the conditions

$$|P_i, P_j| \equiv 0, \text{ (mod. 2)}, \quad \binom{P_i}{P_j} = \binom{P_j}{P_i},$$

we may, without ambiguity, associate with the compound characteristics of the group the $2^r - r$ symbols defined by

$$\epsilon_0 = 1, \quad \epsilon_{i,j} = \epsilon_i \epsilon_j \binom{P_i}{P_j}, \text{ so that } \epsilon_{i,j}^2 = e^{\pi i |P_i| + \pi i |P_j|}, \quad \epsilon_{i,i} = 1,$$

$$\epsilon_{i,j,k} = \epsilon_i \epsilon_{j,k} \binom{P_i}{P_j P_k} = \epsilon_i \epsilon_j \epsilon_k \binom{P_j}{P_k}\binom{P_k}{P_i}\binom{P_i}{P_j} = \epsilon_j \epsilon_{k,i} \binom{P_j}{P_k P_i} = \epsilon_k \epsilon_{i,j} \binom{P_k}{P_i P_j},$$

and

$$\epsilon_j = \epsilon_{i,ij} = \epsilon_i \epsilon_{ij} \binom{P_i P_j}{P_i}, \text{ etc.}$$

Consider now the function* defined by

$$\Phi(u, a; A) = \sum_i \binom{P_i}{A} \epsilon_i \phi(u, a; AP_i),$$

where A is an arbitrary half-integer characteristic, and P_i denotes in turn all the 2^r characteristics of the group (P). Adding to u a half-period Ω_{P_k}, corresponding to a characteristic P_k of the group (P), we obtain

$$\Phi(u + \Omega_{P_k}, a; A) = \sum_i \binom{P_i}{A}\binom{P_k}{AP_i} \epsilon_i e^{2\lambda(u; P_k)} \phi(u, a; AP_iP_k);$$

if then $P_h \equiv P_i P_k$, or $P_i \equiv P_h P_k$, we have

$$\binom{P_i}{A}\binom{P_k}{AP_i}\epsilon_i = \binom{P_h}{A}\binom{P_k}{A}\binom{P_k}{A}\binom{P_k}{P_h}\binom{P_k}{P_k}\epsilon_h\epsilon_k\binom{P_h}{P_k} = \epsilon_k e^{\pi i |P_k|}\binom{P_h}{A}\epsilon_h;$$

now, as P_i becomes in turn all the characteristics of the group (P), $P_h, = P_iP_k$, also becomes all the characteristics of the group, in general in a different order; thus we have

$$\Phi(u + \Omega_{P_k}, a; A) = \epsilon_k e^{\pi i |P_k| + 2\lambda(u; P_k)} \Phi(u, a; A),$$

$$= \epsilon_k^{-1} e^{2\lambda(u; P_k)} \Phi(u, a; A).$$

* If preferred the sign $\binom{P_i}{A}$, whose value is ± 1, may be absorbed in ϵ_i. But there is a certain convenience in writing it explicitly.

If $2\Omega_M$ be any period, we immediately find

$$\Phi(u+2\Omega_M, a;\ A)=e^{2\lambda(u;\,2M)}\,\Phi(u, a;\ A).$$

Thus, $\lambda(u;\ P_k)$ being a linear function of the arguments $u_1, \ldots, u_p$, the function $\Phi(u, a;\ A)$ is a theta function of the second order with zero characteristic, having the additional property that all the partial differential coefficients of its logarithm, of the second order, have the 2^r sets of simultaneous periods denoted by the symbols Ω_{P_k}.

Ex. i. If S be a half-integer characteristic which is syzygetic with every characteristic of the group (P), prove that

$$\Phi(u+\Omega_S, a;\ A)=e^{2\lambda(u;\,S)}\begin{pmatrix}S\\A\end{pmatrix}\Phi(u, a;\ AS)$$

$$\Phi(u, a+\Omega_S;\ A)=e^{2\lambda(a;\,S)+\pi i|S|+\pi i|S,\,A|}\begin{pmatrix}S\\A\end{pmatrix}\Phi(u, a;\ AS)$$

and

$$\Phi(u+\Omega_S, a+\Omega_S;\ A)=e^{2\lambda(u;\,S)+2\lambda(a;\,S)+\pi i|S,\,A|}\,\Phi(u, a;\ A).$$

Ex. ii. If P_k be any characteristic of the group (P), prove that

$$\Phi(u, a;\ AP_k)=\begin{pmatrix}P_k\\A\end{pmatrix}\epsilon_k^{-1}\,\Phi(u, a;\ A).$$

Ex. iii. When, as in Ex. i., S is syzygetic with every characteristic of the group (P), shew that

$$e^{\pi i|SP_k|}\,\Phi(u, a;\ AP_k)\,\Phi(v, b;\ AP_k)=e^{\pi i|S|}\,\Phi(u, a;\ A)\,\Phi(v, b;\ A).$$

Conversely it can be shewn that if a theta function of the second order with zero characteristic, $\Pi(u)$, which, therefore, satisfies the equation

$$\Pi(u+\Omega_m)=e^{2\lambda_m(u)}\,\Pi(u),$$

for integral m, be further such that for each of the two half-periods associated with the characteristics $\frac{1}{2}m=P$, $\frac{1}{2}m=Q$, there exists an equation of the form

$$\Pi(u+\tfrac{1}{2}\Omega_m)=e^{\mu+\nu_1u_1+\ldots+\nu_pu_p}\,\Pi(u),$$

where $\mu, \nu_1, \ldots, \nu_p$ are independent of u, then the characteristics P, Q must be syzygetic. Putting $\nu u=\nu_1u_1+\ldots\ldots+\nu_pu_p$, we infer from the equation just written that

$$\Pi(u+\Omega_m)=e^{\mu+\nu(u+\frac{1}{2}\Omega_m)}\,\Pi(u+\tfrac{1}{2}\Omega_m)=e^{2\mu+2\nu u+\frac{1}{2}\nu\Omega_m}\,\Pi(u);$$

comparing this with the equation

$$\Pi(u+\Omega_m)=e^{2\lambda_m(u)}\,\Pi(u)=e^{2H_m(u+\frac{1}{2}\Omega_m)-2\pi imm'}\,\Pi(u)$$

we infer that $\nu=H_m$, $\mu=k\pi i+\frac{1}{4}H_m\Omega_m-\pi imm'$, where k is integral, and hence

$$\Pi(u+\tfrac{1}{2}\Omega_m)=\pm\,e^{-\frac{1}{2}\pi imm'+2\lambda(u;\,\frac{1}{2}m)}\,\Pi(u).$$

307. In accordance with these indications, let $Q(u)$ denote an analytical integral function of the arguments $u_1, \ldots, u_p$ which satisfies the equations

$$Q(u+\Omega_m)=e^{2\lambda(u;\,m)}\,Q(u);\quad Q(u+\Omega_{P_k})=\epsilon_k e^{\pi i|P_k|+2\lambda(u;\,P_k)}\,Q(u),$$

for every integral m and every half-integer characteristic P_k of the group (P).

We may regard the group (P) as consisting of part of a group of 2^p pairwise syzygetic characteristics formed by all the combinations of the constituents of the group (P) with the constituents of another pairwise syzygetic group (R) of 2^{p-r} characteristics. Then the 2^p characteristics of the compound group are obtainable in the form P_iR_j, wherein P_i has the 2^r values of the group (P), and R_j has the 2^{p-r} values of the group (R). Since every 2^p+1 theta functions of the second order and the same characteristic are connected by a linear equation, we have

$$CQ(u) = \sum_{i,j} C_{i,j}\, \phi(u, a;\ P_iR_j),$$

where C, $C_{i,j}$ are independent of u and are not all zero*. Hence, adding to u the half-period Ω_{P_k}, we have

$$C\epsilon_k e^{\pi i |P_k| + 2\lambda(u;\ P_k)}\, Q(u) = \sum_{i,j} C_{i,j}\, e^{2\lambda(u;\ P_k)} \begin{pmatrix} P_k \\ P_iR_j \end{pmatrix} \phi(u, a;\ P_iP_kR_j),$$

and therefore, as $\epsilon_k e^{\pi i |P_k|} = \epsilon_k^{-1}$,

$$CQ(u) = \sum_{i,j} C_{i,j} \begin{pmatrix} P_k \\ P_iR_j \end{pmatrix} \epsilon_k \phi(u, a;\ P_iP_kR_j);$$

forming this equation for each of the 2^r values of P_k, and adding the results, we have

$$2^r CQ(u) = \sum_{i,j,k} C_{i,j} \begin{pmatrix} P_k \\ P_iR_j \end{pmatrix} \epsilon_k \phi(u, a;\ P_iP_kR_j);$$

herein put $P_h = P_iP_k$, so that as, for any value of i, P_k becomes in turn all the characteristics of the group (P), the characteristic P_h also becomes all the characteristics in turn, in general in a different order; then

$$\epsilon_k \begin{pmatrix} P_k \\ P_iR_j \end{pmatrix} = \epsilon_h \epsilon_i \begin{pmatrix} P_h \\ P_i \end{pmatrix} \begin{pmatrix} P_hP_i \\ P_iR_j \end{pmatrix} = \epsilon_h \epsilon_i \begin{pmatrix} P_h \\ R_j \end{pmatrix} \begin{pmatrix} P_i \\ R_j \end{pmatrix} e^{\pi i |P_i|},$$

and, therefore,

$$2^r CQ(u) = \sum_j \sum_h \epsilon_h \left[\sum_i C_{i,j}\, \epsilon_i \begin{pmatrix} P_i \\ R_j \end{pmatrix} e^{\pi i |P_i|} \right] \begin{pmatrix} P_h \\ R_j \end{pmatrix} \phi(u, a;\ P_hR_j),$$

$$= \sum_j \sum_h C_j \begin{pmatrix} P_h \\ R_j \end{pmatrix} \epsilon_h \phi(u, a;\ P_hR_j),$$

where

$$C_j = \sum_i C_{i,j} \begin{pmatrix} P_i \\ R_j \end{pmatrix} \epsilon_i e^{\pi i |P_i|},$$

and thus

$$2^r CQ(u) = \sum_j C_j \Phi(u, a;\ R_j).$$

Now the 2^{p-r} functions $\Phi(u, a;\ R_j)$ are not in general connected by any linear relation with coefficients independent of u; for such a relation would be of the form

$$\Sigma H_i \vartheta(u+a;\ AQ_i)\, \vartheta(u-a;\ AQ_i) = 0,$$

* It is proved below (§ 308) that the functions $\phi(u, a;\ P_iR_j)$ are linearly independent, so that, in fact, C is not zero.

wherein H_i is independent of u, and Q_i becomes, in turn, all the constituents of a group (Q) of 2^p pairwise syzygetic characteristics, and we shall prove (in § 308) that such a relation is impossible for general values of the arguments a. Hence, *all theta functions of the second order, with zero characteristic, which satisfy the equation*

$$Q\left(u+\Omega_{P_k}\right)=\epsilon_k e^{\pi i|P_k|+2\lambda(u;\, P_k)}\, Q(u)$$

for every half-integer characteristic P_k of the group (P), are representable linearly by 2^{p-r}, $=2^\sigma$, of them, with coefficients independent of u. We have shewn that the functions $\Phi(u, a;\, A)$, defined by the equation

$$\Phi(u, a;\, A)=\sum_i\binom{P_i}{A}\epsilon_i\vartheta(u+a;\, AP_i)\,\vartheta(u-a;\, AP_i),$$

where the summation includes 2^r terms, are a particular case of such theta functions.

308. Suppose there exists a relation of the form

$$\sum_i H_i\vartheta(u+a;\, AQ_i)\,\vartheta(u+b;\, AQ_i)=0,$$

where the summation extends to all the 2^p characteristics Q_i of a Göpel group (Q), and H_i is independent of u. Putting for u, $u+\Omega_{Q_a}$, where Q_a is a characteristic of the group (Q), we obtain

$$\sum_i H_i\binom{Q_a}{Q_i}\vartheta(u+a;\, AQ_iQ_a)\,\vartheta(u+b;\, AQ_iQ_a)=0\,;$$

hence, if $\epsilon_1, \ldots, \epsilon_p$ are fourth roots of unity associated with a basis $Q_1, \ldots, Q_p$ of the group (Q), as before, and this equation be multiplied by ϵ_a, and the equations of this form obtained by taking Q_a to be, in turn, all the 2^p characteristics of the group (Q), be added together, we have

$$\sum_i\sum_a H_i\binom{Q_a}{Q_i}\epsilon_a\vartheta(u+a;\, AQ_iQ_a)\,\vartheta(u+b;\, AQ_iQ_a)=0\,;$$

now let $Q_j\equiv Q_aQ_i$, then for any value of i, as Q_a becomes all the characteristics of the group (Q), Q_j will become all those characteristics; therefore, substituting

$$\binom{Q_a}{Q_i}=\binom{Q_j}{Q_i}\binom{Q_i}{Q_i}, \qquad \epsilon_a=\epsilon_i\epsilon_j\binom{Q_i}{Q_j},$$

we have

$$\sum_i H_i\epsilon_i\binom{Q_i}{Q_i}\sum_j\epsilon_j\vartheta(u+a;\, AQ_j)\,\vartheta(u+b;\, AQ_j)=0\,;$$

hence one at least of the expressions

$$\sum_j\epsilon_j\vartheta(u+a;\, AQ_j)\,\vartheta(u+b;\, AQ_j), \qquad \sum_i H_i\epsilon_i^{-1},$$

must vanish.

Here $\epsilon_1, \epsilon_2, \ldots$ have any one of 2^p possible sets of values. The expression $\sum_i H_i\epsilon_i^{-1}$ cannot vanish for every one of these sets; for, multiplying by ϵ_j^{-1}, we have then

$$\sum_i H_i\binom{Q_i}{Q_j}\epsilon_{i,j}^{-1}=0,$$

where $\epsilon_{i,j}$, like ϵ_i, becomes in turn the symbol associated with every characteristic of the group, and there are 2^p equations of this form; adding these equations we infer $H_j=0$, and, therefore, as j is arbitrary, we infer that all the coefficients are zero.

Hence it follows that there is at least one of the 2^p sets of values for $\epsilon_1, \epsilon_2, \ldots$, for which

$$\sum_j \epsilon_j \vartheta(u+a;\ AQ_j)\,\vartheta(u+b;\ AQ_j)=0.$$

When the arguments $u+a$, $u+b$ are independent, this is impossible; for putting $u+a=U$, $u+b=V$, this is an equation connecting the 2^p functions $\vartheta(U;\ AQ_j)$ in which the coefficients are independent of U (cf. §§ 282, 283, Chap. XV.).

When the arguments $u+a$, $u+b$ are not independent, this equation is not impossible. For instance, if $\epsilon_k=-e^{\frac{1}{2}\pi i|Q_k|}$, it is easy to verify that

$$\epsilon_{h,k}\vartheta(u+\Omega_{Q_k};\ Q_hQ_k)\,\vartheta(u;\ Q_hQ_k)=-\epsilon_h\vartheta(u+\Omega_{Q_k};\ Q_h)\,\vartheta(u;\ Q_h)$$

and hence the equation does hold when $A=0$, $a=\Omega_{Q_k}$, $b=0$, $\epsilon_k=-e^{\frac{1}{2}\pi i|Q_k|}$, for all the values of $\epsilon_1, \ldots, \epsilon_{k-1}, \epsilon_{k+1}, \ldots, \epsilon_p$. For any values of the arguments $u+a$, $u+b$ we infer from the reasoning here given that if the functions $\vartheta(u+a;\ AQ_i)\,\vartheta(u+b;\ AQ_i)$ are connected by a linear equation with coefficients, H_i, independent of u, then (i) they are connected by at least one equation

$$\sum_i \epsilon_i \vartheta(u+a;\ AQ_i)\,\vartheta(u+b;\ AQ_i)=0,$$

for one of the 2^p sets of values of the quantities $\epsilon_1, \epsilon_2, \ldots$, and (ii) similarly, since the 2^p functions $\vartheta(u+a;\ AQ_i)\,\vartheta(u+b;\ AQ_i)$ do not all vanish identically, that the coefficients are connected by at least one equation

$$\sum_i H_i \epsilon_i^{-1}=0.$$

309. The result of § 307 is of great generality; we proceed to give examples of its application (§§ 309—313). The simplest, as well as the most important, case is that in which $\sigma=0$, $r=p$, and to that we give most attention (§§ 309—311).

When $\sigma=0$, any *two* of the functions $\Phi(u,\ a;\ A)$ are connected by a linear equation, in which the coefficients are independent of u. If v, a, b be any arguments, and A, B any half-integer characteristics, introducing the symbol ϵ to put in evidence the fact that $\Phi(u,\ a;\ A)$ is formed with one of 2^p possible selections for the symbols $\epsilon_1, \ldots, \epsilon_p$, and so writing $\Phi(u, a;\ A, \epsilon)$ for $\Phi(u,\ a;\ A)$, we therefore have the fundamental equation

$$\Phi(u,\ v;\ A,\ \epsilon)=\frac{\Phi(u,\ b;\ B,\ \epsilon)\,\Phi(a,\ v;\ A,\ \epsilon)}{\Phi(a,\ b;\ B,\ \epsilon)}.$$

By adding the 2^p equations of this form* which arise by giving all the possible sets of values to the fourth roots of unity $\epsilon_1, \ldots, \epsilon_p$, bearing in mind that every symbol ϵ_i, except ϵ_0, $=1$, occurs as often with the positive as with the negative sign, we obtain

$$2^p\,\vartheta(u+v;\ A)\,\vartheta(u-v;\ A)=\sum_\epsilon\sum_i \binom{P_i}{A}\epsilon_i\,\vartheta(u+v;\ AP_i)\,\vartheta(u-v;\ AP_i)$$
$$=\sum_\epsilon \frac{\Phi(u,\ b;\ B,\ \epsilon)\,\Phi(a,\ v;\ A,\ \epsilon)}{\Phi(a,\ b;\ B,\ \epsilon)},$$

* Wherein it is assumed that a, b have not such special values that any one of the 2^p quantities $\Phi(a,\ b;\ B,\ \epsilon)$ vanishes. Cf. § 308.

whereby the function $\phi(u, v; A)$ is expressed in terms of 2^p functions

$$\Phi(u, b; B, \epsilon).$$

By taking, in the formula

$$\Phi(u, v; A, \epsilon)\Phi(a, b; B, \epsilon) = \Phi(u, b; B, \epsilon)\Phi(a, v; A, \epsilon),$$

or

$$\sum_i \sum_j \binom{P_i}{A}\binom{P_j}{B} \epsilon_i \epsilon_j \phi(u, v; AP_i)\phi(a, b; BP_j)$$
$$= \sum_i \sum_j \binom{P_i}{A}\binom{P_j}{B} \epsilon_i \epsilon_j \phi(u, b; BP_i)\phi(a, v; AP_j),$$

all the 2^p possible sets of values for $\epsilon_1, \ldots, \epsilon_p$, and adding the results, we obtain

$$\sum_i \binom{P_i}{AB} e^{\pi i |P_i|} \phi(u, v; AP_i)\phi(a, b; BP_i)$$
$$= \sum_i \binom{P_i}{AB} e^{\pi i |P_i|} \phi(u, b; BP_i)\phi(a, v; AP_i);$$

increasing u and b each by the half-period Ω_R, we have

$$\sum_i \binom{RP_i}{AB} e^{\pi i |RP_i|} \phi(u, v; ARP_i)\phi(a, b; BRP_i)$$
$$= \sum_i \binom{P_i}{AB} e^{\pi i |P_i| + \pi i |R, P_i|} \phi(u, b; BP_i)\phi(a, v; AP_i);$$

taking R to be all the possible 2^{2p} half-integer characteristics in turn, and adding the resulting equations we deduce*, putting $C = AB$,

$$2^p \phi(u, b; AC)\phi(a, v; A)$$
$$= 2^{-p} \sum_i \sum_R \binom{RP_i}{C} e^{\pi i |RP_i|} \phi(u, v; RAP_i)\phi(a, b; RAP_iC)$$
$$= \sum_S \binom{AS}{C} e^{\pi i |AS|} \phi(u, v; S)\phi(a, b; SC),$$

where A, C are arbitrary half-integer characteristics, and S becomes all 2^{2p} possible half-integer characteristics in turn; for (Ex. ii. § 295), $\sum_R e^{\pi i |R, P_i|} = 2^{2p}$ when $P_i = 0$, and is otherwise zero, while, for any definite characteristic AP_i, as R becomes all possible characteristics, so does RAP_i. The formula can be simplified by adding the half-period Ω_C to the argument b; the result is obtainable directly by taking $C = 0$ in the formula written.

This agrees with a result previously obtained (§ 292, Chap. XVI.); for a generalisation of it, see below, § 314.

* This equation has been called the Riemann theta formula. Cf. Prym, *Untersuchungen über die Riemann'sche Thetaformel*, Leipzig, 1882.

310. The formula just obtained may be regarded as a particular case of another which is immediately deducible therefrom. Let (K) be a group of 2^μ characteristics formed by taking all the combinations of μ independent characteristics $K_1, \dots, K_\mu$; if A be any characteristic whatever, we have

$$\sum_K e^{\pi i|A,\, K|} = (1+e^{\pi i|A,\, K_1|}) \dots (1+e^{\pi i|A,\, K_\mu|}) = 2^\mu, \text{ or } 0,$$

according as $|A, K_i| \equiv 0$ (for $i=1, \dots, \mu$), or not; hence, putting $C=0$ in the formula of § 309, and replacing the A of that formula by K_i, we deduce

$$2^{p-\mu} \sum_{i=1}^{2^\mu} e^{\pi i|AK_i|} \phi(u, b;\, K_i)\, \phi(a, v;\, K_i) = 2^{-\mu} \sum_{i=1}^{2^\mu} e^{\pi i|AK_i|} \sum_S e^{\pi i|K_i S|} \phi(u, v;\, S)\, \phi(a, b;\, S),$$

where S becomes all 2^{2p} characteristics,

$$= 2^{-\mu} \sum_S e^{\pi i|A|+\pi i|S|} \sum_{i=1}^{2^\mu} e^{\pi i|AS,\, K_i|} \phi(u, v;\, S)\, \phi(a, b;\, S)$$

$$= 2^{-\mu} e^{\pi i|A|} \sum_R e^{\pi i|AR|} \left(\sum_{i=1}^{2^\mu} e^{\pi i|R,\, K_i|}\right) \phi(u, v;\, AR)\, \phi(a, b;\, AR),$$

where R becomes all 2^{2p} characteristics,

$$= 2^{-\mu} e^{\pi i|A|}\, 2^\mu \sum_R e^{\pi i|AR|} \phi(u, v;\, AR)\, \phi(a, b;\, AR),$$

where R extends to all the $2^{2p-\mu}$ characteristics for which $|R, K_1| \equiv 0, \dots, |R, K_\mu| \equiv 0$. Putting $u+\Omega_B$, $a+\Omega_B$ for u, a respectively, and replacing AB by C, we obtain

$$2^{p-\mu} \sum_{i=1}^{2^\mu} e^{\pi i|BCK_i|} \phi(u, b;\, BK_i)\, \phi(a, v;\, BK_i)$$

$$= e^{\pi i|BC|} \sum_{j=1}^{2^{2p-\mu}} e^{\pi i|BCL_j|} \phi(u, v;\, CL_j)\, \phi(a, b;\, CL_j);$$

here (K) is any group of 2^μ characteristics, (L) is an adjoint group of $2^{2p-\mu}$ characteristics defined by the conditions $|L, K| \equiv 0$ (mod. 2), and B, C are arbitrary half-integer characteristics. The formula of the previous Article is obtained by taking $\mu=0$. The formula of the present Article may be regarded as a particular case of that given below in § 315.

311. The function $\phi(u, v;\, A)$ is unaffected by the addition of integers to the half-integer characteristic A; we may therefore suppose that in the functions $\phi(u, v;\, AP_i)$ which have frequently occurred in the preceding Articles, the characteristic AP_i is reduced, all its elements being either 0 or $\frac{1}{2}$. In the applications which now immediately follow (§ 311) it is convenient, to avoid the explicit appearance of certain fourth roots of unity (cf. Ex. vii., p. 469), not to use reduced characteristics. Two, or more, characteristics which are to be added without reduction will be placed with a comma between them; thus A, P_i denotes $A + P_i$. The characteristics P_i are still supposed reduced.

Taking the formula (§ 309)

$$2^p\, \vartheta(u+v;\, A)\, \vartheta(u-v;\, A) = \sum_\epsilon \frac{\Phi(u, b;\, A', \epsilon)\, \Phi(a, v;\, A, \epsilon)}{\Phi(a, b;\, A', \epsilon)},$$

where A' replaces the B of § 309, suppose $a = b$, and put, for

$$u-b,\quad a+v,\quad a-v,\quad u+v,\quad u-v,\quad a+b,\quad a-b,\quad u+b,$$

respectively,

$$U,\quad V,\quad W,\quad U+V,\quad U+W,\quad V+W,\quad 0,\quad U+V+W;$$

then we obtain

$$2^p \vartheta(U+V;\, A)\vartheta(U+W;\, A)$$

$$= \sum_{\epsilon} \frac{\sum_i \sum_j \binom{P_i}{A'}\binom{P_j}{A}\epsilon_i\epsilon_j \vartheta(U+V+W;\, A', P_i)\vartheta(U;\, A', P_i)\vartheta(V;\, A, P_j)\vartheta(W;\, A, P_j)}{\sum_k \binom{P_k}{A'}\epsilon_k \vartheta(V+W;\, A', P_k)\vartheta(0;\, A', P_k)};$$

adding to V and W respectively the half-periods Ω_B, Ω_C, this becomes

$$2^p [U, V;\, A, B]\, [U, W;\, A, C]$$

$$= \sum_{\epsilon} \frac{\sum_i \sum_j \nu_i \mu_j t_{i,j} [U, V, W;\, A', B, C, P_i]\, [U;\, A', P_i]\, [V;\, A, B, P_j]\, [W;\, A, C, P_j]}{\sum_k \nu_k s_k [V, W;\, A', B, C, P_k]\, [0;\, A', P_k]}$$

wherein $[U, V;\, A, B]$ denotes $\vartheta\,[U+V;\, A+B]$, etc., $\mu_i = \binom{P_i}{A}\epsilon_i$, $\nu_i = \binom{P_i}{A'}\epsilon_i$, etc., and, if $B = \frac{1}{2}\binom{\beta'}{\beta}$, $C = \frac{1}{2}\binom{\gamma'}{\gamma}$, $P_i = \frac{1}{2}\binom{q_i'}{q_i}$, then $t_{i,j}$, s_k are fourth roots of unity given by $t_{i,j} = e^{-\frac{1}{2}\pi i(\beta'+\gamma')(q_i+q_j)}$, $s_k = e^{-\frac{1}{2}\pi i(\beta'+\gamma')q_k}$.

In connexion with this formula several results may be deduced.

(α) Putting $W = -V$, $A + B = K$, $A + C = D$, $A' = D$, the formula gives an expression of $\vartheta\,[U+V;\, K]\,\vartheta\,[U-V;\, D]$ in terms of the quantities

$$\vartheta\,[U;\, KP_i],\ \vartheta\,[V;\, KP_i],\ \vartheta\,[U;\, DP_i],\ \vartheta\,[V;\, DP_i],\ \vartheta\,[0;\, KP_i],\ \vartheta\,[0;\, DP_i];$$

the expression contains in the denominator only the constants $\vartheta\,[0;\, KP_i]$, $\vartheta\,[0;\, DP_i]$; it has been shewn (§ 299) that not all the characteristics KP_i, DP_i can be odd.

Putting further $K = 0$, we obtain an expression of $\vartheta\,[U+V;\, 0]\ \vartheta\,[U-V;\, D]$ in terms of

$$\vartheta\,[U;\, P_i],\ \vartheta\,[V;\, P_i],\ \vartheta\,[U;\, DP_i],\ \vartheta\,[V;\, DP_i],\ \vartheta\,[0;\, P_i],\ \vartheta\,[0;\, DP_i].$$

Dividing the former result by the latter we obtain an expression for $\vartheta\,[U+V;\, K]/\vartheta\,[U+V;\, 0]$ in terms of theta functions of U and V with the characteristics DP_i, KP_i, P_i, the coefficients being combinations of $\vartheta\,[0;\, P_i]$, $\vartheta\,[0;\, DP_i]$, $\vartheta\,[0;\, KP_i]$ with numerical quantities. In this expression the characteristic D is arbitrary; it may for instance be taken to be zero.

The formulae are very remarkable; replacing, on the right hand, $\epsilon_i e^{\pi i |A, P_i|}$ by ϵ_i, as is clearly allowable, and taking $D=0$, they are both included in the following formula (cf. Ex. viii. § 317)

$$2^p \vartheta[u+v;\ K]\vartheta[u-v;\ 0]$$
$$=\sum_{\epsilon}\frac{[\sum_a \epsilon_a e^{-\frac{1}{2}\pi i k' q_a}\vartheta(u;K+P_a)\vartheta(u;P_a)][\sum_a \epsilon_a e^{-\frac{1}{2}\pi i k' q_a+\pi i|P_a|}\vartheta(v;K+P_a)\vartheta(v;P_a)]}{\sum_a \epsilon_a e^{-\frac{1}{2}\pi i k' q_a}\vartheta(0;\ K+P_a)\vartheta(0;\ P_a)},$$

where $K=\frac{1}{2}\binom{k'}{k}$, $P_a=\frac{1}{2}\binom{q_a'}{q_a}$, and the summation in regard to a extends to all the 2^p characteristics, P_a, of the group (P).

It is assumed that the characteristic K is such that the denominator on the right hand does not vanish for any one of the 2^p sets of values for the quantities ϵ_a. For instance the case when K is one of the characteristics of the group (P), other than zero, is excluded (cf. § 308).

Ex. i. For $p=1$, if P denote any one of the half-integer characteristics other than zero,

$$\vartheta(u+v)\vartheta(u-v)=\frac{[\vartheta^2(u)\vartheta^2(v)+\vartheta_P^2(u)\vartheta_P^2(v)]\vartheta^2(0)-[\vartheta^2(u)\vartheta_P^2(v)+e^{\pi i|P|}\vartheta_P^2(u)\vartheta^2(v)]\vartheta_P^2(0)}{\vartheta^4(0)-e^{\pi i|P|}\vartheta_P^4(0)},$$

where $\vartheta(u)$, $\vartheta_P(u)$ denote $\vartheta(u;\ 0)$, $\vartheta(u;\ P)$, etc.

Ex. ii. By putting, in case $p=2$,

$$K=\tfrac{1}{2}\binom{10}{10},\quad P_1=\tfrac{1}{2}\binom{01}{01},\quad P_2=\tfrac{1}{2}\binom{01}{11},$$

deduce from the formula of the text that

$$4\vartheta_{12}(0)\vartheta_{01}(0)\vartheta_{02}(u+u')\vartheta_5(u-u')=\sum_{\zeta_1,\zeta_2}[i\zeta_1\zeta_2A-\zeta_2B+i\zeta_1C+D][A'-i\zeta_1B'-\zeta_2C'-i\zeta_1\zeta_2D'],$$

wherein $\zeta_1=\pm1$, $\zeta_2=\pm1$, and

$$A=\vartheta_5(u)\vartheta_{02}(u),\quad B=\vartheta_3(u)\vartheta_{14}(u),\quad C=\vartheta_{04}(u)\vartheta_{24}(u),\quad D=\vartheta_{12}(u)\vartheta_{01}(u),$$

A', B', C', D' denoting the same functions of the arguments u'.

Hence obtain the formula given at the bottom of page 457 of this volume.

(β) Putting $B=C$, $V=W=0$, $A'=A$, we obtain

$$2^p\vartheta^2[U;\ A,\ B]=\sum_{\epsilon}\frac{\sum_i\sum_j \mu_i\mu_j t_{i,j}[U;\ A,\ B,\ B,\ P_i][U;\ AP_i][0;\ A,\ B,\ P_j]^2}{\sum_k \mu_k s_k[0;\ A,\ B,\ B,\ P_k][0;\ A,\ P_k]},$$

which shews that the square of *any* theta function is expressible as a linear function of the squares of the theta functions with the characteristics forming the Göpel system (AP). We omit the proof that these 2^p squares, $\vartheta^2(U;\ AP_i)$, are not in general connected* by any linear relation in which the coefficients are independent of U.

* Cf. the concluding remark of § 308, § 291, Ex. iv. and § 283.

Ex. For $p=2$ obtain the formula

$$(\vartheta_2^4-\vartheta_{01}^4)\,\vartheta_{03}^2(u)=\vartheta_{23}^2\,\vartheta_2^2\,\vartheta_0^2(u)+\vartheta_{14}^2\,\vartheta_{01}^2\,\vartheta_{34}^2(u)-\vartheta_{23}^2\,\vartheta_{01}^2\,\vartheta_{12}^2(u)-\vartheta_{14}^2\,\vartheta_2^2\,\vartheta_5^2(u),$$

where $\vartheta_2=\vartheta_2(0)$, etc.

(γ) There is however a biquadratic relation connecting the functions $\vartheta(u;\,AP_i)$ provided p be greater than 1. In the formula (§ 309)

$$\sum_i e^{\pi i|P_i|}\vartheta(u+v;\,A,P_i)\,\vartheta(u-v;\,A,P_i)\,\vartheta(a+b;\,A,P_i)\,\vartheta(a-b;\,A,P_i)$$
$$=\sum_i e^{\pi i|P_i|}\vartheta(u+b;\,A,P_i)\,\vartheta(u-b;\,A,P_i)\,\vartheta(a+v;\,A,P_i)\,\vartheta(a-v;\,A,P_i),$$

supposing the characteristic A to be chosen so that all the characteristics AP_i are even, as is possible (§ 299) by taking A suitably, substitute for

$$u+v,\quad u-v,\quad a+b,\quad a-b,\quad u+b,\quad u-b,\quad a+v,\quad a-v$$

respectively

$$u+v+w,\quad u-v,\quad a+b+w,\quad a-b,\quad u+b+w,\quad u-b,\quad a+v+w,\quad a-v;$$

then, putting $a=b=0$, we have

$$\sum_i e^{\pi i|P_i|}\vartheta(0;\,A,P_i)\,\vartheta(w;\,A,P_i)\,\vartheta(u-v;\,A,P_i)\,\vartheta(u+v+w;\,A,P_i)$$
$$=\sum_i e^{\pi i|P_i|}\vartheta(u;\,A,P_i)\,\vartheta(v;\,A,P_i)\,\vartheta(u+w;\,A,P_i)\,\vartheta(v+w;\,A,P_i);$$

herein put $w=\Omega_{P_1}$, $v=u+\Omega_{P_2}$, where P_1, P_2 are two of the characteristics belonging to the basis $P_1,\dots,P_p$ of the group (P); then we obtain

$$\sum_i\begin{pmatrix}P_1P_2\\P_i\end{pmatrix}e^{\pi i|P_i|}\vartheta(0;\,A,P_i)\vartheta(0;\,A,P_1,P_i)\vartheta(0;\,A,P_2,P_i)\vartheta(2u;\,A,P_1,P_2,P_i)$$
$$=\sum_i\begin{pmatrix}P_1P_2\\P_i\end{pmatrix}e^{\pi i|P_i|}\vartheta(u;\,A,P_i)\vartheta(u;\,A,P_1,P_i)\vartheta(u;\,A,P_2,P_i)\vartheta(u;\,A,P_1,P_2,P_i).$$

Now every characteristic of the group (P) can be given in one of the forms Q_s, Q_sP_1, Q_sP_2, $Q_sP_1P_2$, where Q_s becomes in turn all the characteristics of a group (Q) of 2^{p-2} characteristics; putting

$$\psi(u;\,Q_s)$$
$$=\begin{pmatrix}P_1P_2\\Q_s\end{pmatrix}e^{\pi i|Q_s|}\vartheta(u;\,A,Q_s)\,\vartheta(u;\,A,P_1,Q_s)\,\vartheta(u;\,A,P_2,Q_s)\,\vartheta(u;\,A,P_1,P_2,Q_s),$$

we immediately find

$$\psi(u;\,Q_s)=\psi(u;\,Q_s,P_1)=\psi(u;\,Q_s,P_2)=\psi(u;\,Q_s,P_1,P_2);$$

hence the equation just obtained can be written

$$\sum_{s=1}^{2^{p-2}}\psi(0;\,Q_s)\sum_{m=1}^{4}\frac{\vartheta(2u;\,A,Q_s,R_m)}{\vartheta(0;\,A,Q_s,R_m)}=4\sum_{s=1}^{2^{p-2}}\psi(u;\,Q_s),$$

where R_m has the four values 0, P_1, P_2, P_1+P_2.

Again, if in the formula (§ 309)

$$2^p\,\vartheta(u+v;\,A)\,\vartheta(u-v;\,A)=\sum_\epsilon\frac{\Phi(u,b;\,A,\epsilon)\,\Phi(a,v;\,A,\epsilon)}{\Phi(a,b;\,A,\epsilon)}$$

we add to u the half period Ω_{P_k}, we obtain, after putting $u=v$, $a=b=0$, the result

$$\vartheta(2u;\ A,\ P_k)\,\vartheta(0;\ A,\ P_k)=2^{-p}\binom{P_k}{A}\sum_{\epsilon}\frac{1}{\epsilon_k}\frac{\Phi(u,0;\ A,\epsilon)\,\Phi(0,u;\ A,\epsilon)}{\Phi(0,0;\ A,\epsilon)}$$

$$=2^{-p}\binom{P_k}{A}\sum_{\epsilon}\frac{1}{\epsilon_k}\frac{\Phi^2(u,0;\ A,\epsilon)}{\Phi(0,0;\ A,\epsilon)},$$

where

$$\Phi(u,0;\ A,\epsilon)=\sum_i\binom{P_i}{A}\epsilon_i\vartheta^2(u;\ AP_i);\quad \Phi(0,0;\ A,\epsilon)=\sum_i\binom{P_i}{A}\epsilon_i\vartheta^2(0;\ AP_i).$$

By substitution of the value of $\vartheta(2u;\ A,\ P_k)$ given by this formula, in the formula above, there results the biquadratic relation* connecting the functions $\vartheta(u;\ AP_i)$.

(δ) As an indication of another set of formulae, which are interesting as direct generalizations of the formulae for the elliptic function $\wp(u)$, the following may also be given. Let

$$\delta=\lambda_1\frac{\partial}{\partial v_1}+\ldots+\lambda_p\frac{\partial}{\partial v_p},$$

where $\lambda_1,\ \ldots,\ \lambda_p$ are undetermined quantities, $\delta\vartheta(v)=\vartheta'(v)$, $\delta^2\vartheta(v)=\vartheta''(v)$, and let

$$\wp(v;\ A)=-\delta^2\log\vartheta(v;\ A)=-[\vartheta(v;\ A)\,\vartheta''(v;\ A)-\vartheta'^2(v;\ A)]\div\vartheta^2(v;\ A);$$

then, differentiating the formula

$$2^p\vartheta(u+v;\ A)\,\vartheta(u-v;\ A)=\sum_{\epsilon}\frac{\Phi(u,b;\ A,\epsilon)\,\Phi(a,v;\ A,\epsilon)}{\Phi(a,b;\ A,\epsilon)}$$

twice in regard to v, and afterwards putting $v=0$ and $b=0$, we obtain

$$\wp(u;\ A)=\sum_i C_i\frac{\vartheta^2(u;\ AP_i)}{\vartheta^2(u;\ A)},$$

wherein

$$C_i=\binom{P_i}{A}\sum_{\epsilon}\epsilon_i\frac{\sum_j\binom{P_j}{A}\epsilon_j\vartheta^2(a;\ AP_j)\,\wp(a;\ AP_j)}{\sum_k\binom{P_k}{A}\epsilon_k\vartheta^2(a;\ AP_k)}$$

$$=\sum_{\epsilon}\epsilon_i\frac{\sum_j\epsilon_j\vartheta^2(a;\ AP_j)\,\wp(a;\ AP_j)}{\sum_k\epsilon_k\vartheta^2(a;\ AP_k)},$$

the 2^p quantities C_i being independent of u and of a. By this formula the function $\wp(u;\ A)$ is expressed linearly by the squares of 2^p theta quotients (cf. Chap. XI. § 217).

* Frobenius, *Crelle*, LXXXIX. (1880), p. 204. The general Göpel biquadratic relation has also been obtained algebraically (for Riemann theta functions) by Brioschi, *Annal. d. Mat.*, 2ª Ser., t. x. (1880—1882).

312. These propositions (§§ 309—311) are corollaries from the fact that the functions $\Phi(u, a;\ A, \epsilon)$ are linearly expressible by 2^{p-r} of them; we have considered the case $r=p$ at great length, on account of its importance.

Passing now to the case $r=p-1$, there is a linear relation connecting any three of the functions

$$\Phi(u, a;\ A, \epsilon)=\sum_{i=1}^{2^{p-1}}\binom{P_i}{A}\epsilon_i\vartheta(u+a;\ AP_i)\vartheta(u-a;\ AP_i).$$

There is one case in which we can immediately determine the coefficients in this relation; we have $\sigma=p-r=1$, $2^{2\sigma}=4$; there are thus four characteristics A, whereof three are even and one odd, which are such that all the 2^{p-1} characteristics (AP) are of the same character. Taking the single case in which these are all odd, we have

$$\Phi(u, a;\ A, \epsilon)=-\Phi(a, u;\ A, \epsilon), \quad \text{and} \quad \Phi(a, a;\ A, \epsilon)=0;$$

hence, if, in the existing relation

$$\lambda\Phi(u, a;\ A, \epsilon)+\mu\Phi(u, b;\ A, \epsilon)+\nu\Phi(u, c;\ A, \epsilon)=0,$$

wherein λ, μ, ν are independent of u, we put $u=a$, we infer

$$\mu : \nu=\Phi(c, a;\ A, \epsilon) : \Phi(a, b;\ A, \epsilon);$$

thus the relation is

$$\Phi(b, c;\ A, \epsilon)\Phi(u, a;\ A, \epsilon)+\Phi(c, a;\ A, \epsilon)\Phi(u, b;\ A, \epsilon)$$
$$+\Phi(a, b;\ A, \epsilon)\Phi(u, c;\ A, \epsilon)=0,$$

or

$$\sum_{i=1}^{2^{p-1}}\sum_{j=1}^{2^{p-1}}\binom{P_i}{A}\binom{P_j}{A}\epsilon_i\epsilon_j\psi(i, j)=0,$$

where

$$\begin{aligned}\psi(i, j)=&\vartheta(u+a;\ AP_i)\vartheta(u-a;\ AP_i)\vartheta(b+c;\ AP_j)\vartheta(b-c;\ AP_j)\\ &+\vartheta(u+b;\ AP_i)\vartheta(u-b;\ AP_i)\vartheta(c+a;\ AP_j)\vartheta(c-a;\ AP_j)\\ &+\vartheta(u+c;\ AP_i)\vartheta(u-c;\ AP_i)\vartheta(a+b;\ AP_j)\vartheta(a-b;\ AP_j).\end{aligned}$$

Adding together all the equations thus obtainable, by taking all the 2^{p-1} possible sets of values for the fourth roots of unity $\epsilon_1, \dots, \epsilon_{p-1}$, we obtain

$$\sum_{i=1}^{2^{p-1}} e^{\pi i|P_i|}\psi(i, i)=0.$$

For instance, when $p=1$, this is the so-called equation of three terms, from which all relations connecting the elliptic functions can be derived. When $p=2$, it is an equation of six terms and there are fifteen such equations, all expressed by

$$\sum_{a,b,c}\vartheta(u+a;\ A)\vartheta(u-a;\ A)\vartheta(b+c;\ A)\vartheta(b-c;\ A)$$
$$=-e^{\pi i|AB|}\sum_{a,b,c}\vartheta(u+a;\ B)\vartheta(u-a;\ B)\vartheta(b+c;\ B)\vartheta(b-c;\ B),$$

A and B being any two odd characteristics*.

* Cf. Frobenius, *Crelle*, xcvi. (1884), p. 107.

313. Taking next the case $r=p-2$, every 2^2+1, or 5, functions $\Phi(u, a; A, \epsilon)$ are connected by a linear relation. In this case there are sixteen characteristics A such that all the 2^{p-2} characteristics (AP) are of the same character, six of them being odd. Denoting the six odd characteristics in any order by $A_1, \dots, A_6$, and an even characteristic by A, there is an equation of the form

$$\lambda_1\Phi(u, a; A_1, \epsilon)+\lambda_2\Phi(u, a; A_2, \epsilon)+\lambda_3\Phi(u, a; A_3, \epsilon) \\ =\Phi(u, a; A_4, \epsilon)+\lambda\Phi(u, a; A, \epsilon);$$

putting herein $u=a$, this equation reduces to $\lambda\Phi(a, a; A, \epsilon)=0$, so that $\lambda=0$. The other coefficients can also be determined; for, if $C=A_2A_3$, we have (§ 306, Ex. i.),

$$\Phi(u+\Omega_C, a; A, \epsilon)=e^{2\lambda(u; C)}\begin{pmatrix}A_2A_3\\ A\end{pmatrix}\Phi(u, a; AA_2A_3, \epsilon);$$

putting therefore for u, in the equation above, the value $a+\Omega_C$, where $C=A_2A_3$, and recalling (§ 303) that $A_1A_2A_3$, $A_4A_2A_3$ are even characteristics, we infer

$$\lambda_1\begin{pmatrix}A_2A_3\\ A_1\end{pmatrix}\Phi(a, a; A_1A_2A_3, \epsilon)=\begin{pmatrix}A_2A_3\\ A_4\end{pmatrix}\Phi(a, a; A_4A_2A_3, \epsilon).$$

Proceeding similarly with the characteristics A_3A_1, A_1A_2 in turn, instead of A_2A_3, we finally obtain

$$\begin{pmatrix}A_2A_3\\ A_1A_4\end{pmatrix}\Phi(a, a; A_4A_2A_3)\Phi(u, a; A_1)+\begin{pmatrix}A_3A_1\\ A_2A_4\end{pmatrix}\Phi(a, a; A_4A_3A_1)\Phi(u, a; A_2) \\ +\begin{pmatrix}A_1A_2\\ A_3A_4\end{pmatrix}\Phi(a, a; A_4A_1A_2)\Phi(u, a; A_3)=\Phi(a, a; A_1A_2A_3)\Phi(u, a; A_4),$$

where, for greater brevity, the ϵ is omitted in the sign of the function Φ (cf. Ex. viii., § 289).

Ex. For $p=2$, deduce the result

$$\vartheta_{34}\vartheta_{34}(2v)\,\vartheta_{02}(u+v)\,\vartheta_{02}(u-v)-\vartheta_{03}\vartheta_{03}(2v)\,\vartheta_{24}(u+v)\,\vartheta_{24}(u-v)+\vartheta_{23}\vartheta_{23}(2v)\,\vartheta_{04}(u+v)\,\vartheta_{04}(u-v) \\ =\vartheta_5\vartheta_5(2v)\,\vartheta_1(u+v)\,\vartheta_1(u-v),$$

where $\vartheta_{34}=\vartheta_{34}(0)$, etc. When $v=0$ this is an equation connecting the squares of $\vartheta_{02}(u)$, $\vartheta_{24}(u)$, $\vartheta_{04}(u)$, $\vartheta_1(u)$.

314. The results of §§ 309, 310 are capable of a generalization, obtainable by a repetition of the argument there employed.

A group of 2^k pairwise syzygetic characteristics may be considered as arising by the composition of two such groups. Take $k, =r+s$, characteristics $P_1, \dots, P_r, Q_1, \dots, Q_s$, every two of which are syzygetic; form the groups

$$(P)=0,\ P_1, \dots, P_r,\ P_1P_2, \dots, P_1P_2P_3, \dots$$
$$(Q)=0,\ Q_1, \dots, Q_s,\ Q_1Q_2, \dots, Q_1Q_2Q_3, \dots$$

respectively of 2^r and 2^s characteristics; the 2^{r+s} combinations $R_{i,j}=P_iQ_j$ form a group (R) of 2^{r+s} pairwise syzygetic characteristics; for distinctness the fourth roots of unity associated respectively with $P_1, \ldots, P_r, Q_1, \ldots, Q_s$, may be denoted by $\epsilon_1, \ldots, \epsilon_r, \zeta_1, \ldots, \zeta_s$; then with $P_{i,i_1}, Q_{j,j_1}, R_{i,j}$ will be associated the respective quantities

$$\epsilon_{i,i_1}=\epsilon_i\epsilon_{i_1}\begin{pmatrix}P_i\\P_{i_1}\end{pmatrix},\quad \zeta_{j,j_1}=\zeta_j\zeta_{j_1}\begin{pmatrix}Q_j\\Q_{j_1}\end{pmatrix},\quad E_{i,j}=\epsilon_i\zeta_j\begin{pmatrix}P_i\\Q_j\end{pmatrix};$$

thus if A be any characteristic

$$\begin{pmatrix}R_{i,j}\\A\end{pmatrix}E_{i,j}=\begin{pmatrix}P_iQ_j\\A\end{pmatrix}\epsilon_i\zeta_j\begin{pmatrix}P_i\\Q_j\end{pmatrix}=\begin{pmatrix}Q_j\\A\end{pmatrix}\zeta_j\cdot\begin{pmatrix}P_i\\AQ_j\end{pmatrix}\epsilon_i=\begin{pmatrix}P_i\\A\end{pmatrix}\epsilon_i\cdot\begin{pmatrix}Q_j\\AP_i\end{pmatrix}\zeta_j.$$

Therefore, using the symbol Ψ for a sum extending to the whole group (PQ),

$$\begin{aligned}\Psi(u, a;\ A, E)&=\sum_{i,j}\begin{pmatrix}R_{i,j}\\A\end{pmatrix}E_{i,j}\,\vartheta(u+a;\ AR_{i,j})\,\vartheta(u-a;\ AR_{i,j})\\&=\sum_j\begin{pmatrix}Q_j\\A\end{pmatrix}\zeta_j\sum_i\begin{pmatrix}P_i\\AQ_j\end{pmatrix}\epsilon_i\,\vartheta(u+a;\ AQ_jP_i)\,\vartheta(u-a;\ AQ_jP_i)\\&=\sum_j\begin{pmatrix}Q_j\\A\end{pmatrix}\zeta_j\,\Phi(u, a;\ AQ_j, \epsilon),\end{aligned}$$

where Φ denotes a sum extending to the 2^r terms corresponding to the characteristics of the group (P).

By the theorem of § 307 the functions obtainable from $\Psi(u, a;\ A, E)$ by taking different values of a and A, and the same group (PQ), are linearly expressible by $2^{p-r-s}=2^{\sigma-s}$ of them, if $\sigma=p-r$, with coefficients independent of u. The 2^s functions $\Phi(u, a;\ AQ_j, \epsilon)$, obtained by varying a and Q_j, are themselves expressible by 2^σ of them.

Thus, taking $r+s=p$, or $s=\sigma$, we have

$$\Psi(u, v;\ A, E)\,\Psi(a, b;\ A, E)=\Psi(u, b;\ A, E)\,\Psi(a, v;\ A, E)$$

or

$$\begin{aligned}&\sum_{j,j_1}\begin{pmatrix}Q_j\\A\end{pmatrix}\begin{pmatrix}Q_{j_1}\\A\end{pmatrix}\zeta_j\zeta_{j_1}\Phi(u, v;\ AQ_j, \epsilon)\,\Phi(a, b;\ AQ_{j_1}, \epsilon)\\&\qquad=\sum_{j,j_1}\begin{pmatrix}Q_j\\A\end{pmatrix}\begin{pmatrix}Q_{j_1}\\A\end{pmatrix}\zeta_j\zeta_{j_1}\Phi(u, b;\ AQ_j, \epsilon)\,\Phi(a, v;\ AQ_{j_1}, \epsilon);\end{aligned}$$

taking for $\zeta_1, \ldots, \zeta_s$ all the possible 2^s values, and adding the 2^s equations of this form, we obtain

$$\sum_{j=1}^{2^s}e^{\pi i|Q_j|}\Phi(u, v;\ AQ_j, \epsilon)\,\Phi(a, b;\ AQ_j, \epsilon)=\sum_{j=1}^{2^s}e^{\pi i|Q_j|}\Phi(u, b;\ AQ_j, \epsilon)\,\Phi(a, v;\ AQ_j, \epsilon).$$

Suppose now that $A_1, \ldots, A_\lambda$ are the $2^{2\sigma}$ characteristics satisfying the r relations $|X, P_i|\equiv|P_i|$, (mod. 2), and let $C_m=A_1A_m$; then $|C_m, P_i|\equiv 0$; hence, by the formulae of § 306, Ex. i., adding the half period Ω_{C_m} to u and b, and dividing by the factor $e^{\pi i|C_m, A|}$, we have

$$\begin{aligned}&\sum_{j=1}^{2^\sigma}e^{\pi i|C_mQ_j|}\Phi(u, v;\ AC_mQ_j, \epsilon)\,\Phi(a, b;\ AC_mQ_j, \epsilon)\\&\qquad=\sum_{j=1}^{2^\sigma}e^{\pi i|Q_j|+\pi i|C_m, Q_j|}\Phi(u, b;\ AQ_j, \epsilon)\,\Phi(a, v;\ AQ_j, \epsilon);\end{aligned}$$

taking, here, all the $2^{2\sigma}$ values of C_m in turn, and adding the equations, noticing that

$$\sum_{m=1}^{2^{2\sigma}}e^{\pi i|C_m, Q_j|},\quad =e^{\pi i|A_1, Q_j|}\sum_{m=1}^{2^{2\sigma}}e^{\pi i|A_m, Q_j|},$$

is zero because Q_j is not a characteristic of the group (P), except for the special value $Q_j=0$, when its value is $2^{2\sigma}$ (§ 300), we derive the formula

$$2^{2\sigma}\Phi(u, b;\ A, \epsilon)\,\Phi(a, v;\ A, \epsilon)=\sum_{j=1}^{2^\sigma}\sum_{m=1}^{2^{2\sigma}}e^{\pi i|C_mQ_j|}\Phi(u, v;\ AC_mQ_j, \epsilon)\,\Phi(a, b;\ AC_mQ_j, \epsilon);$$

now, as already remarked (§ 298, Ex.), if a characteristic S which is syzygetic with every characteristic of the group (P) be added to each of the $2^{2\sigma}$ characteristics $A_1, \dots, A_\lambda$, the result is another set of $2^{2\sigma}$ characteristics satisfying the same congruences, $|X, P_i| \equiv |P_i|$, as the set $A_1, \dots, A_\lambda$, and incongruent mod. (P); thus, taking a fixed value of j, we have $C_m Q_j \equiv C_n P_i$, where, as C_m takes its $2^{2\sigma}$ values, C_n also takes the same values in another order, and P_i varies with m. Hence (Ex. iii. § 306) we have

$$e^{\pi i|C_m Q_j|}\Phi(u, v; AC_mQ_j, \epsilon)\Phi(a, b; AC_mQ_j, \epsilon) = e^{\pi i|C_nP_i|}\Phi(u, v; AC_nP_i, \epsilon)\Phi(a, b; AC_nP_i, \epsilon),$$
$$= e^{\pi i|C_n|}\Phi(u, v; AC_n, \epsilon)\Phi(a, b; AC_n, \epsilon),$$

and

$$\sum_{m=1}^{2^{2\sigma}} e^{\pi i|C_mQ_j|}\Phi(u, v; AC_mQ_j, \epsilon)\Phi(a, b; AC_mQ_j, \epsilon)$$
$$= \sum_{m=1}^{2^{2\sigma}} e^{\pi i|C_m|}\Phi(u, v; AC_m, \epsilon)\Phi(a, b; AC_m, \epsilon),$$

and therefore, finally, dividing by a factor 2^σ (there being 2^σ characteristics in (Q)), we have

$$2^\sigma\Phi(u, b; A, \epsilon)\Phi(a, v; A, \epsilon) = \sum_{m=1}^{2^{2\sigma}} e^{\pi i|A_1A_m|}\Phi(u, v; AA_1A_m, \epsilon)\Phi(a, b; AA_1A_m, \epsilon).$$

When $\sigma = p$, this becomes the formula of § 309. We infer that the functions $\Phi(u, a; A, \epsilon)$ are connected by the same relations as the functions of the form $\vartheta(u+a; A)\vartheta(u-a; A)$ when the number of variables (in the latter functions) is σ.

Ex. Prove that, with the notation of the text,

$$2^\sigma\Phi(u, v; A, \epsilon) = \sum_\zeta \frac{\Psi(u, b; A, E)\Psi(a, v; A, E)}{\Psi(a, b; A, E)}.$$

315. The formula of the last Article is capable of a further generalization. Let (R) be a group of 2^μ characteristics, formed with $R_1, \dots, R_\mu$ as basis, which satisfy the conditions

$$|R, P_1| \equiv 0, \dots, |R, P_r| \equiv 0.$$

Thus (P) is a sub-group of (R); the group (R) consists of (P), together with groups (RP), whereof the characteristics R form a group of $2^{\mu-r}$ characteristics, whose constituents are incongruent for the modulus (P). The basis of this sub-group of $2^{\mu-r}$ characteristics will be denoted by $R_1, \dots, R_{\mu-r}$. The total number of characteristics satisfying the prescribed conditions is 2^{2p-r}; thus $\mu \not> 2p-r$, and, when $\mu < 2p-r$ the given conditions are not enough to ensure that a characteristic belongs to the group (R).

Then, if F, G be arbitrary characteristics, and R_i become in turn all the characteristics of a group of $2^{\mu-r}$ characteristics of the group (R) which are incongruent mod. (P), we have

$$2^{p-\mu}\sum_{i=1}^{2^{\mu-r}} e^{\pi i|FGR_i|}\Phi(u, b; GR_i, \epsilon)\Phi(a, v; GR_i, \epsilon)$$
$$= 2^{p-\mu-\sigma}\sum_{i=1}^{2^{\mu-r}} e^{\pi i|FGR_i|}\sum_{m=1}^{2^{2\sigma}} e^{\pi i|C_m|}\Phi(u, v; GR_iC_m, \epsilon)\Phi(a, b; GR_iC_m, \epsilon),$$

where $C_m = A_1A_m$. Since $|R_i, P| \equiv 0$, the constituents of the set R_iC_m, where R_i is a fixed characteristic and $m = 1, 2, \dots, 2^{2\sigma}$, are in some order congruent (mod. (P)) to the constituents of the set C_m; hence (§ 306, Ex. iii.) the series is equal to

$$2^{r-\mu}\sum_{m=1}^{2^{2\sigma}}\sum_{i=1}^{2^{\mu-r}} e^{\pi i|FGR_i|+\pi i|R_iC_m|}\Phi(u, v; GC_m, \epsilon)\Phi(a, b; GC_m, \epsilon),$$

$$= 2^{r-\mu}\sum_{m=1}^{2^{2\sigma}} e^{\pi i|FG|+\pi i|C_m|}\left(\sum_{i=1}^{2^{\mu-r}} e^{\pi i|FGC_m, R_i|}\right)\Phi(u, v; GC_m, \epsilon)\Phi(a, b; GC_m, \epsilon);$$

now $\sum_{i=1}^{2^{\mu-r}} e^{\pi i|L, R_i}$ is zero, unless $|L, R_i| \equiv 0$ (mod. 2) for every characteristic R_i, in which case its value is $2^{\mu-r}$; thus the series is equal to

$$\Sigma e^{\pi i|FG|+\pi i|FGS_m|}\,\Phi(u, v;\ FS_m, \epsilon)\,\Phi(a, b;\ FS_m, \epsilon),$$

where S_m satisfies the conditions involved in $|S_m, R_i| \equiv 0$, $FGC_m \equiv S_m$, namely the conditions

$$|S_m, R_1| \equiv 0, \ldots, |S_m, R_{\mu-r}| \equiv 0, |FGS_m, P_1| \equiv 0, \ldots, |FGS_m, P_r| \equiv 0;$$

the number of characteristics satisfying these μ conditions is $2^{2p-\mu}$; the number of these which are incongruent for the modulus (P) is $2^{2p-\mu-r} = 2^{2\sigma+r-\mu}$.

Suppose now that $|FG, P_1| \equiv 0, \ldots, |FG, P_r| \equiv 0$; then the characteristics S_m constitute a group satisfying the conditions $|S_m, R| \equiv 0$, where R becomes in turn all the 2^μ characteristics of the group (R). The group (S) of the characteristics S_m may be obtained by combining the characteristics of the group (P) with the characteristics of a group of $2^{2\sigma-\mu+r}$ characteristics which also satisfy these conditions and are incongruent for the modulus (P); putting $\mu = r + \rho$, we have therefore*

$$2^{\sigma-\rho} \sum_{i=1}^{2^\rho} e^{\pi i|FGR_i|}\,\Phi(u, b;\ GR_i, \epsilon)\,\Phi(a, v;\ GR_i, \epsilon)$$
$$= e^{\pi i|FG|} \sum_{m=1}^{2^{2\sigma-\rho}} e^{\pi i|FGS_m|}\,\Phi(u, v;\ FS_m, \epsilon)\,\Phi(a, b;\ FS_m, \epsilon).$$

In this equation each of R_i, S_m represents the characteristics, respectively of the groups (R), (S), which are incongruent mod. (P). But it is easy to see (§ 306, Ex. iii.) that we may also regard R_i, S_m as becoming equal to *all* the characteristics, respectively, of the groups (R), (S).

316. We have shewn in Chap. XV. (§ 286, Ex. i.) that a certain addition formula can be obtained for the cases $p = 1, 2, 3$ by the application of one rule. We give now a generalization of that rule, which furnishes results for any value of p.

Suppose that among the $2^{2\sigma}$ characteristics $A_1, A_2, \ldots, A_\lambda$ which, for any Göpel system (P) of 2^r characteristics, satisfy the conditions

$$|X, P_1| \equiv |P_1|, \ldots, |X, P_r| \equiv |P_r|,$$

we have $k + 1 = 2^\sigma + 1$ characteristics $B_1, \ldots, B_k, B$, of which B is even, which are such that, when i is not equal to j, BB_iB_j is an odd characteristic; as follows from § 302 of this chapter, and § 286, Ex. i., Chap. XV., this is certainly possible when $\sigma = 1$, or 2, or 3; and, since

$$|BB_iB_j, P| \equiv |B, P| + |B_i, P| + |B_j, P| \equiv |P|,$$

* The formula is given by Frobenius, *Crelle*, XCVI. p. 95, being there obtained from the formula of § 310, which is a particular case of it. The formula is generalised by Braunmühl to theta functions whose characteristics are n-th parts of integers in *Math. Annal.* XXXVII. (1890), p. 98. The formula includes previous formulae of this chapter.

the characteristics BB_iB_j will be among the set $A_1, \ldots, A_\lambda$, so that all characteristics congruent to BB_iB_j (mod. (P)) are also odd. Then by § 307 there exists an equation of the form*

$$\lambda\Phi(u, c;\ B, \epsilon) = \sum_{m=1}^{k} \lambda_m \Phi(u, a;\ B_m, \epsilon),$$

wherein the coefficients $\lambda, \lambda_1, \ldots, \lambda_k$, are independent of u. Put in this equation $u = a + \Omega_{BB_i}$; then we infer (§ 306, Ex. i.)

$$\lambda\Phi(a, c;\ B_i, \epsilon) = \lambda_i \Phi(a, a;\ B, \epsilon);$$

hence we have

$$\Phi(a, a;\ B, \epsilon)\,\Phi(u, c;\ B, \epsilon) = \sum_{m=1}^{k} e^{\pi i |BB_m|} \Phi(a, c;\ B_m, \epsilon)\,\Phi(u, a;\ B_m, \epsilon),$$

which is the formula in question†.

Adding the 2^r equations obtainable from this formula by taking the different sets of values for the fourth roots of unity $\epsilon_1, \ldots, \epsilon_r$, there results

$$\sum_{i=1}^{2^r} e^{\pi i |P_i|} \psi_0(BP_i) = \sum_{m=1}^{2^\sigma} \sum_{i=1}^{2^r} e^{\pi i |BB_m| + \pi i |P_i|} \psi(B_m P_i),$$

where

$$\psi_0(BP_i) = \vartheta(0;\ BP_i)\,\vartheta(2a;\ BP_i)\,\vartheta(u+c;\ BP_i)\,\vartheta(u-c;\ BP_i),$$

$$\psi(B_mP_i) = \vartheta(a+c;\ B_mP_i)\,\vartheta(a-c;\ B_mP_i)\,\vartheta(u+a;\ B_mP_i)\,\vartheta(u-a;\ B_mP_i).$$

Herein we may replace the arguments

$$2a,\quad u+c,\quad u-c,\quad a+c,\quad a-c,\quad u+a,\quad u-a$$

respectively by

$$U,\ V,\ W,\ \tfrac{1}{2}(U+V-W),\ \tfrac{1}{2}(U-V+W),\ \tfrac{1}{2}(U+V+W),\ \tfrac{1}{2}(-U+V+W),$$

and thence, in case $p = 2$, or $p = 3$, obtain the formula of Ex. xi., § 286, Chap. XV.

Or we may put $a = 0$, and so obtain

$$\sum_{i=1}^{2^r} e^{\pi i |P_i|} \vartheta^2(0;\ BP_i)\,\vartheta(u+c;\ BP_i)\,\vartheta(u-c;\ BP_i) = \sum_{m=1}^{2^\sigma} \sum_{i=1}^{2^r} e^{\pi i |B_m, BP_i|} \vartheta^2(u;\ B_mP_i)\,\vartheta^2(c;\ B_mP_i).$$

Other developments are clearly possible, as in § 286, Chap. XV.

Ex. When $\sigma = 1$ there are three even Göpel systems, and one odd; let (BP), (B_1P), (B_2P) be the three even Göpel systems; then we have

$$\Phi(a, a;\ B, \epsilon)\,\Phi(u, c;\ B, \epsilon) = e^{\pi i |BB_1|} \Phi(a, c;\ B_1, \epsilon)\,\Phi(u, a;\ B_1, \epsilon) + e^{\pi i |BB_2|} \Phi(a, c;\ B_2, \epsilon)\,\Phi(u, a;\ B_2, \epsilon),$$

* We may, if we wish, take, instead of the characteristic B on the left hand, any characteristic A such that $|A, P_i| \equiv |P_i|$, $(i = 1, \ldots, 2^r)$.

† For similar results, cf. Frobenius, *Crelle*, LXXXIX. (1880), pp. 219, 220, and Noether, *Math. Annal.* XVI. (1880), p. 327.

where $\Phi(u, a; B, \epsilon)$ consists of 2^{p-1} terms; for instance when $p=1$ we obtain

$$\vartheta(0; B)\vartheta(2a; B)\vartheta(u+c; B)\vartheta(u-c; B)$$
$$=e^{\pi i|BB_1|}\vartheta(a+c; B_1)\vartheta(a-c; B_1)\vartheta(u+a; B_1)\vartheta(u-a; B_1)$$
$$+e^{\pi i|BB_2|}\vartheta(a+c; B_2)\vartheta(a-c; B_2)\vartheta(u+a; B_2)\vartheta(u-a; B_2).$$

317. *Ex.* i. If P be a fixed characteristic and $\Psi(u; A)$ denote the function $\vartheta(u; A)\vartheta(u; A+P)$, prove that

$$\Psi(u+\Omega_P; A)=e^{\frac{1}{2}\pi i|P|+2\lambda(u; P)}\Psi(u; A),$$

and

$$\Psi(u+\Omega_Q; A)/\Psi(u+\Omega_Q; B)=\begin{pmatrix}Q\\AB\end{pmatrix}\Psi(u; A+Q)/\Psi(u; B+Q).$$

Hence, if $B_1, \ldots, B_k, B$ be $k+1=2^{p-1}+1$ characteristics each satisfying the condition $|X, P|\equiv|P|$, such that, when i is not equal to j, BB_iB_j is odd, we have (§ 307) an equation

$$\lambda\Psi(u; A)=\sum_{m=1}^{2^{p-1}}\lambda_m\Psi(u; B_m),$$

where A is any other even characteristic such that $|A, P|\equiv|P|$; putting $u=\Omega_B+\Omega_{B_i}$, we obtain

$$\lambda\begin{pmatrix}BB_i\\AB_i\end{pmatrix}\Psi(0; A+B+B_i)=\lambda_i\Psi(0; B+2B_i)=\lambda_i\begin{pmatrix}P\\B_i\end{pmatrix}\Psi(0; B);$$

therefore

$$\Psi(0; B)\Psi(u; A)=\sum_{m=1}^{2^{p-1}}\begin{pmatrix}BB_m\\AB_m\end{pmatrix}\begin{pmatrix}P\\B_m\end{pmatrix}\Psi(0; A+B+B_m)\Psi(u; B_m).$$

Ex. ii. Obtain applications of the formula of Ex. i. when $p=2, 3, 4$; in these cases $\sigma, =p-1, =1, 2, 3$ respectively, so that we know how to choose the characteristics $B_1, \ldots, B_k, B$ (Ex. i., § 286, Chap. XV., and § 302 of this Chap.).

Ex. iii. From the formula (§ 309)

$$\vartheta(u+b; A)\vartheta(u-b; A)\vartheta(a+v; A)\vartheta(a-v; A)$$
$$=\frac{1}{2^p}\sum_R e^{\pi i|AR|}\vartheta(u+v; R)\vartheta(u-v; R)\vartheta(a+b; R)\vartheta(a-b; R),$$

by putting $a+\Omega_P$ for a, and $b=v=0$, we deduce

$$\vartheta^2(u; A)\vartheta^2(a; AP)=2^{-p}\sum_R e^{\pi i|AR|}\begin{pmatrix}P\\AR\end{pmatrix}\vartheta^2(u; R)\vartheta^2(a; PR),$$

where A, P are any half-integer characteristics and R becomes all the 2^{2p} half-integer characteristics in turn; putting RP for R we also have, from this equation,

$$\vartheta^2(u; A)\vartheta^2(a; AP)=2^{-p}\sum_R e^{\pi i|AR|}\begin{pmatrix}P\\AR\end{pmatrix}e^{\pi i|AR, P|}\vartheta^2(u; RP)\vartheta^2(a; R);$$

therefore

$$[1+e^{\pi i|A, P|+\pi i|P|}]\vartheta^2(0; A)\vartheta^2(0; AP)$$
$$=2^{-p}\sum_R e^{\pi i|AR|}\begin{pmatrix}P\\AR\end{pmatrix}[1+e^{\pi i|P|+\pi i|R, P|}]\vartheta^2(0; R)\vartheta^2(0; PR).$$

The values of R may be divided into two sets, according as $|R, P|+|P|\equiv 1$ (mod. 2), or $\equiv 0$; for the values of the former set the corresponding terms vanish; the values of R for which $|R, P|+|P|\equiv 0$ (mod. 2) may be either odd or even; for the odd values the zero values of the corresponding theta functions are zero; there remain then (§ 299) only $2 . 2^{p-2}(2^{p-1}+1)$ terms on the right hand corresponding to values of R which satisfy the conditions $|R|\equiv|RP|\equiv 0$ (mod. 2); these values are divisible into pairs denoted by $R=E$, $R=EP$; for such values $1+e^{\pi i|R,\,P|+\pi i|P|}=2$, and

$$e^{\pi i|AE|}\begin{pmatrix}P\\AE\end{pmatrix}+e^{\pi i|AEP|}\begin{pmatrix}P\\AEP\end{pmatrix}$$
$$=e^{\pi i|AE|}\begin{pmatrix}P\\AE\end{pmatrix}[1+e^{\pi i|AE,\,P|}]=e^{\pi i|AE|}\begin{pmatrix}P\\AE\end{pmatrix}[1+e^{\pi i|A,\,P|+\pi i|P|}];$$

thus, provided $|A,\,P|+|P|\equiv 0$ (mod. 2),

$$\vartheta^2(\,;\,A)\,\vartheta^2(\,;\,AP)=2^{-(p-1)}\sum_E e^{\pi i|AE|}\begin{pmatrix}P\\AE\end{pmatrix}\vartheta^2(\,;\,E)\,\vartheta^2(\,;\,EP), \qquad \text{(i)},$$

wherein $\vartheta^2(\,;\,A)$ denotes $\vartheta^2(0;\,A)$, etc., and, on the right hand there are $2^{p-2}(2^{p-1}+1)$ terms corresponding to values of E for which $|E|\equiv|EP|\equiv 0$ (mod. 2), only one of the two values, E, EP, satisfying these conditions being taken.

Putting $P=0$, $u=a$, in the second equation of this example, we deduce in order

$$\vartheta^4(u;\,A)=2^{-p}\sum_R e^{\pi i|AR|}\vartheta^4(u;\,R);\quad \vartheta^4(u;\,AP)=2^{-p}\sum_R e^{\pi i|APR|}\vartheta^4(u;\,R);$$

so that, by addition,

$$\vartheta^4(u;\,A)+e^{\pi i|A,\,P|}\vartheta^4(u;\,AP)=2^{-p}\sum_R e^{\pi i|AR|}[1+e^{\pi i|P|+\pi i|R,\,P|}]\,\vartheta^4(u;\,R);$$

thus, as before,

$$\vartheta^4(\,;\,A)+e^{\pi i|A,\,P|}\vartheta^4(\,;\,AP)=2^{-(p-1)}\sum_E e^{\pi i|AE|}\{\vartheta^4(\,;\,E)+e^{\pi i|A,\,P|}\vartheta^4(\,;\,EP)\}, \qquad \text{(ii)}.$$

Ex. iv. Taking $p=2$, let $(P)=0$, P_1, P_2, P_1P_2 be a Göpel group of even characteristics*; let B_1, B_2, B_1B_2 be such characteristics (§ 297) that the Göpel systems (P), (B_1P), (B_2P), (B_1B_2P) constitute all the sixteen characteristics; each of the systems (B_1P), (B_2P), (B_1B_2P) contains two odd characteristics and two even characteristics. Then, in the formulae (i), (ii) of Ex. iii., if P denote any one of the three characteristics P_1, P_2, P_1P_2, the conditions for the characteristics E are $|E,\,P|\equiv|P|\equiv 0$, $|E|\equiv 0$; the $2\,.\,2^{p-2}(2^{p-1}+1)$, $=6$, solutions of these conditions must consist of 0, Q, B and P, QP, BP, where Q is defined by the condition that the characteristics 0, Q, P, QP constitute the group (P), and B is a certain even characteristic chosen from one of the systems (B_1P), (B_2P), (B_1B_2P). Hence, when $P=P_1$, we may, without loss of generality, take for the $2^{p-2}(2^{p-1}+1)=3$ values of E which give rise to different terms in the series (i), (ii), the values 0, P_2, B_1; similarly, when $P=P_2$, we have, for the values of E, $E=0$, P_1, B_2; and when $P=P_1P_2$, $E=0$, P_1, B_1B_2; taking A to be respectively† B_1, B_2, B_1B_2 in these cases, we obtain the six equations

$$\begin{pmatrix}P_1\\B_1\end{pmatrix}\vartheta^2(\,;\,0)\,\vartheta^2(\,;\,P_1)+e^{\pi i|B_1P_2|}\begin{pmatrix}P_1\\B_1P_2\end{pmatrix}\vartheta^2(\,;\,P_2)\,\vartheta^2(\,;\,P_1P_2)-\vartheta^2(\,;\,B_1)\,\vartheta^2(\,;\,B_1P_1)=0,$$

$$\vartheta^4(\,;\,0)+\vartheta^4(\,;\,P_1)+e^{\pi i|B_1P_2|}[\vartheta^4(\,;\,P_2)+\vartheta^4(\,;\,P_2P_1)]-[\vartheta^4(\,;\,B_1)+\vartheta^4(\,;\,B_1P_1)]=0,$$

$$\begin{pmatrix}P_2\\B_2\end{pmatrix}\vartheta^2(\,;\,0)\,\vartheta^2(\,;\,P_2)+e^{\pi i|B_2P_1|}\begin{pmatrix}P_2\\B_2P_1\end{pmatrix}\vartheta^2(\,;\,P_1)\,\vartheta^2(\,;\,P_1P_2)-\vartheta^2(\,;\,B_2)\,\vartheta^2(\,;\,B_2P_2)=0,$$

$$\vartheta^4(\,;\,0)+\vartheta^4(\,;\,P_2)+e^{\pi i|B_2P_1|}[\vartheta^4(\,;\,P_1)+\vartheta^4(\,;\,P_1P_2)]-[\vartheta^4(\,;\,B_2)+\vartheta^4(\,;\,B_2P_2)]=0,$$

$$\begin{pmatrix}P_1P_2\\B_1B_2\end{pmatrix}\vartheta^2(\,;\,0)\,\vartheta^2(\,;\,P_1P_2)+e^{\pi i|B_1B_2P_1|}\begin{pmatrix}P_1P_2\\B_1B_2P_1\end{pmatrix}\vartheta^2(\,;\,P_1)\,\vartheta^2(\,;\,P_2)$$
$$-\vartheta^2(\,;\,B_1B_2)\,\vartheta^2(\,;\,B_1B_2P_1P_2)=0,$$

$$\vartheta^4(\,;0)+\vartheta^4(\,;P_1P_2)+e^{\pi i|B_1B_2P_1|}[\vartheta^4(\,;P_1)+\vartheta^4(\,;P_2)]-[\vartheta^4(\,;B_1B_2)+\vartheta^4(;B_1B_2P_1P_2)]=0,$$

* There are six such groups (Ex. iv. § 289).

† We easily find $|B_1B_2P_1|\equiv|B_1B_2P_2|\equiv-|B_1B_2|$. Thus the case when B_1B_2 is odd is included by writing B_1P_1 in place of B_1.

wherein $e^{\pi i|B_1P_2|}=e^{\pi i|B_2P_1|}=e^{\pi i|B_1B_2P_1}=-1$. These formulae express the zero values of all the even theta functions in terms of the four $\vartheta(;\,0)$, $\vartheta(;\,P_1)$, $\vartheta(;\,P_2)$, $\vartheta(;\,P_1P_2)$. Thus for instance they can be expressed in terms of ϑ_5, ϑ_{34}, ϑ_{12}, ϑ_0; the equations have been given in Ex. iii., § 289, Chap. XV.

Ex. v. We have in Chap. XVI. (§ 291) obtained the formula

$$\vartheta(u-v;\,q)\,\vartheta(u+v;\,r)=\vartheta\left[u-v;\,\binom{q'}{q}\right]\vartheta\left[u+v;\,\binom{r'}{r}\right]$$
$$=\sum_{\epsilon'}\vartheta_1\left[u;\,\begin{matrix}\tfrac{1}{2}(\epsilon'+q'+r')\\ q+r\end{matrix}\right]\vartheta_1\left[v;\,\begin{matrix}\tfrac{1}{2}(\epsilon'-q'+r')\\ -q+r\end{matrix}\right],$$

where ϵ' represents a set of p integers, each either 0 or 1, and has therefore 2^p values.

Suppose now that q, r represent the same half-integer characteristic, $=\tfrac{1}{2}\binom{c'}{c}+\tfrac{1}{2}\binom{0}{k_\alpha}$, $=C+K_\alpha$, say; then we immediately find

$$\vartheta_1\left[u;\,\begin{matrix}\tfrac{1}{2}(\epsilon'+q'+r')\\ q+r\end{matrix}\right]\vartheta_1\left[v;\,\begin{matrix}\tfrac{1}{2}(\epsilon'-q'+r')\\ -q+r\end{matrix}\right]=\vartheta_1\left[u;\,\begin{matrix}\tfrac{1}{2}\epsilon'c'\\ k_\alpha\end{matrix}\right].\,e^{\pi icc'}\vartheta_1\left[v;\,\begin{matrix}\tfrac{1}{2}\epsilon'\\ c\end{matrix}\right],$$

where $\epsilon'c'$ denotes the row of p integers, each either 0 or 1, which are given by $(\epsilon'c')_i\equiv\epsilon_i'+c_i'$ (mod. 2); herein the factor $e^{\pi icc'}\vartheta_1\left[v;\,\begin{matrix}\tfrac{1}{2}\epsilon'\\ c\end{matrix}\right]$ is independent of k_α. For K_α we take now, in turn, the constituents

$$0,\ K_1,\ K_2,\ \dots,\ K_p,\ K_1K_2,\ \dots,\ K_1K_2K_3,\ \dots$$

of a Göpel set of 2^p characteristics, in which

$$K_1=\tfrac{1}{2}\begin{pmatrix}0,\,0,\,0,\,\dots\\ 1,\,0,\,0,\,\dots\end{pmatrix},\quad K_2=\tfrac{1}{2}\begin{pmatrix}0,\,0,\,0,\,\dots\\ 0,\,1,\,0,\,\dots\end{pmatrix},\ \dots,\quad K_p=\tfrac{1}{2}\begin{pmatrix}0,\,\dots,\,0,\,0\\ 0,\,\dots,\,0,\,1\end{pmatrix};$$

then denoting $\vartheta[u+v;\,CK_\alpha]\,\vartheta[u-v;\,CK_\alpha]$ by $[CK_\alpha]$, we obtain 2^p equations which are all included in the equation

$$([CK_1],\,\dots,\,[CK_s])=J\left(e^{\pi icc'}\vartheta_1\left[v;\,\begin{matrix}\tfrac{1}{2}\epsilon_1'\\ c\end{matrix}\right],\,\dots,\,e^{\pi icc'}\vartheta_1\left[v;\,\begin{matrix}\tfrac{1}{2}\epsilon_s'\\ c\end{matrix}\right]\right),$$

wherein $s=2^p$, ϵ_1', ..., ϵ_s' represent the different values of ϵ', and J is a matrix wherein the β-th element of the α-th row is $\vartheta_1\left[u;\,\begin{matrix}\tfrac{1}{2}\epsilon'_\beta c'\\ k_\alpha\end{matrix}\right]$.

The 2^p various values of $\epsilon'_\beta c'$, for an assigned value of c', are, in general in a different order, the same as the various values of ϵ'_β; we may suppose the order of the columns of J to be so altered that the various values of $\epsilon'_\beta c'$ become the values of ϵ'_β *in an assigned order*, the order of the elements $e^{\pi icc'}\vartheta_1\left[v;\,\begin{matrix}\tfrac{1}{2}\epsilon_1'\\ c\end{matrix}\right]$, ..., $e^{\pi icc'}\vartheta_1\left[v;\,\begin{matrix}\tfrac{1}{2}\epsilon_s'\\ c\end{matrix}\right]$ being correspondingly altered. When this is done the matrix J is independent of the characteristic C. Now it is possible to choose 2^p characteristics C, say C_1, ..., C_s such that the Göpel systems (C_iK) give, together, all the 2^p possible characteristics; then the 2^p equations obtainable from that just written by replacing C in turn by C_1, ..., C_s, are all included, using the notation of matrices, in the one equation*

$$\left(\vartheta[u+v;\,C_\alpha K_\beta]\,\vartheta[u-v;\,C_\alpha K_\beta]\right)=\left(e^{\pi ic_\alpha c'_\alpha}\vartheta_1\left[v;\,\begin{matrix}\tfrac{1}{2}\zeta'_\beta c'_\alpha\\ c_\alpha\end{matrix}\right]\right)\left(\vartheta_1\left[u;\,\begin{matrix}\tfrac{1}{2}\zeta'_\alpha\\ k_\beta\end{matrix}\right]\right),$$

wherein ζ'_α denotes a row of p integers, each either 0 or 1, and has 2^p values. In each matrix the element written down is the β-th element of the α-th row.

* We can obviously obtain a more general equation by taking 2^{2p} different sets of arguments, the general element of the matrix on the left hand being $\vartheta[u^{(\alpha)}+v^{(\beta)};\,C_\alpha K_\beta]\,\vartheta[u^{(\alpha)}-v^{(\beta)};\,C_\alpha K_\beta]$. Cf. Chap. XV. § 291, Ex. v., and Caspary, *Crelle*, XCVI. (1884), pp. 182, 324; Frobenius, *Crelle*, XCVI. (1884), p. 100. Also Weierstrass, *Sitzungsber. der Ak. d. Wiss. zu Berlin*, 1882, I.—XXVI. p. 506.

Ex. vi. If in Ex. v., $p=2$, and the group (K) consists of the characteristics

$$\tfrac{1}{2}\begin{pmatrix}00\\00\end{pmatrix},\ \tfrac{1}{2}\begin{pmatrix}00\\10\end{pmatrix},\ \tfrac{1}{2}\begin{pmatrix}00\\01\end{pmatrix},\ \tfrac{1}{2}\begin{pmatrix}00\\11\end{pmatrix},$$

while the characteristics C consist of

$$\tfrac{1}{2}\begin{pmatrix}00\\00\end{pmatrix},\ \tfrac{1}{2}\begin{pmatrix}10\\00\end{pmatrix},\ \tfrac{1}{2}\begin{pmatrix}01\\00\end{pmatrix},\ \tfrac{1}{2}\begin{pmatrix}11\\00\end{pmatrix},$$

and the values of ζ' are, in order,

$$(0, 0),\quad (0, 1),\quad (1, 0),\quad (1, 1),$$

shew that the sixteen equations expressed by the final equation of Ex. v. are equivalent to

$$\begin{pmatrix} \begin{bmatrix}00\\11\end{bmatrix}, & \begin{bmatrix}10\\00\end{bmatrix}, & -\begin{bmatrix}01\\10\end{bmatrix}, & \begin{bmatrix}11\\01\end{bmatrix} \\ -\begin{bmatrix}10\\01\end{bmatrix}, & \begin{bmatrix}00\\10\end{bmatrix}, & \begin{bmatrix}11\\00\end{bmatrix}, & \begin{bmatrix}01\\11\end{bmatrix} \\ \begin{bmatrix}01\\00\end{bmatrix}, & -\begin{bmatrix}11\\11\end{bmatrix}, & \begin{bmatrix}00\\01\end{bmatrix}, & \begin{bmatrix}10\\10\end{bmatrix} \\ -\begin{bmatrix}11\\10\end{bmatrix}, & -\begin{bmatrix}01\\01\end{bmatrix}, & -\begin{bmatrix}10\\11\end{bmatrix}, & \begin{bmatrix}00\\00\end{bmatrix} \end{pmatrix} = \begin{pmatrix} a_4, & a_3, & -a_2, & a_1 \\ -a_3, & a_4, & a_1, & a_2 \\ a_1, & a_2, & a_3, & -a_4 \\ a_2, & -a_1, & a_4, & a_3 \end{pmatrix} \begin{pmatrix} \beta_1, & -\beta_4, & \beta_3, & \beta_2 \\ \beta_2, & \beta_3, & \beta_4, & -\beta_1 \\ -\beta_3, & \beta_2, & \beta_1, & \beta_4 \\ \beta_4, & \beta_1, & -\beta_2, & \beta_3 \end{pmatrix}$$

wherein, on the left hand, $\begin{bmatrix}00\\11\end{bmatrix}$ denotes $\vartheta\left[u+v;\ \tfrac{1}{2}\begin{pmatrix}00\\11\end{pmatrix}\right]\vartheta\left[u-v;\ \tfrac{1}{2}\begin{pmatrix}00\\11\end{pmatrix}\right]$, etc., and on the right hand,

$$a_1=\vartheta_1\left[u;\ \tfrac{1}{2}\begin{pmatrix}00\\00\end{pmatrix}\right],\ a_2=\vartheta_1\left[u;\ \tfrac{1}{2}\begin{pmatrix}10\\00\end{pmatrix}\right],\ a_3=\vartheta_1\left[u;\ \tfrac{1}{2}\begin{pmatrix}01\\00\end{pmatrix}\right],\ a_4=\vartheta_1\left[u;\ \tfrac{1}{2}\begin{pmatrix}11\\00\end{pmatrix}\right],$$

$\beta_1, \beta_2, \beta_3, \beta_4$ being respectively the same theta functions with the argument v.

Now if A, B denote respectively the first and second matrices on the right hand, the linear equations

$$(y_1, y_2, y_3, y_4)=A\,(x_1, x_2, x_3, x_4),\quad (x_1, x_2, x_3, x_4)=B\,(z_1, z_2, z_3, z_4)$$

are immediately seen to lead to the results

$$y_1^2+y_2^2+y_3^2+y_4^2=(a_1^2+a_2^2+a_3^2+a_4^2)\,(x_1^2+x_2^2+x_3^2+x_4^2),$$
$$x_1^2+x_2^2+x_3^2+x_4^2=(\beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2)\,(z_1^2+z_2^2+z_3^2+z_4^2)\,;$$

hence if the j-th element of the i-th row of the compound matrix AB, which is the matrix on the left-hand side of the equation, be denoted by $\gamma_{i,j}$, we have

$$\sum_{i=1}^{4}\gamma_{i,s}^2=\sum_{i=1}^{4}\gamma_{i,r}^2,\ \sum_{i=1}^{4}\gamma_{i,r}\gamma_{i,s}=0,\qquad (r\neq s,\ r, s=1, 2, 3, 4),$$

and these equations lead to

$$\sum_{j=1}^{4}\gamma_{s,j}^2=\sum_{j=1}^{4}\gamma_{r,j}^2,\ \sum_{j=1}^{4}\gamma_{r,j}\gamma_{s,j}=0.$$

Denoting $\begin{bmatrix}00\\11\end{bmatrix}$, $\begin{bmatrix}10\\00\end{bmatrix}$, by $[a_1c_2]$, $[a_1c_1]$, etc., as in the table of § 204, and interchanging the second and third rows of the matrix on the left-hand side, we may express the result by saying that the matrix

$$\begin{pmatrix} [a_1c_2], & [a_1c_1], & -[a_1c], & [a_2] \\ [a_2c_2], & -[a_2c_1], & [a_2c], & [a_1] \\ -[c_2], & [c_1], & [c], & [a_1a_2] \\ -[cc_1], & -[cc_2], & -[c_1c_2], & [0] \end{pmatrix}$$

gives an *orthogonal* linear substitution of four variables*.

* An algebraic proof may be given ; cf. Brioschi, *Ann. d. Mat.* XIV.

Ex. vii. Deduce from § 309 that

$$2^p \vartheta(u+v;\ AP_i)\,\vartheta(u-v;\ AP_i)=\sum_{\epsilon} \frac{\epsilon_i^{-1}\left[\sum_{\alpha}\epsilon_\alpha \vartheta^2(u;\ AP_\alpha)\right]\left[\sum_{\alpha}\epsilon_\alpha e^{\pi i|AP_\alpha|}\vartheta^2(v;\ AP_\alpha)\right]}{\sum_{\alpha}\epsilon_\alpha \vartheta^2(0;\ AP_\alpha)},$$

where P_i, P_α are characteristics of a Göpel group (P), of 2^p characteristics. Infer that, *if n be any positive integer, and AP_i be an even characteristic, $\vartheta(nv;\ AP_i)$ is expressible as an integral polynomial of order n^2 in the 2^p functions $\vartheta(v;\ AP_\alpha)$.*

Ex. viii. If $K=\frac{1}{2}\binom{k'}{k}$, $P_\alpha=\frac{1}{2}\binom{q'_\alpha}{q_\alpha}$, deduce from § 309, putting

$$a=b=u-U=v-V=\tfrac{1}{2}\Omega_k, \quad .$$

that

$$\chi(U+V,\ U-V)\,\chi(0,\ 0)=\chi(U,\ U)\,\chi(V,\ -V),$$

where

$$\chi(u,\ v)=\sum_{\alpha}\epsilon_\alpha e^{-\frac{1}{2}\pi i k' q_\alpha}\,\vartheta(u;\ K+P_\alpha)\,\vartheta(v;\ P_\alpha).$$