CHAPTER XV

THE CALCULUS OF VARIATIONS

155. The treatment of the simplest case. The integral

$$I = {}_C\int_A^B F(x, y, y')\,dx = {}_C\int_A^B \Phi(x, y, dx, dy), \tag{1}$$

where Φ is homogeneous of the first degree in dx and dy, may be evaluated along any curve C between the limits A and B by reduction to an ordinary integral. For if C is given by $y = f(x)$,

$$I = {}_C\int_A^B F(x, y, y')\,dx = \int_{x_0}^{x_1} F(x, f(x), f'(x))\,dx;$$

and if C is given by $x = \phi(t)$, $y = \psi(t)$,

$$I = {}_C\int_A^B \Phi(x, y, dx, dy) = \int_{t_0}^{t_1} \Phi(\phi, \psi, \phi', \psi')\,dt.$$

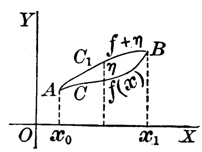
The ordinary line integral (§ 122) is merely the special case in which $\Phi = Pdx + Qdy$ and $F = P + Qy'$. In general the value of I will depend on the path C of integration; *the problem of the calculus of variations is to find that path which will make I a maximum or minimum relative to neighboring paths.*

If a second path C_1 be $y = f(x) + \eta(x)$, where $\eta(x)$ is a small quantity which vanishes at x_0 and x_1, a whole family of paths is given by

$$y = f(x) + \alpha\eta(x), \qquad -1 \leqq \alpha \leqq 1, \qquad \eta(x_0) = \eta(x_1) = 0,$$

and the value of the integral

$$I(\alpha) = \int_{x_0}^{x_1} F(x, f + \alpha\eta, f' + \alpha\eta')\,dx, \tag{1'}$$



taken along the different paths of the family, becomes a function of α; in particular $I(0)$ and $I(1)$ are the values along C and C_1. Under appropriate assumptions as to the continuity of F and its partial derivatives F'_x, F'_y, $F'_{y'}$, the function $I(\alpha)$ will be continuous and have a continuous derivative which may be found by differentiating under the sign (§ 119); then

$$I'(\alpha) = \int_{x_0}^{x_1} [\eta F'_y(x, f + \alpha\eta, f' + \alpha\eta') + \eta' F'_{y'}(x, f + \alpha\eta, f' + \alpha\eta')]\,dx.$$

If the curve C is to give $I(\alpha)$ a maximum or minimum value for all the curves of this family, it is necessary that

$$I'(0)=\int_{x_0}^{x_1}[\eta F'_y(x,\,y,\,y')+\eta' F'_{y'}(x,\,y,\,y')]\,dx=0\,; \tag{2}$$

and if C is to make I a maximum or minimum relative to all neighboring curves, it is necessary that (2) shall hold for any function $\eta(x)$ which is small. It is more usual and more suggestive to write $\eta(x)=\delta y$, and to say that δy is the *variation of* y in passing from the curve C or $y=f(x)$ to the neighboring curve C' or $y=f(x)+\eta(x)$. From the relations

$$y'=f'(x), \qquad y'=f'(x)+\eta'(x), \qquad \delta y'=\eta'(x)=\frac{d}{dx}\delta y,$$

connecting the slope of C with the slope of C_1, it is seen that *the variation of the derivative is the derivative of the variation.* In differential notation this is $d\delta y=\delta dy$, where it should be noted that the sign δ applies to changes which occur on passing from one curve C to another curve C_1, and the sign d applies to changes taking place along a particular curve.

With these notations the condition (2) becomes

$$\int_{x_0}^{x_1}(F'_y\delta y+F'_{y'}\delta y')\,dx=\int_{x_0}^{x_1}\delta F dx=0, \tag{3}$$

where δF is computed from F, δy, $\delta y'$ by the same rule as the differential dF is computed from F and the differentials of the variables which it contains. The condition (3) is not sufficient to distinguish between a maximum and a minimum or to insure the existence of either; neither is the condition $g'(x)=0$ in elementary calculus sufficient to answer these questions relative to a function $g(x)$; in both cases additional conditions are required (§ 9). It should be remembered, however, that these additional conditions were seldom actually applied in discussing maxima and minima of $g(x)$ in practical problems, because in such cases the distinction between the two was usually obvious; so in this case the discussion of sufficient conditions will be omitted altogether, as in §§ 58 and 61, and (3) alone will be applied.

An integration by parts will convert (3) into a differential equation of the second order. In fact

$$\int_{x_0}^{x_1}F'_{y'}\delta y' dx=\int_{x_0}^{x_1}F'_{y'}\frac{d}{dx}\,\delta y dx=\Big[F'_{y'}\delta y\Big]_{x_0}^{x_1}-\int_{x_0}^{x_1}\delta y\,\frac{d}{dx}\,F'_{y'}dx.$$

Hence $$\int_{x_0}^{x_1}(F'_y\delta y+F'_{y'}\delta y')\,dx=\int_{x_0}^{x_1}\left(F'_y-\frac{d}{dx}\,F'_{y'}\right)\delta y dx=0, \tag{3'}$$

since the assumption that $\delta y = \eta(x)$ vanishes at x_0 and x_1 causes the integrated term $[F'_{y'}\delta y]$ to drop out. Then

$$F'_y - \frac{d}{dx}F'_{y'} = \frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial x\partial y'} - \frac{\partial^2 F}{\partial y\partial y'}y' - \frac{\partial^2 F}{\partial y'^2}y'' = 0. \tag{4}$$

For it must be remembered that the function $\delta y = \eta(x)$ is *any* function that is small, and if $F'_y - \frac{d}{dx}F'_{y'}$ in (3′) did not vanish at every point of the interval $x_0 \leqq x \leqq x_1$, the arbitrary function δy could be chosen to agree with it in sign, so that the integral of the product would necessarily be positive instead of zero as the condition demands.

156. *The method of rendering an integral* (1) *a minimum or maximum is therefore to set up the differential equation* (4) *of the second order and solve it.* The solution will contain two arbitrary constants of integration which may be so determined that one particular solution shall pass through the points A and B, which are the initial and final points of the path C of integration. In this way a path C which connects A and B and which satisfies (4) is found; under ordinary conditions the integral will then be either a maximum or minimum. An example follows.

Let it be required to render $I = \int_{x_0}^{x_1}\frac{1}{y}\sqrt{1+y'^2}dx$ a maximum or minimum.

$$F(x, y, y') = \frac{1}{y}\sqrt{1+y'^2}, \qquad \frac{\partial F}{\partial y} = -\frac{1}{y^2}\sqrt{1+y'^2}, \qquad \frac{\partial F}{\partial y'} = \frac{y'}{y}\frac{1}{\sqrt{1+y'^2}}.$$

Hence $$-\frac{1}{y^2}\sqrt{1+y'^2} + \frac{y'}{y^2}\frac{1}{\sqrt{1+y'^2}}y' - \frac{1}{y}\frac{1}{(1+y'^2)^{\frac{3}{2}}}y'' = 0 \quad \text{or} \quad yy'' + y'^2 + 1 = 0$$

is the desired equation (4). It is exact and the integration is immediate.

$$(yy')' + 1 = 0, \qquad yy' + x = c_1, \qquad y^2 + (x - c_1)^2 = c_2.$$

The curves are circles with their centers on the x-axis. From this fact it is easy by a geometrical construction to determine the curve which passes through two given points $A(x_0, y_0)$ and $B(x_1, y_1)$; the analytical determination is not difficult. The two points A and B must lie on the same side of the x-axis or the integral I will not converge and the problem will have no meaning. The question of whether a maximum or a minimum has been determined may be settled by taking a curve C_1 which lies under the circular arc from A to B and yet has the same length. The integrand is of the form ds/y and the integral along C_1 is greater than along the circle C if y is positive, but less if y is negative. It therefore appears that the integral is rendered a minimum if A and B are above the axis, but a maximum if they are below.

For many problems it is more convenient not to make the choice of x or y as independent variable in the first place, but to operate symmetrically with both variables upon the second form of (1). Suppose that the integral of the variation of Φ be set equal to zero, as in (3).

$$\int_A^B \delta\Phi = \int_A^B [\Phi'_x\delta x + \Phi'_y\delta y + \Phi'_{dx}\delta dx + \Phi'_{dy}\delta dy] = 0.$$

Let the rules $\delta dx = d\delta x$ and $\delta dy = d\delta y$ be applied and let the terms which contain $d\delta x$ and $d\delta y$ be integrated by parts as before.

$$\int_A^B \delta\Phi = \int_A^B [(\Phi'_x - d\Phi'_{dx})\,\delta x + (\Phi'_y - d\Phi'_{dy})\delta y] + \Big[\Phi'_{dx}\delta x + \Phi'_{dy}\delta y\Big]_A^B = 0.$$

As A and B are fixed points, the integrated term disappears. As the variations δx and δy may be arbitrary, reasoning as above gives

$$\Phi'_x - d\Phi'_{dx} = 0, \qquad \Phi'_y - d\Phi'_{dy} = 0. \tag{4'}$$

If these two equations can be shown to be essentially identical and to reduce to the condition (4) previously obtained, the justification of the second method will be complete and either of (4′) may be used to determine the solution of the problem.

Now the identity $\Phi(x, y, dx, dy) = F(x, y, dy/dx)\,dx$ gives, on differentiation,

$$\Phi'_x = F'_x dx, \qquad \Phi'_y = F'_y dx, \qquad \Phi'_{dy} = F'_{y'}, \qquad \Phi'_{dx} = -F'_{y'}\frac{dy}{dx} + F$$

by the ordinary rules for partial derivatives. Substitution in each of (4′) gives

$$\Phi'_y - d\Phi'_{dy} = F'_y dx - dF'_{y'} = \left(F'_y - \frac{d}{dx}F'_{y'}\right)dx = 0,$$

$$\begin{aligned}\Phi'_x - d\Phi'_{dx} &= F'_x dx - d(F - F'_{y'}y') = F'_x dx - dF + F'_{y'}dy' + y'dF'_{y'}\\ &= F'_x dx - F'_x dx - F'_y dy - F'_{y'}dy' + F'_{y'}dy' + y'dF'_{y'}\\ &= -F'_y dy + y'dF'_{y'} = -\left(F'_y - \frac{d}{dx}F'_{y'}\right)dy = 0.\end{aligned}$$

Hence each of (4′) reduces to the original condition (4), as was to be proved.

Suppose this method be applied to $\displaystyle\int \frac{ds}{y} = \int \frac{\sqrt{dx^2 + dy^2}}{y}$. Then

$$\begin{aligned}\int \delta\frac{ds}{y} = \int \delta\frac{\sqrt{dx^2+dy^2}}{y} &= \int\left[\frac{dx\delta dx + dy\delta dy}{yds} - \frac{ds}{y^2}\delta y\right]\\ &= -\int\left[d\frac{dx}{yds}\delta x + \left(d\frac{dy}{yds} + \frac{ds}{y^2}\right)\delta y\right],\end{aligned}$$

where the transformation has been integration by parts, including the discarding of the integrated term which vanishes at the limits. The two equations are

$$d\frac{dx}{yds} = 0, \qquad d\frac{dy}{yds} + \frac{ds}{y^2} = 0; \quad \text{and} \quad \frac{dx}{yds} = \frac{1}{c_1}$$

is the obvious first integral of the first. The integration may then be completed to find the circles as before. The integration of the second equation would not be so simple. In some instances *the advantage of the choice of one of the two equations offered by this method of direct operation is marked.*

EXERCISES

1. *The shortest distance.* Treat $\int (1 + y'^2)^{\frac{1}{2}} dx$ for a minimum.

2. Treat $\int \sqrt{dr^2 + r^2 d\phi^2}$ for a minimum in polar coördinates.

3. *The brachistochrone.* If a particle falls along any curve from A to B, the velocity acquired at a distance h below A is $v = \sqrt{2gh}$ regardless of the path followed. Hence the time spent in passing from A to B is $T = \int ds/v$. The path of quickest descent from A to B is called the brachistochrone. Show that the curve is a cycloid. Take the origin at A.

4. The minimum surface of revolution is found by revolving a catenary.

5. The curve of constant density which joins two points of the plane and has a minimum moment of inertia with respect to the origin is $c_1 r^3 = \sec(3\phi + c_2)$. Note that the two points must subtend an angle of less than 60° at the origin.

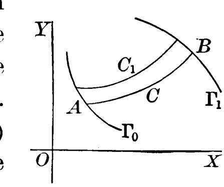
6. Upon the sphere the minimum line is the great circle (polar coördinates).

7. Upon the circular cylinder the minimum line is the helix.

8. Find the minimum line on the cone of revolution.

9. Minimize the integral $\int \left[\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2} n^2 x^2\right] dt$.

157. Variable limits and constrained minima. This second method of operation has also the advantage that it suggests the solution of *the problem of making an integral between variable end-points a maximum or minimum.* Thus suppose that the curve C which shall join some point A of one curve Γ_0 to some point B of another curve Γ_1, and which shall make a given integral a minimum or maximum, is desired. In the first place C must satisfy the condition (4) or (4′) for fixed end-points because C will not give a maximum or minimum value as compared with all other curves unless it does as compared merely with all other curves which join its end-points. There must, however, be additional conditions which shall serve to determine the points A and B which C connects. These conditions are precisely that *the integrated terms,*



$$\left[\Phi'_{dx}\delta x + \Phi'_{dy}\delta y\right]_A^B = 0, \qquad \text{for } A \text{ and for } B, \tag{5}$$

which vanish identically when the end-points are fixed, *shall vanish at each point A or B* provided δx and δy are interpreted as differentials along the curves Γ_0 and Γ_1.

For example, in the case of $\int \frac{ds}{y} = \int \frac{\sqrt{dx^2 + dy^2}}{y}$ treated above, the integrated terms, which were discarded, and the resulting conditions are

$$\left[\frac{dx\delta x}{yds} + \frac{dy\delta y}{yds}\right]_A^B, \qquad \left.\frac{dx\delta x + dy\delta y}{yds}\right]^B = 0, \qquad \left.\frac{dx\delta x + dy\delta y}{yds}\right]_A = 0.$$

Here dx and dy are differentials along the circle C and δx and δy are to be interpreted as differentials along the curves Γ_0 and Γ_1 which respectively pass through A and B. The conditions therefore show that the tangents to C and Γ_0 at A are perpendicular, and similarly for C and Γ_1 at B. In other words the curve which renders the integral a minimum and has its extremities on two fixed curves is the circle which has its center on the x-axis and cuts both the curves orthogonally.

To prove the rule for finding the conditions at the end points it will be sufficient to prove it for one variable point. Let the equations

$$C : x = \phi(t), \qquad y = \psi(t), \qquad C_1 : x = \phi(t) + \zeta(t), \qquad y = \psi(t) + \eta(t),$$

$$\zeta(t_0) = \eta(t_0) = 0, \qquad \zeta(t_1) = a, \qquad \eta(t_1) = b; \qquad \delta x = \zeta(t), \qquad \delta y = \eta(t),$$

determine C and C_1 with the common initial point A and different terminal points B and B' upon Γ_1. As parametric equations of Γ_1, take

$$x = x_B + al(s), \qquad y = y_B + bm(s); \qquad \frac{\delta x}{\delta s} = al'(s), \qquad \frac{\delta y}{\delta s} = bm'(s),$$

where s represents the arc along Γ_1 measured from B, and the functions $l(s)$ and $m(s)$ vary from 0 at B to 1 at B'. Next form the family

$$x = \phi(t) + l(s)\zeta(t), \qquad y = \psi(t) + m(s)\eta(t), \qquad x' = \phi' + l\zeta', \qquad y' = \psi' + m\eta',$$

which all pass through A for $t = t_0$ and which for $t = t_1$ describe the curve Γ_1. Consider

$$g(s) = \int_{t_0}^{t_1} \Phi(x + l(s)\zeta,\ y + m(s)\eta,\ x' + l\zeta',\ y' + m\eta')\,dt, \tag{6}$$

which is the integral taken from A to Γ_1 along the curves of the family, where x, y, x', y' are on the curve C corresponding to $s = 0$. Differentiate. Then

$$g'(s) = \int_{t_0}^{t_1} \left[l'(s)\zeta\Phi'_x + m'(s)\eta\Phi'_y + l'(s)\zeta'\Phi'_{x'} + m'(s)\eta'\Phi'_{y'}\right] dt,$$

where the accents mean differentiation with regard to s when upon g, l, or m, but with regard to t when on x or y, and partial differentiation when on Φ, and where the argument of Φ is as in (6). Now if $g(s)$ has a maximum or minimum when $s = 0$, then

$$g'(0) = \int_{t_0}^{t_1} \left[l'(0)\zeta\Phi'_x(x, y, x', y') + m'(0)\eta\Phi'_y + l'(0)\zeta'\Phi'_{x'} + m'(0)\eta'\Phi'_{y'}\right] dt = 0;$$

$$\Big[l'(0)\zeta\Phi'_{x'} + m'(0)\eta\Phi'_{y'}\Big]_{t_0}^{t_1} + \int_{t_0}^{t_1} \left[l'(0)\zeta\left(\Phi'_x - \frac{d}{dt}\Phi'_{x'}\right) + m'(0)\eta\left(\Phi'_y - \frac{d}{dt}\Phi'_{y'}\right)\right] dt = 0.$$

The change is made as usual by integration by parts. Now as

$$\Phi(x, y, x', y')\,dt = \Phi(x, y, dx, dy), \quad \text{so} \quad \Phi'_x dt = \Phi'_x, \qquad \Phi'_{x'} = \Phi'_{dx}, \text{ etc.}$$

Hence the parentheses under the integral sign, when multiplied by dt, reduce to (4′) and vanish ; they could be seen to vanish also for the reason that $\dot{\zeta}$ and η are arbitrary functions of t except at $t = t_0$ and $t = t_1$, and the integrated term is a constant. There remains the integrated term which must vanish,

$$l'(0)\,\zeta(t_1)\,\Phi'_{x'} + m'(0)\,\eta(t_1)\,\Phi'_{y'} = \left[\frac{\delta x}{\delta s}\Phi'_{x'} + \frac{\delta y}{\delta s}\Phi'_{y'}\right]^{t_1} = \left[\Phi'_{dx}\,\delta x + \Phi'_{dy}\,\delta y\right]^{t_1} = 0.$$

The condition therefore reduces to its appropriate half of (5), provided that, in interpreting it, the quantities δx and δy be regarded not as $a = \zeta(t_1)$ and $b = \eta(t_1)$ but as the differentials along Γ_1 at B.

158. In many cases one integral is to be made a maximum or minimum subject to the condition that another integral shall have a fixed value,

$$I = \int_{x_0}^{x_1} F(x, y, y')\,dx \;\frac{\text{min.}}{\text{max.}}, \qquad J = \int_{x_0}^{x_1} G(x, y, y')\,dx = \text{const.} \tag{7}$$

For instance a curve of given length might run from A to B, and the form of the curve which would make the area under the curve a maximum or minimum might be desired; to make the area a maximum or minimum without the restriction of constant length of arc would be useless, because by taking a curve which dropped sharply from A, inclosed a large area below the x-axis, and rose sharply to B the area could be made as small as desired. Again the curve in which a chain would hang might be required. The length of the chain being given, the form of the curve is that which will make the potential energy a minimum, that is, will bring the center of gravity lowest. The problems in constrained maxima and minima are called *isoperimetric problems* because it is so frequently the perimeter or length of the curve which is given as constant.

If the method of determining constrained maxima and minima by means of undetermined multipliers be recalled (§§ 58, 61), it will appear that the solution of the isoperimetric problem might reasonably be sought by rendering the integral

$$I + \lambda J = \int_{x_0}^{x_1} [F(x, y, y') + \lambda G(x, y, y')]\,dx \tag{8}$$

a maximum or minimum. The solution of this problem would contain three constants, namely, λ and two constants c_1, c_2 of integration. The constants c_1, c_2 could be determined so that the curve should pass through A and B and the value of λ would still remain to be determined in such a manner that the integral J should have the desired value. This is the method of solution.

To justify the method in the case of fixed end-points, which is the only case that will be considered, the procedure is like that of § 155. Let C be given by $y = f(x)$; consider

$$y = f(x) + \alpha\eta(x) + \beta\zeta(x), \qquad \eta_0 = \eta_1 = \zeta_0 = \zeta_1 = 0,$$

a two-parametered family of curves near to C. Then

$$g(\alpha, \beta) = \int_{x_0}^{x_1} F(x, y + \alpha\eta + \beta\zeta, y' + \alpha\eta' + \beta\zeta')\,dx, \qquad g(0, 0) = I$$

$$h(\alpha, \beta) = \int_{x_0}^{x_1} G(x, y + \alpha\eta + \beta\zeta, y' + \alpha\eta' + \beta\zeta')\,dx = J = \text{const.}$$

would be two functions of the two variables α and β. The conditions for the minimum or maximum of $g(\alpha, \beta)$ at $(0, 0)$ subject to the condition that $h(\alpha, \beta) = \text{const.}$ are required. Hence

$$g'_\alpha(0, 0) + \lambda h'_\alpha(0, 0) = 0, \qquad g'_\beta(0, 0) + \lambda h'_\beta(0, 0) = 0,$$

or
$$\int_{x_0}^{x_1} \eta(F'_y + \lambda G'_y) + \eta'(F'_{y'} + \lambda G'_{y'})\,dx = 0,$$

$$\int_{x_0}^{x_1} \zeta(F'_y + \lambda G'_y) + \zeta'(F'_{y'} + \lambda G'_{y'})\,dx = 0.$$

By integration by parts either of these equations gives

$$(F + \lambda G)'_y - \frac{d}{dx}(F + \lambda G)'_{y'} = 0; \tag{9}$$

the rule is justified, and will be applied to an example.

Required the curve which, when revolved about an axis, will generate a given volume of revolution bounded by the least surface. The integrals are

$$I = 2\pi \int_{x_0}^{x_1} y\,ds, \text{ min.}, \qquad J = \pi \int_{x_0}^{x_1} y^2 dx, \text{ const.}$$

Make
$$\int_{x_0}^{x_1} (y\,ds + \lambda y^2 dx) \text{ min. } \quad \text{or} \quad \int_{x_0}^{x_1} \delta(y\,ds + \lambda y^2 dx) = 0.$$

$$\int_{x_0}^{x_1} \delta(y\,ds + \lambda y^2 dx) = \int_{x_0}^{x_1} \left[\delta y\,ds + y\frac{dx\delta dx + dy\delta dy}{ds} + 2\lambda y\delta y dx + \lambda y^2 \delta dx\right] = 0$$
$$= \int_{x_0}^{x_1} \left[\delta x\left(-\lambda d(y^2) - d\frac{y\,dx}{ds}\right) + \delta y\left(ds - d\frac{y\,dy}{ds} + 2\lambda y\,dx\right)\right].$$

Hence
$$\lambda d(y^2) + d\frac{y\,dx}{ds} = 0 \quad \text{or} \quad ds - d\frac{y\,dy}{ds} + 2\lambda y\,dx = 0.$$

The second method of computation has been used and the vanishing integrated terms have been discarded. The first equation is simplest to integrate.

$$\lambda y^2 + y\frac{1}{\sqrt{1 + y'^2}} = c_1\lambda, \qquad \pm\frac{\lambda(c_1 - y^2)\,dy}{\sqrt{y^2 - \lambda^2(c_1 - y^2)^2}} = dx.$$

The variables are separated, but the integration cannot be executed in terms of elementary functions. If, however, one of the end-points is on the x-axis, the values x_0, 0, y_0' or x_1, 0, y_1' must satisfy the equation and, as no term of the equation can become infinite, c_1 must vanish. The integration may then be performed.

$$\pm\frac{\lambda y dy}{\sqrt{1-\lambda^2 y^2}}=dx, \qquad 1-\lambda^2 y^2=\lambda^2(x-c_2)^2 \quad \text{or} \quad (x-c_2)^2+y^2=\frac{1}{\lambda^2}.$$

In this special case the curve is a circle. The constants c_1 and λ may be determined from the other point (x_1, y_1) through which the curve passes and from the value of $J=v$; the equations will also determine the abscissa x_0 of the point on the axis. It is simpler to suppose $x_0=0$ and leave x_1 to be determined. With this procedure the equations are

$$c_2^2=\frac{1}{\lambda^2}, \qquad (x_1-c_2)^2+y_1^2=\frac{1}{\lambda^2}, \qquad \frac{v}{\pi}=\frac{x_1}{\lambda^2}-\frac{1}{3}(x_1^3-3c_2x_1^2+3c_2^2x_1),$$

or
$$x_1^3+3y_1^2x_1-\frac{6v}{\pi}=0, \qquad c_2=\frac{x_1^2+y_1^2}{2x_1},$$

and
$$x_1=\pi^{-\frac{1}{3}}\left[\left(3v+\sqrt{9v^2+\pi^2y_1^6}\right)^{\frac{1}{3}}+\left(3v-\sqrt{9v^2+\pi^2y_1^6}\right)^{\frac{1}{3}}\right].$$

EXERCISES

1. Show that (α) the minimum line from one curve to another in the plane is their common normal; (β) if the ends of the catenary which generates the minimum surface of revolution are constrained to lie on two curves, the catenary shall be perpendicular to the curves; (γ) the brachistochrone from a fixed point to a curve is the cycloid which cuts the curve orthogonally.

2. Generalize to show that if the end-points of the curve which makes any integral of the form $\int F(x, y)ds$ a maximum or a minimum are variable upon two curves, the solution shall cut the curves orthogonally.

3. Show that if the integrand $\Phi(x, y, dx, dy, x_1)$ depends on the limit x_1, the condition for the limit B becomes $\left[\Phi'_{dx}\delta x+\Phi'_{dy}\delta y+\delta x\int_{x_0}^{x_1}\Phi'_{x_1}\right]^B=0.$

4. Show that the cycloid which is the brachistochrone from a point A, constrained to lie on one curve Γ_0, to another curve Γ_1 must leave Γ_0 at the point A where the tangent to Γ_0 is parallel to the tangent to Γ_1 at the point of arrival.

5. Prove that the curve of given length which generates the minimum surface of revolution is still the catenary.

6. If the area under a curve of given length is to be a maximum or minimum, the curve must be a circular arc connecting the two points.

7. In polar coördinates the sectorial area bounded by a curve of given length is a maximum or minimum when the curve is a circle.

8. A curve of given length generates a maximum or minimum volume of revolution. The elastic curve

$$R=\frac{(1+y'^2)^{\frac{3}{2}}}{y''}=-\frac{\lambda}{2y} \quad \text{or} \quad dx=\frac{(y^2-c_1)\,dy}{\sqrt{\lambda^2-(y^2-c_1)^2}}.$$

9. A chain lies in a central field of force of which the potential per unit mass is $V(r)$. If the constant density of the chain is ρ, show that the form of the curve is

$$\phi + c_2 = \int \frac{dr}{r\left[c_1^2(\rho V + \lambda)^2 - 1\right]^{\frac{1}{2}}}.$$

10. Discuss the reciprocity of I and J, that is, the questions of making I a maximum or minimum when J is fixed, and of making J a minimum or maximum when I is fixed.

11. A solid of revolution of given mass and uniform density exerts a maximum attraction on a point at its axis. *Ans.* $2\lambda(x^2 + y^2)^{\frac{3}{2}} + x = 0$, if the point is at the origin.

159. Some generalizations. Suppose that an integral

$$I = \int_A^B F(x, y, y', z, z', \cdots)\,dx = \int_A^B \Phi(x, dx, y, dy, z, dz, \cdots) \qquad (10)$$

(of which the integrand contains two or more dependent variables $y, z, \cdots$ and their derivatives $y', z', \cdots$ with respect to the independent variable x, or in the symmetrical form contains three or more variables and their differentials) were to be made a maximum or minimum. In case there is only one additional variable, the problem still has a geometric interpretation, namely, to find

$$y = f(x), \qquad z = g(x), \quad \text{or} \quad x = \phi(t), \qquad y = \psi(t), \qquad z = \chi(t),$$

a curve in space, which will make the value of the integral greater or less than all neighboring curves. A slight modification of the previous reasoning will show that necessary conditions are

$$F'_y - \frac{d}{dx}F'_{y'} = 0 \quad \text{and} \quad F'_z - \frac{d}{dx}F'_{z'} = 0 \qquad (11)$$

or $$\Phi'_x - d\Phi'_{dx} = 0, \qquad \Phi'_y - d\Phi'_{dy} = 0, \qquad \Phi'_z - d\Phi'_{dz'} = 0,$$

where of the last three conditions only two are independent. Each of (11) is a differential equation of the second order, and the solution of the two simultaneous equations will be a family of curves in space dependent on four arbitrary constants of integration which may be so determined that one curve of the family shall pass through the endpoints A and B.

Instead of following the previous method to establish these facts, an older and perhaps less accurate method will be used. Let the varied values of y, z, y', z', be denoted by

$$y + \delta y, \quad z + \delta z, \quad y' + \delta y', \quad z' + \delta z', \quad \delta y' = (\delta y)', \quad \delta z' = (\delta z)'.$$

The difference between the integral along the two curves is

$$\Delta I = \int_{x_0}^{x_1} [F(x,\, y + \delta y,\, y' + \delta y',\, z + \delta z,\, z' + \delta z') - F(x,\, y,\, y',\, z,\, z')]\,dx$$

$$= \int_{x_0}^{x_1} \Delta F dx = \int_{x_0}^{x_1} (F'_y \delta y + F'_{y'}\delta y' + F'_z \delta z + F'_{z'}\delta z')\,dx + \cdots,$$

where F has been expanded by Taylor's Formula* for the four variables y, y', z, z' which are varied, and "$+ \cdots$" refers to the remainder or the subsequent terms in the development which contain the higher powers of δy, $\delta y'$, δz, $\delta z'$.

For sufficiently small values of the variations the terms of higher order may be neglected. Then if ΔI is to be either positive or negative for all small variations, the terms of the first order which change in sign when the signs of the variations are reversed must vanish and the condition becomes

$$\int_{x_0}^{x_1} (F'_y \delta y + F'_{y'}\delta y' + F'_z \delta z + F'_{z'}\delta z')\,dx = \int_{x_0}^{x_1} \delta F dx = 0. \qquad (12)$$

Integrate by parts and discard the integrated terms. Then

$$\int_{x_0}^{x_1} \left[\left(F'_y - \frac{d}{dx} F'_{y'}\right)\delta y + \left(F'_z - \frac{d}{dx} F'_{z'}\right)\delta z\right] = 0. \qquad (13)$$

* In the simpler case of § 155 this formal development would run as

$$\Delta I = \int_{x_0}^{x_1} (F'_y \delta y + F'_{y'}\delta y')\,dx + \frac{1}{2!}\int_{x_0}^{x_1} (F''_{yy}\delta y^2 + 2F''_{yy'}\delta y \delta y' + F''_{y'y'}\delta y'^2)\,dx + \text{higher terms},$$

and with the expansion $\Delta I = \delta I + \frac{1}{2!}\delta^2 I + \frac{1}{3!}\delta^3 I + \cdots$ it would appear that

$$\delta I = \int_{x_0}^{x_1} (F'_y \delta y + F'_{y'}\delta y')\,dx, \qquad \delta^2 I = \int_{x_0}^{x_1} (F''_{yy}\delta y^2 + 2F''_{yy'}\delta y \delta y' + F''_{y'y'}\delta y'^2)\,dx,$$

$$\delta^3 I = \int_{x_0}^{x_1} (F'''_{y^3}\delta y^3 + 3F'''_{y^2y'}\delta y^2 \delta y' + 3F'''_{yy'^2}\delta y \delta y'^2 + F'''_{y'^3}\delta y'^3)\,dx, \cdots.$$

The terms δI, $\delta^2 I$, $\delta^3 I$, $\cdots$ are called the *first, second, third,* $\cdots$ *variations* of the integral I in the case of fixed limits. The condition for a maximum or minimum then becomes $\delta I = 0$, just as $dg = 0$ is the condition in the case of $g(x)$. In the case of variable limits there are some modifications appropriate to the limits. This method of procedure suggests the reason that δx, δy are frequently to be treated exactly as differentials. It also suggests that $\delta^2 I > 0$ and $\delta^2 I < 0$ would be criteria for distinguishing between maxima and minima. The same results can be had by differentiating (1') repeatedly under the sign and expanding $I(\alpha)$ into series; in fact, $\delta I = I'(0)$, $\delta^2 I = I''(0)$, $\cdots$. No emphasis has been laid in the text on the suggestive relations $\delta I = \int \delta F dx$ for fixed limits or $\delta I = \int \delta \Phi$ for variable limits (variable in x, y, but not in t) because only the most elementary results were desired, and the treatment given has some advantages as to modernity.

As δy and δz are arbitrary, either may in particular be taken equal to 0 while the other is assigned the same sign as its coefficient in the parenthesis; and hence the integral would not vanish unless that coefficient vanished. Hence the conditions (11) are derived, and it is seen that there would be precisely similar conditions, one for each variable $y, z, \cdots$, no matter how many variables might occur in the integrand.

Without going at all into the matter of proof it will be stated as a fact that the condition for the maximum or minimum of

$$\int \Phi(x,\ dx,\ y,\ dy,\ z,\ dz, \ldots) \quad \text{is} \quad \int \delta\Phi = 0,$$

which may be transformed into the set of differential equations

$$\Phi'_x - d\Phi'_{dx} = 0, \qquad \Phi'_y - d\Phi'_{dy} = 0, \qquad \Phi'_z - d\Phi'_{dz} = 0, \qquad \cdots,$$

of which any one may be discarded as dependent on the rest; and

$$\Phi'_{dx}\delta x + \Phi'_{dy}\delta y + \Phi'_{dz}\delta z + \cdots = 0, \qquad \text{at } A \text{ and at } B,$$

where the variations are to be interpreted as differentials along the loci upon which A and B are constrained to lie.

It frequently happens that the variables in the integrand of an integral which is to be made a maximum or minimum are connected by an equation. For instance

$$\int \Phi(x,\ dx,\ y,\ dy,\ z,\ dz) \text{ min.}, \qquad S(x,\ y,\ z) = 0. \tag{14}$$

It is possible to eliminate one of the variables and its differential by means of $S = 0$ and proceed as before; but it is usually better to introduce an undetermined multiplier (§§ 58, 61). From

$$S(x,\ y,\ z) = 0 \quad \text{follows} \quad S'_x\delta x + S'_y\delta y + S'_z\delta z = 0$$

if the variations be treated as differentials. Hence if

$$\int [(\Phi'_x - d\Phi'_{dx})\,\delta x + (\Phi'_y - d\Phi'_{dy})\,\delta y + (\Phi'_z - d\Phi'_{dz})\,\delta z] = 0,$$

$$\int [(\Phi'_x - d\Phi'_{dx} + \lambda S'_x)\,\delta x + (\Phi'_y - d\Phi'_{dy} + \lambda S'_y)\,\delta y \\ + (\Phi'_z - d\Phi'_{dz} + \lambda S'_z)\,\delta z] = 0$$

no matter what the value of λ. Let the value of λ be so chosen as to annul the coefficient of δz. Then as the two remaining variations are independent, the same reasoning as above will cause the coefficients of δx and δy to vanish and

$$\Phi'_x - d\Phi'_{dx} + \lambda S'_x = 0, \quad \Phi'_y - d\Phi'_{dy} + \lambda S'_y = 0, \quad \Phi'_z - d\Phi'_{dz} + \lambda S'_z = 0 \tag{15}$$

will hold. These equations, taken with $S = 0$, will determine y and z as functions of x and also incidentally will fix λ.

Consider the problem of determining *the shortest lines upon a surface* $S(x, y, z) = 0$. These lines are called *the geodesics.* Then

$$\int \delta ds = 0 = \frac{dx\delta x + dy\delta y + dz\delta z}{ds}\Bigg| - \int \left[d\frac{dx}{ds}\delta x + d\frac{dy}{ds}\delta y + d\frac{dz}{ds}\delta z \right], \quad (16)$$

$$\int \left(d\frac{dx}{ds} + \lambda S'_x \right)\delta x + \left(d\frac{dy}{ds} + \lambda S'_y \right)\delta y + \left(d\frac{dz}{ds} + \lambda S'_z \right)\delta z = 0,$$

$$d\frac{dx}{ds} + \lambda S'_x = d\frac{dy}{ds} + \lambda S'_y = d\frac{dz}{ds} + \lambda S'_z = 0, \quad \text{and} \quad \frac{d\frac{dx}{ds}}{S'_x} = \frac{d\frac{dy}{ds}}{S'_y} = \frac{d\frac{dz}{ds}}{S'_z}.$$

In the last set of equations λ has been eliminated and the equations, taken with $S = 0$, may be regarded as *the differential equations of the geodesics.* The denominators are proportional to the direction cosines of the normal to the surface, and the numerators are the components of the differential of the unit tangent to the curve and are therefore proportional to the direction cosines of the normal to the curve in its osculating plane. Hence it appears that *the osculating plane of a geodesic curve contains the normal to the surface.*

The integrated terms $dx\delta x + dy\delta y + dz\delta z = 0$ show that the least geodesic which connects two curves on the surface will cut both curves orthogonally. These terms will also suffice to prove a number of interesting theorems which establish an analogy between geodesics on a surface and straight lines in a plane. For instance: The locus of points whose geodesic distance from a fixed point is constant (a geodesic circle) cuts the geodesic lines orthogonally. To see this write

$$\int_O^P ds = \text{const.}, \quad \Delta\int_O^P ds = 0, \quad \delta\int_O^P ds = 0, \quad \int_O^P \delta ds = 0 = dx\delta x + dy\delta y + dz\delta z\Big|^P.$$

The integral in (16) drops out because taken along a geodesic. This final equality establishes the perpendicularity of the lines. The fact also follows from the statement that the geodesic circle and its center can be regarded as two curves between which the shortest distance is the distance measured along any of the geodesic radii, and that the radii must therefore be perpendicular to the curve.

160. The most fundamental and important single theorem of mathematical physics is Hamilton's Principle, which is expressed by means of the calculus of variations and affords a necessary and sufficient condition for studying the elements of this subject. Let T be the kinetic energy of any dynamical system. Let X_i, Y_i, Z_i be the forces which act at any point x_i, y_i, z_i of the system, and let δx_i, δy_i, δz_i represent displacements of that point. Then the work is

$$\delta W = \sum (X_i\delta x_i + Y_i\delta y_i + Z_i\delta z_i).$$

Hamilton's Principle states that the *time integral*

$$\int_{t_0}^{t_1}(\delta T + \delta W)\,dt = \int_{t_0}^{t_1}[\delta T + \sum (X\delta x + Y\delta y + Z\delta z)]\,dt = 0 \quad (17)$$

vanishes for the actual motion of the system. If in particular there is a potential function V, then $\delta W = -\delta V$ and

$$\int_{t_0}^{t_1}\delta(T - V)\,dt = \delta\int_{t_0}^{t_1}(T - V)\,dt = 0, \qquad (17')$$

and *the time integral of the difference between the kinetic and potential energies is a maximum or minimum for the actual motion of the system* as compared with any neighboring motion.

Suppose that the position of a system can be expressed by means of n independent variables or coördinates $q_1, q_2, \cdots, q_n$. Let the kinetic energy be expressed as

$$T = \sum \tfrac{1}{2} m_i v_i^2 = \int \tfrac{1}{2} v^2 dm = T(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n),$$

a function of the coördinates and their derivatives with respect to the time. Let the work done by displacing the single coördinate q_r be $\delta W = Q_r\delta q_r$, so that the total work, in view of the independence of the coördinates, is $Q_1\delta q_1 + Q_2 dq_2 + \cdots + Q_n dq_n$. Then

$$0 = \int_{t_0}^{t_1}(\delta T + \delta W)\,dt = \int_{t_0}^{t_1}(T'_{q_1}\delta q_1 + T'_{q_2}\delta q_2 + \cdots + T'_{q_n}\delta q_n + T'_{\dot{q}_1}\delta\dot{q}_1 + T'_{\dot{q}_2}\delta\dot{q}_2$$
$$+ \cdots + T'_{\dot{q}_n}\delta q_n + Q_1\delta q_1 + Q_2\delta q_2 + \cdots + Q_n\delta q_n)\,dt.$$

Perform the usual integration by parts and discard the integrated terms which vanish at the limits $t = t_0$ and $t = t_1$. Then

$$0 = \int_{t_0}^{t_1}\Big[\Big(T'_{q_1} + Q_1 - \frac{d}{dt}T'_{\dot{q}_1}\Big)\delta q_1 + \Big(T'_{q_2} + Q_2 - \frac{d}{dt}T'_{\dot{q}_2}\Big)\delta q_2$$
$$+ \cdots + \Big(T'_{q_n} + Q_n - \frac{d}{dt}T'_{\dot{q}_n}\Big)\delta q_n\Big]\,dt.$$

In view of the independence of the variations $\delta q_1, \delta q_2, \cdots, \delta q_n$,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} = Q_1, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2} - \frac{\partial T}{\partial q_2} = Q_2, \quad \cdots, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_n} - \frac{\partial T}{\partial q_n} = Q_n. \quad (18)$$

These are the *Lagrangian equations* for the motion of a dynamical system.* If there is a potential function $V(q_1, q_2, \cdots, q_n)$, then by definition

$$Q_1 = -\frac{\partial V}{\partial q_1}, \quad Q_2 = -\frac{\partial V}{\partial q_2}, \quad \cdots, \quad Q_n = -\frac{\partial V}{\partial q_n}, \quad \frac{\partial V}{\partial \dot{q}_1} = \frac{\partial V}{\partial \dot{q}_2} = \cdots = \frac{\partial V}{\partial \dot{q}_n} = 0.$$

Hence $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = 0, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = 0, \quad \cdots, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} = 0, \quad L = T - V.$$

The equations of motion have been expressed in terms of a single function L, which is the difference between the kinetic energy T and potential function V. By

* Compare Ex. 19, p. 112, for a deduction of (18) by transformation.

comparing the equations with (17′) it is seen that the dynamics of a system which may be specified by n coördinates, and which has a potential function, may be stated as the problem of rendering the integral $\int L dt$ a maximum or a minimum; both the kinetic energy T and potential function V may contain the time t without changing the results.

For example, let it be required to derive the equations of motion of a lamina lying in a plane and acted upon by any forces in the plane. Select as coördinates the ordinary coördinates (x, y) of the center of gravity and the angle ϕ through which the lamina may turn about its center of gravity. The kinetic energy of the lamina (p. 318) will then be the sum $\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$. Now if the lamina be moved a distance δx to the right, the work done by the forces will be $X\delta x$, where X denotes the sum of all the components of force along the x-axis no matter at what points they act. In like manner $Y\delta y$ will be the work for a displacement δy. Suppose next that the lamina is rotated about its center of gravity through the angle $\delta\phi$; the actual displacement of any point is $r\delta\phi$ where r is its distance from the center of gravity. The work of any force will then be $Rrd\phi$ where R is the component of the force perpendicular to the radius r; but $Rr = \Phi$ is the moment of the force about the center of gravity. Hence

$$T = \tfrac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\dot{\phi}^2, \qquad \delta W = X\delta x + Y\delta y + \Phi\delta\phi$$

and
$$M\frac{d^2x}{dt^2} = X, \qquad M\frac{d^2y}{dt^2} = Y, \qquad I\frac{d^2\phi}{dt^2} = \Phi,$$

by substitution in (18), are the desired equations, where X and Y are the *total* components along the axis and Φ is the *total* moment about the center of gravity.

A particle glides without friction on the interior of an inverted cone of revolution; determine the motion. Choose the distance r of the particle from the vertex and the meridional angle ϕ as the two coördinates. If l be the sine of the angle between the axis of the cone and the elements, then $ds^2 = dr^2 + r^2l^2d\phi^2$ and $v^2 = \dot{r}^2 + r^2l^2\dot{\phi}^2$. The pressure of the cone against the particle does no work; it is normal to the motion. For a change $\delta\phi$ gravity does no work; for a change δr it does work to the amount $-mg\sqrt{1-l^2}\delta r$. Hence

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2l^2\dot{\phi}^2), \qquad \delta W = -mg\sqrt{1-l^2}\delta r \quad \text{or} \quad V = mg\sqrt{1-l^2}r.$$

Then
$$\frac{d^2r}{dt^2} - rl^2\left(\frac{d\phi}{dt}\right)^2 = -g\sqrt{1-l^2}, \qquad \frac{d}{dt}\left(r^2l^2\frac{d\phi}{dt}\right) = 0 \quad \text{or} \quad r^2\frac{d\phi}{dt} = C.$$

The remaining integrations cannot all be effected in terms of elementary functions.

161. Suppose the double integral

$$I = \iint F(x, y, z, p, q)\,dxdy, \qquad p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}, \tag{19}$$

extended over a certain area of the xy-plane were to be made a maximum or minimum by a surface $z = z(x, y)$, which shall pass through a given curve upon the cylinder which stands upon the bounding curve of the area. This problem is analogous to the problem of § 155 with fixed limits; the procedure for finding the partial differential equation which z shall satisfy is also analogous. Set

$$\iint \delta F dxdy = \iint (F'_z \delta z + F'_p \delta p + F'_q \delta q)\, dxdy = 0.$$

Write $\delta p = \frac{\partial \delta z}{\partial x}$, $\delta q = \frac{\partial \delta z}{\partial y}$ and integrate by parts.

$$\iint F'_p \frac{\partial \delta z}{\partial x} dxdy = \int F'_p \delta z \Big|_A^B dy - \iint \frac{dF'_p}{dx} \delta z dxdy.$$

The limits A and B for which the first term is taken are points upon the bounding contour of the area, and $\delta z = 0$ for A and B by virtue of the assumption that the surface is to pass through a fixed curve above that contour. The integration of the term in δq is similar. Hence the condition becomes

$$\iint \delta F dxdy = \iint \left(F'_z - \frac{d}{dx}\frac{\partial F}{\partial p} - \frac{d}{dy}\frac{\partial F}{\partial q} \right) \delta z dxdy = 0 \qquad (20)$$

or

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial p} - \frac{d}{dy}\frac{\partial F}{\partial q} = 0, \qquad (20')$$

by the familiar reasoning. The total differentiations give

$$F'_z - F''_{xp} - F''_{yp} - F''_{zp} p - F''_{zq} q - F''_{pp} r - 2 F''_{pq} s - F''_{qq} t = 0.$$

The stock illustration introduced at this point is the minimum surface, that is, the surface which spans a given contour with the least area and which is physically represented by a soap film. The real use, however, of the theory is in connection with Hamilton's Principle. To study the motion of a chain hung up and allowed to vibrate, or of a piano wire stretched between two points, compute the kinetic and potential energies and apply Hamilton's Principle. Is the motion of a vibrating elastic body to be investigated? Apply Hamilton's Principle. And so in electrodynamics. In fact, with the very foundations of mechanics sometimes in doubt owing to modern ideas on electricity, the one refuge of many theorists is Hamilton's Principle. Two problems will be worked in detail to exhibit the method.

Let a uniform chain of density ρ and length l be suspended by one extremity and caused to execute small oscillations in a vertical plane. At any time the shape of the curve is $y = y(x)$, and $y = y(x, t)$ will be taken to represent the shape of the curve at all times. Let $y' = \partial y/\partial x$ and $\dot{y} = \partial y/\partial t$. As the oscillations are small, the chain will rise only slightly and the main part of the kinetic energy will be in the whipping motion from side to side; the assumption $dx = ds$ may be made and the kinetic energy may be taken as

$$T = \int_0^l \frac{1}{2}\rho \left(\frac{\partial y}{\partial t}\right)^2 dx.$$

The potential energy is a little harder to compute, for it is necessary to obtain the slight rise in the center of gravity due to the bending of the chain. Let λ be the shortened length. The position of the center of gravity is

$$\bar{x} = \frac{\int_0^\lambda x(1+\frac{1}{2}y'^2)\,dx}{\int_0^\lambda (1+\frac{1}{2}y'^2)\,dx} = \frac{\frac{1}{2}\lambda^2 + \int_0^\lambda \frac{1}{2}xy'^2dx}{\lambda + \int_0^\lambda \frac{1}{2}y'^2dx} = \frac{1}{2}\lambda - \frac{1}{\lambda}\int_0^\lambda \left(\frac{1}{4}\lambda - \frac{1}{2}x\right) y'^2dx.$$

Here $ds = \sqrt{1+y'^2}dx$ has been expanded and terms higher than y'^2 have been omitted.

$$l = \lambda + \int_0^\lambda \frac{1}{2}y'^2dx, \qquad \frac{1}{2}l - \bar{x} = \frac{1}{\lambda}\int_0^\lambda \frac{1}{2}(\lambda - x)\,y'^2dx, \qquad V = l\rho g\left(\frac{1}{2}l - \bar{x}\right).$$

Then
$$\int_{t_0}^{t_1}(T-V)\,dt = \int_{t_0}^{t_1}\int_0^l \left[\frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2 dx - \frac{1}{2}\rho g(l-x)\left(\frac{\partial y}{\partial x}\right)^2\right] dxdt, \tag{21}$$

provided λ be now replaced in V by l which differs but slightly from it.

Hamilton's Principle states that (21) must be a maximum or minimum and the integrand is of precisely the form (19) except for a change of notation. Hence

$$-\frac{d}{dx}\left[-\rho g(l-x)\frac{\partial y}{\partial x}\right] - \frac{d}{dt}\left(\rho\frac{\partial y}{\partial t}\right) = 0 \quad \text{or} \quad \frac{1}{g}\frac{\partial^2 y}{\partial t^2} = (l-x)\frac{\partial^2 y}{\partial x^2} - \frac{\partial y}{\partial x}.$$

The change of variable $l - x = u^2$, which brings the origin to the end of the chain and reverses the direction of the axis, gives the differential equation

$$\frac{\partial^2 y}{\partial u^2} + \frac{1}{u}\frac{\partial y}{\partial u} = \frac{4}{g}\frac{\partial^2 y}{\partial t^2} \quad \text{or} \quad \frac{d^2P}{du^2} + \frac{1}{u}\frac{dP}{du} + \frac{4n^2}{g}P = 0 \quad \text{if} \quad y = P(u)\cos nt.$$

As the equation is a partial differential equation the usual device of writing the dependent variable as the product of two functions and trying for a special type of solution has been used (§ 194). The equation in P is a Bessel equation (§ 107) of which one solution $P(u) = AJ_0(2ng^{-\frac{1}{2}}u)$ is finite at the origin $u = 0$, while the other is infinite and must be discarded as not representing possible motions. Thus

$$y(x, t) = AJ_0(2ng^{-\frac{1}{2}}u)\cos nt, \quad \text{with} \quad y(l, t) = AJ_0(2ng^{-\frac{1}{2}}l^{\frac{1}{2}}) = 0$$

as the condition that the chain shall be tied at the original origin, is a possible mode of motion for the chain and consists of whipping back and forth in the periodic time $2\pi/n$. The condition $J_0(2ng^{-\frac{1}{2}}l^{\frac{1}{2}}) = 0$ limits n to one of an infinite set of values obtained from the roots of J_0.

Let there be found the equations for the motion of a medium in which

$$T = \frac{1}{2}A\iiint\left[\left(\frac{\partial\xi}{\partial t}\right)^2 + \left(\frac{\partial\eta}{\partial t}\right)^2 + \left(\frac{\partial\zeta}{\partial t}\right)^2\right] dxdydz,$$

$$V = \frac{1}{2}B\iiint\left(f^2 + g^2 + h^2\right) dxdydz$$

are the kinetic and potential energies, where A and B are constants and

$$4\pi f = \frac{\partial\zeta}{\partial y} - \frac{\partial\eta}{\partial z}, \qquad 4\pi g = \frac{\partial\xi}{\partial z} - \frac{\partial\zeta}{\partial x}, \qquad 4\pi h = \frac{\partial\eta}{\partial x} - \frac{\partial\xi}{\partial y}$$

are relations connecting f, g, h with the displacements ξ, η, ζ along the axes of x, y, z. Then

$$\iiiint \delta[\tfrac{1}{2}A(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) - \tfrac{1}{2}B(f^2 + g^2 + h^2)]\,dxdydzdt = 0 \tag{22}$$

is the expression of Hamilton's Principle. These integrals are more general than (19), for there are three dependent variables ξ, η, ζ and four independent variables x, y, z, t of which they are functions. It is therefore necessary to apply the method of variations directly.

After taking the variations an integration by parts will be applied to the variation of each derivative and the integrated terms will be discarded.

$$\iiiint \delta\,\tfrac{1}{2}A(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)\,dxdydzdt = \iiiint A(\dot{\xi}\delta\dot{\xi} + \dot{\eta}\delta\dot{\eta} + \dot{\zeta}\delta\dot{\zeta})\,dxdydzdt$$

$$= -\iiiint A(\ddot{\xi}\delta\xi + \ddot{\eta}\delta\eta + \ddot{\zeta}\delta\zeta)\,dxdydzdt.$$

$$\iiiint \delta\,\tfrac{1}{2}B(f^2 + g^2 + h^2)\,dxdydzdt = \iiiint B(f\delta f + g\delta g + h\delta h)\,dxdydzdt$$

$$= \iiiint \frac{B}{4\pi}\left[f\left(\frac{\partial\delta\zeta}{\partial y} - \frac{\partial\delta\eta}{\partial z}\right) + g\left(\frac{\partial\delta\xi}{\partial z} - \frac{\partial\delta\zeta}{\partial x}\right) + h\left(\frac{\partial\delta\eta}{\partial x} - \frac{\partial\delta\xi}{\partial y}\right)\right]dxdydzdt$$

$$= -\iiiint \frac{B}{4\pi}\left[\left(\frac{\partial g}{\partial z} - \frac{\partial h}{\partial y}\right)\delta\xi + \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\delta\eta + \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right)\delta\zeta\right]dxdydzdt.$$

After substitution in (22) the coefficients of $\delta\xi, \delta\eta, \delta\zeta$ may be severally equated to zero because $\delta\xi, \delta\eta, \delta\zeta$ are each arbitrary. Hence the equations

$$4\pi A\frac{\partial^2\xi}{\partial t^2} = -B\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right),\quad 4\pi A\frac{\partial^2\eta}{\partial t^2} = -B\left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right),\quad 4\pi A\frac{\partial^2\zeta}{\partial t^2} = -B\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right).$$

With the proper determination of A and B and the proper interpretation of $\xi, \eta, \zeta, f, g, h$, these are the equations of electromagnetism for the free ether.

EXERCISES

1. Show that the straight line is the shortest line in space and that the shortest distance between two curves or surfaces will be normal to both.

2. If at each point of a curve on a surface a geodesic be erected perpendicular to the curve, the locus of its extremity is perpendicular to the geodesic.

3. With any two points of a surface as foci construct a geodesic ellipse by taking the distances $FP + F'P = 2a$ along the geodesics. Show that the tangent to the ellipse is equally inclined to the two geodesic focal radii.

4. Extend Ex. 2, p. 408, to space. If $\int_O^P F(x, y, z)\,ds = \text{const.}$, show that the locus of P is a surface normal to the radii, provided the radii be curves which make the integral a maximum or minimum.

5. Obtain the polar equations for the motion of a particle in a plane.

6. Find the polar equations for the motion of a particle in space.

7. A particle glides down a helicoid ($z = k\phi$ in cylindrical coördinates). Find the equations of motion in (r, ϕ), (r, z), or (z, ϕ), and carry the integration as far as possible toward expressing the position as a function of the time.

8. If $z = ax^2 + by^2 + \cdots$, with $a > 0$, $b > 0$, is the Maclaurin expansion of a surface tangent to the plane $z = 0$ at $(0, 0)$, find and solve the equations for the motion of a particle gliding about on the surface and remaining near the origin.

9. Show that $r(1+q^2) + t(1+p^2) - 2pqs = 0$ is the partial differential equation of a minimum surface; test the helicoid.

10. If ρ and S are the density and tension in a uniform piano wire, show that the approximate expressions for the kinetic and potential energies are

$$T = \frac{1}{2}\int_0^l \rho\left(\frac{\partial y}{\partial t}\right)^2 dx, \qquad V = \frac{1}{2}\int_0^l S\left(\frac{\partial y}{\partial x}\right)^2 dx.$$

Obtain the differential equation of the motion and try for solutions $y = P(x)\cos nt$.

11. If ξ, η, ζ are the displacements in a uniform elastic medium, and

$$a = \frac{\partial \xi}{\partial x},\quad b = \frac{\partial \eta}{\partial y},\quad c = \frac{\partial \zeta}{\partial z},\quad f = \left(\frac{\partial \zeta}{\partial y} + \frac{\partial \eta}{\partial z}\right),\quad g = \left(\frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x}\right),\quad h = \left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right)$$

are six combinations of the nine possible first partial derivatives, it is assumed that $V = \iiint F dx dy dz$, where F is a homogeneous quadratic function of a, b, c, f, g, h, with constant coefficients. Establish the equations of the motion of the medium.

$$\rho\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial^2 F}{\partial x \partial a} + \frac{\partial^2 F}{\partial y \partial h} + \frac{\partial^2 F}{\partial z \partial g}, \qquad \rho\frac{\partial^2 \eta}{\partial t^2} = \frac{\partial^2 F}{\partial x \partial h} + \frac{\partial^2 F}{\partial y \partial b} + \frac{\partial^2 F}{\partial z \partial f},$$

$$\rho\frac{\partial^2 \zeta}{\partial t^2} = \frac{\partial^2 F}{\partial x \partial g} + \frac{\partial^2 F}{\partial y \partial f} + \frac{\partial^2 F}{\partial z \partial c}.$$

12. Establish the conditions (11) by the method of the text in § 155.

13. By the method of § 159 and footnote establish the conditions at the end points for a minimum of $\int F(x, y, y')\,dx$ in terms of F instead of Φ.

14. Prove Stokes's Formula $I = \oint \mathbf{F}\cdot d\mathbf{r} = \iint \nabla\times\mathbf{F}\cdot d\mathbf{S}$ of p. 345 by the calculus of variations along the following lines: First compute the variation of I on passing from one closed curve to a neighboring (larger) one.

$$\delta I = \delta \oint \mathbf{F}\cdot d\mathbf{r} = \oint (\delta\mathbf{F}\cdot d\mathbf{r} - d\mathbf{F}\cdot\delta\mathbf{r}) + \oint d(\mathbf{F}\cdot\delta\mathbf{r}) = \oint (\nabla\times\mathbf{F})\cdot(\delta\mathbf{r}\times d\mathbf{r}),$$

where the integral of $d(\mathbf{F}\cdot\delta\mathbf{r})$ vanishes. Second interpret the last expression as the integral of $\nabla\times\mathbf{F}\cdot d\mathbf{S}$ over the ring formed by one position of the closed curve and a neighboring position. Finally sum up the variations δI which thus arise on passing through a succession of closed curves expanding from a point to final coincidence with the given closed curve.

15. In case the integrand contains y'' show by successive integrations by parts that

$$\delta\int_{x_0}^{x_1} F(x, y, y', y'')\,dx = \left[Y'\omega + Y''\omega' - \frac{dY''}{dx}\omega\right]_0^1 + \int_{x_0}^{x_1}\left(Y - \frac{dY'}{dx} + \frac{d^2Y''}{dx^2}\right)\omega\,dx,$$

where $\qquad Y = \frac{\partial F}{\partial y}, \qquad Y' = \frac{\partial F}{\partial y'}, \qquad Y'' = \frac{\partial F}{\partial y''}, \qquad \omega = \delta y.$