CHAPTER XX

FUNCTIONS OF REAL VARIABLES

194. Partial differential equations of physics. In the solution of physical problems partial differential equations of higher order, particularly the second, frequently arise. With very few exceptions these equations are linear, and if they are solved at all, are solved by assuming the solution as a product of functions each of which contains only one of the variables. The determination of such a solution offers only a particular solution of the problem, but the combination of different particular solutions often suffices to give a suitably general solution. For instance

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \text{or} \quad \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0 \tag{1}$$

is Laplace's equation in rectangular and polar coördinates. For a solution in rectangular coördinates the assumption $V = X(x)\,Y(y)$ would be made, and the assumption $V = R(r)\,\Phi(\phi)$ for a solution in polar coördinates. The equations would then become

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \quad \text{or} \quad \frac{r^2 R''}{R} + r\frac{R'}{R} + \frac{\Phi''}{\Phi} = 0. \tag{2}$$

Now each equation as written is a sum of functions of a single variable. But a function of x cannot equal a function of y and a function of r cannot equal a function of ϕ unless the functions are constant and have the same value. Hence

$$\begin{aligned} &\frac{X''}{X} = -m^2, && \frac{\Phi''}{\Phi} = -m^2, \\ & && \text{or} \\ &\frac{Y''}{Y} = +m^2, && \frac{r^2 R''}{R} + r\frac{R'}{R} = +m^2. \end{aligned} \tag{2'}$$

These are ordinary equations of the second order and may be solved as such. The second case will be treated in detail.

The solution corresponding to any value of m is

$$\Phi = a_m \cos m\phi + b_m \sin m\phi, \qquad R = A_m r^m + B_m r^{-m}$$

and $$V = R\Phi = (A_m r^m + B_m r^{-m})(a_m \cos m\phi + b_m \sin m\phi)$$

or $$V = \sum_m (A_m r^m + B_m r^{-m})(a_m \cos m\phi + b_m \sin m\phi). \tag{3}$$

That any number of solutions corresponding to different values of m may be added together to give another solution is due to the *linearity* of the given equation (§ 96). It may be that a single term will suffice as a solution of a given problem. But it may be seen in general that: A solution for V may be found in the form of a Fourier series which shall give V any assigned values on a unit circle and either be convergent for all values within the circle or be convergent for all values outside the circle. In fact let $f(\phi)$ be the values of V on the unit circle. Expand $f(\phi)$ into its Fourier series

$$f(\phi) = \tfrac{1}{2} a_0 + \sum_m (a_m \cos m\phi + b_m \sin m\phi).$$

Then $$V = \tfrac{1}{2} a_0 + \sum_m r^m (a_m \cos m\phi + b_m \sin m\phi) \tag{3'}$$

will be a solution of the equation which reduces to $f(\phi)$ on the circle and, as it is a power series in r, converges at every point within the circle. In like manner a solution convergent outside the circle is

$$V = \tfrac{1}{2} a_0 + \sum_m r^{-m} (a_m \cos m\phi + b_m \sin m\phi). \tag{3''}$$

The infinite series for V have been called solutions of Laplace's equation. As a matter of fact they have not been proved to be solutions. The finite sum obtained by taking any number of terms of the series would surely be a solution; but the limit of that sum when the series becomes infinite is not thereby proved to be a solution even if the series is convergent. For theoretical purposes it would be necessary to give the proof, but the matter will be passed over here as having a negligible bearing on the practical solution of many problems. For in practice the values of $f(\phi)$ on the circle could not be exactly known and could therefore be adequately represented by a finite and in general not very large number of terms of the development of $f(\phi)$, and these terms would give only a finite series for the desired function V.

In some problems it is better to keep the particular solutions separate, discuss each possible particular solution, and then imagine them compounded physically. Thus in the motion of a drumhead, the most general solution obtainable is not so instructive as the particular solution corresponding to particular notes; and in the motion of the surface of the ocean it is preferable to discuss individual types of waves and compound them according to the law of superposition of small vibrations (p. 226). For example if

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}, \qquad \frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}, \qquad z = XYT,$$

be taken as the equation of motion of a rectangular drumhead,

$$X = \begin{cases}\sin \alpha x, \\ \cos \alpha x,\end{cases} \qquad Y = \begin{cases}\sin \beta x, \\ \cos \beta x,\end{cases} \qquad T = \begin{cases}\sin c\sqrt{\alpha^2+\beta^2}t \\ \cos c\sqrt{\alpha^2+\beta^2}t\end{cases}$$

are particular solutions which may be combined in any way desired. As the edges of the drumhead are supposed to be fixed at all times,

$$z = 0 \quad \text{if} \quad x = 0, \quad x = a, \quad y = 0, \quad y = b, \quad t = \text{anything},$$

where the dimensions of the head are a by b. Then the solution

$$z = XYT = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\,t \tag{4}$$

is a possible type of vibration satisfying the given conditions at the perimeter of the head for any integral values of m, n. The solution is periodic in t and represents a particular note which may be omitted. A sum of such expressions multiplied by any constants would also be a solution and would represent a possible mode of motion, but would not be periodic in t and would represent no note.

195. For three dimensions Laplace's equation becomes

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0 \tag{5}$$

in polar coördinates. Substitute $V = R(r)\Theta(\theta)\Phi(\phi)$; then

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0.$$

Here the first term involves r alone and no other term involves r Hence the first term must be a constant, say, $n(n+1)$. Then

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - n(n+1)R = 0, \qquad R = Ar^n + Br^{-n-1}.$$

Next consider the last term after multiplying through by $\sin^2\theta$. It ap pears that $\Phi^{-1}\Phi''$ is a constant, say, $-m^2$. Hence

$$\Phi'' = -m^2\Phi, \qquad \Phi = a_m\cos m\phi + b_m\sin m\phi.$$

Moreover the equation for Θ now reduces to the simple form

$$\frac{d}{d\cos\theta}\left[(1-\cos^2\theta)\frac{d\Theta}{d\cos\theta}\right] + \left[n(n+1) - \frac{m^2}{1-\cos^2\theta}\right]\Theta = 0.$$

The problem is now separated into that of the integration of three differential equations of which the first two are readily integrable. The third equation is a generalization of Legendre's (Exs. 13–17, p. 252), and in case n, m are positive integers the solution may be expressed in terms of polynomials $P_{n,m}(\cos\theta)$ in $\cos\theta$. Any expression

$$\sum_{n,m}(A_n r^n + B_n r^{-n-1})(a_m \cos m\phi + b_m \sin m\phi)\, P_{n,m}(\cos\theta)$$

is therefore a solution of Laplace's equation, and it may be shown that by combining such solutions into infinite series, a solution may be obtained which takes on any desired values on the unit sphere and converges for all points within or outside.

Of particular simplicity and importance is the case in which V is supposed independent of ϕ so that $m = 0$ and the equation for Θ is soluble in terms of Legendre's polynomials $P_n(\cos\theta)$ if n is integral. As the potential V of any distribution of matter attracting according to the inverse square of the distance satisfies Laplace's equation at all points exterior to the mass (§ 201), the potential of any mass symmetric with respect to revolution about the polar axis $\theta = 0$ may be expressed if its expression for points on the axis is known. For instance, the potential of a mass M distributed along a circular wire of radius a is

$$V = \frac{M}{\sqrt{a^2 + r^2}} = \begin{cases} \dfrac{M}{a}\left(1 - \dfrac{1}{2}\dfrac{r^2}{a^2} + \dfrac{1\cdot 3}{2\cdot 4}\dfrac{r^4}{a^4} - \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\dfrac{r^6}{a^6} + \cdots, \right. & r < a, \\[2ex] \dfrac{M}{a}\left(\dfrac{a}{r} - \dfrac{1}{2}\dfrac{a^3}{r^3} + \dfrac{1\cdot 3}{2\cdot 4}\dfrac{a^5}{r^5} - \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\dfrac{a^7}{r^7} + \cdots, \right. & r > a, \end{cases}$$

at a point distant r from the center of the wire along a perpendicular to the plane of the wire. The two series

$$V = \begin{cases} \dfrac{M}{a}\left(P_0 - \dfrac{1}{2}\dfrac{r^2}{a^2}P_2 + \dfrac{1\cdot 3}{2\cdot 4}\dfrac{r^4}{a^4}P_4 - \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\dfrac{r^6}{a^6}P_6 + \cdots, \right. & r < a, \\[2ex] \dfrac{M}{a}\left(\dfrac{a}{r}P_0 - \dfrac{1}{2}\dfrac{a^3}{r^3}P_2 + \dfrac{1\cdot 3}{2\cdot 4}\dfrac{a^5}{r^5}P_4 - \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\dfrac{a^7}{r^7}P_6 + \cdots, \right. & r > a, \end{cases}$$

are then precisely of the form $\Sigma A_n r^n P_n$, $\Sigma A_n r^{-n-1} P_n$ admissible for solutions of Laplace's equation and reduce to the known value of V along the axis $\theta = 0$ since $P_n(1) = 1$. They give the values of V at all points of space.

To this point the method of combining solutions of the given differential equations was to add them into a finite or infinite series. It is also possible to combine them by integration and to obtain a solution as a *definite integral* instead of as an infinite series. It should be noted in this case, too, that a limit of a sum has replaced a sum and that it would theoretically be necessary to demonstrate that the limit of the sum was really a solution of the given equation. It will be sufficient at this point to illustrate the method without any rigorous attempt to justify it. Consider (2′) in rectangular coördinates. The solutions for X, Y are

$$\frac{X''}{X}=-m^2,\quad \frac{Y''}{Y}=m^2,\quad X=a_m\cos mx+b_m\sin mx,\quad Y=A_me^{my}+B_me^{-my},$$

where Y may be expressed in terms of hyperbolic functions. Now

$$\begin{aligned} V&=\int_{m_0}^{m_1}e^{-my}[a(m)\cos mx+b(m)\sin mx]\,dm \\ &=\lim\sum e^{-m_iy}[a(m_i)\cos m_ix+b(m_i)\sin m_ix]\,\Delta m_i \end{aligned}\tag{6}$$

is the limit of a sum of terms each of which is a solution of the given equation; for $a(m_i)$ and $b(m_i)$ are constants for any given value $m=m_i$, no matter what functions $a(m)$ and $b(m)$ are of m. It may be assumed that V is a solution of the given equation. Another solution could be found by replacing e^{-my} by e^{my}.

It is sometimes possible to determine a (m), b (m) so that V shall reduce to assigned values on certain lines. In fact (p. 466)

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^{+\infty}f(\lambda)\cos m(\lambda-x)\,d\lambda dm.\tag{7}$$

Hence if the limits for m be 0 and ∞ and if the choice

$$a(m)=\frac{1}{\pi}\int_{-\infty}^{+\infty}f(\lambda)\cos m\lambda d\lambda,\qquad b(m)=\frac{1}{\pi}\int_{-\infty}^{+\infty}f(\lambda)\sin m\lambda d\lambda$$

is taken for $a(m)$, $b(m)$, the expression (6) for V becomes

$$V=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^{+\infty}e^{-my}f(\lambda)\cos m(\lambda-x)\,d\lambda dm\tag{8}$$

and reduces to $f(x)$ when $y=0$. Hence a solution V is found which takes on any assigned values $f(x)$ along the x-axis. This solution clearly becomes zero when y becomes infinite. When $f(x)$ is given it is sometimes possible to perform one or more of the integrations and thus simplify the expression for V.

For instance if

$$f(x)=1 \text{ when } x>0 \quad\text{and}\quad f(x)=0 \text{ when } x<0,$$

the integral from $-\infty$ to 0 drops out and

$$\begin{aligned} V&=\frac{1}{\pi}\int_0^\infty\int_0^\infty e^{-my}\cdot 1\cdot\cos m(\lambda-x)\,d\lambda dm=\frac{1}{\pi}\int_0^\infty\int_0^\infty e^{-my}\cos m(\lambda-x)\,dmd\lambda \\ &=\frac{1}{\pi}\int_0^\infty\frac{yd\lambda}{y^2+(\lambda-x)^2}=\frac{1}{\pi}\left(\frac{\pi}{2}+\tan^{-1}\frac{x}{y}\right)=1-\frac{1}{\pi}\tan^{-1}\frac{y}{x}. \end{aligned}$$

It may readily be shown that when $y > 0$ the reversal of the order of integration is permissible; but as V is determined completely, it is simpler to substitute the value as found in the equation and see that $V''_{xx} + V''_{yy} = 0$, and to check the fact that V reduces to $f(x)$ when $y = 0$. It may perhaps be superfluous to state that the proved correctness of an answer does not show the justification of the steps by which that answer is found; but on the other hand as those steps were taken solely to obtain the answer, there is no practical need of justifying them if the answer is clearly right.

EXERCISES

1. Find the indicated particular solutions of these equations:

$$(\alpha)\ c^2\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}, \qquad V = \sum A_m e^{-m^2 t}(a_m \cos cmx + b_m \sin cmx),$$

$$(\beta)\ \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \frac{\partial^2 V}{\partial x^2}, \qquad V = \sum (A_m \cos cmt + B_m \sin cmt)(a_m \cos mx + b_m \sin mx),$$

$$(\gamma)\ c^2\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}, \qquad X = \begin{cases}\sin c\alpha x\\ \cos c\alpha x,\end{cases} \qquad Y = \begin{cases}\sin c\beta y\\ \cos c\beta y,\end{cases} \qquad T = e^{-(\alpha^2+\beta^2)t}.$$

2. Determine the solutions of Laplace's equation in the plane that have $V = 1$ for $0 < \phi < \pi$ and $V = -1$ for $0 < \phi < 2\pi$ on a unit circle.

3. If $V = |\pi - \phi|$ on the unit circle, find the expansion for V.

4. Show that $V = \Sigma a_i \sin m\pi x/l \cdot \cos cm\pi t/l$ is the solution of Ex. 1 (β) which vanishes at $x = 0$ and $x = l$. Determine the coefficients a_i so that for $t = 0$ the value of V shall be an assigned function $f(x)$. This is the problem of the violin string started from any assigned configuration.

5. If the string of Ex. 4 is started with any assigned velocity $\partial V/\partial t = f(x)$ when $t = 0$, show that the solution is $\Sigma a_i \sin m\pi x/l \cdot \sin cm\pi t/l$ and make the proper determination of the constants a_i.

6. If the drumhead is started with the shape $z = f(x, y)$, show that

$$z = \sum_{m,n} A_{m,n} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \cos c\pi t\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}},$$

$$A_{m,n} = \frac{4}{ab}\int_0^a\int_0^b f(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dydx.$$

7. In hydrodynamics it is shown that $\dfrac{\partial^2 y}{\partial t^2} = \dfrac{g}{b}\dfrac{\partial}{\partial x}\left(hb\dfrac{\partial y}{dx}\right)$ is the differential equation for the surface of the sea in an estuary or on a beach of breadth b and depth h measured perpendicularly to the x-axis which is supposed to run seaward. Find

$$(\alpha)\ y = AJ_0(kx)\cos nt, \quad k^2 = n^2/gh, \qquad (\beta)\ y = AJ_0(2\sqrt{kx})\cos nt, \quad k = n^2/gm,$$

as particular solutions of the equation when (α) the depth is uniform but the breadth is proportional to the distance out to sea, and when (β) the breadth is uniform but the depth is mx. Discuss the shape of the waves that may thus stand on the surface of the estuary or beach.

8. If a series of parallel waves on an ocean of constant depth h is cut perpendicularly by the xy-plane with the axes horizontal and vertical so that $y = -h$ is the ocean bed, the equations for the velocity potential ϕ are known to be

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \qquad \left[\frac{\partial\phi}{\partial y}\right]_{y=-h} = 0, \qquad \left[\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y}\right]_{y=0} = 0.$$

Find and combine particular solutions to show that ϕ may have the form

$$\phi = A\cosh k(y+h)\cos(kx - nt), \qquad n^2 = gk\tanh kh.$$

9. Obtain the solutions or types of solutions for these equations.

(α) $\dfrac{\partial^2 V}{\partial z^2} + \dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial\phi^2} = 0,$ *Ans.* $e^{\pm kz}\begin{Bmatrix}\cos m\phi\\ \sin m\phi\end{Bmatrix}J_m(kr),$

(β) $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial\phi^2} + V = 0,$ *Ans.* $\sum(a_m\cos\phi + b_m\sin\phi)J_m(r),$

(γ) $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} + V = 0,$ *Ans.* $r^{-\frac{1}{2}}J_{m+\frac{1}{2}}(r)P_{n,m}(\cos\theta)\times$
$(a_{n,m}\cos m\phi + b_{n,m}\sin m\phi),$

(δ) $\dfrac{\partial^2 V}{\partial t^2} + 2\dfrac{\partial V}{\partial t} = \dfrac{\partial^2 V}{\partial x^2},$ (ϵ) $\dfrac{1}{c^2}\dfrac{\partial^2 V}{\partial t^2} = \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2}.$

10. Find the potential of a homogeneous circular disk as (Ex. 22, p. 68; Ex. 23, p. 332)

$$V = \frac{2M}{a}\left[\frac{1}{2}\frac{a}{r} - \frac{1\cdot 1}{2\cdot 4}\frac{a^3}{r^3}P_2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\frac{a^5}{r^5}P_4 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\frac{a^7}{r^7}P_6 + \cdots\right], \qquad r > a,$$

$$= \frac{2M}{a}\left[1 \mp \frac{r}{a}P_1 + \frac{1}{2}\frac{r^2}{a^2}P_2 - \frac{1\cdot 1}{2\cdot 4}\frac{r^4}{a^4}P_4 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\frac{r^6}{a^6}P_6 - \cdots\right], \qquad r < a,$$

where the negative sign before P_1 holds for $\theta < \frac{1}{2}\pi$ and the positive for $\theta > \frac{1}{2}\pi$.

11. Find the potential of a homogeneous hemispherical shell.

12. Find the potential of (α) a homogeneous hemisphere at all points outside the hemisphere, and (β) a homogeneous circular cylinder at all external points.

13. Assume $\dfrac{Q}{2a}\cos^{-1}\dfrac{x^2 - a^2}{x^2 + a^2}$ is the potential at a point of the axis of a conducting disk of radius a charged with Q units of electricity. Find the potential anywhere.

196. Harmonic functions; general theorems. A function which satisfies Laplace's equation $V''_{xx} + V''_{yy} = 0$ or $V''_{xx} + V''_{yy} + V''_{zz} = 0$, whether in the plane or in space, is called a *harmonic function.* It is assumed that the first and second partial derivatives of a harmonic function are continuous except at specified points called singular points. There are many similarities between harmonic functions in the plane and harmonic functions in space, and some differences. The fundamental theorem is that: *If a function is harmonic and has no singularities upon or within a simple closed curve (or surface), the line integral of its normal derivative along the curve (respectively, surface) vanishes; and conversely if a function $V(x, y)$, or $V(x, y, z)$, has continuous first and second* *partial derivatives and the line integral (or surface integral) along every closed curve (or surface) in a region vanishes, the function is harmonic.* For by Green's Formula, in the respective cases of plane and space (Ex. 10, p. 349),

$$\int_{\circ} \frac{dV}{dn} ds = \int_{\circ} \frac{\partial V}{\partial x} dy - \frac{\partial V}{\partial y} dx = \iint \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) dxdy,$$
$$\int_{\circ} \frac{dV}{dn} dS = \int_{\circ} d\mathbf{S} \cdot \nabla V = \iiint \nabla \cdot \nabla V \, dxdydz. \tag{9}$$

Now if the function is harmonic, the right-hand side vanishes and so must the left; and conversely if the left-hand side vanishes for all closed curves (or surfaces), the right-hand side must vanish for every region, and hence the integrand must vanish.

If in particular the curve or surface be taken as a circle or sphere of radius a and polar coördinates be taken at the center, the normal derivative becomes $\partial V/\partial r$ and the result is

$$\int_0^{2\pi} \frac{\partial V}{\partial r} d\phi = 0 \quad \text{or} \quad \int_0^{2\pi} \int_0^{\pi} \frac{\partial V}{\partial r} \sin\theta d\theta d\phi = 0,$$

where the constant a or a^2 has been discarded from the element of arc $ad\phi$ or the element of surface $a^2 \sin\theta d\theta d\phi$. If these equations be integrated with respect to r from 0 to a, the integrals may be evaluated by reversing the order of integration. Thus

$$0 = \int_0^a dr \int_0^{2\pi} \frac{\partial V}{\partial r} d\phi = \int_0^{2\pi} \int_0^a \frac{\partial V}{\partial r} dr d\phi = \int_0^{2\pi} (V_a - V_0)\, d\phi,$$

and $$\int_0^{2\pi} V_a d\phi = V_0 \int_0^{2\pi} d\phi, \quad \text{or} \quad \overline{V}_a = V_0, \tag{10}$$

where V_a is the value of V on the circle of radius a and V_0 is the value at the center and $\overline{V}_a$ is the average value along the perimeter of the circle. Similar analysis would hold in space. The result states the important theorem: *The average value of a harmonic function over a circle (or sphere) is equal to the value at the center.*

This theorem has immediate corollaries of importance. *A harmonic function which has no singularities within a region cannot become maximum or minimum at any point within the region.* For if the function were a maximum at any point, that point could be surrounded by a circle or sphere so small that the value of the function at every point of the contour would be less than at the assumed maximum and hence the average value on the contour could not be the value at the center.

A harmonic function which has no singularities within a region and is constant on the boundary is constant throughout the region. For the maximum and minimum values must be on the boundary, and if these have the same value, the function must have that same value throughout the included region. *Two harmonic functions which have identical values upon a closed contour and have no singularities within, are identical throughout the included region.* For their difference is harmonic and has the constant value 0 on the boundary and hence throughout the region. These theorems are equally true if the region is allowed to grow until it is infinite, provided the values which the function takes on at infinity are taken into consideration. Thus, if two harmonic functions have no singularities in a certain infinite region, take on the same values at all points of the boundary of the region, and approach the same values as the point (x, y) or (x, y, z) in any manner recedes indefinitely in the region, the two functions are identical.

If Green's Formula be applied to a product UdV/dn, then

$$\int_{\bigcirc} U\frac{dV}{dn}ds = \int_{\bigcirc} U\frac{dV}{dx}dy - U\frac{dV}{dy}dx$$

$$= \iint U(V''_{xx} + V''_{yy})\,dxdy + \iint (U'_x V'_x + U'_y V'_y)\,dxdy,$$

or

$$\int_{\bigcirc} Ud\mathbf{S}\cdot\nabla V = \int U\nabla\cdot\nabla V dv + \int \nabla U\cdot\nabla V dv \tag{11}$$

in the plane or in space. In this relation let V be harmonic without singularities within and upon the contour, and let $U = V$. The first integral on the right vanishes and the second is necessarily positive unless the relations $V'_x = V'_y = 0$ or $V'_x = V'_y = V'_z = 0$, which is equivalent to $\nabla V = 0$, are fulfilled at all points of the included region. Suppose further that the normal derivative dV/dn is zero over the entire boundary. The integral on the left will then vanish and that on the right must vanish. Hence V contains none of the variables and is constant. *If the normal derivative of a function harmonic and devoid of singularities at all points on and within a given contour vanishes identically upon the contour, the function is constant.* As a corollary: If two functions are harmonic and devoid of singularities upon and within a given contour, and if their normal derivatives are identically equal upon the contour, the functions differ at most by an additive constant. In other words, *a harmonic function without singularities not only is determined by its values on a contour but also (except for an additive constant) by the values of its normal derivative upon a contour.*

Laplace's equation arises directly upon the statement of some problems in physics in mathematical form. In the first place consider the flow of heat or of electricity in a conducting body. The physical law is that heat flows along the direction of most rapid decrease of temperature T, and that the amount of the flow is proportional to the rate of decrease. As $-\nabla T$ gives the direction and magnitude of the most rapid decrease of temperature, the flow of heat may be represented by $-k\nabla T$, where k is a constant. The rate of flow in any direction is the component of this vector in that direction. The rate of flow across any boundary is therefore the integral along the boundary of the normal derivative of T. Now the flow is said to be *steady* if there is no increase or decrease of heat within any closed boundary, that is

$$k\oint d\mathbf{S}\cdot\nabla T = 0 \quad \text{or} \quad T \text{ is harmonic.}$$

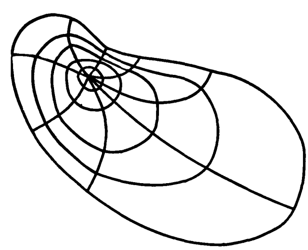
Hence the problem of the distribution of the temperature in a body supporting a steady flow of heat is the problem of integrating Laplace's equation. In like manner, the laws of the flow of electricity being identical with those for the flow of heat except that the potential V replaces the temperature T, the problem of the distribution of potential in a body supporting a steady flow of electricity will also be that of solving Laplace's equation.

Another problem which gives rise to Laplace's equation is that of the irrotational motion of an incompressible fluid. If $\mathbf{v}$ is the velocity of the fluid, the motion is called irrotational when $\nabla\times\mathbf{v} = 0$, that is, when the line integral of the velocity about any closed curve is zero. In this case the negative of the line integral from a fixed limit to a variable limit defines a function $\Phi(x, y, z)$ called the velocity potential, and the velocity may be expressed as $\mathbf{v} = -\nabla\Phi$. As the fluid is incompressible, the flow across any closed boundary is necessarily zero. Hence

$$\oint d\mathbf{S}\cdot\nabla\Phi = 0 \quad \text{or} \quad \int\nabla\cdot\nabla\Phi dv = 0 \quad \text{or} \quad \nabla\cdot\nabla\Phi = 0,$$

and the velocity potential Φ is a harmonic function. Both these problems may be stated without vector notation by carrying out the ideas involved with the aid of ordinary coördinates. The problems may also be solved for the plane instead of for space in a precisely analogous manner.

197. The conception of the flow of electricity will be advantageous in discussing the singularities of harmonic functions and a more general conception of steady flow. Suppose an electrode is set down on a sheet of zinc of which the perimeter is grounded. The equipotential lines and the lines of flow which are orthogonal to them may be sketched in. Electricity passes steadily from the electrode to the rim of the sheet and off to the ground. Across any circuit which does not surround the electrode the



flow of electricity is zero as the flow is steady, but across any circuit surrounding the electrode there will be a certain definite flow; the circuit integral of the normal derivative of the potential V around such a circuit is not zero. This may be compared with the fact that the circuit integral of a function of a complex variable is not necessarily zero about a singularity, although it is zero if the circuit contains no singularity. Or the electrode may not be considered as corresponding to a singularity but to a portion cut out from the sheet so that the sheet is no longer simply connected, and the comparison would then be with a circuit which could not be shrunk to nothing. Concerning this latter interpretation little need be said; the facts are readily seen. It is the former conception which is interesting.

For mathematical purposes the electrode will be idealized by assuming its diameter to shrink down to a point. It is physically clear that the smaller the electrode, the higher must be the potential at the electrode to force a given flow of electricity into the plate. Indeed it may be seen that V must become infinite as $-C\log r$, where r is the distance from the point electrode. For note in the first place that $\log r$ is a solution of Laplace's equation in the plane; and let $U = V + C\log r$ or $V = U - C\log r$, where U is a harmonic function which remains finite at the electrode. The flow across any small circle concentric with the electrode is

$$-\int_0^{2\pi}\frac{\partial V}{\partial r}rd\phi = -\int_0^{2\pi}\frac{\partial U}{\partial r}rd\phi + 2\pi C = 2\pi C,$$

and is finite. The constant C is called the strength of the source situated at the point electrode. A similar discussion for space would show that the potential in the neighborhood of a source would become infinite as C/r. The particular solutions $-\log r$ and $1/r$ of Laplace's equation in the respective cases may be called the *fundamental solutions.*

The physical analogy will also suggest a method of obtaining higher singularities by combining fundamental singularities. For suppose that a powerful positive electrode is placed near an equally powerful negative electrode, that is, suppose a strong source and a strong sink near together. The greater part of the flow will be nearly in a straight line from the source to the sink, but some part of it will spread out over the sheet. The value of V obtained by adding together the two values for source and sink is

$$\begin{aligned}
V &= -\tfrac{1}{2}C\log(r^2+l^2-2rl\cos\phi) + \tfrac{1}{2}C\log(r^2+l^2+2rl\cos\phi)\\
&= -\frac{1}{2}C\log\left(1-\frac{2l}{r}\cos\phi+\frac{l^2}{r^2}\right)+\frac{1}{2}C\log\left(1+\frac{2l}{r}\cos\phi+\frac{l^2}{r^2}\right)\\
&= \frac{2lC}{r}\cos\phi + \text{higher powers} = \frac{M}{r}\cos\phi + \cdots.
\end{aligned}$$

Thus if the strength C be allowed to become infinite as the distance $2l$ becomes zero, and if M denote the limit of the product $2lC$, the limiting form of V is $Mr^{-1}\cos\phi$ and is itself a solution of the equation, becoming infinite more strongly than $-\log r$. In space the corresponding solution would be $Mr^{-2}\cos\phi$.

It was seen that a harmonic function which had no singularities on or within a given contour was determined by its values on the contour and determined except for an additive constant by the values of its normal derivative upon the contour. If now there be actually within the contour certain singularities at which the function becomes infinite as certain particular solutions $V_1, V_2, \cdots$, the function $U = V - V_1 - V_2 - \cdots$ is harmonic without singularities and may be determined as before. Moreover, the values of $V_1, V_2, \cdots$ or their normal derivatives may be considered as known upon the contour inasmuch as these are definite particular solutions. Hence it appears, as before, that *the harmonic function V is determined by its values on the boundary of the region or (except for an additive constant) by the values of its normal derivative on the boundary, provided the singularities are specified in position and their mode of becoming infinite is given in each case as some particular solution of Laplace's equation.*

Consider again the conducting sheet with its perimeter grounded and with a single electrode of strength unity at some interior point of the sheet. The potential thus set up has the properties that: 1° the potential is zero along the perimeter because the perimeter is grounded; 2° at the position P of the electrode the potential becomes infinite as $-\log r$; and 3° at any other point of the sheet the potential is regular and satisfies Laplace's equation. This particular distribution of potential is denoted by $G(P)$ and is called the Green Function of the sheet relative to P. In space the Green Function of a region would still satisfy 1° and 3°, but in 2° the fundamental solution $-\log r$ would have to be replaced by the corresponding fundamental solution $1/r$. It should be noted that the Green Function is really a function

$$G(P) = G(a, b;\ x, y) \quad \text{or} \quad G(P) = G(a, b, c;\ x, y, z)$$

of four or six variables if the position $P(a, b)$ or $P(a, b, c)$ of the electrode is considered as variable. The function is considered as known only when it is known for any position of P.

If now the symmetrical form of Green's Formula

$$-\iint (u\Delta v - v\Delta u)\, dxdy + \int_{\bigcirc}\left(u\frac{dv}{dn} - v\frac{du}{dn}\right) ds = 0, \tag{12}$$

where Δ denotes the sum of the second derivatives, be applied to the entire sheet with the exception of a small circle concentric with P and if the choice $u = G$ and $v = V$ be made, then as G and V are harmonic the double integral drops out and

$$\int_{\bigcirc} - V\frac{dG}{dn}\, ds - \int_0^{2\pi} G\frac{dV}{dr}\, rd\phi + \int_0^{2\pi} V\frac{dG}{dr}\, rd\phi = 0. \tag{13}$$

Now let the radius r of the small circle approach 0. Under the assumption that V is devoid of singularities and that G becomes infinite as $-\log r$, the middle integral approaches 0 because its integrand does, and the final integral approaches $2\pi V(P)$. Hence

$$V(P) = \frac{-1}{2\pi}\int_{\bigcirc} V\frac{dG}{dn}\,ds. \tag{13'}$$

This formula expresses the values of V at any interior point of the sheet in terms of the values of V upon the contour and of the normal derivative of G along the contour. It appears, therefore, that *the determination of the value of a harmonic function devoid of singularities within and upon a contour may be made in terms of the values on the contour provided the Green Function of the region is known.* Hence the particular importance of the problem of determining the Green Function for a given region. This theorem is analogous to Cauchy's Integral (§ 126).

EXERCISES

1. Show that any linear function $ax + by + cz + d = 0$ is harmonic. Find the conditions that a quadratic function be harmonic.

2. Show that the real and imaginary parts of any function of a complex variable are each harmonic functions of (x, y).

3. Why is the sum or difference of any two harmonic functions multiplied by any constants itself harmonic? Is the power of a harmonic function harmonic?

4. Show that the product UV of two harmonic functions is harmonic when and only when $U'_xV'_x + U'_yV'_y = 0$ or $\nabla U\cdot\nabla V = 0$. In this case the two functions are called conjugate or orthogonal. What is the significance of this condition geometrically?

5. Prove the average value theorem for space as for the plane.

6. Show for the plane that if V is harmonic, then

$$U = \int \frac{dV}{dn}\,ds = \int \frac{\partial V}{\partial x}\,dy - \frac{\partial V}{\partial y}\,dx$$

is independent of the path and is the conjugate or orthogonal function to V, and that U is devoid of singularities over any region over which V is devoid of them. Show that $V + iU$ is a function of $z = x + iy$.

7. State the problems of the steady flow of heat or electricity in terms of ordinary coördinates for the case of the plane.

8. Discuss for space the problem of the source, showing that C/r gives a finite flow $4\pi C$, where C is called the strength of the source. Note the presence of the factor 4π in the place of 2π as found in two dimensions.

9. Derive the solution $Mr^{-2}\cos\phi$ for the source-sink combination in space.

10. Discuss the problem of the small magnet or the electric doublet in view of Ex. 9. Note that as the attraction is inversely as the square of the distance, the potential of the force satisfies Laplace's equation in space.

11. Let equal infinite sources and sinks be located alternately at the vertices of an infinitesimal square. Find the corresponding particular solution (α) in the case of the plane, and (β) in the case of space. What combination of magnets does this represent if the point of view of Ex. 10 be taken, and for what purpose is the combination used?

12. Express $V(P)$ in terms of $G(P)$ and the boundary values of V in space.

13. If an analytic function has no singularities within or on a contour, Cauchy's Integral gives the value at any interior point. If there are within the contour certain poles, what must be known in addition to the boundary values to determine the function? Compare with the analogous theorem for harmonic functions.

14. Why were the solutions in § 194 as series the only possible solutions provided they were really solutions? Is there any difficulty in making the same inference relative to the problem of the potential of a circular wire in § 195?

15. Let $G(P)$ and $G(Q)$ be the Green Functions for the same sheet but relative to two different points P and Q. Apply Green's symmetric theorem to the sheet from which two small circles about P and Q have been removed, making the choice $u = G(P)$ and $v = G(Q)$. Hence show that $G(P)$ at Q is equal to $G(Q)$ at P. This may be written as

$$G(a, b;\ x, y) = G(x, y;\ a, b) \quad \text{or} \quad G(a, b, c;\ x, y, z) = G(x, y, z;\ a, b, c).$$

16. Test these functions for the harmonic property, determine the conjugate functions and the allied functions of a complex variable:

$$(\alpha)\ xy, \qquad (\beta)\ x^2y - \tfrac{1}{3}y^3, \qquad (\gamma)\ \tfrac{1}{2}\log(x^2 + y^2),$$

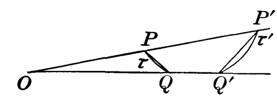
$$(\delta)\ e^x \sin x, \qquad (\epsilon)\ \sin x \cosh y, \qquad (\zeta)\ \tan^{-1}(\cot x \tanh y).$$

198. Harmonic functions; special theorems. For the purposes of the next paragraphs it is necessary to study the properties of the geometric transformation known as *inversion.* The definition of inversion will be given so as to be applicable either to space or to the plane. The transformation which replaces each point P by a point P' such that $OP \cdot OP' = k^2$ where O is a given fixed point, k a constant, and P' is on the line OP, is called *inversion with the center O and the radius k.* Note that if P is thus carried into P', then P' will be carried into P; and hence if any geometrical configuration is carried into another, that other will be carried into the first. Points very near to O are carried off to a great distance; for the point O itself the definition breaks down and O corresponds to no point of space. If desired, one may add to space a fictitious point called the point at infinity and may then say that the center O of the inversion corresponds to the point at infinity (p. 481). A pair of points P, P' which go over into each other, and another pair Q, Q' satisfy the equation $OP \cdot OP' = OQ \cdot OQ'$.

A curve which cuts the line OP at an angle τ is carried into a curve which cuts the line at the angle $\tau' = \pi - \tau$. For by the relation $OP \cdot OP' = OQ \cdot OQ'$, the triangles OPQ, $OQ'P'$ are similar and

$$\angle\, OPQ = \angle\, OQ'P' = \pi - \angle\, O - \angle\, OP'Q'.$$

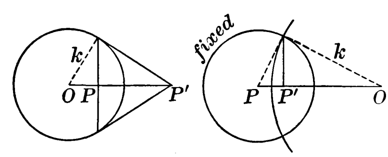
Now if $Q \doteq P$ and $Q' \doteq P'$, then $\angle\, O \doteq 0$, $\angle\, OPQ \doteq \tau$, $\angle\, OP'Q' \doteq \tau$ and it is seen that $\tau = \pi - \tau'$ or $\tau' = \pi - \tau$. An immediate extension of the argument will show that the magnitude of the angle between two intersecting curves will be unchanged by the transformation; *the transformation is therefore conformal.* (In the plane where it is possible to distinguish between positive and negative angles, the sign of the angle is reversed by the transformation.)



If polar coördinates relative to the point O be introduced, the equations of the transformation are simply $rr' = k^2$ with the understanding that the angle ϕ in the plane or the angles ϕ, θ in space are unchanged. The locus $r = k$, which is a circle in the plane or a sphere in space, becomes $r' = k$ and is therefore unchanged. This is called the circle or the sphere of inversion. Relative to this locus a simple construction for a pair of inverse points P and P' may be made as indicated in the figure. The locus

$$r^2 + k^2 = 2\sqrt{a^2 + k^2}\, r \cos\phi \quad \text{becomes} \quad k^2 + r'^2 = 2\sqrt{a^2 + k^2}\, r' \cos\phi$$

and is therefore unchanged as a whole. This locus represents a circle or a sphere of radius a orthogonal to the circle or sphere of inversion. A construction may now be made for finding an inversion which carries a given circle into itself and the center P of the circle into any assigned point P' of the circle; the construction holds for space by revolving the figure about the line OP.



To find what figure a line in the plane or a plane in space becomes on inversion, let the polar axis $\phi = 0$ or $\theta = 0$ be taken perpendicular to the line or plane as the case may be. Then

$$r = p \sec\phi, \qquad r' \sec\phi = k^2/p \quad \text{or} \quad r = p \sec\theta, \qquad r' \sec\theta = k^2/p$$

are the equations of the line or plane and the inverse locus. The locus is seen to be a circle or sphere through the center of inversion. This may also be seen directly by applying the geometric definition of inversion. In a similar manner, or analytically, it may be shown that any circle in the plane or any sphere in space inverts into a circle or into a sphere, unless it passes through the center of inversion and becomes a line or a plane.

If d be the distance of P from the circle or sphere of inversion, the distance of P from the center is $k-d$, the distance of P' from the center is $k^2/(k-d)$, and from the circle or sphere it is $d'=dk/(k-d)$. Now if the radius k is very large in comparison with d, the ratio $k/(k-d)$ is nearly 1 and d' is nearly equal to d. If k is allowed to become infinite so that the center of inversion recedes indefinitely and the circle or sphere of inversion approaches a line or plane, the distance d' approaches d as a limit. As the transformation which replaces each point by a point equidistant from a given line or plane and perpendicularly opposite to the point is the ordinary inversion or reflection in the line or plane such as is familiar in optics, it appears that reflection in a line or plane may be regarded as the limiting case of inversion in a circle or sphere.

The importance of inversion in the study of harmonic functions lies in two theorems applicable respectively to the plane and to space. First, *if V is harmonic over any region of the plane and if that region be inverted in any circle, the function $V'(P')=V(P)$ formed by assigning the same value at P' in the new region as the function had at the point P which inverted into P' is also harmonic.* Second, *if V is harmonic over any region in space, and if that region be inverted in a sphere of radius k, the function $V'(P')=kV(P)/r'$ formed by assigning at P' the value the function had at P multiplied by k and divided by the distance $OP'=r'$ of P' from the center of inversion is also harmonic.* The significance of these theorems lies in the fact that if one distribution of potential is known, another may be derived from it by inversion; and conversely it is often possible to determine a distribution of potential by inverting an unknown case into one that is known. The proof of the theorems consists merely in making the changes of variable

$$r=k^2/r' \quad \text{or} \quad r'=k^2/r, \qquad \phi'=\phi, \qquad \theta'=\theta$$

in the polar forms of Laplace's equation (Exs. 21, 22, p. 112).

The method of using inversion to determine distribution of potential in electrostatics is often called the method of *electric images*. As a charge e located at a point exerts on other point charges a force proportional to the inverse square of the distance, the potential due to e is as $1/\rho$, where ρ is the distance from the charge (with the proper units it may be taken as e/ρ), and satisfies Laplace's equation. The potential due to any number of point charges is the sum of the individual potentials due to the charges. Thus far the theory is essentially the same as if the charges were attracting particles of matter. In electricity, however, the question of the distribution of potential is further complicated when there are in the neighborhood of the charges certain conducting surfaces. For 1° a conducting surface in an electrostatic field must everywhere be at a constant potential or there would be a component force along the surface and the electricity upon it would move, and 2° there is the phenomenon of induced electricity whereby a variable surface charge is induced upon the conductor by other charges in the neighborhood. If the potential $V(P)$ due to any distribution of charges be inverted in any sphere, the new potential is $kV(P)/r'$. As the potential $V(P)$

becomes infinite as e/ρ at the point charges e, the potential $kV(P)/r'$ will become infinite at the inverted positions of the charges. As the ratio $ds' : ds$ of the inverted and original elements of length is r'^2/k^2, the potential $kV(P)/r'$ will become infinite as $k/r' \cdot e/\rho' \cdot r'^2/k^2$, that is, as $r'e/k\rho'$. Hence it appears that the charge e inverts into a charge $e' = r'e/k$; the charge $-e'$ is called the electric image of e. As the new potential is $kV(P)/r'$ instead of $V(P)$, it appears that an equipotential surface $V =$ const. will not invert into an equipotential surface $V'(P') =$ const. unless $V = 0$ or r' is constant. But if to the inverted system there be added the charge $e = -kV$ at the center O of inversion, the inverted equipotential surface becomes a surface of zero potential.

With these preliminaries, consider the question of the distribution of potential due to an external charge e at a distance r from the center of a conducting spherical surface of radius k which has been grounded so as to be maintained at zero potential. If the system be inverted with respect to the sphere of radius k, the potential of the spherical surface remains zero and the charge e goes over into a charge $e' = r'e/k$ at the inverse point. Now if ρ, ρ' are the distances from e, e' to the sphere, it is a fact of elementary geometry that $\rho : \rho' =$ const. $= r' : k$. Hence the potential

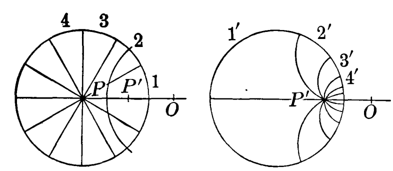
$$V = \frac{e}{\rho} - \frac{e'}{\rho'} = e\left(\frac{1}{\rho} - \frac{r'}{k\rho'}\right) = e\,\frac{k\rho' - r'\rho}{k\rho\rho'},$$

due to the charge e and to its image $-e'$, actually vanishes upon the sphere; and as it is harmonic and has only the singularity e/ρ outside the sphere (which is the same as the singularity due to e), this value of V throughout all space must be precisely the value due to the charge and the grounded sphere. The distribution of potential in the given system is therefore determined. The potential outside the sphere is as if the sphere were removed and the two charges e, $-e'$ left alone. By Gauss's Integral (Ex. 8, p. 348) the charge within any region may be evaluated by a surface integral around the region. This integral over a surface surrounding the sphere is the same as if over a surface shrunk down around the charge $-e'$, and hence the total charge induced on the sphere is $-e' = -r'e/k$.

199. Inversion will transform the average value theorem

$$V(P) = \frac{1}{2\pi}\int_0^{2\pi} V d\phi \quad \text{into} \quad V'(P') = \frac{1}{2\pi}\int_0^{2\pi} V' d\psi, \tag{14}$$

a form applicable to determine the value of V at any point of a circle in terms of the value upon the circumference. For suppose the circle with center at P and with the set of radii spaced at angles $d\phi$, as implied in the computation of the average value, be inverted upon an orthogonal circle so chosen that P shall go over into P'. The given



circle goes over into itself and the series of lines goes over into a series of circles through P' and the center O of inversion. (The figures are drawn separately instead of superposed.) From the conformal property the angles between the circles of the series are equal to the angles between the radii, and the circles cut the given circle orthogonally just as the radii did Let V' along the arcs $1', 2', 3', \cdots$ be equal to V along the corresponding arcs $1, 2, 3, \cdots$ and let $V(P) = V'(P')$ as required by the theorem on inversion of harmonic functions. Then the two integrals are equal element for element and their values $V(P)$ and $V'(P')$ are equal. Hence the desired form follows from the given form as stated. (It may be observed that $d\phi$ and $d\psi$, strictly speaking, have opposite signs, but in determining the average value $V'(P')$, $d\psi$ is taken positively.) The derived form of integral may be written

$$V'(P') = \frac{1}{2\pi}\int_0^{2\pi} V' d\psi = \frac{1}{2\pi}\int_0^{2\pi a} V' \frac{d\psi}{ds'} ds', \tag{14'}$$

as a line integral along the arc of the circle. If P' is at the distance r from the center, and if a be the radius, the center of inversion O is at the distance a^2/r from the center of the circle, and the value of k is seen to be $k^2 = (a^2 - r^2)a^2/r^2$. Then, if Q and Q' be points on the circle,

$$ds' = ds\,\frac{\overline{OQ'}^2}{k^2} = \frac{r^2(a^2 - 2\,a^3r^{-1}\cos\phi' + a^4r^{-2})}{(a^2 - r^2)\,a^2}\,ad\phi.$$

Now $d\psi/ds'$ may be obtained, because of the equality of $d\psi$ and $d\phi$, and ds' may be written as $ad\phi'$. Hence

$$V'(P') = \frac{1}{2\pi}\int_0^{2\pi} V' \frac{a^2 - r^2}{a^2 - 2\,ar\cos\phi' + r^2}\,d\phi'.$$

Finally the primes may be dropped from V' and P', the position of P' may be expressed in terms of its coördinates (r, ϕ), and

$$V(r, \phi) = \frac{1}{2\pi}\int_0^{2\pi} V \frac{(a^2 - r^2)\,d\phi'}{a^2 - 2\,ar\cos(\phi' - \phi) + r^2} = \frac{1}{2\pi}\int_0^{2\pi} V d\psi \tag{15}$$

is the expression of V in terms of its boundary values.

The integral (15) is called *Poisson's Integral.* It should be noted particularly that the form of Poisson's Integral first obtained by inversion represents the average value of V along the circumference, provided that average be computed for each point by considering the values along the circumference as distributed relative to the angle ψ as independent variable. That V as defined by the integral actually approaches the value on the circumference when the point approaches the circumference is clear from the figure, which shows that all except an infinitesimal fraction of the orthogonal circles cut the circle within infinitesimal limits when the point is infinitely near to the circumference. Poisson's Integral may be obtained in another way. For if P and P' are now two inverse points relative to the circle, the equation of the circle may be written as

$$\rho/\rho' = \text{const.} = r/a, \quad \text{and} \quad G(P) = -\log\rho + \log\rho' + \log(r/a) \quad (16)$$

is then the Green Function of the circular sheet because it vanishes along the circumference, is harmonic owing to the fact that the logarithm of the distance from a point is a solution of Laplace's equation, and becomes infinite at P as $-\log\rho$. Hence

$$V = \frac{-1}{2\pi}\int V\frac{dG}{dn}ds = \frac{-1}{2\pi}\int V\frac{d}{dn}(\log\rho' - \log\rho)\,ds. \qquad (16')$$

It is not difficult to reduce this form of the integral to (15).

If a harmonic function is defined in a region abutting upon a segment of a straight line or an arc of a circle, and if the function vanishes along the segment or arc, the function may be extended across the segment or arc by assigning to the inverse point P' the value $V(P') = -V(P)$, which is the negative of the value at P; the conjugate function

$$U = \int\frac{dV}{dn}ds + C = \int\frac{\partial V}{\partial x}dy - \frac{\partial V}{\partial y}dx + C \qquad (17)$$

takes on the same values at P and P'. It will be sufficient to prove this theorem in the case of the straight line because, by the theorem on inversion, the arc may be inverted into a line by taking the center of inversion at any point of the arc or the arc produced. As the Laplace operator $D_x^2 + D_y^2$ is independent of the axes (Ex. 25, p. 112), the line may be taken as the x-axis without restricting the conclusion.

Now the extended function $V(P')$ satisfies Laplace's equation since

$$\frac{\partial^2 V(P')}{\partial x'^2} + \frac{\partial^2 V(P')}{\partial y'^2} = -\frac{\partial^2 V(P)}{\partial x^2} - \frac{\partial^2 V(P)}{\partial y^2} = 0.$$

Therefore $V(P')$ is harmonic. By the definition $V(P') = -V(P)$ and the assumption that V vanishes along the segment it appears that the function V on the two sides of the line pieces on to itself in a continuous manner, and it remains merely to show that it pieces on to itself in a harmonic manner, that is, that the function V and its extension form a function harmonic at points of the line. This follows from Poisson's Integral applied to a circle centered on the line. For let

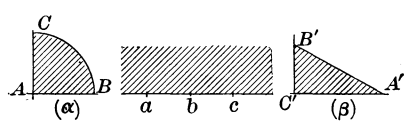
$$H(x, y) = \int_0^{2\pi} V d\psi; \quad \text{then} \quad H(x, 0) = 0$$

because V takes on equal and opposite values on the upper and lower semicircumferences. Hence $H = V(P) = V(P') = 0$ along the axis. But $H = V(P)$ along the upper arc and $H = V(P')$ along the lower arc because Poisson's Integral takes on the boundary values as a limit when the point approaches the boundary. Now as H is harmonic and agrees with $V(P)$ upon the whole perimeter of the upper semicircle it must be identical with $V(P)$ throughout that semicircle. In like manner it is identical with $V(P')$ throughout the lower semicircle. As the functions $V(P)$ and $V(P')$ are identical with the single harmonic function H, they must piece together harmonically across the axis. The theorem is thus completely proved. The statement about the conjugate function may be verified by taking the integral along paths symmetric with respect to the axis.

200. *If a function $w = f(z) = u + iv$ of a complex variable becomes real along the segment of a line or the arc of a circle, the function may be extended analytically across the segment or arc by assigning to the inverse point P' the value $w = u - iv$ conjugate to that at P.* This is merely a corollary of the preceding theorem. For if w be real, the harmonic function v vanishes on the line and may be assigned equal and opposite values on the opposite sides of the line; the conjugate function u then takes on equal values on the opposite sides of the line. The case of the circular arc would again follow from inversion as before.

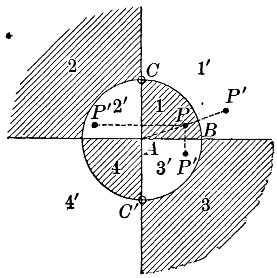
The method employed to identify functions in §§ 185–187 was to map the halves of the w-plane, or rather the several repetitions of these halves which were required to complete the map of the w-surface, on a region of the z-plane. By virtue of the theorem just obtained the converse process may often be carried out and the function $w = f(z)$ which maps a given region of the z-plane upon the half of the w-plane may be obtained. The method will apply only to regions of the z-plane which are bounded by rectilinear segments and circular arcs; for it is only for such that the theorems on inversion and the theorem on the extension of harmonic functions have been proved. To identify the function it is necessary to extend the given region of the z-plane by inversions across its boundaries until the w-surface is completed. The method is not satisfactory if the successive extensions of the region in the z-plane result in overlapping.

The method will be applied to determining the function (α) which maps the first quadrant of the unit circle in the z-plane upon the upper half of the w-plane, and (β) which maps a 30°–60°–90° triangle upon the upper half of the w-plane. Suppose the sector ABC mapped on the w-half-plane so that the perimeter ABC corresponds to the real axis abc. When the perimeter is described in the order written and the interior is on the left, the real axis must, by the principle of conformality, be described in such an order that the upper half-plane which is to correspond to the interior shall also lie on the left. The points a, b, c correspond to points



A, B, C. At these points the correspondence required is such that the conformality must break down. As angles are doubled, each of the points A, B, C must be a critical point of the first order for $w = f(z)$ and a, b, c must be branch points. To map the triangle, similar considerations apply except that whereas C' is a critical point of the first order, the points A', B' are critical of orders 5, 2 respectively. Each case may now be treated separately in detail.

Let it be assumed that the three vertices A, B, C of the sector go into the points* $w = 0, 1, \infty$. As the perimeter of the sector is mapped on the real axis, the function $w = f(z)$ takes on real values for points z along the perimeter. Hence if the sector be inverted over any of its sides, the point P' which corresponds to P may be given a value conjugate to w at P, and the image of P' in the w-plane is symmetrical to the image of P with respect to the real axis. The three regions 1′, 2′, 3′ of the z-plane correspond to the lower half of the w-plane; and the perimeters of these regions correspond also to the real axis. These regions may now be inverted across their boundaries and give rise to the regions 2, 3, 4 which must correspond to the upper half of the w-plane. Finally by inversion from one of these regions the region 4′ may be obtained as corresponding to the lower half of the w-plane. In this manner the inversion has been carried on until the entire z-plane is covered. Moreover there is no overlapping of the regions and the figure may be inverted in any of its lines without producing any overlapping; it will merely invert into itself. If a Riemann surface were to be constructed over the w-plane, it would clearly require four sheets. The surface could be connected up by studying the correspondence; but this is not necessary. Note merely that the function $f(z)$ becomes infinite at C when $z = i$ by hypothesis and at C' when $z = -i$ by inversion; and at no other point. The values $\pm i$ will therefore be taken as poles of $f(z)$ and as poles of the second order because angles are doubled. Note again that the function $f(z)$ vanishes at A when $z = 0$ by hypothesis and at $z = \infty$ by inversion. These will be assumed to be zeros of the second order because the points are critical points at which angles are doubled. The function



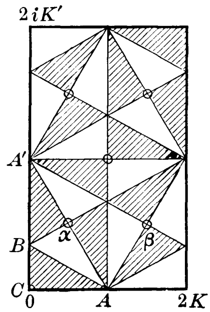
$$w = f(z) = Cz^2(z-i)^{-2}(z+i)^{-2} = Cz^2(z^2+1)^{-2}$$

has the above zeros and poles and must be identical with the desired function when the constant C is properly chosen. As the correspondence is such that $f(1) = 1$ by hypothesis, the constant C is 4. The determination of the function is complete as given.

Consider next the case of the triangle. The same process of inversion and repeated inversion may be followed, and never results in overlapping except as one

* It may be observed that the linear transformation $(\gamma w + \delta)w' = \alpha w + \beta$ (Ex. 15, p. 157) has three arbitrary constants $\alpha:\beta:\gamma:\delta$, and that by such a transformation any three points of the w-plane may be carried into any three points of the w'-plane. It is therefore a proper and trivial restriction to assume that 0, 1, ∞ are the points of the w-plane which correspond to A, B, C.

region falls into absolute coincidence with one previously obtained. To cover the whole z-plane the inversion would have to be continued indefinitely; but it may be observed that the rectangle inclosed by the heavy line is repeated indefinitely. Hence $w = f(z)$ is a doubly periodic function with the periods $2K$, $2iK'$ if $2K$, $2K'$ be the length and breadth of the rectangle. The function has a pole of the second order at C or $z = 0$ and at the points, marked with circles, into which the origin is carried by the successive inversions. As there are six poles of the second order, the function is of order twelve. When $z = K$ at A or $z = iK'$ at A' the function vanishes and each of these zeros is of the sixth order because angles are increased 6-fold. Again it appears that the function is of order 12. It is very simple to write the function down in terms of the theta functions constructed with the periods $2K$, $2iK'$.



$$w = f(z) = C\frac{H_1^6(z)\,\Theta^6(z)}{H^2(z)\,\Theta_1^2(z)\,H^2(z-\alpha)\,\Theta_1^2(z-\alpha)\,H^2(z-\beta)^2\,\Theta_1^2(z-\beta)}.$$

For this function is really doubly periodic, it vanishes to the sixth order at K, iK', and has poles of the second order at the points

$$0,\quad K + iK',\quad \alpha = \tfrac{1}{2}K + \tfrac{1}{2}iK',\quad \alpha + K + iK',\quad \beta = 2K - \alpha,\quad \beta + K + iK'.$$

As $\beta = 2K - \alpha$ the reduction $H^2(z-\beta) = H^2(z+\alpha)$, $\Theta_1(z-\beta) = \Theta_1(z+\alpha)$ may be made.

$$w = f(z) = C\frac{H_1^6(z)\,\Theta^6(z)}{H^2(z)\,\Theta_1^2(z)\,H^2(z-\alpha)\,H^2(z+\alpha)\,\Theta_1^2(z-\alpha)\,\Theta_1^2(z+\alpha)}.$$

The constant C may be determined, and the expression for $f(z)$ may be reduced further by means of identities; it might be expressed in terms of $\operatorname{sn}(z, k)$ and $\operatorname{cn}(z, k)$, with properly chosen k, or in terms of $p(z)$ and $p'(z)$. For the purposes of computations that might be involved in carrying out the details of the map, it would probably be better to leave the expression of $f(z)$ in terms of the theta functions, as the value of q is about 0.01.

EXERCISES

1. Show geometrically that a plane inverts into a sphere through the center of inversion, and a line into a circle through the center of inversion.

2. Show geometrically or analytically that in the plane a circle inverts into a circle and that in space a sphere inverts into a sphere.

3. Show that in the plane angles are reversed in sign by inversion. Show that in space the magnitude of an angle between two curves is unchanged.

4. If ds, dS, dv are elements of arc, surface, and volume, show that

$$ds' = \frac{r'}{r}ds = \frac{r'^2}{k^2}ds,\qquad dS' = \frac{r'^2}{r^2}dS = \frac{r'^4}{k^4}dS,\qquad dv' = \frac{r'^3}{r^3}dv = \frac{r'^6}{k^6}dv.$$

Note that in the plane an area and its inverted area are of opposite sign, and that the same is true of volumes in space.

5. Show that the system of circles through any point and its inverse with respect to a given circle cut that circle orthogonally. Hence show that if two points are inverse with respect to any circle, they are carried into points inverse with respect to the inverted position of the circle if the circle be inverted in any manner. In particular show that if a circle be inverted with respect to an orthogonal circle, its center is carried into the point which is inverse with respect to the center of inversion.

6. Obtain Poisson's Integral (15) from the form (16′). Note that

$$r^2 = \rho^2 + a^2 - 2a\rho\cos(\rho, n), \qquad \frac{dG}{dn} = \frac{\cos(\rho, n)}{\rho} - \frac{\cos(\rho', n)}{\rho'} = \frac{a^2 - r^2}{a^2\rho^2}.$$

7. From the equation $\rho/\rho' = \text{const.} = r/a$ of the sphere obtain

$$G(P) = \frac{1}{\rho} - \frac{a}{r}\frac{1}{\rho'}, \qquad V = \frac{1}{4\pi a}\int \frac{V(a^2 - r^2)\,dS}{[a^2 + r^2 - 2ar\cos(r, a)]^{\frac{3}{2}}},$$

the Green Function and Poisson's Integral for the sphere.

8. Obtain Poisson's Integral in space by the method of inversion.

9. Find the potential due to an insulated spherical conductor and an external charge (by placing at the center of the sphere a charge equal to the negative of that induced on the grounded sphere).

10. If two spheres intersect at right angles, and charges proportional to the diameters are placed at their centers with an opposite charge proportional to the diameter of the common circle at the center of the circle, then the potential over the two spheres is constant. Hence determine the effect throughout external space of two orthogonal conducting spheres maintained at a given potential.

11. A charge is placed at a distance h from an infinite conducting plane. Determine the potential on the supposition that the plane is insulated with no charge or maintained at zero potential.

12. Map the quadrantal sector on the upper half-plane so that the vertices C, A, B correspond to 1, ∞, 0.

13. Determine the constant C occurring in the map of the triangle on the plane. Find the point into which the median point of the triangle is carried.

14. With various selections of correspondences of the vertices to the three points $0, 1, \infty$ of the w-plane, map the following configurations upon the upper half-plane:

(α) a sector of 60°, (β) an isosceles right triangle,

(γ) a sector of 45°, (δ) an equilateral triangle.

201. The potential integrals. If $\rho(x, y, z)$ is a function defined at different points of a region of space, the integral

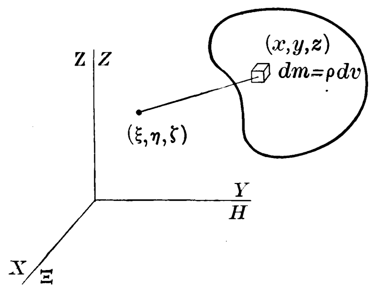
$$U(\xi, \eta, \zeta) = \iiint \frac{\rho(x, y, z)\,dxdydz}{\sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}} = \int \frac{\rho dv}{r} \tag{18}$$

evaluated over that region is called the potential of ρ at the point (ξ, η, ζ). The significance of the integral may be seen by considering the attraction and the potential energy at the point (ξ, η, ζ) due to a distribution of matter of density $\rho(x, y, z)$ in some region of space. If μ be a mass at (ξ, η, ζ) and m a mass at (x, y, z), the component forces exerted by m upon μ are

$$X = c\frac{\mu m}{r^2}\frac{x-\xi}{r}, \qquad Y = c\frac{\mu m}{r^2}\frac{y-\eta}{r}, \qquad Z = c\frac{\mu m}{r^2}\frac{z-\zeta}{r},$$

and $$F = c\frac{\mu m}{r^2}, \qquad V = -c\mu\frac{m}{r} + C \tag{19}$$

are respectively the total force on μ and the potential energy of the two masses. The potential energy may be considered as the work done by F or X, Y, Z on μ in bringing the mass μ from a fixed point to the point (ξ, η, ζ) under the action of m at (x, y, z) or it may be regarded as the function such that the negative of the derivatives of V by x, y, z give the forces X, Y, Z, or in vector notation $\mathbf{F} = -\nabla V$. Hence if the units be so chosen that $c = 1$, and if the forces and potential at (ξ, η, ζ) be measured per unit mass by dividing by μ, the results are (after disregarding the arbitrary constant C)



$$X = \frac{m}{r^2}\frac{x-\xi}{r}, \qquad Y = \frac{m}{r^2}\frac{y-\eta}{r}, \qquad Z = \frac{m}{r^2}\frac{z-\zeta}{r}, \qquad V = -\frac{m}{r}. \tag{19'}$$

Now if there be a region of matter of density $\rho(x, y, z)$, the forces and potential energy at (ξ, η, ζ) measured per unit mass there located may be obtained by summation or integration and are

$$X = \iiint \frac{\rho(x, y, z)(x-\xi)\,dxdydz}{[(\xi-x)^2 + (\eta-y)^2 + (\zeta-z)^2]^{\frac{3}{2}}}, \cdots, \quad V = -\int\frac{\rho dv}{r}. \tag{19''}$$

It therefore appears that the potential U defined by (18) is the negative of the potential energy V due to the distribution of matter.* Note further that in evaluating the integrals to determine X, Y, Z, and $U = -V$, the variables x, y, z with respect to which the integrations are performed will drop out on substituting the limits which determine the region, and will therefore leave X, Y, Z, U as functions of the parameters ξ, η, ζ which appear in the integrand. And finally

$$X = \frac{\partial U}{\partial \xi}, \qquad Y = \frac{\partial U}{\partial \eta}, \qquad Z = \frac{\partial U}{\partial \zeta} \tag{20}$$

*In electric and magnetic theory, where like *repels* like, the potential and potential energy have the same sign.

are consequences either of differentiating U under the sign of integration or of integrating the expressions (19′) for X, Y, Z expressed in terms of the derivatives of U, over the whole region.

THEOREM. The potential integral U satisfies the equations

$$\frac{\partial^2 U}{\partial \xi^2}+\frac{\partial^2 U}{\partial \eta^2}+\frac{\partial^2 U}{\partial \zeta^2}=0 \quad \text{or} \quad \frac{\partial^2 U}{\partial \xi^2}+\frac{\partial^2 U}{\partial \eta^2}+\frac{\partial^2 U}{\partial \zeta^2}=-4\pi\rho, \qquad (21)$$

known respectively as *Laplace's* and *Poisson's Equations*, according as the point (ξ, η, ζ) lies outside or within the body of density $\rho(x, y, z)$.

In case (ξ, η, ζ) lies outside the body, the proof is very simple. For the second derivatives of U may be obtained by differentiating with respect to ξ, η, ζ under the sign of integration, and the sum of the results is then zero. In case (ξ, η, ζ) lies within the body, the value for r vanishes when (ξ, η, ζ) coincides with (x, y, z) during the integration, and hence the integrals for U, X, Y, Z become infinite integrals for which differentiation under the sign is not permissible without justification. Suppose therefore that a small sphere of radius r concentric with (ξ, η, ζ) be cut out of the body, and the contributions $\mathbf{F}'$ of this sphere and $\mathbf{F}^*$ of the remainder of the body to the force $\mathbf{F}$ be considered separately. For convenience suppose the origin moved up to the point (ξ, η, ζ). Then

$$\mathbf{F}=\nabla U=\mathbf{F}^*+\mathbf{F}'=\int_* \rho\nabla\frac{1}{r}\,dv+\mathbf{F}'.$$

Now as the sphere is small and the density ρ is supposed continuous, the attraction F' of the sphere at any point of its surface may be taken as $\frac{4}{3}\pi r^3\rho_0/r^2$, the quotient of the mass by the square of the distance to the center, where ρ_0 is the density at the center. The force $\mathbf{F}'$ then reduces to $-\frac{4}{3}\pi\rho_0\mathbf{r}$ in magnitude and direction. Hence

$$\nabla\cdot\mathbf{F}=\nabla\cdot\nabla U=\nabla\cdot\mathbf{F}^*+\nabla\cdot\mathbf{F}'=\int_* \rho\nabla\cdot\nabla\frac{1}{r}\,dv+\nabla\cdot\mathbf{F}'.$$

The integral vanishes as in the first case, and $\nabla\cdot\mathbf{F}'=-4\pi\rho_0$. Hence if the suffix 0 be now dropped, $\nabla\cdot\nabla U=-4\pi\rho$, and Poisson's Equation is proved. Gauss's Integral (p. 348) affords a similar proof.

A rigorous treatment of the potential U and the forces X, Y, Z and their derivatives requires the discussion of convergence and allied topics. A detailed treatment will not be given, but a few of the most important facts may be pointed out. Consider the ordinary case where the volume density ρ remains finite and the body itself does not extend to infinity. The integrand ρ/r becomes infinite when $r=0$. But as dv is an infinitesimal of the third order around the point where $r=0$, the term $\rho dv/r$ in the integral U will be infinitesimal, may be disregarded, and the integral U converges. In like manner the integrals for X, Y, Z will converge because $\rho(\xi - x)/r^3$, etc., become infinite at $r = 0$ to only the second order. If $\partial X/\partial \xi$ were obtained by differentiation under the sign, the expressions ρ/r^3 and $\rho(\xi - x)^2/r^5$ would become infinite to the third order, and the integrals

$$\int \frac{\rho}{r^3}\, dv = \iiint \frac{\rho}{r^3} r^2 \sin\theta\, dr d\phi d\theta, \text{ etc.,}$$

as expressed in polar coördinates with origin at $r = 0$, are seen to diverge. Hence the derivatives of the forces and the second derivatives of the potential, as obtained by differentiating under the sign, are valueless.

Consider therefore the following device:

$$\frac{\partial}{\partial \xi}\frac{1}{r} = -\frac{\partial}{\partial x}\frac{1}{r}, \qquad \frac{\partial U}{\partial \xi} = \int \rho \frac{\partial}{\partial \xi}\frac{1}{r} dv = -\int \rho \frac{\partial}{\partial x}\frac{1}{r} dv,$$

$$\frac{\partial}{\partial x}\frac{\rho}{r} = \frac{\partial \rho}{\partial x}\frac{1}{r} + \rho\frac{\partial}{\partial x}\frac{1}{r}, \qquad -\int \rho \frac{\partial}{\partial x}\frac{1}{r} dv = \int \frac{1}{r}\frac{\partial \rho}{\partial x} dv - \int \frac{\partial}{\partial x}\frac{\rho}{r} dv.$$

The last integral may be transformed into a surface integral so that

$$\frac{\partial U}{\partial \xi} = \int \frac{1}{r}\frac{\partial \rho}{\partial x} dv - \int \frac{\rho}{r}\cos\alpha dS = \iiint \frac{1}{r}\frac{\partial \rho}{\partial x} dx dy dz - \iint \frac{\rho}{r} dy dz. \tag{22}$$

It should be remembered, however, that if $r = 0$ within the body, the transformation can only be made after cutting out the singularity $r = 0$, and the surface integral must extend over the surface of the excised region as well as over the surface of the body. But in this case, as dS is of the second order of infinitesimals while r is of the first order, the integral over the surface of the excised region vanishes when $r \doteq 0$ and the equation is valid for the whole region. In vectors

$$\nabla U = \int \frac{\nabla \rho}{r} dv - \int \frac{\rho}{r} d\mathbf{S}. \tag{22'}$$

It is noteworthy that the first integral gives the potential of $\nabla \rho$, that is, the integral is formed for $\nabla\rho$ just as (18) was from ρ. As $\nabla\rho$ is a vector, the summation is vector addition. It is further noteworthy that in $\nabla\rho$ the differentiation is with respect to x, y, z, whereas in ∇U it is with respect to ξ, η, ζ. Now differentiate (22) under the sign. (Distinguish ∇ as formed for ξ, η, ζ and x, y, z by ∇_ξ and ∇_x.)

$$\frac{\partial^2 U}{\partial \xi^2} = \int \frac{\partial}{\partial \xi}\frac{1}{r}\frac{\partial \rho}{\partial x} dv - \int \rho \cos\alpha \frac{\partial}{\partial \xi}\frac{1}{r} dS \text{ or } \nabla_\xi \cdot \nabla_\xi U = \int \nabla_\xi \frac{1}{r} \cdot \nabla_x \rho dv - \int \rho \nabla_\xi \frac{1}{r} \cdot d\mathbf{S},$$

or again $$\nabla_\xi \cdot \nabla_\xi U = -\int \nabla_x \frac{1}{r} \cdot \nabla_x \rho dv + \int \rho \nabla_x \frac{1}{r} \cdot d\mathbf{S}. \tag{23}$$

This result is valid for the whole region. Now by Green's Formula (Ex. 10, p. 349)

$$\int \rho \nabla_x \cdot \nabla_x \frac{1}{r} dv + \int \nabla_x \frac{1}{r} \cdot \nabla_x \rho dv = \int \nabla_x \cdot \left(\rho \nabla_x \frac{1}{r}\right) dv = \int \rho \nabla_x \frac{1}{r} \cdot d\mathbf{S} = \int \rho \frac{d}{dn}\frac{1}{r} dS.$$

Here the small region about $r = 0$ must again be excised and the surface integral must extend over its surface. If the region be taken as a sphere, the normal dn, being exterior to the body, is directed along $-dr$. Thus for the sphere

$$\int \rho \frac{d}{dn}\frac{1}{r} dS = \iint \rho \frac{1}{r^2} r^2 \sin\theta d\phi d\theta = \iint \rho \sin\theta d\phi d\theta = 4\pi\bar{\rho},$$

where $\bar{\rho}$ is the average of ρ upon the surface. If now r be allowed to approach 0 and $\nabla\cdot\nabla r^{-1}$ be set equal to zero, Green's Formula reduces to

$$\int \nabla_x \frac{1}{r}\cdot\nabla_x \rho dv = \int \rho\nabla_x \frac{1}{r}\cdot d\mathbf{S} + 4\pi\rho,$$

where the volume integrals extend over the whole volume and the surface integral extends like that of (23) over the surface of the body but not over the small sphere. Hence (23) reduces to $\nabla\cdot\nabla U = -4\pi\rho$.

Throughout this discussion it has been assumed that ρ and its derivatives are continuous throughout the body. In practice it frequently happens that a body consists really of several, say two, bodies of different nature (separated by a bounding surface S_{12}) in each of which ρ and its derivatives are continuous. Let the suffixes 1, 2 serve to distinguish the bodies. Then

$$U = \int \frac{\rho_1}{r} dv_1 + \int \frac{\rho_2}{r} dv_2 = \int \frac{\rho}{r} dv.$$

The discontinuity in ρ along a surface S_{12} does not affect a triple integral.

$$\nabla U = \int \frac{\nabla\rho_1}{r} dv_1 - \int \frac{\rho_1}{r} d\mathbf{S}_{1,12} + \int \frac{\nabla\rho_2}{r} dv_2 - \int \frac{\rho_2}{r} d\mathbf{S}_{2,21}.$$

Here the first surface integral extends over the boundary of the region 1 which includes the surface S_{12} between the regions. For the interface S_{12} the direction of $d\mathbf{S}$ is from 1 into 2 in the first case, but from 2 into 1 in the second. Hence

$$\nabla U = \int \frac{\nabla\rho}{r} dv - \int \frac{\rho}{r} d\mathbf{S} - \int \frac{\rho_1 - \rho_2}{r} d\mathbf{S}_{12}.$$

It may be noted that the first and second surface integrals are entirely analogous because the first may be regarded as extended over the surface separating a body of density ρ from one of density 0. Now $\nabla\cdot\nabla U$ may be found, and if the proper modifications be introduced in Green's Formula, it is seen that $\nabla\cdot\nabla U = -4\pi\rho$ still holds provided the point lies entirely within either body. The fact that ρ comes from the average value $\bar{\rho}$ upon the surface of an infinitesimal sphere shows that if the point lies on the interface S_{12} at a regular point, $\nabla\cdot\nabla U = -4\pi(\frac{1}{2}\rho_1 + \frac{1}{2}\rho_2)$.

The application of Green's Formula in its symmetric form (Ex. 10, p. 349) to the two functions r^{-1} and U, and the calculation of the integral over the infinitesimal sphere about $r = 0$, gives

$$\int \left(\frac{1}{r}\nabla\cdot\nabla U - U\nabla\cdot\nabla\frac{1}{r}\right) dv = \int \left(\frac{1}{r}\frac{dU}{dn} - U\frac{d}{dn}\frac{1}{r}\right) dS - 4\pi U$$

or

$$\begin{aligned}\int \frac{\nabla\cdot\nabla U}{r} dv &= \sum\int \frac{\left(\frac{dU}{dn}\right)_1 - \left(\frac{dU}{dn}\right)_2}{r} dS_{12} \\ &\quad - \sum\int (U_1 - U_2)\frac{d}{dn}\frac{1}{r} dS_{12} - 4\pi U,\end{aligned} \tag{24}$$

where Σ extends over all the surfaces of discontinuity, including the boundary of the whole body where the density changes to 0. Now $\nabla\cdot\nabla U = -4\pi\rho$ and if the definitions be given that

$$\left(\frac{dU}{dn}\right)_1 - \left(\frac{dU}{dn}\right)_2 = -4\pi\sigma, \qquad U_1 - U_2 = 4\pi\tau,$$

then $$U = \int \frac{\rho}{r} dv + \int \frac{\sigma}{r} dS + \int \tau \frac{d}{dn} \frac{1}{r} dS, \tag{25}$$

where the surface integrals extend over all surfaces of discontinuity. This form of U appears more general than the initial form (18), and indeed it is more general, for it takes into account the discontinuities of U and its derivative, which cannot arise when ρ is an ordinary continuous function representing a volume distribution of matter. The two surface integrals may be interpreted as due to surface distributions. For suppose that along some surface there is a surface density σ of matter. Then the first surface integral represents the potential of the matter in the surface. Strictly speaking, a surface distribution of matter with σ units of matter per unit surface is a physical impossibility, but it is none the less a convenient mathematical fiction when dealing with thin sheets of matter or with the charge of electricity upon a conducting surface. The surface distribution may be regarded as a limiting case of volume distribution where ρ becomes infinite and the volume throughout which it is spread becomes infinitely thin. In fact if dn be the thickness of the sheet of matter $\rho dndS = \sigma dS$. The second surface integral may likewise be regarded as a limit. For suppose that there are two surfaces infinitely near together upon one of which there is a surface density $-\sigma$, and upon the other a surface density σ. The potential due to the two equal superimposed elements dS is the

$$\frac{\sigma_1 dS_1}{r_1} + \frac{\sigma_2 dS_2}{r_2} = \sigma dS\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \sigma dS \frac{d}{dn}\frac{1}{r} \cdot dn = \sigma dn \frac{d}{dn}\frac{1}{r} dS.$$

Hence if $\sigma dn = \tau$, the potential takes the form $\tau dr^{-1}/dndS$. Just this sort of distribution of magnetism arises in the case of a magnetic shell, that is, a surface covered on one side with positive poles and on the other with negative poles. The three integrals in (25) are known respectively as volume potential, surface potential, and double surface potential.

202. The potentials may be used to obtain particular integrals of some differential equations. In the first place the equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = f(x, y, z) \quad \text{has} \quad U = \frac{-1}{4\pi}\int \frac{fdv}{r}$$

as its solution, when the integral is extended over the region throughout which f is defined. To this particular solution for U may be added any solution of Laplace's equation, but the particular solution is frequently precisely that particular solution which is desired. If the functions $\mathbf{U}$ and $\mathbf{f}$ were vector functions so that $\mathbf{U} = \mathbf{i}U_1 + \mathbf{j}U_2 + \mathbf{k}U_3$, and $\mathbf{f} = \mathbf{i}f_1 + \mathbf{j}f_2 + \mathbf{k}f_3$, the results would be

$$\frac{\partial^2 \mathbf{U}}{\partial x^2} + \frac{\partial^2 \mathbf{U}}{\partial y^2} + \frac{\partial^2 \mathbf{U}}{\partial z^2} = \mathbf{f}(x, y, z) \quad \text{and} \quad \mathbf{U} = \frac{-1}{4\pi}\int \frac{\mathbf{f}dv}{r},$$

where the integration denotes vector summation, as may be seen by adding the results for $\nabla\cdot\nabla U_1 = f_1$, $\nabla\cdot\nabla U_2 = f_2$, $\nabla\cdot\nabla U_3 = f_3$ after multiplication by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. If it is desired to indicate the vectorial nature of $\mathbf{U}$ and $\mathbf{f}$, the potential $\mathbf{U}$ may be called a vector potential.

In evaluating the potential and the forces at (ξ, η, ζ) due to an element dm at (x, y, z), it has been assumed that the action depends solely on the distance r. Now suppose that the distribution $\rho(x, y, z, t)$ is a function of the time and that the action of the element ρdv at (x, y, z) does not make its effect felt instantly at (ξ, η, ζ) but is propagated toward (ξ, η, ζ) from (x, y, z) at a velocity $1/a$ so as to arrive at the time $(t + ar)$. The potential and the forces at (ξ, η, ζ) as calculated by (18) will then be those there transpiring at the time $t + ar$ instead of at the time t. To obtain the effect at the time t it would therefore be necessary to calculate the potential from the distribution $\rho(x, y, z, t - ar)$ at the time $t - ar$. The potential

$$\begin{aligned} U(x, y, z, t) &= \int \frac{\rho(x, y, z, t - ar)\, dxdydz}{\sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}} \\ &= \int \frac{\rho(t)}{r} dv + \int \frac{\rho(t - ar) - \rho(t)}{r} dv, \end{aligned} \tag{26}$$

where for brevity the variables x, y, z have been dropped in the second form, is called a *retarded potential* as the time has been set back from t to $t - ar$. *The retarded potential satisfies the equation*

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + \frac{\partial^2 U}{\partial \zeta^2} - a^2 \frac{\partial^2 U}{\partial t^2} = -4\pi\rho(\xi, \eta, \zeta, t) \quad \text{or} \quad 0 \tag{27}$$

according as (ξ, η, ζ) *lies within or outside the distribution* ρ. There is really no need of the alternative statements because if (ξ, η, ζ) is outside, ρ vanishes. Hence a solution of the equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - a^2 \frac{\partial^2 U}{\partial t^2} = f(x, y, z, t)$$

is

$$U = \frac{-1}{4\pi} \int \frac{f(x, y, z, t - ar)}{r} dv.$$

The proof of the equation (27) is relatively simple. For in vector notation,

$$\begin{aligned} \nabla\cdot\nabla U &= \nabla\cdot\nabla \int \frac{\rho(t)}{r} dv + \nabla\cdot\nabla \int \frac{\rho(t - ar) - \rho(t)}{r} dv \\ &= -4\pi\rho + \nabla\cdot\nabla \int \frac{\rho(t - ar) - \rho(t)}{r} dv. \end{aligned}$$

The first reduction is made by Poisson's Equation. The second expression may be evaluated by differentiation under the sign. For it should be remarked that $\rho(t - ar) - \rho(t)$ vanishes when $r = 0$, and hence the order of the infinite in the integrand before and after differentiation is less by unity than it was in the corresponding steps of § 201. Then

$$\nabla_\xi \int \frac{\rho(t - ar) - \rho(t)}{r} dv = \int \left\{ \frac{(-a)\rho'(t - ar)\nabla_\xi r}{r} + [\rho(t - ar) - \rho(t)]\nabla_\xi \frac{1}{r} \right\} dv,$$

$$\nabla_\xi\cdot\nabla_\xi\int\frac{\rho(t-ar)-\rho(t)}{r}dv=\int\left\{\frac{(-a)^2\rho''\nabla_\xi r\cdot\nabla_\xi r}{r}+\frac{(-a)\rho'\nabla_\xi\cdot\nabla_\xi r}{r}\right.$$
$$\left.+(-a)\rho'\nabla_\xi r\cdot\nabla_\xi\frac{1}{r}+(-a)\rho'\nabla_\xi r\cdot\nabla_\xi\frac{1}{r}+[\rho(t-ar)-\rho(t)]\nabla_\xi\cdot\nabla_\xi\frac{1}{r}\right\}dv.$$

But $\quad \nabla_\xi=-\nabla_x \quad$ and $\quad \nabla r=\mathbf{r}/r$ and $\nabla r^{-1}=-\mathbf{r}/r^3 \quad$ and $\quad \nabla\cdot\nabla r^{-1}=0$.

Hence $\quad \nabla_\xi r\cdot\nabla_\xi r=1, \quad \nabla_\xi r\cdot\nabla_\xi r^{-1}=-r^{-2}, \quad \nabla_\xi\cdot\nabla_\xi r=2r^{-1}$

and $\quad \nabla\cdot\nabla\int\frac{\rho(t-ar)-\rho(t)}{r}dv=\int\frac{a^2\rho''}{r}dv=\int\frac{a^2}{r}\frac{\partial^2\rho(t-ar)}{\partial t^2}dv=a^2\frac{\partial^2 U}{\partial t^2}.$

It was seen (p. 345) that if $\mathbf{F}$ is a vector function with no curl, that is, if $\nabla\times\mathbf{F}=0$, then $\mathbf{F}\cdot d\mathbf{r}$ is an exact differential $d\phi$; and $\mathbf{F}$ may be expressed as the gradient of ϕ, that is, as $\mathbf{F}=\nabla\phi$. This problem may also be solved by potentials. For suppose

$$\mathbf{F}=\nabla\phi,\quad\text{then}\quad\nabla\cdot\mathbf{F}=\nabla\cdot\nabla\phi,\qquad\phi=\frac{-1}{4\pi}\int\frac{\nabla\cdot\mathbf{F}}{r}dv. \tag{28}$$

It appears therefore that ϕ may be expressed as a potential. This solution for ϕ is less general than the former because it depends on the fact that the potential integral of $\nabla\cdot\mathbf{F}$ shall converge. Moreover as the value of ϕ thus found is only a particular solution of $\nabla\cdot\mathbf{F}=\nabla\cdot\nabla\phi$, it should be proved that for this ϕ the relation $\mathbf{F}=\nabla\phi$ is actually satisfied. The proof will be given below. A similar method may now be employed to show that if $\mathbf{F}$ is a vector function with no divergence, that is, if $\nabla\cdot\mathbf{F}=0$, then $\mathbf{F}$ may be written as the curl of a vector function $\mathbf{G}$, that is, as $\mathbf{F}=\nabla\times\mathbf{G}$. For suppose

$$\mathbf{F}=\nabla\times\mathbf{G},\quad\text{then}\quad\nabla\times\mathbf{F}=\nabla\times\nabla\times\mathbf{G}=\nabla\nabla\cdot\mathbf{G}-\nabla\cdot\nabla\mathbf{G}.$$

As $\mathbf{G}$ is to be determined, let it be supposed that $\nabla\cdot\mathbf{G}=0$.

Then $$\mathbf{F}=\nabla\times\mathbf{G}\quad\text{gives}\quad\mathbf{G}=\frac{1}{4\pi}\int\frac{\nabla\times\mathbf{F}}{r}dv. \tag{29}$$

Here again the solution is valid only when the vector potential integral of $\nabla\times\mathbf{F}$ converges, and it is further necessary to show that $\mathbf{F}=\nabla\times\mathbf{G}$. The conditions of convergence are, however, satisfied for the functions that usually arise in physics.

To amplify the treatment of (28) and (29), let it be shown that

$$\nabla\phi=-\frac{1}{4\pi}\nabla\int\frac{\nabla\cdot\mathbf{F}}{r}dv=\mathbf{F},\qquad\nabla\times\mathbf{G}=\frac{1}{4\pi}\nabla\times\int\frac{\nabla\times\mathbf{F}}{r}dv=\mathbf{F}.$$

By use of (22) it is possible to pass the differentiations under the sign of integration and apply them to the functions $\nabla\cdot\mathbf{F}$ and $\nabla\times\mathbf{F}$, instead of to $1/r$ as would be required by Leibniz's Rule (§ 119). Then

$$\nabla\phi=-\frac{1}{4\pi}\int\frac{\nabla\nabla\cdot\mathbf{F}}{r}dv+\frac{1}{4\pi}\int\frac{\nabla\cdot\mathbf{F}}{r}d\mathbf{S}.$$

The surface integral extends over the surfaces of discontinuity of $\nabla\cdot\mathbf{F}$, over a large (infinite) surface, and over an infinitesimal sphere surrounding $r = 0$. It will be assumed that $\nabla\cdot\mathbf{F}$ is such that the surface integral is infinitesimal. Now as $\nabla\times\mathbf{F} = 0$, $\nabla\times\nabla\times\mathbf{F} = 0$ and $\nabla\nabla\cdot\mathbf{F} = \nabla\cdot\nabla\mathbf{F}$. Hence if $\mathbf{F}$ and its derivatives are continuous, a reference to (24) shows that

$$\nabla\phi = -\frac{1}{4\pi}\int\frac{\nabla\cdot\nabla\mathbf{F}}{r}dv = \mathbf{F}.$$

In like manner

$$\nabla\times\mathbf{G} = \frac{1}{4\pi}\int\frac{\nabla\times\nabla\times\mathbf{F}}{r}dv - \frac{1}{4\pi}\int\frac{\nabla\times\mathbf{F}}{r}\times d\mathbf{S} = \frac{-1}{4\pi}\int\frac{\nabla\cdot\nabla\mathbf{F}}{r}dv = \mathbf{F}.$$

Questions of continuity and the significance of the vanishing of the neglected surface integrals will not be further examined. The elementary facts concerning potentials are necessary knowledge for students of physics (especially electromagnetism); the detailed discussion of the subject, whether from its physical or mathematical side, may well be left to special treatises.

EXERCISES

1. Discuss the potential U and its derivative ∇U for the case of a uniform sphere, both at external and internal points, and upon the surface.

2. Discuss the second derivatives of the potential, that is, the derivatives of the forces, at a surface of discontinuity of density.

3. If a distribution of matter is external to a sphere, the average value of the potential on the spherical surface is the value at the center; if it is internal, the average value is the value obtained by concentrating all the mass at the center.

4. What density of distribution is indicated by the potential e^{-r^2}? What density of distribution gives a potential proportional to itself?

5. In a space free of matter the determination of a potential which shall take assigned values on the boundary is equivalent to the problem of minimizing

$$\frac{1}{2}\iiint\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2\right]dxdydz = \frac{1}{2}\int\nabla U\cdot\nabla U dv.$$

6. For Laplace's equation in the plane and for the logarithmic potential $-\log r$, develop the theory of potential integrals analogously to the work of § 201 for Laplace's equation in space and for the fundamental solution $1/r$.