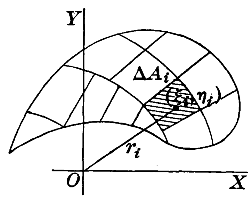
CHAPTER XII

ON MULTIPLE INTEGRALS

129. Double sums and double integrals. Suppose that a body of matter is so thin and flat that it can be considered to lie in a plane. If any small portion of the body surrounding a given point $P(x, y)$ be considered, and if its mass be denoted by Δm and its area by ΔA, the average (surface) density of the portion is the quotient $\Delta m/\Delta A$, and the actual density at the point P is defined as the limit of this quotient when $\Delta A \doteq 0$, that is,

$$D(x, y) = \lim_{\Delta A \doteq 0} \frac{\Delta m}{\Delta A}.$$

The density may vary from point to point. Now conversely suppose that the density $D(x, y)$ of the body is a known function of (x, y) and that it be required to find the total mass of the body. Let the body be considered as divided up into a large number of pieces each of which is *small in every direction*, and let ΔA_i be the area of any piece. If (ξ_i, η_i) be any point in ΔA_i, the density at that point is $D(\xi_i, \eta_i)$ and the amount of matter in the piece is approximately $D(\xi_i, \eta_i)\Delta A_i$ provided the density be regarded as continuous, that is, as not varying much over so small an area. Then the sum



$$D(\xi_1, \eta_1)\Delta A_1 + D(\xi_2, \eta_2)\Delta A_2 + \cdots + D(\xi_n, \eta_n)\Delta A_n = \sum D(\xi_i, \eta_i)\Delta A_i,$$

extended over all the pieces, is an approximation to the total mass, and may be sufficient for practical purposes if the pieces be taken tolerably small.

The process of dividing a body up into a large number of small pieces of which it is regarded as the sum is a device often resorted to; for the properties of the small pieces may be known approximately, so that the corresponding property for the whole body can be obtained approximately by summation. Thus by definition the moment of inertia of a small particle of matter relative to an axis is mr^2, where m is the mass of the particle and r its distance from the axis. If therefore the moment of inertia of a plane body with respect to an axis perpendicular to its plane were required, the body would be divided into a large number of small portions as above. The mass of each portion would be approximately $D(\xi_i, \eta_i)\Delta A_i$ and the distance of the portion from the axis might be considered as approximately the distance r_i from the point where the axis cut the plane to the point (ξ_i, η_i) in the portion. The moment of inertia would be

$$D(\xi_1, \eta_1)r_1^2\Delta A_1 + \cdots + D(\xi_n, \eta_n)r_n^2\Delta A_n = \sum D(\xi_i, \eta_i)r_i^2\Delta A_i,$$

or nearly this, where the sum is extended over all the pieces.

These sums may be called *double* sums because they extend over two dimensions. To pass from the approximate to the actual values of the mass or moment of inertia or whatever else might be desired, the underlying idea of a division into parts and a subsequent summation is kept, but there is added to this the idea of passing to a limit. Compare §§ 16–17. Thus

$$\underset{n=\infty,\, \Delta A_i \doteq 0}{\text{limit}} \sum D(\xi_i, \eta_i)\Delta A_i \quad \text{and} \quad \underset{n=\infty,\, \Delta A_i \doteq 0}{\text{limit}} \sum D(\xi_i, \eta_i)r_i^2\Delta A_i$$

would be taken as the total mass or inertia, where the sum over n divisions is replaced by the limit of that sum as the number of divisions becomes infinite and each becomes small in every direction. The limits are indicated by a sign of integration, as

$$\lim \sum D(\xi_i, \eta_i)\Delta A_i = \int D(x, y)\,dA, \qquad \lim \sum D(\xi_i, \eta_i)r_i^2\Delta A_i = \int Dr^2 dA.$$

The use of the limit is of course dependent on the fact that the limit is actually approached, and for practical purposes it is further dependent on the invention of some way of evaluating the limit. Both these questions have been treated when the sum is a simple sum (§§ 16–17, 28–30, 35); they must now be treated for the case of a double sum like those above.

130. Consider again the problem of finding the mass and let D_i be used briefly for $D(\xi_i, \eta_i)$. Let M_i be the maximum value of the density in the piece ΔA_i and let m_i be the minimum value. Then

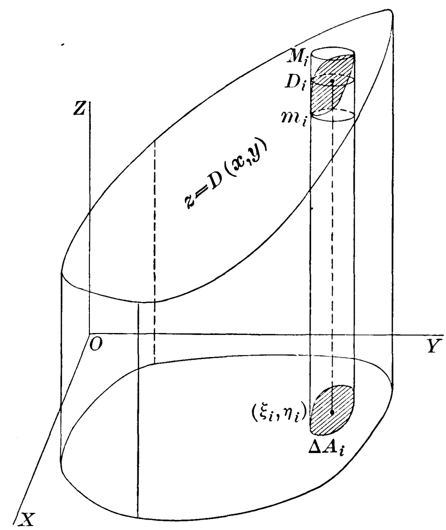
$$m_i\Delta A_i \leqq D_i\Delta A_i \leqq M_i\Delta A_i.$$

In this way any approximate expression $D_i\Delta A_i$ for the mass is shut in between two values, of which one is surely not greater than the true mass and the other surely not less. Form the sums

$$s = \sum m_i\Delta A_i \leqq \sum D_i\Delta A_i \leqq \sum M_i\Delta A_i = S$$

extended over all the elements ΔA_i. Now if the sums s and S approach the same limit when $\Delta A_i \doteq 0$, the sum $\Sigma D_i\Delta A_i$ which is constantly included between s and S must also approach that limit independently of how the points (ξ_i, η_i) are chosen in the areas ΔA_i.

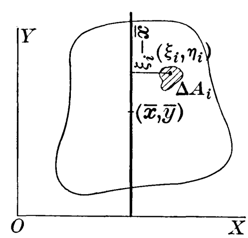
That s and S do approach a common limit in the usual case of a continuous function $D(x, y)$ may be shown strikingly if the surface $z = D(x, y)$ be drawn. The term $D_i \Delta A_i$ is then represented by the volume of a small cylinder upon the base ΔA_i and with an altitude equal to the height of the surface $z = D(x, y)$ above some point of ΔA_i. The sum $\Sigma D_i \Delta A_i$ of all these cylinders will be approximately the volume under the surface $z = D(x, y)$ and over the total area $A = \Sigma \Delta A_i$. The term $M_i \Delta A_i$ is represented by the volume of a small cylinder upon the base ΔA_i and circumscribed about the surface; the term $m_i \Delta A_i$, by a cylinder inscribed in the surface. When the number of elements ΔA_i is increased without limit so that each becomes indefinitely small, the three sums s, S, and $\Sigma D_i \Delta A_i$ all approach as their limit the volume under the surface and over the area A. Thus the notion of volume does for the double sum the same service as the notion of area for a simple sum.



Let the notion of the integral be applied to find *the formula for the center of gravity of a plane lamina.* Assume that the rectangular coördinates of the center of gravity are $(\bar{x}, \bar{y})$. Consider the body as divided into small areas ΔA_i. If (ξ_i, η_i) is any point in the area ΔA_i, the approximate moment of the approximate mass $D_i \Delta A_i$ in that area with respect to the line $x = \bar{x}$ is the product $(\xi_i - \bar{x}) D_i \Delta A_i$ of the mass by its distance from the line. The total exact moment would therefore be

$$\lim \sum (\xi_i - \bar{x}) D_i \Delta A_i = \int (x - \bar{x}) D(x, y) dA = 0,$$

and must vanish if the center of gravity lies on the line $x = \bar{x}$ as assumed. Then



$$\int x D(x, y) dA - \int \bar{x} D(x, y) dA = 0 \quad \text{or} \quad \int x D dA = \bar{x} \int D(x, y) dA.$$

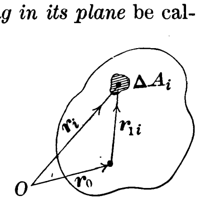
These formal operations presuppose the facts that the difference of two integrals is the integral of the difference and that the integral of a constant $\bar{x}$ times a function D

is the product of the constant by the integral of the function. It should be immediately apparent that as these rules are applicable to sums, they must be applicable to the limits of the sums. The equation may now be solved for $\bar{x}$. Then

$$\bar{x} = \frac{\int xDdA}{\int DdA} = \frac{\int xdm}{m}, \qquad \bar{y} = \frac{\int yDdA}{\int DdA} = \frac{\int ydm}{m}, \tag{1}$$

where m stands for the mass of the body and dm for DdA, just as Δm_i might replace $D_i\Delta A_i$; the result for y may be written down from symmetry.

As another example let *the kinetic energy of a lamina moving in its plane* be calculated. The use of vectors is advantageous. Let $\mathbf{r}_0$ be the vector from a fixed origin to a point which is fixed in the body, and let $\mathbf{r}_1$ be the vector from this point to any other point of the body so that



$$\mathbf{r}_i = \mathbf{r}_0 + \mathbf{r}_{1i}, \qquad \frac{d\mathbf{r}_i}{dt} = \frac{d\mathbf{r}_0}{dt} + \frac{d\mathbf{r}_{1i}}{dt} \quad \text{or} \quad \mathbf{v}_i = \mathbf{v}_0 + \mathbf{v}_{1i}.$$

The kinetic energy is $\Sigma \frac{1}{2} v_i^2 \Delta m_i$ or better the integral of $\frac{1}{2} v^2 dm$. Now

$$v_i^2 = \mathbf{v}_i \cdot \mathbf{v}_i = \mathbf{v}_0 \cdot \mathbf{v}_0 + \mathbf{v}_{1i} \cdot \mathbf{v}_{1i} + 2\,\mathbf{v}_0 \cdot \mathbf{v}_{1i} = v_0^2 + r_{1i}^2 \omega^2 + 2\,\mathbf{v}_0 \cdot \mathbf{v}_{1i}.$$

That $\mathbf{v}_{1i} \cdot \mathbf{v}_{1i} = r_{1i}^2 \omega^2$, where $r_{1i} = |\mathbf{r}_{1i}|$ and ω is the angular velocity of the body about the point $\mathbf{r}_0$, follows from the fact that $\mathbf{r}_{1i}$ is a vector of constant length r_{1i} and hence $|d\mathbf{r}_{1i}| = r_{1i} d\theta$, where $d\theta$ is the angle that r_{1i} turns through, and consequently $\omega = d\theta/dt$. Next integrate over the body.

$$\int \tfrac{1}{2} v^2 dm = \int \tfrac{1}{2} v_0^2 dm + \int \tfrac{1}{2} r_1^2 \omega^2 dm + \int \mathbf{v}_0 \cdot \mathbf{v}_1 dm$$
$$= \tfrac{1}{2} v_0^2 M + \tfrac{1}{2} \omega^2 \int r_1^2 dm + \mathbf{v}_0 \cdot \int \mathbf{v}_1 dm; \tag{2}$$

for v_0^2 and ω^2 are constants relative to the integration over the body. Note that

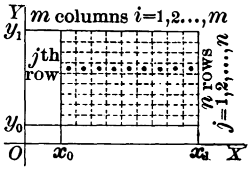
$$\mathbf{v}_0 \cdot \int \mathbf{v}_1 dm = 0 \quad \text{if} \quad \mathbf{v}_0 = 0 \quad \text{or if} \quad \int \mathbf{v}_1 dm = \int \frac{d}{dt} \mathbf{r}_1 dm = \frac{d}{dt} \int \mathbf{r}_1 dm = 0.$$

But $\mathbf{v}_0 = 0$ holds only when the point $\mathbf{r}_0$ is at rest, and $\int \mathbf{r}_1 dm = 0$ is the condition that $\mathbf{r}_0$ be the center of gravity. In the last case

$$T = \int \tfrac{1}{2} v^2 dm = \tfrac{1}{2} v_0^2 M + \tfrac{1}{2} \omega^2 I, \qquad I = \int r_1^2 dm.$$

As I is the integral which has been called the moment of inertia relative to an axis through the point $\mathbf{r}_0$ perpendicular to the plane of the body, the kinetic energy is seen to be the sum of $\frac{1}{2} M v_0^2$, which would be the kinetic energy if all the mass were concentrated at the center of gravity, and of $\frac{1}{2} I \omega^2$, which is the kinetic energy of rotation about the center of gravity; in case $\mathbf{r}_0$ indicated a point at rest (even if only instantaneously as in § 39) the whole kinetic energy would reduce to the kinetic energy of rotation $\frac{1}{2} I \omega^2$. In case $\mathbf{r}_0$ indicated neither the center of gravity nor a point at rest, the third term in (2) would not vanish and the expression for the kinetic energy would be more complicated owing to the presence of this term.

131. *To evaluate the double integral in case the region is a rectangle parallel to the axes of coördinates,* let the division be made into small rectangles by drawing lines parallel to the axes. Let there be m equal divisions on one side and n on the other. There will then be mn small pieces. It will be convenient to introduce a double index and denote by ΔA_{ij} the area of the rectangle in the ith column and jth row. Let (ξ_{ij}, η_{ij}) be any point, say the middle point in the area $\Delta A_{ij} = \Delta x_i \Delta y_i$. Then the sum may be written



$$\sum_{i,j} D(\xi_{ij}, \eta_{ij})\Delta A_{ij} = D_{11}\Delta x_1 \Delta y_1 + D_{21}\Delta x_2 \Delta y_1 + \cdots + D_{m1}\Delta x_m \Delta y_1$$
$$+ D_{12}\Delta x_1 \Delta y_2 + D_{22}\Delta x_2 \Delta y_2 + \cdots + D_{m2}\Delta x_m \Delta y_2$$
$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$+ D_{1n}\Delta x_1 \Delta y_n + D_{2n}\Delta x_2 \Delta y_n + \cdots + D_{mn}\Delta x_m \Delta y_n.$$

Now the terms in the first row are the sum of the contributions to $\Sigma_{i,j}$ of the rectangles in the first row, and so on. But

$$(D_{1j}\Delta x_1 + D_{2j}\Delta x_2 + \cdots + D_{mj}\Delta x_m)\Delta y_j = \Delta y_j \sum_i D(\xi_i, \eta_j)\Delta x_i$$

and
$$\Delta y_i \sum_i D(\xi_i, \eta_j)\Delta x_i = \left[\int_{x_0}^{x_1} D(x, \eta_j)\,dx + \zeta_j\right]\Delta y_j.$$

That is to say, by taking m sufficiently large so that the individual increments Δx_i are sufficiently small, the sum can be made to differ from the integral by as little as desired because the integral is by definition the limit of the sum. In fact

$$|\zeta_j| \leqq \sum_i |M_{ij} - m_{ij}|\Delta x_i \leqq \epsilon(x_1 - x_0)$$

if ϵ be the maximum variation of $D(x, y)$ over one of the little rectangles. After thus summing up according to rows, sum up the rows. Then

$$\sum_{i,j} D_{ij}\Delta A_{ij} = \int_{x_0}^{x_1} D(x, \eta_1)\,dx\Delta y_1 + \int_{x_0}^{x_1} D(x, \eta_2)\,dx\Delta y_2$$
$$+ \cdots + \int_{x_0}^{x_1} D(x, \eta_n)\,dx\Delta y_n + \lambda,$$

$$|\lambda| = |\zeta_1\Delta y_1 + \zeta_2\Delta y_2 + \cdots + \zeta_n\Delta y_n| \leqq \epsilon(x - x_0)\sum\Delta y = \epsilon(x - x_0)(y - y_0).$$

If
$$\int_{x_0}^{x_1} D(x, y)\,dx = \phi(y),$$

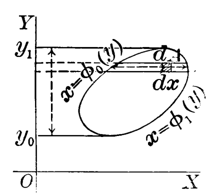
then
$$\sum_{i,j} D_{ij}\Delta A_{ij} = \phi(\eta_1)\Delta y_1 + \phi(\eta_2)\Delta y_2 + \cdots + \phi(\eta_n)\Delta y_n + \lambda$$
$$= \int_{y_0}^{y_1} \phi(y)\,dy + \kappa + \lambda, \qquad \kappa, \lambda \text{ small}.$$

Hence * $$\lim \sum_{i,j} D_{ij}\Delta A_{ij} = \int DdA = \int_{y_0}^{y_1}\int_{x_0}^{x_1} D(x, y)\,dxdy. \qquad (3)$$

It is seen that the double integral is equal to the result obtained by first integrating with respect to x, regarding y as a parameter, and then, after substituting the limits, integrating with respect to y. If the summation had been first according to columns and second according to rows, then by symmetry

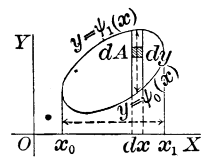
$$\int DdA = \int_{y_0}^{y_1}\int_{x_0}^{x_1} D(x, y)\,dxdy = \int_{x_0}^{x_1}\int_{y_0}^{y_1} D(x, y)\,dydx. \qquad (3')$$

This is really nothing but an integration under the sign (§ 120).



If the region over which the summation is extended is not a rectangle parallel to the axes, the method could still be applied. But after summing or rather integrating according to rows, the limits would not be constants as x_0 and x_1, but would be those functions $x = \phi_0(y)$ and $x = \phi_1(y)$ of y which represent the left-hand and right-hand curves which bound the region. Thus

$$\int DdA = \int_{y_0}^{y_1}\int_{\phi_0(y)}^{\phi_1(y)} D(x, y)\,dxdy. \qquad (3'')$$



And if the summation or integration had been first with respect to columns, the limits would not have been the constants y_0 and y_1, but the functions $y = \psi_0(x)$ and $y = \psi_1(x)$ which represent the lower and upper bounding curves of the region. Thus

$$\int DdA = \int_{x_0}^{x_1}\int_{\psi_0(x)}^{\psi_1(x)} D(x, y)\,dydx. \qquad (3''')$$

The order of the integrations cannot be inverted without making the corresponding changes in the limits, the first set of limits being such functions (of the variable with regard to which the second integration is to be performed) as to sum up according to strips reaching from one side of the region to the other, and the second set of limits being constants which determine the extreme limits of the second variable so as to sum up all the strips. Although the results (3″) and (3‴) are equal, it frequently happens that one of them is decidedly easier to evaluate than the other. Moreover, it has clearly been assumed that a line parallel to the

* The result may also be obtained as in Ex. 8 below.

axis of the first integration cuts the bounding curve in only two points; if this condition is not fulfilled, the area must be divided into subareas for which it is fulfilled, and the results of integrating over these smaller areas must be added algebraically to find the complete value.

To apply these rules for evaluating a double integral, consider the problem of finding the moment of inertia of a rectangle of constant density with respect to one vertex. Here

$$I = \int Dr^2 dA = D\int (x^2 + y^2)\,dA = D\int_0^b \int_0^a (x^2 + y^2)\,dxdy$$
$$= D\int_0^b \Big[\tfrac{1}{3}x^3 + xy^2\Big]_0^a dy = D\int_0^b (\tfrac{1}{3}a^3 + ay^2)\,dy = \tfrac{1}{3}Dab(a^2 + b^2).$$

If the problem had been to find the moment of inertia of an ellipse of uniform density with respect to the center, then

$$I = D\int (x^2 + y^2)\,dA = D\int_{-b}^{b} \int_{-\frac{a}{b}\sqrt{b^2 - y^2}}^{+\frac{a}{b}\sqrt{b^2 - y^2}} (x^2 + y^2)\,dxdy$$
$$= D\int_{-a}^{+a} \int_{-\frac{b}{a}\sqrt{a^2 - x^2}}^{+\frac{b}{a}\sqrt{a^2 - x^2}} (x^2 + y^2)\,dxdy.$$

Either of these forms might be evaluated, but the moment of inertia of the whole ellipse is clearly four times that of a quadrant, and hence the simpler results

$$I = 4D\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2 - y^2}} (x^2 + y^2)\,dxdy$$
$$= 4D\int_0^a \int_0^{\frac{a}{b}\sqrt{a^2 - x^2}} (x^2 + y^2)\,dydx = \frac{\pi}{4}Dab(a^2 + b^2).$$

It is highly advisable to make use of symmetry, wherever possible, to reduce the region over which the integration is extended.

132. With regard to *the more careful consideration of the limits involved in the definition of a double integral* a few observations will be sufficient. Consider the sums S and s and let $M_i\Delta A_i$ be any term of the first and $m_i\Delta A_i$ the corresponding term of the second. Suppose the area ΔA_i divided into two parts ΔA_{1i} and ΔA_{2i}, and let M_{1i}, M_{2i} be the maxima in the parts and m_{1i}, m_{2i} the minima. Then since the maximum in the whole area ΔA_i cannot be less than that in either part, and the minimum in the whole cannot be greater than that in either part, it follows that $m_{1i} \geqq m_i$, $m_{2i} \geqq m_i$, $M_{1i} \leqq M_i$, $M_{2i} \leqq M_i$, and

$$m_i\Delta A_i \leqq m_{1i}\Delta A_{1i} + m_{2i}\Delta A_{2i}, \qquad M_{1i}\Delta A_{1i} + M_{2i}\Delta A_{2i} \leqq M_i\Delta A_i.$$

Hence when one of the pieces ΔA_i is subdivided the sum S cannot increase nor the sum s decrease. Then continued inequalities may be written as

$$mA \leqq \sum m_i\Delta A_i \leqq \sum D(\xi_i, \eta_i)\Delta A_i \leqq \sum M_i\Delta A_i \leqq MA.$$

If then the original divisions ΔA_i be subdivided indefinitely, both S and s will approach limits (§§ 21–22); and if those limits are the same, the sum $\Sigma D_i\Delta A_i$ will approach that common limit as its limit independently of how the points (ξ_i, η_i) are chosen in the areas ΔA_i.

It has not been shown, however, that the limits of S and s are independent of the method of division and subdivision of the whole area. Consider therefore not only the sums S and s due to some particular mode of subdivision, but consider all such sums due to all possible modes of subdivision. As the sums S are limited below by mA they must have a lower frontier L, and as the sums s are limited above by MA they must have an upper frontier l. It must be shown that $l \leqq L$. To see this consider any pair of sums S and s corresponding to one division and any other pair of sums S' and s' corresponding to another method of division; also the sums S'' and s'' corresponding to the division obtained by combining, that is, by superposing the two methods. Now

$$S' \geqq S'' \geqq s'' \geqq s, \quad S \geqq S'' \geqq s'' \geqq s', \quad S \geqq L, \quad S' \geqq L, \quad s \leqq l, \quad s' \leqq l.$$

It therefore is seen that any S is greater than any s, whether these sums correspond to the same or to different methods of subdivision. Now if $L < l$, some S would have to be less than some s; for as L is the frontier for the sums S, there must be some such sums which differ by as little as desired from L; and in like manner there must be some sums s which differ by as little as desired from l. Hence as no S can be less than any s, the supposition $L < l$ is untrue and $L \geqq l$.

Now if for any method of division the limit of the difference

$$\lim (S - s) = \lim \sum (M_i - m_i) \Delta A_i = \lim \sum O_i \Delta A_i = 0$$

of the two sums corresponding to that method is zero, the frontiers L and l must be the same and both S and s approach that common value as their limit; and if the difference $S - s$ approaches zero for every method of division, the sums S and s will approach the same limit $L = l$ for all methods of division, and the sum $\Sigma D_i \Delta A_i$ will approach that limit independently of the method of division as well as independently of the selection of (ξ_i, η_i). This result follows from the fact that $L - l \leqq S - s$, $S - L \leqq S - s$, $l - s \leqq S - s$, and hence if the limit of $S - s$ is zero, then $L = l$ and S and s must approach the limit $L = l$. One case, which covers those arising in practice, in which these results are true is that in which $D(x, y)$ is continuous over the area A except perhaps upon a finite number of curves, each of which may be inclosed in a strip of area as small as desired and upon which $D(x, y)$ remains finite though it be discontinuous. For let the curves over which $D(x, y)$ is discontinuous be inclosed in strips of total area a. The contribution of these areas to the difference $S - s$ cannot exceed $(M - m)a$. Apart from these areas, the function $D(x, y)$ is continuous, and it is possible to take the divisions ΔA_i so small that the oscillation of the function over any one of them is less than an assigned number ϵ. Hence the contribution to $S - s$ is less than $\epsilon(A - a)$ for the remaining undeleted regions. The total value of $S - s$ is therefore less than $(M - m)a + \epsilon(A - a)$ and can certainly be made as small as desired.

The proof of the existence and uniqueness of the limit of $\Sigma D_i \Delta A_i$ is therefore obtained in case D is continuous over the region A except for points along a finite number of curves where it may be discontinuous provided it remains finite. Throughout the discussion the term "area" has been applied; this is justified by the previous work (§ 128). Instead of dividing the area A into elements ΔA, one may rule the area with lines parallel to the axes, as done in § 128, and consider the sums $\Sigma M \Delta x \Delta y$, $\Sigma m \Delta x \Delta y$, $\Sigma D \Delta x \Delta y$, where the first sum is extended over all the rectangles which lie within or upon the curve, where the second sum is extended over all the rectangles within the curve, and where the last extends over all rectangles within the curve and over an arbitrary number of those upon it. In a certain sense this method is simpler, in that the area then falls out as the integral of the special function which reduces to 1 within the curve and to 0 outside the curve, and to either upon the curve. The reader who desires to follow this method through may do so for himself. It is not within the range of this book to do more in the way of rigorous analysis than to treat the simpler questions and to indicate the need of corresponding treatment for other questions.

The justification for the method of evaluating a definite double integral as given above offers some difficulties in case the function $D(x, y)$ is discontinuous. The proof of the rule may be obtained by a careful consideration of the integration of a function defined by an integral containing a parameter. Consider

$$\phi(y) = \int_{x_0}^{x_1} D(x, y)\,dx, \qquad \int_{y_0}^{y_1} \phi(y)\,dy = \int_{y_0}^{y_1}\int_{x_0}^{x_1} D(x, y)\,dxdy. \tag{4}$$

It was seen (§ 118) that $\phi(y)$ is a continuous function of y if $D(x, y)$ is a continuous function of (x, y). Suppose that $D(x, y)$ were discontinuous, but remained finite, on a finite number of curves each of which is cut by a line parallel to the x-axis in only a finite number of points. Form $\Delta\phi$ as before. Cut out the short intervals in which discontinuities may occur. As the number of such intervals is finite and as each can be taken as short as desired, their total contribution to $\phi(y)$ or $\phi(y + \Delta y)$ can be made as small as desired. For the remaining portions of the interval $x_0 \leqq x \leqq x_1$ the previous reasoning applies. Hence the difference $\Delta\phi$ can still be made as small as desired and $\phi(y)$ is continuous. If $D(x, y)$ be discontinuous along a line $y = \beta$ parallel to the x-axis, then $\phi(y)$ might not be defined and might have a discontinuity for the value $y = \beta$. But there can be only a finite number of such values if $D(x, y)$ satisfies the conditions imposed upon it in considering the double integral above. Hence $\phi(y)$ would still be integrable from y_0 to y_1. Hence

$$\int_{y_0}^{y_1}\int_{x_0}^{x_1} D(x, y)\,dxdy \qquad \text{exists}$$

and $$m(x_1 - x_0)(y_1 - y_0) \leqq \int_{y_0}^{y_1}\int_{x_0}^{x_1} D(x, y)\,dxdy \leqq M(x_1 - x_0)(y_1 - y_0)$$

under the conditions imposed for the double integral.

Now let the rectangle $x_0 \leqq x \leqq x_1$, $y_0 \leqq y \leqq y_1$ be divided up as before. Then

$$m_{ij}\Delta x_i \Delta y_j \leqq \int_y^{y+\Delta_j y}\int_x^{x+\Delta_i x} D(x, y)\,dxdy \leqq M_{ij}\Delta_i x \Delta_j y.$$

Add: $$\sum m_{ij}\Delta x_i \Delta y_j \leqq \sum \int_y^{y+\Delta_j y}\int_x^{x+\Delta_i x} D(x, y)\,dxdy \leqq \sum M_{ij}\Delta_i x \Delta_j y$$

and $$\sum \int_y^{y+\Delta_j y}\int_x^{x+\Delta_i x} D(x, y)\,dxdy = \int_{y_0}^{y_1}\int_{x_0}^{x_1} D(x, y)\,dxdy.$$

Now if the number of divisions is multiplied indefinitely, the limit is

$$\int_{y_0}^{y_1}\int_{x_0}^{x_1} D(x, y)\,dxdy = \lim \sum m_{ij}\Delta A_{ij} = \lim \sum M_{ij}\Delta A_{ij} = \int D(x, y)\,dA.$$

Thus the previous rule for the rectangle is proved with proper allowance for possible discontinuities. In case the area A did not form a rectangle, a rectangle could be described about it and the function $D(x, y)$ could be defined for the whole rectangle as follows: For points within A the value of $D(x, y)$ is already defined, for points of the rectangle outside of A take $D(x, y) = 0$. The discontinuities across the boundary of A which are thus introduced are of the sort allowable for either integral in (4), and the integration when applied to the rectangle would then clearly give merely the integral over A. The limits could then be adjusted so that

$$\int_{y_0}^{y_1}\int_{x_0}^{x_1} D(x, y)\,dxdy = \int_{y_0}^{y_1}\int_{x=\phi_0(y)}^{x=\phi_1(y)} D(x, y)\,dxdy = \int D(x, y)\,dA.$$

The rule for evaluating the double integral by repeated integration is therefore proved.

EXERCISES

1. The sum of the moments of inertia of a plane lamina about two perpendicular lines in its plane is equal to the moment of inertia about an axis perpendicular to the plane and passing through their point of intersection.

2. The moment of inertia of a plane lamina about any point is equal to the sum of the moment of inertia about the center of gravity and the product of the total mass by the square of the distance of the point from the center of gravity.

3. If upon every line issuing from a point O of a lamina there is laid off a distance OP such that OP is inversely proportional to the square root of the moment of inertia of the lamina about the line OP, the locus of P is an ellipse with center at O.

4. Find the moments of inertia of these uniform laminas:

(α) segment of a circle about the center of the circle,
(β) rectangle about the center and about either side,
(γ) parabolic segment bounded by the latus rectum about the vertex or diameter,
(δ) right triangle about the right-angled vertex and about the hypotenuse.

5. Find by double integration the following areas:

(α) quadrantal segment of the ellipse, (β) between $y^2 = x^3$ and $y = x$,
(γ) between $3y^2 = 25x$ and $5x^2 = 9y$,
(δ) between $x^2 + y^2 - 2x = 0,\ x^2 + y^2 - 2y = 0$,
(ϵ) between $y^2 = 4ax + 4a^2,\ y^2 = -4bx + 4b^2$,
(ζ) within $(y - x - 2)^2 = 4 - x^2$,
(η) between $x^2 = 4ay,\ y(x^2 + 4a^2) = 8a^3$,
(θ) $y^2 = ax,\ x^2 + y^2 - 2ax = 0$.

6. Find the center of gravity of the areas in Ex. 5 (α), (β), (γ), (δ), and

(α) quadrant of $a^4y^2 = a^2x^4 - x^6$, (β) quadrant of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$,
(γ) between $x^{\frac{1}{2}} = y^{\frac{1}{2}} + a^{\frac{1}{2}},\ x + y = a$, ($\delta$) segment of a circle.

7. Find the volumes under the surfaces and over the areas given:

(α) sphere $z = \sqrt{a^2 - x^2 - y^2}$ and square inscribed in $x^2 + y^2 = a^2$,
(β) sphere $z = \sqrt{a^2 - x^2 - y^2}$ and circle $x^2 + y^2 - ax = 0$,
(γ) cylinder $z = \sqrt{4a^2 - y^2}$ and circle $x^2 + y^2 - 2ax = 0$,
(δ) paraboloid $z = kxy$ and rectangle $0 \leqq x \leqq a,\ 0 \leqq y \leqq b$,
(ϵ) paraboloid $z = kxy$ and circle $x^2 + y^2 - 2ax - 2ay = 0$,
(ζ) plane $x/a + y/b + z/c = 1$ and triangle $xy(x/a + y/b - 1) = 0$,
(η) paraboloid $z = 1 - x^2/4 - y^2/9$ above the plane $z = 0$,
(θ) paraboloid $z = (x + y)^2$ and circle $x^2 + y^2 = a^2$.

8. Instead of choosing (ξ_i, η_j) as particular points, namely the middle points, of the rectangles and evaluating $\Sigma D(\xi_i, \eta_j)\Delta x_i \Delta y_j$ subject to errors λ, κ which vanish in the limit, assume the function $D(x, y)$ continuous and resolve the double integral into a double sum by repeated use of the Theorem of the Mean, as

$$\phi(y) = \int_{x_0}^{x_1} D(x, y)\,dx = \sum_i D(\xi_i, y)\Delta x_i, \qquad \xi\text{'s properly chosen,}$$

$$\int_{y_0}^{y_1} \phi(y)\,dy = \sum_j \phi(\eta_j)\Delta y_j = \sum_j \Big[\sum_i D(\xi_i, \eta_j)\Delta x_i\Big]\Delta y_j = \sum_{i,j} D(\xi_i, \eta_j)\Delta A_{ij}.$$

9. Consider the generalization of Osgood's Theorem (§ 35) to apply to double integrals and sums, namely: If α_{ij} are infinitesimals such that

$$\alpha_{ij} = D(\xi_i, \eta_j)\Delta A_{ij} + \zeta_{ij}\Delta A_{ij},$$

where ζ_{ij} is uniformly an infinitesimal, then

$$\lim \sum_{i,j} \alpha_{ij} = \int D(x, y)\,dA = \int_{y_0}^{y_1}\int_{x_0}^{x_1} D(x, y)\,dxdy.$$

Discuss the statement and the result in detail in view of § 34.

10. Mark the region of the xy-plane over which the integration extends: *

$(\alpha)\ \int_0^a \int_0^x Ddydx,$ $\qquad (\beta)\ \int_1^2 \int_x^{x^2} Ddydx,$ $\qquad (\gamma)\ \int_0^1 \int_{y^2}^{y} Ddxdy,$

$(\delta)\ \int_1^2 \int_{\frac{\sqrt{2}}{x}}^{\sqrt{3-x^2}} Ddydx,$ $\qquad (\epsilon)\ \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_a^{a\sqrt{2\cos 2\phi}} Ddrd\phi,$ $\qquad (\zeta)\ \int_a^{2a} \int_{-\frac{\pi}{6}}^{\frac{1}{2}\cos^{-1}\frac{r}{2a}} Dd\phi dr.$

11. The density of a rectangle varies as the square of the distance from one vertex. Find the moment of inertia about that vertex, and about a side through the vertex.

12. Find the mass and center of gravity in Ex. 11.

13. Show that the moments of momentum (§ 80) of a lamina about the origin and about the point at the extremity of the vector $\mathbf{r}_0$ satisfy

$$\int \mathbf{r}\times\mathbf{v}dm = \mathbf{r}_Q\times\int \mathbf{v}dm + \int \mathbf{r}'\times\mathbf{v}dm,$$

or the difference between the moments of momentum about P and Q is the moment about P of the total momentum considered as applied at Q.

14. Show that the formulas (1) for the center of gravity reduce to

$$\bar{x} = \frac{\int_0^a xyDdx}{\int_0^a yDdx}, \qquad \bar{y} = \frac{\int_0^a \frac{1}{2}yyDdx}{\int yDdx} \qquad \text{or} \qquad \bar{x} = \frac{\int_{x_0}^{x_1} x(y_1 - y_0)Ddx}{\int_{x_0}^{x_1}(y_1 - y_0)Ddx},$$

$$\bar{y} = \frac{\int_{x_0}^{x_1} \frac{1}{2}(y_1 + y_0)(y_1 - y_0)Ddx}{\int_{x_0}^{x_1}(y_1 - y_0)Ddx}$$

* Exercises involving polar coördinates may be postponed until § 134 is reached, unless the student is already somewhat familiar with the subject.

when $D(x, y)$ reduces to a function $D(x)$, it being understood that for the first two the area is bounded by $x = 0$, $x = a$, $y = f(x)$, $y = 0$, and for the second two by $x = x_0$, $x = x_1$, $y_1 = f_1(x)$, $y_0 = f_0(x)$.

15. A rectangular hole is cut through a sphere, the axis of the hole being a diameter of the sphere. Find the volume cut out. Discuss the problem by double integration and also as a solid with parallel bases.

16. Show that the moment of momentum of a plane lamina about a fixed point or about the instantaneous center is $I\omega$, where ω is the angular velocity and I the moment of inertia. Is this true for the center of gravity (not necessarily fixed)? Is it true for other points of the lamina?

17. Invert the order of integration in Ex. 10 and in $\int_{-1}^{1}\int_{\sqrt{4-y^2}}^{\sqrt{3}y+2\sqrt{3}} Ddydx$.

18. In these integrals cut down the region over which the integral must be extended to the smallest possible by using symmetry, and evaluate if possible:

(α) the integral of Ex. 17 with $D = y^3 - 2x^2y$,
(β) the integral of Ex. 17 with $D = (x - 2\sqrt{3})^2 y$ or $D = (x - 2\sqrt{3})y^2$,
(γ) the integral of Ex. 10(ϵ) with $D = r(1 + \cos\phi)$ or $D = \sin\phi\cos\phi$.

19. The curve $y = f(x)$ between $x = a$ and $x = b$ is constantly increasing. Express the volume obtained by revolving the curve about the x-axis as $\pi[f(a)]^2(b - a)$ plus a double integral, in rectangular and in polar coördinates.

20. Express the area of the cardioid $r = a(1 - \cos\phi)$ by means of double integration in rectangular coördinates with the limits for both orders of integration.

133. Triple integrals and change of variable. In the extension from double to triple and higher integrals there is little to cause difficulty. For the discussion of the triple integral the same foundation of mass and density may be made fundamental. If $D(x, y, z)$ is the density of a body at any point, the mass of a small volume of the body surrounding the point (ξ_i, η_i, ζ_i) will be approximately $D(\xi_i, \eta_i, \zeta_i)\Delta V_i$, and will surely lie between the limits $M_i\Delta V_i$ and $m_i\Delta V_i$, where M_i and m_i are the maximum and minimum values of the density in the element of volume ΔV_i. The total mass of the body would be taken as

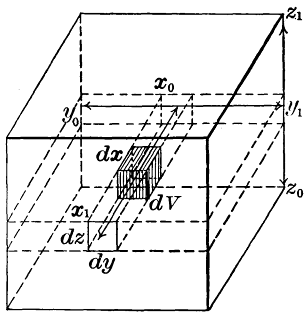
$$\lim_{\Delta V_i \doteq 0} \sum D(\xi_i, \eta_i, \zeta_i)\Delta V_i = \int D(x, y, z)\,dV, \tag{5}$$

where the sum is extended over the whole body. That the limit of the sum exists and is independent of the method of choice of the points (ξ_i, η_i, ζ_i) and of the method of division of the total volume into elements ΔV_i, provided $D(x, y, z)$ is continuous and the elements ΔV_i approach zero in such a manner that they become small in every direction, is tolerably apparent.

The evaluation of the triple integral by repeated or iterated integration is the immediate generalization of the method used for the double integral. If the region over which the integration takes place is a rectangular parallelepiped with its edges parallel to the axes, the integral is

$$\int D(x, y, z)\,dV = \int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} D(x, y, z)\,dxdydz. \qquad (5')$$

The integration with respect to x adds up the mass of the elements in the column upon the base $dydz$, the integration with respect to y then adds these columns together into a lamina of thickness dz, and the integration with respect to z finally adds together the laminas and obtains the mass in the entire parallelepiped. This could be done in other orders; in fact the integration might be performed first with regard to any of the three variables, second with either of the others, and finally with the last. There are, therefore, six equivalent methods of integration.



If the region over which the integration is desired is not a rectangular parallelepiped, the only modification which must be introduced is to adjust the limits in the successive integrations so as to cover the entire region. Thus if the first integration is with respect to x and the region is bounded by a surface $x = \psi_0(y, z)$ on the side nearer the yz-plane and by a surface $x = \psi_1(y, z)$ on the remoter side, the integration

$$\int_{x=\psi_0(y, z)}^{x=\psi_1(y, z)} D(x, y, z)\,dxdydz = \Omega(y, z)\,dydz$$

will add up the mass in elements of the column which has the cross section $dydz$ and is intercepted between the two surfaces. The problem of adding up the columns is merely one in double integration over the region of the yz-plane upon which they stand; this region is the projection of the given volume upon the yz-plane. The value of the integral is then

$$\int D\,dV = \int_{z_0}^{z_1}\int_{y=\phi_0(z)}^{y=\phi_1(z)} \Omega\,dydz = \int_{z_0}^{z_1}\int_{\phi_0(z)}^{\phi_1(z)}\int_{\psi_0(x, y)}^{\psi_1(x, y)} D\,dxdydz. \qquad (5'')$$

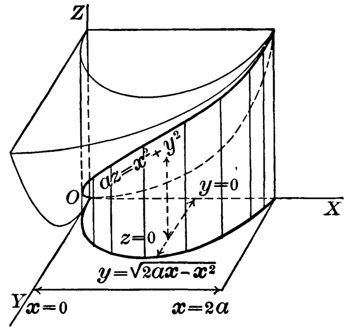
Here again the integrations may be performed in any order, provided the limits of the integrals are carefully adjusted to correspond to that order. The method may best be learned by example.

Find the mass, center of gravity, and moment of inertia about the axes of the volume of the cylinder $x^2 + y^2 - 2ax = 0$ which lies in the first octant and under paraboloid $x^2 + y^2 = az$, if the density be assumed constant. The integrals to evaluate are:

$$m = \int D dV, \quad \bar{x} = \frac{\int x dm}{m}, \quad \bar{y} = \frac{\int y dm}{m}, \quad \bar{z} = \frac{\int z dm}{m}, \tag{6}$$

$$I_x = \int D(y^2 + z^2) dV, \quad I_y = D\int (x^2 + z^2) dV, \quad I_z = D\int (x^2 + y^2) dV.$$

The consideration of how the figure looks shows that the limits for z are $z = 0$ and $z = (x^2 + y^2)/a$ if the first integration be with respect to z; then the double integral in x and y has to be evaluated over a semicircle, and the first integration is more simple if made with respect to y with limits $y = 0$ and $y = \sqrt{2ax - x^2}$, and final limits $x = 0$ and $x = 2a$ for x. If the attempt were made to integrate first with respect to y, there would be difficulty because a line parallel to the y-axis will give different limits according as it cuts both the paraboloid and cylinder or the xz-plane and cylinder; the total integral would be the sum of two integrals. There would be a similar difficulty with respect to an initial integration by x. The order of integration should therefore be z, y, x.



$$m = D\int_{x=0}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}}\int_{z=0}^{(x^2+y^2)/a} dz dy dx = D\int_{x=0}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}} \frac{x^2+y^2}{a} dy dx$$

$$= \frac{D}{a}\int_0^{2a}\left[x^2\sqrt{2ax - x^2} + \frac{1}{3}(2ax - x^2)^{\frac{3}{2}}\right]dx \qquad \begin{cases} x = a(1-\cos\theta) \\ \sqrt{2ax - x^2} = a\sin\theta \\ dx = a\sin\theta d\theta \end{cases}$$

$$\doteq Da^3\int_0^{\pi}\left[(1-\cos\theta)^2\sin^2\theta + \frac{1}{3}\sin^4\theta\right]d\theta = \frac{3}{4}\pi a^3 D$$

$$m\bar{x} = \int_{x=0}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}}\int_{z=0}^{(x^2+y^2)/a} x dz dy dx = D\int_{x=0}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}} \frac{x^3+xy^2}{a} dy dx$$

$$= \frac{D}{a}\int_0^{2a}\left[x^3\sqrt{2ax - x^2} + \frac{1}{3}x(2ax - x^2)^{\frac{3}{2}}\right]dx = \pi a^4 D.$$

Hence $\bar{x} = 4a/3$. The computation of the other integrals may be left as an exercise.

134. Sometimes the region over which a multiple integral is to be evaluated is such that the evaluation is relatively simple in one kind of coördinates but entirely impracticable in another kind. In addition to the rectangular coördinates the most useful systems are polar coördinates in the plane (for double integrals) and polar and cylindrical coördinates in space (for triple integrals). It has been seen (§ 40) that the element of area or of volume in these cases is

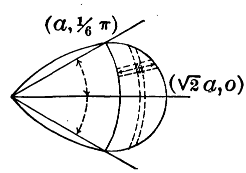
$$dA = r dr d\phi, \qquad dV = r^2 \sin\theta dr d\theta d\phi, \qquad dV = r dr d\phi dz, \tag{7}$$

except for infinitesimals of higher order. These quantities may be substituted in the double or triple integral and the evaluation may be made by successive integration. The proof that the substitution can be made is entirely similar to that given in §§ 34–35. The proof that the integral may still be evaluated by successive integration, with a proper choice of the limits so as to cover the region, is contained in the statement that the formal work of evaluating a multiple integral by repeated integration is independent of what the coördinates actually represent, for the reason that they could be interpreted if desired as representing rectangular coördinates.

Find the area of the part of one loop of the lemniscate $r^2 = 2a^2\cos 2\phi$ which is exterior to the circle $r = a$; also the center of gravity and the moment of inertia relative to the origin under the assumption of constant density. Here the integrals are

$$A = \int dA, \qquad A\bar{x} = \int x dA, \qquad A\bar{y} = \int y dA, \qquad I = D\int r^2 dA, \qquad m = DA.$$

The integrations may be performed first with respect to r so as to add up the elements in the little radial sectors, and then with regard to ϕ so as to add the sectors; or first with regard to ϕ so as to combine the elements of the little circular strips, and then with regard to r so as to add up the strips. Thus



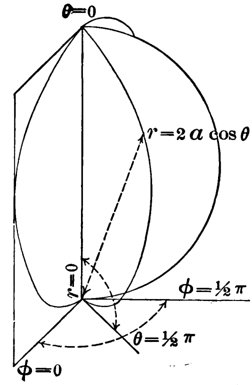
$$A = 2\int_{\phi=0}^{\frac{\pi}{6}}\int_{r=a}^{a\sqrt{2\cos^2\phi}} r dr d\phi = \int_0^{\frac{\pi}{6}}(2a^2\cos 2\phi - a^2)d\phi = \left(\frac{1}{2}\sqrt{3} - \frac{\pi}{6}\right)a^2 = .343\,a^2,$$

$$A\bar{x} = 2\int_{\phi=0}^{\frac{\pi}{6}}\int_{r=a}^{a\sqrt{2\cos^2\phi}} r\cos\phi \cdot r dr d\phi = \frac{2}{3}\int_0^{\frac{\pi}{6}}(2\sqrt{2}\,a^3\cos^{\frac{3}{2}}2\phi - a^3)\cos\phi d\phi$$

$$= \frac{2}{3}a^3\int_0^{\frac{\pi}{6}}\left[2\sqrt{2}\,(1 - 2\sin^2\phi)^{\frac{3}{2}}\,d\sin\phi - \cos\phi d\phi\right] = \frac{\pi}{8}a^3 = .393\,a^3.$$

Hence $\bar{x} = 3\pi a/(12\sqrt{3} - 4\pi) = 1.15\,a$. The symmetry of the figure shows that $\bar{y} = 0$. The calculation of I may be left as an exercise.

Given a sphere of which the density varies as the distance from some point of the surface; required the mass and the center of gravity. If polar coördinates with the origin at the given point and the polar axis along the diameter through that point be assumed, the equation of the sphere reduces to $r = 2a\cos\theta$ where a is the radius. The center of gravity from reasons of symmetry will fall on the diameter. To cover the volume of the sphere r must vary from $r = 0$ at the origin to $r = 2a\cos\theta$ upon the sphere. The polar angle must range from $\theta = 0$ to $\theta = \frac{1}{2}\pi$, and the longitudinal angle from $\phi = 0$ to $\phi = 2\pi$. Then



$$m = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{2a\cos\theta} kr\cdot r^2\sin\theta dr d\theta d\phi,$$

$$m\bar{z} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{r=2a\cos\theta} kr\cdot r\cos\theta\cdot r^2\sin\theta dr d\theta d\phi,$$

$$m = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} 4ka^4\cos^4\theta\sin\theta d\theta d\phi = \int_0^{2\pi}\frac{4}{5}ka^4 d\phi = \frac{8\pi ka^4}{5},$$

$$m\bar{z} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} \frac{32ka^5}{5}\cos^6\theta\sin\theta d\theta d\phi = \int_0^{2\pi}\frac{32ka^5}{35}d\phi = \frac{64\pi ka^5}{35}.$$

The center of gravity is therefore $\bar{z} = 8a/7$.

Sometimes it is necessary to make a change of variable

$$x = \phi(u, v), \qquad y = \psi(u, v)$$

or $$x = \phi(u, v, w), \qquad y = \psi(u, v, w), \qquad z = \omega(u, v, w) \tag{8}$$

in a double or a triple integral. The element of area or of volume has been seen to be (§ 63, and Ex. 7, p. 135)

$$dA = \left|J\left(\frac{x, y}{u, v}\right)\right| dudv \quad \text{or} \quad dV = \left|J\left(\frac{x, y, z}{u, v, w}\right)\right| dudvdw. \tag{8'}$$

Hence $$\int D(x, y)dA = \int D(\phi, \psi)\left|J\left(\frac{x, y}{u, v}\right)\right| dudv \tag{8''}$$

and $$\int D(x, y, z)dV = \int D(\phi, \psi, \omega)\left|J\left(\frac{x, y, z}{u, v, w}\right)\right| dudvdw.$$

It should be noted that the Jacobian may be either positive or negative but should not vanish; the difference between the case of positive and the case of negative values is of the same nature as the difference between an area or volume and the reflection of the area or volume. As the elements of area or volume are considered as positive when the increments of the variables are positive, the absolute value of the Jacobian is taken.

EXERCISES

1. Show that (6) are the formulas for the center of gravity of a solid body.

2. Show that $I_x = \int (y^2 + z^2)\,dm$, $I_y = \int (x^2 + z^2)\,dm$, $I_z = \int (x^2 + y^2)dm$ are the formulas for the moment of inertia of a solid about the axes.

3. Prove that the difference between the moments of inertia of a solid about any line and about a parallel line through the center of gravity is the product of the mass of the body by the square of the perpendicular distance between the lines.

4. Find the moment of inertia of a body about a line through the origin in the direction determined by the cosines l, m, n, and show that if a distance OP be laid off along this line inversely proportional to the square root of the moment of inertia, the locus of P is an ellipsoid with O as center.

5. Find the moments of inertia of these solids of uniform density:
(α) rectangular parallelepiped abc, about the edge a,
(β) ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, about the z-axis,
(γ) circular cylinder, about a perpendicular bisector of its axis,
(δ) wedge cut from the cylinder $x^2 + y^2 = r^2$ by $z = \pm mx$, about its edge.

6. Find the volume of the solids of Ex. 5 (β), (δ), and of the:
(α) trirectangular tetrahedron between $xyz = 0$ and $x/a + y/b + z/c = 1$,
(β) solid bounded by the surfaces $y^2 + z^2 = 4ax, y^2 = ax, x = 3a$,
(γ) solid common to the two equal perpendicular cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$,
(δ) octant of $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} + \left(\frac{z}{c}\right)^{\frac{1}{2}} = 1$, ($\epsilon$) octant of $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1$.

7. Find the center of gravity in Ex. 5 (δ), Ex. 6 (α), (β), (δ), (ϵ), density uniform.

8. Find the area in these cases: (α) between $r = a \sin 2\phi$ and $r = \frac{1}{2}a$.
(β) between $r^2 = 2a^2 \cos 2\phi$ and $r = 3^{\frac{1}{4}}a$, (γ) between $r = a \sin\phi$ and $r = b\cos\phi$,
(δ) $r^2 = 2a^2 \cos 2\phi$, $r\cos\phi = \frac{1}{2}\sqrt{3}\,a$, ($\epsilon$) $r = a(1 + \cos\phi)$, $r = a$.

9. Find the moments of inertia about the pole for the cases in Ex. 8, density uniform.

10. Assuming uniform density, find the center of gravity of the area of one loop:
(α) $r^2 = 2a^2\cos 2\phi$, (β) $r = a(1 - \cos\phi)$, (γ) $r = a\sin 2\phi$,
(δ) $r = a\sin^3\frac{1}{3}\phi$ (small loop), (ϵ) circular sector of angle 2α.

11. Find the moments of inertia of the areas in Ex. 10 (α), (β), (γ) about the initial line.

12. If the density of a sphere decreases uniformly from D_0 at the center to D_1 at the surface, find the mass and the moment of inertia about a diameter.

13. Find the total volume of:
(α) $(x^2 + y^2 + z^2)^2 = axyz$, ($\beta$) $(x^2 + y^2 + z^2)^3 = 27a^3xyz$.

14. A spherical sector is bounded by a cone of revolution; find the center of gravity and the moment of inertia about the axis of revolution if the density varies as the nth power of the distance from the center.

15. If a cylinder of liquid rotates about the axis, the shape of the surface is a paraboloid of revolution. Find the kinetic energy.

16. Compute $J\left(\frac{x,\ y}{r,\ \phi}\right)$, $J\left(\frac{x,\ y,\ z}{r,\ \phi,\ z}\right)$, $J\left(\frac{x,\ y,\ z}{r,\ \phi,\ \theta}\right)$ and hence verify (7).

17. Sketch the region of integration and the curves $u =$ const., $v =$ const.; hence show:

(α) $\int_0^c\int_{y=0}^{c-x} f(x,\ y)\,dxdy = \int_0^1\int_{u=0}^{c} f(u - uv,\ uv)\,ududv$, if $u = y + x$, $y = uv$,

(β) $\int_0^a\int_{y=0}^{x} f(x,\ y)\,dxdy$
$$= \int^1\int_{v=0}^{a(1+u)} f\left(\frac{v}{1+u},\ \frac{uv}{1+u}\right)\frac{v}{(1+u)^2}\,dvdu \text{ if } y = xu,\ x = \frac{v}{1+u},$$

(γ) or $= \int_0^a\int_{u=0}^{1} f\frac{v}{(1+u)^2}\,dudv - \int_a^{2a}\int_{u=1}^{\frac{v}{a}-1} f\frac{v}{(1+u)^2}\,dudv.$

18. Find the volume of the cylinder $r = 2a\cos\phi$ between the cone $z = r$ and the plane $z = 0$.

19. Same as Ex. 18 for cylinder $r^2 = 2a^2\cos 2\phi$; and find the moment of inertia about $r = 0$ if the density varies as the distance from $r = 0$.

20. Assuming the law of the inverse square of the distance, show that the attraction of a homogeneous sphere at a point outside the sphere is as though all the mass were concentrated at the center.

21. Find the attraction of a right circular cone for a particle at the vertex.

22. Find the attraction of (α) a solid cylinder, (β) a cylindrical shell upon a point on its axis; assume homogeneity.

23. Find the potentials, along the axes only, in Ex. 22. The potential may be defined as $\Sigma r^{-1}dm$ or as the integral of the force.

24. Obtain the formulas for the center of gravity of a sectorial area as

$$\bar{x} = \frac{1}{A}\int_{\phi_0}^{\phi_1}\frac{1}{3}r^3\cos\phi d\phi, \qquad \bar{y} = \frac{1}{A}\int_{\phi_0}^{\phi_1}\frac{1}{3}r^3\sin\phi d\phi,$$

and explain how they could be derived from the fact that the center of gravity of a uniform triangle is at the intersection of the medians.

25. Find the total illumination upon a circle of radius a, owing to a light at a distance h above the center. The illumination varies inversely as the square of the distance and directly as the cosine of the angle between the ray and the normal to the surface.

26. Write the limits for the examples worked in §§ 133 and 134 when the integrations are performed in various other orders.

27. A theorem of Pappus. If a closed plane curve be revolved about an axis which does not cut it, the volume generated is equal to the product of the area of the curve by the distance traversed by the center of gravity of the area. Prove either analytically or by infinitesimal analysis. Apply to various figures in which two of the three quantities, volume, area, position of center of gravity, are known, to find the third. Compare Ex. 3, p. 346.

135. Average values and higher integrals. The value of some special interpretation of integrals and other mathematical entities lies in the concreteness and suggestiveness which would be lacking in a purely analytical handling of the subject. For the simple integral $\int f(x)\,dx$ the curve $y = f(x)$ was plotted and the integral was interpreted as an area; it would have been possible to remain in one dimension by interpreting $f(x)$ as the density of a rod and the integral as the mass. In the case of the double integral $\int f(x, y)\,dA$ the conception of density and mass of a lamina was made fundamental; as was pointed out, it is possible to go into three dimensions and plot the surface $z = f(x, y)$

and interpret the integral as a volume. In the treatment of the triple integral $\int f(x, y, z)\, dV$ the density and mass of a body in space were made fundamental; here it would not be possible to plot $u = f(x, y, z)$ as there are only three dimensions available for plotting.

Another important interpretation of an integral is found in the conception of *average value*. If $q_1, q_2, \cdots, q_n$ are n numbers, the average of the numbers is the quotient of their sum by n.

$$\bar{q} = \frac{q_1 + q_2 + \cdots + q_n}{n} = \frac{\Sigma q_i}{n}. \tag{9}$$

If a set of numbers is formed of w_1 numbers q_1, and w_2 numbers $q_2, \cdots$, and w_n numbers q_n, so that the total number of the numbers is $w_1 + w_2 + \cdots + w_n$, the average is

$$\bar{q} = \frac{w_1 q_1 + w_2 q_2 + \cdots + w_n q_n}{w_1 + w_2 + \cdots + w_n} = \frac{\Sigma w_i q_i}{\Sigma w_i}. \tag{9'}$$

The coefficients $w_1, w_2, \cdots, w_n$, or any set of numbers which are proportional to them, are called the *weights* of $q_1, q_2, \cdots, q_n$. These definitions of average will not apply to finding the average of an infinite number of numbers because the denominator n would not be an arithmetical number. Hence it would not be possible to apply the definition to finding the average of a function $f(x)$ in an interval $x_0 \leqq x \leqq x_1$.

A slight change in the point of view will, however, lead to a definition for *the average value of a function*. Suppose that the interval $x_0 \leqq x \leqq x_1$ is divided into a number of intervals Δx_i, and that it be imagined that the number of values of $y = f(x)$ in the interval Δx_i is proportional to the length of the interval. Then the quantities Δx_i would be taken as the weights of the values $f(\xi_i)$ and the average would be

$$\bar{y} = \frac{\Sigma \Delta x_i f(\xi_i)}{\Sigma \Delta x_i}, \quad \text{or better} \quad \bar{y} = \frac{\int_{x_0}^{x_1} f(x)\, dx}{\int_{x_0}^{x_1} dx} \tag{10}$$

by passing to the limit as the Δx_i's approach zero. Then

$$\bar{y} = \frac{\int_{x_0}^{x_1} f(x)\, dx}{x_1 - x_0} \quad \text{or} \quad \int_{x_0}^{x_1} f(x)\, dx = (x_1 - x_0)\, \bar{y}. \tag{10'}$$

In like manner if $z = f(x, y)$ be a function of two variables or $u = f(x, y, z)$ a function of three variables, the averages over an area or volume would be defined by the integrals

$$\bar{z} = \frac{\int f(x, y)\, dA}{\int dA = A} \quad \text{and} \quad \bar{u} = \frac{\int f(x, y, z)\, dV}{\int dV = V}. \tag{10''}$$

It should be particularly noticed that *the value of the average is defined with reference to the variables of which the function averaged is a function; a change of variable will in general bring about a change in the value of the average.* For

$$\text{if} \qquad y = f(x), \qquad \overline{y(x)} = \frac{1}{x_1 - x_0}\int_{x_0}^{x_1} f(x)\, dx;$$

$$\text{but if} \qquad y = f(\phi(t)), \quad \overline{y(t)} = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} f(\phi(t))\, dt;$$

and there is no reason for assuming that these very different expressions have the same numerical value. Thus let

$$y = x^2, \qquad 0 \leqq x \leqq 1, \qquad x = \sin t, \qquad 0 \leqq t \leqq \tfrac{1}{2}\pi,$$

$$\overline{y(x)} = \frac{1}{1}\int_0^1 x^2 dx = \frac{1}{3}, \qquad \overline{y(t)} = \frac{1}{\frac{1}{2}\pi}\int_0^{\frac{\pi}{2}} \sin^2 t dt = \frac{1}{2}.$$

The average values of x and y over a plane area are

$$\bar{x} = \frac{1}{A}\int x dA, \qquad \bar{y} = \frac{1}{A}\int y dA,$$

when the weights are taken proportional to the elements of area; but if the area be occupied by a lamina and the weights be assigned as proportional to the elements of mass, then

$$\bar{x} = \frac{1}{m}\int x dm, \qquad \bar{y} = \frac{1}{m}\int y dm,$$

and the average values of x and y are the coördinates of the center of gravity. These two averages cannot be expected to be equal unless the density is constant. The first would be called an area-average of x and y; the second, a mass-average of x and y. The mass average of the square of the distance from a point to the different points of a lamina would be

$$\overline{r^2} = k^2 = \frac{1}{M}\int r^2 dm = I/M, \tag{11}$$

and is defined as the radius of gyration of the lamina about that point; it is the quotient of the moment of inertia by the mass.

As a problem in averages consider the determination of the average value of a proper fraction; also the average value of a proper fraction subject to the condition that it be one of two proper fractions of which the sum shall be less than or equal to 1. Let x be the proper fraction. Then in the first case

$$\bar{x} = \frac{1}{1}\int_0^1 x dx = \frac{1}{2}.$$

In the second case let y be the other fraction so that $x + y \leqq 1$. Now if (x, y) be taken as coördinates in a plane, the range is over a triangle, the number of points (x, y) in the element $dxdy$ would naturally be taken as proportional to the area of the element, and the average of x over the region would be

$$\bar{x} = \frac{\int x dA}{\int dA} = \frac{\int_0^1 \int_0^{1-y} x dx dy}{\int_0^1 \int_0^{1-y} dx dy} = \frac{\int_0^1 (1 - 2y + y^2) dy}{2\int_0^1 (1-y) dy} = \frac{1}{3}.$$

Now if x were one of four proper fractions whose sum was not greater than 1, the problem would be to average x over all sets of values (x, y, z, u) subject to the relation $x + y + z + u \leqq 1$. From the analogy with the above problems, the result would be

$$\bar{x} = \lim \frac{\Sigma x \Delta x \Delta y \Delta z \Delta u}{\Sigma \Delta x \Delta y \Delta z \Delta u} = \frac{\int_{u=0}^1 \int_{z=0}^{1-u} \int_{y=0}^{1-u-z} \int_{x=0}^{1-u-z-y} x dx dy dz du}{\int_{u=0}^1 \int_{z=0}^{1-u} \int_{y=0}^{1-u-z} \int_{x=0}^{1-u-z-y} dx dy dz du}.$$

The evaluation of the quadruple integral gives $\bar{x} = 1/5$.

136. The foregoing problem and other problems which may arise lead to the consideration of integrals of greater multiplicity than three. It will be sufficient to mention the case of a quadruple integral. In the first place let the four variables be

$$x_0 \leqq x \leqq x_1, \qquad y_0 \leqq y \leqq y_1, \qquad z_0 \leqq z \leqq z_1, \qquad u_0 \leqq u \leqq u_1, \quad (12)$$

included in intervals with constant limits. This is analogous to the case of a rectangle or rectangular parallelepiped for double or triple integrals. The range of values of x, y, z, u in (12) may be spoken of as a rectangular volume in four dimensions, if it be desired to use geometrical as well as analytical analogy. Then the product $\Delta x_i \Delta y_i \Delta z_i \Delta u_i$ would be an element of the region. If

$$x_i \leqq \xi_i \leqq x_i + \Delta x_i, \cdots, u_i \leqq \theta_i \leqq u_i + \Delta u_i,$$

the point $(\xi_i, \eta_i, \zeta_i, \theta_i)$ would be said to lie in the element of the region. The formation of a quadruple sum

$$\sum f(\xi_i, \eta_i, \zeta_i, \theta_i) \Delta x_i \Delta y_i \Delta z_i \Delta u_i$$

could be carried out in a manner similar to that of double and triple sums, and the sum could readily be shown to have a limit when

Δx_i, Δy_i, Δz_i, Δu_i approach zero, provided f is continuous. The limit of this sum could be evaluated by iterated integration

$$\lim \sum f_i \Delta x_i \Delta y_i \Delta z_i \Delta u_i = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \int_{u_0}^{u_1} f(x, y, z, u)\, du dz dy dx$$

where the order of the integrations is immaterial.

It is possible to define regions other than by means of inequalities such as arose above. Consider

$$F(x, y, z, u) = 0 \quad \text{and} \quad F(x, y, z, u) \leqq 0,$$

where it may be assumed that when three of the four variables are given the solution of $F = 0$ gives not more than two values for the fourth. The values of x, y, z, u which make $F < 0$ are separated from those which make $F > 0$ by the values which make $F = 0$. If the sign of F is so chosen that large values of x, y, z, u make F positive, the values which give $F > 0$ will be said to be outside the region and those which give $F < 0$ will be said to be inside the region. The value of the integral of $f(x, y, z, u)$ over the region $F \leqq 0$ could be found as

$$\int_{x_0}^{x_1} \int_{y=\phi_0(x)}^{y=\phi_1(x)} \int_{z=\psi_0(x, y)}^{z=\psi_1(x, y)} \int_{u=\omega_0(x, y, z)}^{u=\omega_1(x, y, z)} f(x, y, z, u)\, du dz dy dx,$$

where $u = \omega_1(x, y, z)$ and $u = \omega_0(x, y, z)$ are the two solutions of $F = 0$ for u in terms of x, y, z, and where the triple integral remaining after the first integration must be evaluated over the range of all possible values for (x, y, z). By first solving for one of the other variables, the integrations could be arranged in another order with properly changed limits.

If a change of variable is effected such as

$$x = \phi(x', y', z', u'), \quad y = \psi(x', y', z', u'), \quad z = \chi(x', y', z', u'), \quad u = \omega(x', y', z', u') \quad (13)$$

the integrals in the new and old variables are related by

$$\iiiint f(x, y, z, u)\, dx dy dz du = \iiiint f(\phi, \psi, \chi, \omega) \left| J\left(\frac{x, y, z, u}{x', y', z', u'}\right) \right| dx' dy' dz' du'. \quad (14)$$

The result may be accepted as a fact in view of its analogy with the results (8) for the simpler cases. A proof, however, may be given which will serve equally well as another way of establishing those results, — a way which does not depend on the somewhat loose treatment of infinitesimals and may therefore be considered as more satisfactory. In the first place note that from the relation (33) of p. 134 involving Jacobians, and from its generalization to several variables, it appears that if the change (14) is possible for each of two transformations, it is possible for the succession of the two. Now for the simple transformation

$$x = x', \quad y = y', \quad z = z', \quad u = \omega(x', y', z', u') = \omega(x, y, z, u'), \quad (13')$$

which involves only one variable, $J = \partial\omega/\partial u'$, and here

$$\int f(x, y, z, u)\,du = \int f(x, y, z, u')\left|\frac{\partial u}{\partial u'}\right| du' = \int f(x', y', z', u')\left| J \right| du',$$

and each side may be integrated with respect to x, y, z. Hence (14) is true in this case. For the transformation

$$x = \phi(x', y', z', u'), \quad y = \psi(x', y', z', u'), \quad z = \chi(x', y', z', u'), \quad u = u', \qquad (13'')$$

which involves only three variables, $J\left(\frac{x, y, z, u}{x', y', z', u'}\right) = J\left(\frac{x, y, z}{x', y', z'}\right)$ and

$$\iiint f(x, y, z, u)\,dxdydz = \iiint f(\phi, \psi, \chi, u)\,|J|\,dx'dy'dz',$$

and each side may be integrated with respect to u. The rule therefore holds in this case. It remains therefore merely to show that any transformation (13) may be resolved into the succession of two such as (13′), (13″). Let

$$x_1 = x', \quad y_1 = y', \quad z_1 = z', \quad u_1 = \omega(x', y', z', u') = \omega(x_1, y_1, z_1, u').$$

Solve the equation $u_1 = \omega(x_1, y_1, z_1, u')$ for $u' = \omega_1(x_1, y_1, z_1, u_1)$ and write

$$x = \phi(x_1, y_1, z_1, \omega_1), \quad y = \psi(x_1, y_1, z_1, \omega_1), \quad z = \chi(x_1, y_1, z_1, \omega_1), \quad u = u_1.$$

Now by virtue of the value of ω_1, this is of the type (13″), and the substitution of x_1, y_1, z_1, u_1 in it gives the original transformation.

EXERCISES

1. Determine the average values of these functions over the intervals:

(α) x^2, $0 \leqq x \leqq 10$, (β) $\sin x$, $0 \leqq x \leqq \frac{1}{2}\pi$,
(γ) x^n, $0 \leqq x \leqq n$, (δ) $\cos^n x$, $0 \leqq x \leqq \frac{1}{2}\pi$.

2. Determine the average values as indicated:

(α) ordinate in a semicircle $x^2 + y^2 = a^2$, $y > 0$, with x as variable,
(β) ordinate in a semicircle, with the arc as variable,
(γ) ordinate in semiellipse $x = a\cos\phi$, $y = b\sin\phi$, with ϕ as variable,
(δ) focal radius of ellipse, with equiangular spacing about focus,
(ϵ) focal radius of ellipse, with equal spacing along the major axis,
(ζ) chord of a circle (with the most natural assumption).

3. Find the average height of so much of these surfaces as lies above the xy-plane:

(α) $x^2 + y^2 + z^2 = a^2$, (β) $z = a^4 - p^2x^2 - q^2y^2$, (γ) $e^z = 4 - x^2 - y^2$.

4. If a man's height is the average height of a conical tent, on how much of the floor space can he stand erect?

5. Obtain the average values of the following:

(α) distance of a point in a square from the center, (β) ditto from vertex,
(γ) distance of a point in a circle from the center, (δ) ditto for sphere,
(ϵ) distance of a point in a sphere from a fixed point on the surface.

6. From the S.W. corner of a township persons start in random directions between N. and E. to walk across the township. What is their average walk? Which has it?

7. On each of the two legs of a right triangle a point is selected and the line joining them is drawn. Show that the average of the area of the square on this line is $\frac{1}{3}$ the square on the hypotenuse of the triangle.

8. A line joins two points on opposite sides of a square of side a. What is the ratio of the average square on the line to the given square?

9. Find the average value of the sum of the squares of two proper fractions. What are the results for three and for four fractions?

10. If the sum of n proper fractions cannot exceed 1, show that the average value of any one of the fractions is $1/(n+1)$.

11. The average value of the product of k proper fractions is 2^{-k}.

12. Two points are selected at random within a circle. Find the ratio of the average area of the circle described on the line joining them as diameter to the area of the circle.

13. Show that $J = r^3 \sin^2\theta \sin\phi$ for the transformation

$$x = r\cos\theta, \quad y = r\sin\theta\cos\phi, \quad z = r\sin\theta\sin\phi\cos\psi, \quad u = r\sin\theta\sin\phi\sin\psi,$$

and prove that all values of x, y, z, u defined by $x^2 + y^2 + z^2 + u^2 \leqq a^2$ are covered by the range $0 \leqq r \leqq a$, $0 \leqq \theta \leqq \pi$, $0 \leqq \phi \leqq \pi$, $0 \leqq \psi \leqq 2\pi$. What range will cover all positive values of x, y, z, u?

14. The sum of the squares of two proper fractions cannot exceed 1. Find the average value of one of the fractions.

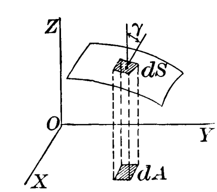
15. The same as Ex. 14 where three or four fractions are involved.

16. Note that the solution of $u_1 = \omega(x_1, y_1, z_1, u')$ for $u' = \omega_1(x_1, y_1, z_1, u_1)$ requires that $\partial\omega/\partial u'$ shall not vanish. Show that the hypothesis that J does not vanish in the region, is sufficient to show that at and in the neighborhood of each point (x, y, z, u) there must be at least one of the 16 derivatives of ϕ, ψ, χ, ω by x, y, z, u which does not vanish; and thus complete the proof of the text that in case $J \neq 0$ and the 16 derivatives exist and are continuous the change of variable is as given.

17. The intensity of light varies inversely as the square of the distance. Find the average intensity of illumination in a hemispherical dome lighted by a lamp at the top.

18. If the data be as in Ex. 12, p. 331, find the average density.

137. Surfaces and surface integrals. Consider a surface which has at each point a tangent plane that changes continuously from point to point of the surface. Consider also the projection of the surface upon a plane, say the xy-plane, and assume that a line perpendicular to the plane cuts the surface in only one point. Over any element dA of the projection there will be a small portion of the surface. If this small portion were plane and if its normal made an angle γ with the z-axis, the area of the surface (p. 167) would be to its projection as 1 is to



$\cos\gamma$ and would be $\sec\gamma dA$. The value of $\cos\gamma$ may be read from (9) on page 96. This suggests that the quantity

$$S = \int \sec\gamma dA = \iint \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} dxdy \tag{15}$$

be taken as *the definition of the area of the surface*, where the double integral is extended over the projection of the surface; and this definition will be adopted. This definition is really dependent on the particular plane upon which the surface is projected; that the value of the area of the surface would turn out to be the same no matter what plane was used for projection is tolerably apparent, but will be proved later.

Let the area cut out of a hemisphere by a cylinder upon the radius of the hemisphere as diameter be evaluated. Here (or by geometry directly)

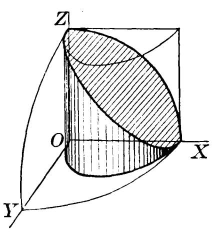
$$x^2 + y^2 + z^2 = a^2, \qquad \frac{\partial z}{\partial x} = -\frac{x}{z}, \qquad \frac{\partial z}{\partial y} = -\frac{y}{z},$$

$$S = \int \left[1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}\right]^{\frac{1}{2}} dA = 2\int_{x=0}^{a}\int_{y=0}^{\sqrt{ax-x^2}} \frac{a}{\sqrt{a^2-x^2-y^2}}\, dydx.$$

This integral may be evaluated directly, but it is better to transform it to polar coördinates in the plane. Then

$$S = 2\int_{\phi=0}^{\frac{1}{2}\pi}\int_{r=0}^{a\cos\phi} \frac{a}{\sqrt{a^2-r^2}}\, rdrd\phi = 2\int_0^{\frac{1}{2}\pi} a^2(1-\sin\phi)\, d\phi = (\pi - 2)\, a^2.$$

It is clear that the half area which lies in the first octant could be projected upon the xz-plane and thus evaluated. The region over which the integration would extend is that between $x^2 + z^2 = a^2$ and the projection $z^2 + ax = a^2$ of the curve of intersection of the sphere and cylinder. The projection could also be made on the yz-plane. If the area of the cylinder between $z = 0$ and the sphere were desired, projection on $z = 0$ would be useless, projection on $x = 0$ would be involved owing to the overlapping of the projection on itself, but projection on $y = 0$ would be entirely feasible.



To show that the definition of area does not depend, except apparently, upon the plane of projection consider any second plane which makes an angle θ with the first. Let the line of intersection be the y-axis; then from a figure the new coördinate x' is

$$x' = x\cos\theta + z\sin\theta, \quad y = y, \quad \text{and} \quad J\frac{(x', y)}{(x, y)} = \frac{\partial x'}{\partial x} = \cos\theta + \frac{\partial z}{\partial x}\sin\theta,$$

$$S = \iint \frac{dxdy}{\cos\gamma} = \iint J\frac{(x, y)}{(x', y)}\frac{dx'dy}{\cos\gamma} = \iint \frac{dx'dy}{\cos\gamma(\cos\theta + p\sin\theta)}.$$

It remains to show that the denominator $\cos\gamma(\cos\theta + p\sin\theta) = \cos\gamma'$. Referred to the original axes the direction cosines of the normal are $-p : -q : 1$, and of the z'-axis are $-\sin\theta : 0 : \cos\theta$. The cosine of the angle between these lines is therefore

$$\cos\gamma' = \frac{p\sin\theta + 0 + \cos\theta}{\sqrt{1+p^2+q^2}} = \frac{p\sin\theta + \cos\theta}{\sec\gamma} = \cos\gamma\,(\cos\theta + p\sin\theta).$$

Hence the new form of the area is the integral of $\sec\gamma' dA'$ and equals the old form.

The integrand $dS = \sec\gamma dA$ is called *the element of surface.* There are other forms such as $dS = \sec(r, n)\, r^2 \sin\theta d\theta d\phi$, where (r, n) is the angle between the radius vector and the normal; but they are used comparatively little. The possession of an expression for the element of surface affords a means of computing *averages over surfaces.* For if $u = u(x, y, z)$ be any function of (x, y, z), and $z = f(x, y)$ any surface, the integral

$$\bar{u} = \frac{1}{S}\int u(x, y, z)\, dS = \frac{1}{S}\iint u(x, y, f)\sqrt{1+p^2+q^2}\,dxdy \qquad (16)$$

will be the average of u over the surface S. Thus the average height of a hemisphere is (for the surface average)

$$\bar{z} = \frac{1}{2\pi a^2}\int z dS = \frac{1}{2\pi a^2}\iint z\cdot\frac{a}{z}\,dxdy = \frac{1}{2\pi a^2}\cdot \pi a^2 = \frac{1}{2};$$

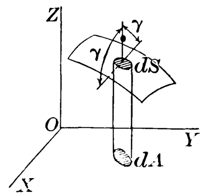
whereas the average height over the diametral plane would be $2/3$. This illustrates again the fact that the value of an average depends on the assumption made as to the weights.

138. If a surface $z = f(x, y)$ be divided into elements ΔS_i, and the function $u(x, y, z)$ be formed for any point (ξ_i, η_i, ζ_i) of the element, and the sum $\Sigma u_i \Delta S_i$ be extended over all the elements, the limit of the sum as the elements become small in every direction is defined as the *surface integral* of the function over the surface and may be evaluated as

$$\lim \sum u(\xi_i, \eta_i, \zeta_i)\Delta S_i = \int u(x\ y, z)\, dS$$
$$= \iint u[x, y, f(x, y)]\sqrt{1 + f_x'^2 + f_y'^2}\,dxdy. \qquad (17)$$

That the sum approaches a limit independently of how (ξ_i, η_i, ζ_i) is chosen in ΔS_i and how ΔS_i approaches zero follows from the fact that the element $u(\xi_i, \eta_i, \zeta_i)\Delta S_i$ of the sum differs uniformly from the integrand of the double integral by an infinitesimal of higher order, provided $u(x, y, z)$ be assumed continuous in (x, y, z) for points near the surface and $\sqrt{1 + f_x'^2 + f_y'^2}$ be continuous in (x, y) over the surface.

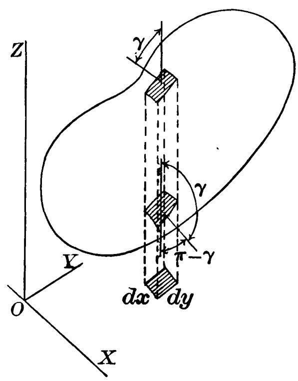
For many purposes it is more convenient to take as the normal form of the integrand of a surface integral, instead of udS, the product $R\cos\gamma dS$ of a function $R(x, y, z)$ by the cosine of the inclination of the surface to the z-axis by the element dS of the surface. Then the integral may be evaluated *over either side* of the surface; for $R(x, y, z)$ has a definite value on the surface, dS is a positive quantity, but $\cos\gamma$ is positive or negative according as the normal is drawn on the upper or lower side of the surface. The value of the integral over the surface will be



$$\int R(x, y, z)\cos\gamma dS = \iint Rdxdy \quad \text{or} \quad -\iint Rdxdy \tag{18}$$

according as the evaluation is made over the upper or lower side. If the function $R(x, y, z)$ is continuous over the surface, these integrands will be finite even when the surface becomes perpendicular to the xy-plane, which might not be the case with an integrand of the form $u(x, y, z)dS$.

An integral of this sort may be evaluated over a closed surface. Let it be assumed that the surface is cut by a line parallel to the z-axis in a finite number of points, and for convenience let that number be two. Let the normal to the surface be taken constantly as the exterior normal (some take the interior normal with a resulting change of sign in some formulas), so that for the upper part of the surface $\cos\gamma > 0$ and for the lower part $\cos\gamma < 0$. Let $z = f_1(x, y)$ and $z = f_0(x, y)$ be the upper and lower values of z on the surface. Then the exterior integral over the closed surface will have the form



$$\int R\cos\gamma dS = \iint R[x, y, f_1(x, y)]\,dxdy - \iint R[x, y, f_0(x, y)]\,dxdy, \tag{18'}$$

where the double integrals are extended over the area of the projection of the surface on the xy-plane.

From this form of the surface integral over a closed surface it appears that a surface integral over a closed surface may be expressed as a volume integral over the volume inclosed by the surface.*

*** Certain restrictions upon the functions and derivatives, as regards their becoming infinite and the like, must hold upon and within the surface. It will be quite sufficient if the functions and derivatives remain finite and continuous, but such extreme conditions are by no means necessary.**

For by the rule for integration,

$$\iiint_{z=f_0(x,y)}^{z=f_1(x,y)} \frac{\partial R}{\partial z}\, dz dx dy = \iint R(x, y, z)\Big|_{z=f_0(x,y)}^{z=f_1(x,y)} dx dy.$$

Hence
$$\int_{\bigcirc} R \cos \gamma dS = \int \frac{\partial R}{\partial z}\, dV$$
or
$$\iint_{\bigcirc} R dx dy = \iiint \frac{\partial R}{\partial z}\, dx dy dz \tag{19}$$

if the symbol O be used to designate a closed surface, and if the double integral on the left of (19) be understood to stand for either side of the equality (18′). In a similar manner

$$\int P \cos \alpha dS = \iint_{\bigcirc} P dy dz = \iiint \frac{\partial P}{\partial x}\, dx dy dz = \int \frac{\partial P}{\partial x}\, dV,$$
$$\int Q \cos \beta dS = \iint_{\bigcirc} Q dx dz = \iiint \frac{\partial Q}{\partial y}\, dy dx dz = \int \frac{\partial Q}{\partial y}\, dV. \tag{19'}$$

Then
$$\int_{\bigcirc} (P \cos \alpha + Q \cos \beta + R \cos \gamma)\, dS = \int \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV$$
or
$$\iint_{\bigcirc} (P dy dz + Q dz dx + R dx dy) = \iiint \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx dy dz \tag{20}$$

follows immediately by merely adding the three equalities. Any one of these equalities (19), (20) is sometimes called *Gauss's Formula*, sometimes *Green's Lemma*, sometimes *the divergence formula* owing to the interpretation below.

The interpretation of Gauss's Formula (20) by vectors is important. From the viewpoint of vectors the element of surface is a vector $d\mathbf{S}$ directed along the exterior normal to the surface (§ 76). Construct the vector function

$$\mathbf{F}(x, y, z) = \mathbf{i}P(x, y, z) + \mathbf{j}Q(x, y, z) + \mathbf{k}R(x, y, z).$$

Let $\quad d\mathbf{S} = (\mathbf{i} \cos \alpha + \mathbf{j} \cos \beta + \mathbf{k} \cos \gamma)\, dS = \mathbf{i} dS_x + \mathbf{j} dS_y + \mathbf{k} dS_z,$

where dS_x, dS_y, dS_z are the projections of dS on the coördinate planes

Then
$$P \cos \alpha dS + Q \cos \beta dS + R \cos \gamma dS = \mathbf{F} \cdot d\mathbf{S}$$

and
$$\iint (P dy dz + Q dx dz + R dx dy) = \int \mathbf{F} \cdot d\mathbf{S},$$

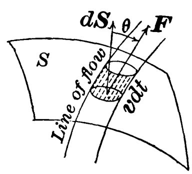
where dS_x, dS_y, dS_z have been replaced by the elements $dydz$, $dxdz$, $dxdy$, which would be used to evaluate the integrals in rectangular coördinates, without at all implying that the projections dS_x, dS_y, dS_z are actually rectangular. The combination of partial derivatives

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \text{div } \mathbf{F} = \nabla \cdot \mathbf{F}, \qquad (21)$$

where $\nabla \cdot \mathbf{F}$ is the symbolic scalar product of ∇ and $\mathbf{F}$ (Ex. 9 below), is called the *divergence* of $\mathbf{F}$. Hence (20) becomes

$$\int \text{div } \mathbf{F} dV = \int \nabla \cdot \mathbf{F} dV = \int \mathbf{F} \cdot d\mathbf{S}. \qquad (20')$$

Now the function $\mathbf{F}(x, y, z)$ is such that at each point (x, y, z) of space a vector is defined. Such a function is seen in the velocity in a moving fluid such as air or water. The picture of a scalar function $u(x, y, z)$ was by means of the surfaces $u = \text{const.}$; the picture of a vector function $\mathbf{F}(x, y, z)$ may be found in the system of curves tangent to the vector, the stream lines in the fluid if $\mathbf{F}$ be the velocity. For the immediate purposes it is better to consider the function $\mathbf{F}(x, y, z)$ as the flux $D\mathbf{v}$, the product of the density in the fluid by the velocity. With this interpretation the rate at which the fluid flows through an element of surface $d\mathbf{S}$ is $D\mathbf{v} \cdot d\mathbf{S} = \mathbf{F} \cdot d\mathbf{S}$. For in the time dt the fluid will advance along a stream line by the amount $\mathbf{v}dt$ and the volume of the cylindrical volume of fluid which advances through the surface will be $\mathbf{v} \cdot d\mathbf{S} dt$. Hence $\Sigma D\mathbf{v} \cdot d\mathbf{S}$ will be the rate of diminution of the amount of fluid within the closed surface.



As the amount of fluid in an element of volume dV is DdV, the rate of diminution of the fluid in the element of volume is $-\partial D/\partial t$ where $\partial D/\partial t$ is the rate of increase of the density D at a point within the element. The total rate of diminution of the amount of fluid within the whole volume is therefore $-\Sigma \partial D/\partial t dV$. Hence, by virtue of the principle of the indestructibility of matter,

$$\int_{\bigcirc} \mathbf{F} \cdot d\mathbf{S} = \int_{\bigcirc} D\mathbf{v} \cdot d\mathbf{S} = -\int \frac{\partial D}{\partial t} dV. \qquad (20'')$$

Now if v_x, v_y, v_z be the components of $\mathbf{v}$ so that $P = Dv_x$, $Q = Dv_y$, $R = Dv_z$ are the components of $\mathbf{F}$, a comparison of (21), (20′), (20″) shows that the integrals of $-\partial D/\partial t$ and div $\mathbf{F}$ are always equal, and hence the integrands,

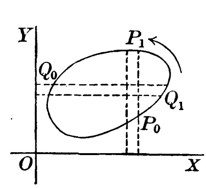
$$-\frac{\partial D}{\partial t} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial Dv_x}{\partial x} + \frac{\partial Dv_y}{\partial y} + \frac{\partial Dv_z}{\partial z},$$

are equal; that is, the sum $P'_x + Q'_y + R'_z$ represents the rate of diminution of density when $\mathbf{i}P + \mathbf{j}Q + \mathbf{k}R$ is the flux vector; this combination is called the divergence of the vector, no matter what the vector $\mathbf{F}$ really represents.

139. Not only may a surface integral be stepped up to a volume integral, but a line integral around a closed curve may be stepped up into a surface integral over a surface which spans the curve. To begin with the simple case of a line integral in a plane, note that by the same reasoning as above

$$\int_{\bigcirc} Pdx = \iint -\frac{\partial P}{\partial y}\,dxdy, \qquad \int_{\bigcirc} Qdy = \iint \frac{\partial Q}{\partial x}\,dxdy,$$
$$\int_{\bigcirc} [P(x, y)\,dx + Q(x, y)\,dy] = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy. \tag{22}$$

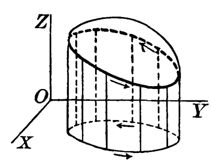
This is sometimes called *Green's Lemma for the plane* in distinction to the general Green's Lemma for space. The opposite signs must be taken to preserve the direction of the line integral about the contour. This result may be used to establish the rule for transforming a double integral by the change of variable $x = \phi(u, v)$, $y = \psi(u, v)$. For



$$A = \int_{\bigcirc} xdy = \pm \int_{\bigcirc} x\frac{\partial y}{\partial u}\,du + x\frac{\partial y}{\partial v}\,dv$$
$$= \pm \iint \left[\frac{\partial}{\partial u}\left(x\frac{\partial y}{\partial v}\right) - \frac{\partial}{\partial v}\left(x\frac{\partial y}{\partial u}\right)\right] dudv$$
$$= \pm \iint \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) dudv$$
$$= \pm \iint J\left(\frac{x, y}{u, v}\right) dudv = \iint |J|\,dudv.$$

(The double signs have to be introduced at first to allow for the case where J is negative.) The element of area $dA = |J|\,dudv$ is therefore established.

To obtain the formula for the conversion of a line integral in space to a surface integral, let $P(x, y, z)$ be given and let $z = f(x, y)$ be a surface spanning the closed curve $\bigcirc$. Then by virtue of $z = f(x, y)$, the function $P(x, y, z) = P_1(x, y)$ and



$$\int_{\bigcirc} Pdx = \int_{\bigcirc'} P_1 dx = \iint -\frac{\partial P_1}{\partial y}\,dxdy = -\iint \left(\frac{\partial P}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial P}{\partial z}\right) dxdy,$$

where $\bigcirc'$ denotes the projection of $\bigcirc$ on the xy-plane. Now the final double integral may be transformed by the introduction of the cosines of the normal direction to $z = f(x, y)$.

$$\cos\beta = \cos\gamma = -q : 1, \quad dxdy = \cos\gamma dS, \quad qdxdy = -\cos\beta dS = -dxdz.$$

Then $-\iint\left(\frac{\partial P}{\partial y}+q\frac{\partial P}{\partial z}\right)dxdy=\iint\left(\frac{\partial P}{\partial z}dxdz-\frac{\partial P}{\partial y}dxdy\right)=\int_{\bigcirc}Pdx.$

If this result and those obtained by permuting the letters be added,

$$\int_{\bigcirc}(Pdx+Qdy+Rdz)$$
$$=\iint\left[\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dydz+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dxdz+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy\right]. \quad (23)$$

This is known as *Stokes's Formula* and is of especial importance in hydromechanics and the theory of electromagnetism. Note that the line integral is carried around the rim of the surface in the direction which appears positive to one standing upon that side of the surface over which the surface integral is extended.

Again the vector interpretation of the result is valuable. Let

$$\mathbf{F}(x, y, z)=\mathbf{i}P(x, y, z)+\mathbf{j}Q(x, y, z)+\mathbf{k}R(x, y, z),$$

$$\text{curl}\,\mathbf{F}=\mathbf{i}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+\mathbf{j}\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+\mathbf{k}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right). \quad (24)$$

Then
$$\int_{\bigcirc}\mathbf{F}\cdot d\mathbf{r}=\int\text{curl}\,\mathbf{F}\cdot d\mathbf{S}=\int\nabla\times\mathbf{F}\cdot d\mathbf{S}, \quad (23')$$

where $\nabla\times\mathbf{F}$ is the symbolic vector product of ∇ and $\mathbf{F}$ (Ex. 9, below), is the form of Stokes's Formula; that is, the line integral of a vector around a closed curve is equal to the surface integral of the curl of the vector, as defined by (24), around any surface which spans the curve. If the line integral is zero about every closed curve, the surface integral must vanish over every surface. It follows that curl $\mathbf{F}=0$. For if the vector curl $\mathbf{F}$ failed to vanish at any point, a small plane surface $d\mathbf{S}$ perpendicular to the vector might be taken at that point and the integral over the surface would be approximately $|\text{curl}\,\mathbf{F}|dS$ and would fail to vanish, — thus contradicting the hypothesis. Now the vanishing of the vector curl $\mathbf{F}$ requires the vanishing

$$R'_y-Q'_z=0, \qquad P'_z-R'_x=0, \qquad Q'_x-P'_y=0$$

of each of its components. Thus may be derived the condition that $Pdx+Qdy+Rdz$ be an exact differential.

If $\mathbf{F}$ be interpreted as the velocity $\mathbf{v}$ in a fluid, the integral

$$\int\mathbf{v}\cdot d\mathbf{r}=\int v_xdx+v_ydy+v_zdz$$

of the component of the velocity along a curve, whether open or closed, is called the *circulation* of the fluid along the curve; it might be more natural to define the integral of the flux $D\mathbf{v}$ along the curve as the circulation, but this is not the convention. Now if the velocity be that due to rotation with the angular velocity $\mathbf{a}$ about a line through the origin, the circulation in a closed curve is readily computed. For

$$\mathbf{v} = \mathbf{a}\times\mathbf{r}, \qquad \int_O \mathbf{v}\cdot d\mathbf{r} = \int_O \mathbf{a}\times\mathbf{r}\cdot d\mathbf{r} = \int_O \mathbf{a}\cdot\mathbf{r}\times d\mathbf{r} = \mathbf{a}\cdot\int_O \mathbf{r}\times d\mathbf{r} = 2\,\mathbf{a}\cdot\mathbf{A}.$$

The circulation is therefore the product of twice the angular velocity and the area of the surface inclosed by the curve. If the circuit be taken indefinitely small, the integral is $2\,\mathbf{a}\cdot d\mathbf{S}$ and a comparison with (23′) shows that $\operatorname{curl}\mathbf{v} = 2\,\mathbf{a}$; that is, the curl of the velocity due to rotation about an axis is twice the angular velocity and is constant in magnitude and direction all over space. The general motion of a fluid is not one of uniform rotation about any axis; in fact if a small element of fluid be considered and an interval of time δt be allowed to elapse, the element will have moved into a new position, will have been somewhat deformed owing to the motion of the fluid, and will have been somewhat rotated. The vector $\operatorname{curl}\mathbf{v}$, as defined in (24), may be shown to give twice the instantaneous angular velocity of the element at each point of space.

EXERCISES

1. Find the areas of the following surfaces:

(α) cylinder $x^2 + y^2 - ax = 0$ included by the sphere $x^2 + y^2 + z^2 = a^2$,
(β) $x/a + y/b + z/c = 1$ in first octant, (γ) $x^2 + y^2 + z^2 = a^2$ above $r = a\cos n\phi$,
(δ) sphere $x^2 + y^2 + z^2 = a^2$ above a square $|x| \leqq b$, $|y| \leqq b$, $b < \frac{1}{2}\sqrt{2}\,a$,
(ϵ) $z = xy$ over $x^2 + y^2 = a^2$, (ζ) $2\,az = x^2 - y^2$ over $r^2 = a^2\cos\phi$,
(η) $z^2 + (x\cos\alpha + y\sin\alpha)^2 = a^2$ in first octant, (θ) $z = xy$ over $r^2 = \cos 2\,\phi$,
(ι) cylinder $x^2 + y^2 = a^2$ intercepted by equal cylinder $y^2 + z^2 = a^2$.

2. Compute the following superficial averages:

(α) latitude of places north of the equator, *Ans.* $32\frac{7}{10}^\circ$.
(β) ordinate in a right circular cone $h^2(x^2 + y^2) - a^2(z - h)^2 = 0$,
(γ) illumination of a hollow spherical surface by a light at a point of it,
(δ) illumination of a hemispherical surface by a distant light,
(ϵ) rectilinear distance of points north of equator from north pole.

3. A theorem of Pappus: If a closed or open plane curve be revolved about an axis in its plane, the area of the surface generated is equal to the product of the length of the curve by the distance described by the center of gravity of the curve. The curve shall not cut the axis. Prove either analytically or by infinitesimal analysis. Apply to various figures in which two of the three quantities, length of curve, area of surface, position of center of gravity, are known, to find the third. Compare Ex. 27, p. 332.

4. The surface integrals are to be evaluated over the closed surfaces by expressing them as volume integrals. Try also direct calculation:

(α) $\iint (x^2dydz + xydxdy + xzdxdz)$ over the spherical surface $x^2 + y^2 + z^2 = a^2$,

(β) $\iint (x^2dydz + y^2dxdz + z^2dxdy)$, cylindrical surface $x^2 + y^2 = a^2$, $z = \pm b$,

(γ) $\iint [(x^2 - yz)\,dydz - 2\,xydxdz + dxdy]$ over the cube $0 \leqq x,\ y,\ z \leqq a$,

(δ) $\iint xdydz = \iint ydxdz = \iint zdxdy = \frac{1}{3}\iint (xdydz + ydxdz + zdxdy) = V$,

(ϵ) Calculate the line integrals of Ex. 8, p. 297, around a closed path formed by two paths there given, by applying Green's Lemma (22) and evaluating the resulting double integrals.

5. If $x = \phi_1(u, v)$, $y = \phi_2(u, v)$, $z = \phi_3(u, v)$ are the parametric equations of a surface, the direction ratios of the normal are (see Ex. 15, p. 135)

$$\cos\alpha : \cos\beta : \cos\gamma = J_1 : J_2 : J_3 \quad \text{if} \quad J_i = J\left(\frac{\phi_{i+1},\ \phi_{i+2}}{u,\ v}\right).$$

Show 1° that the area of a surface may be written as

$$S = \iint \frac{\sqrt{J_1^2 + J_2^2 + J_3^2}}{|J_3|}\,dxdy = \iint \sqrt{J_1^2 + J_2^2 + J_3^2}\,dudv = \iint \sqrt{EG - F^2}dudv,$$

where $$E = \sum\left(\frac{\partial\phi_i}{\partial u}\right)^2, \quad G = \sum\left(\frac{\partial\phi_i}{\partial v}\right)^2, \quad F = \sum\frac{\partial\phi_i}{\partial u}\frac{\partial\phi_i}{\partial v},$$

and $$ds^2 = Edu^2 + 2\,Fdudv + Gdv^2.$$

Show 2° that the surface integral of the first type becomes merely

$$\iint f(x, y, z)\sec\gamma dxdy = \iint f(\phi_1, \phi_2, \phi_3)\sqrt{EG - F^2}dudv,$$

and determine the integrand in the case of the developable surface of Ex. 17, p. 143.

Show 3° that if $x = f_1(\xi, \eta, \zeta)$, $y = f_2(\xi, \eta, \zeta)$, $z = f_3(\xi, \eta, \zeta)$ is a transformation of space which transforms the above surface into a new surface $\xi = \psi_1(u, v)$, $\eta = \psi_2(u, v)$, $\zeta = \psi_3(u, v)$, then

$$J\left(\frac{x,\ y}{u,\ v}\right) = J\left(\frac{x,\ y}{\xi,\ \eta}\right)J\left(\frac{\xi,\ \eta}{u,\ v}\right) + J\left(\frac{x,\ y}{\eta,\ \zeta}\right)J\left(\frac{\eta,\ \zeta}{u,\ v}\right) + J\left(\frac{x,\ y}{\zeta,\ \xi}\right)J\left(\frac{\zeta,\ \xi}{u,\ v}\right).$$

Show 4° that the surface integral of the second type becomes

$$\iint Rdxdy = \iint RJ\left(\frac{x,\ y}{u,\ v}\right)dudv$$
$$= \iint R\left[J\left(\frac{x,\ y}{\eta,\ \zeta}\right)d\eta d\zeta + J\left(\frac{x,\ y}{\zeta,\ \xi}\right)d\zeta d\xi + J\left(\frac{x,\ y}{\xi,\ \eta}\right)d\xi d\eta\right],$$

where the integration is now in terms of the new variables ξ, η, ζ in place of x, y, z.

Show 5° that when $R = z$ the double integral above may be transformed by Green's Lemma in such a manner as to establish the formula for change of variables in triple integrals.

6. Show that for vector surface integrals $\int_{\bigcirc} Ud\mathbf{S} = \int \nabla UdV$.

7. *Solid angle as a surface integral.* The area cut out from the unit sphere by a cone with its vertex at the center of the sphere is called the *solid angle* ω subtended at the vertex of the cone. The solid angle may also be defined as the ratio of the area cut out upon any sphere concentric with the vertex of the cone, to the square of the radius of the sphere (compare the definition of the angle between two lines in radians). Show geometrically (compare Ex. 16, p. 297) that the infinitesimal solid angle $d\omega$ of the cone which joins the origin $r = 0$ to the periphery of the element dS of a surface is $d\omega = \cos(r, n)\,dS/r^2$, where (r, n) is the angle between the radius produced and the outward normal to the surface. Hence show

$$\omega = \int \frac{\cos(r, n)}{r^2}\,dS = \int \frac{\mathbf{r}\cdot d\mathbf{S}}{r^3} = \int \frac{1}{r^2}\frac{dr}{dn}\,dS = -\int \frac{d}{dn}\frac{1}{r}\,dS = -\int d\mathbf{S}\cdot\nabla\frac{1}{r},$$

where the integrals extend over a surface, is the solid angle subtended at the origin by that surface. Infer further that

$$-\int_O \frac{d}{dn}\frac{1}{r}\,dS = 4\pi \quad \text{or} \quad -\int_O \frac{d}{dn}\frac{1}{r}\,dS = 0 \quad \text{or} \quad -\int_O \frac{d}{dn}\frac{1}{r}\,dS = \theta$$

according as the point $r = 0$ is within the closed surface or outside it or upon it at a point where the tangent planes envelop a cone of solid angle θ (usually 2π). Note that the formula may be applied at any point (ξ, η, ζ) if

$$r^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2$$

where (x, y, z) is a point of the surface.

8. *Gauss's Integral.* Suppose that at $\mathbf{r} = 0$ there is a particle of mass m which *attracts* according to the Newtonian Law $F = m/r^2$. Show that the potential is $V = -m/r$ so that $\mathbf{F} = -\nabla V$. The induction or flux (see Ex. 19, p. 308) of the force $\mathbf{F}$ outward across the element $d\mathbf{S}$ of a surface is by definition $-F\cos(F, n)\,dS = \mathbf{F}\cdot d\mathbf{S}$. Show that the total induction or flux of $\mathbf{F}$ across a surface is the surface integral

$$\int \mathbf{F}\cdot d\mathbf{S} = -\int d\mathbf{S}\cdot\nabla V = -\int \frac{dV}{dn}\,dS = m\int d\mathbf{S}\cdot\nabla\frac{1}{r};$$

and

$$m = \frac{-1}{4\pi}\int_O \mathbf{F}\cdot d\mathbf{S} = \frac{1}{4\pi}\int_O d\mathbf{S}\cdot\nabla V = \frac{-1}{4\pi}\int_O \frac{d}{dn}\frac{m}{r}\,dS,$$

where the surface integral extends over a surface surrounding a point $\mathbf{r} = 0$, is the formula for obtaining the mass m within the surface from the field of force $\mathbf{F}$ which is set up by the mass. If there are several masses $m_1, m_2, \cdots$ situated at points $(\xi_1, \eta_1, \zeta_1), (\xi_2, \eta_2, \zeta_2), \cdots$, let

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots, \qquad V = V_1 + V_2 + \cdots,$$
$$V_i = -m[(\xi_i - x_i)^2 + (\eta_i - y_i)^2 + (\zeta_i - z_i)^2]^{-\frac{1}{2}}$$

be the force and potential at (x, y, z) due to the masses. Show that

$$\frac{-1}{4\pi}\int_O \mathbf{F}\cdot d\mathbf{S} = \frac{1}{4\pi}\int_O d\mathbf{S}\cdot\nabla V = -\frac{1}{4\pi}\sum\int_O \frac{d}{dn}\frac{1}{r_i}\,dS = \sum{}' m_i = M, \tag{25}$$

where Σ extends over all the masses and Σ' over all the masses within the surface (none being on it), gives the total mass M within the surface. The integral (25) which gives the mass within a surface as a surface integral is known as Gauss's Integral. If the force were repulsive (as in electricity and magnetism) instead of attracting (as in gravitation), the results would be $V = m/r$ and

$$\frac{1}{4\pi}\int_O \mathbf{F}\cdot d\mathbf{S} = \frac{-1}{4\pi}\int_O d\mathbf{S}\cdot\nabla V = \frac{-1}{4\pi}\sum\int_O \frac{d}{dn}\frac{m_i}{r_i}\,dS = \sum{}' m_i = M. \tag{25'}$$

9. If $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ be the operator defined on page 172, show

$$\nabla\cdot\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}, \qquad \nabla\times\mathbf{F} = \mathbf{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \mathbf{j}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \mathbf{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

by formal operation on $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Show further that

$$\nabla\times\nabla U = 0, \qquad \nabla\cdot\nabla\times\mathbf{F} = 0, \qquad (\nabla\cdot\nabla)(*) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(*),$$

$$\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - (\nabla\cdot\nabla)\mathbf{F} \qquad \text{(write the Cartesian form).}$$

Show that $(\nabla\cdot\nabla)\, U = \nabla\cdot(\nabla U)$. If $\mathbf{u}$ is a constant unit vector, show

$$(\mathbf{u}\cdot\nabla)\,\mathbf{F} = \frac{\partial\mathbf{F}}{\partial x}\cos\alpha + \frac{\partial\mathbf{F}}{\partial y}\cos\beta + \frac{\partial\mathbf{F}}{\partial z}\cos\gamma = \frac{d\mathbf{F}}{ds}$$

is the directional derivative of $\mathbf{F}$ in the direction $\mathbf{u}$. Show $(d\mathbf{r}\cdot\nabla)\,\mathbf{F} = d\mathbf{F}$.

10. *Green's Formula* (space). Let $F(x, y, z)$ and $G(x, y, z)$ be two functions so that ∇F and ∇G become two vector functions and $F\nabla G$ and $G\nabla F$ two other vector functions. Show

$$\nabla\cdot(F\nabla G) = \nabla F\cdot\nabla G + F\nabla\cdot\nabla G, \qquad \nabla\cdot(G\nabla F) = \nabla F\cdot\nabla G + G\nabla\cdot\nabla G,$$

or

$$\frac{\partial}{\partial x}\left(F\frac{\partial G}{\partial x}\right) + \frac{\partial}{\partial y}\left(F\frac{\partial G}{\partial y}\right) + \frac{\partial}{\partial z}\left(F\frac{\partial G}{\partial z}\right)$$
$$= \frac{\partial F}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial G}{\partial z} + F\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2}\right),$$

and the similar expressions which are the Cartesian equivalents of the above vector forms. Apply Green's Lemma or Gauss's Formula to show

$$\int_{\bigcirc} F\nabla G\cdot d\mathbf{S} = \int \nabla F\cdot\nabla G dV + \int F\nabla\cdot\nabla G dV, \tag{26}$$

$$\int_{\bigcirc} G\nabla F\cdot d\mathbf{S} = \int \nabla F\cdot\nabla G dV + \int G\nabla\cdot\nabla F dV, \tag{26'}$$

$$\int_{\bigcirc} (F\nabla G - G\nabla F)\cdot d\mathbf{S} = \int (F\nabla\cdot\nabla G - G\nabla\cdot\nabla F)\, dV, \tag{26''}$$

or

$$\int_{\bigcirc} F\frac{dG}{dn}\, dS = \int\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial G}{\partial z}\right) dV + \int F\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2}\right) dV,$$

$$\int_{\bigcirc}\left(F\frac{dG}{dn} - G\frac{dF}{dn}\right) dS = \int\left[F\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2}\right) - G\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}\right)\right] dV.$$

The formulas (26), (26′), (26″) are known as *Green's Formulas;* in particular the first two are asymmetric and the third symmetric. The ordinary Cartesian forms of (26) and (26″) are given. The expression $\partial^2F/\partial x^2 + \partial^2F/\partial y^2 + \partial^2F/\partial z^2$ is often written as ΔF for brevity ; the vector form is $\nabla\cdot\nabla F$.

11. From the fact that the integral of $\mathbf{F}\cdot d\mathbf{r}$ has opposite values when the curve is traced in opposite directions, show that the integral of $\nabla\times\mathbf{F}$ over a closed surface vanishes and that the integral of $\nabla\cdot\nabla\times\mathbf{F}$ over a volume vanishes. Infer that $\nabla\cdot\nabla\times\mathbf{F} = 0$.

12. Reduce the integral of $\nabla\times\nabla U$ over any (open) surface to the difference in the values of U at two same points of the bounding curve. Hence infer $\nabla\times\nabla U = 0$.

13. Comment on the remark that the line integral of a vector, integral of $\mathbf{F}\cdot d\mathbf{r}$, is around a curve and *along* it, whereas the surface integral of a vector, integral of $\mathbf{F}\cdot d\mathbf{S}$, is over a surface but *through* it. Compare Ex. 7 with Ex. 16 of p. 297. In particular give vector forms of the integrals in Ex. 16, p. 297, analogous to those of Ex. 7 by using as the element of the curve a normal $d\mathbf{n}$ equal in length to $d\mathbf{r}$, instead of $d\mathbf{r}$.

14. If in $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, the functions P, Q depend only on x, y and the function $R = 0$, apply Gauss's Formula to a cylinder of unit height upon the xy-plane to show that

$$\int \nabla\cdot\mathbf{F}dV = \int \mathbf{F}\cdot d\mathbf{S} \quad \text{becomes} \quad \int\int\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)dxdy = \int \mathbf{F}\cdot d\mathbf{n},$$

where $d\mathbf{n}$ has the meaning given in Ex. 13. Show that numerically $\mathbf{F}\cdot d\mathbf{n}$ and $\mathbf{F}\times d\mathbf{r}$ are equal, and thus obtain Green's Lemma for the plane (22) as a special case of (20). Derive Green's Formula (Ex. 10) for the plane.

15. If $\int \mathbf{F}\cdot d\mathbf{r} = \int \mathbf{G}\cdot d\mathbf{S}$, show that $\int (\mathbf{G} - \nabla\times\mathbf{F})\cdot d\mathbf{S} = 0$. Hence infer that if these relations hold for every surface and its bounding curve, then $\mathbf{G} = \nabla\times\mathbf{F}$. Ampère's Law states that the integral of the magnetic force **H** about any circuit is equal to 4π times the flux of the electric current **C** through the circuit, that is, through any surface spanning the circuit. Faraday's Law states that the integral of the electromotive force **E** around any circuit is the negative of the time rate of flux of the magnetic induction **B** through the circuit. Phrase these laws as integrals and convert into the form

$$4\pi\mathbf{C} = \text{curl}\,\mathbf{H}, \qquad -\dot{\mathbf{B}} = \text{curl}\,\mathbf{E}.$$

16. By formal expansion prove $\nabla\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{H}\cdot\nabla\times\mathbf{E} - \mathbf{E}\cdot\nabla\times\mathbf{H}$. Assume $\nabla\times\mathbf{E} = -\dot{\mathbf{H}}$ and $\nabla\times\mathbf{H} = \dot{\mathbf{E}}$ and establish Poynting's Theorem that

$$\int (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S} = -\frac{\partial}{\partial t}\int \frac{1}{2}(\mathbf{E}\cdot\mathbf{E} + \mathbf{H}\cdot\mathbf{H})\,dV.$$

17. The "equation of continuity" for fluid motion is

$$\frac{\partial D}{\partial t} + \frac{\partial Dv_x}{\partial x} + \frac{\partial Dv_y}{\partial y} + \frac{\partial Dv_z}{\partial z} = 0 \quad \text{or} \quad \frac{dD}{dt} + D\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = 0,$$

where D is the density, $\mathbf{v} = \mathbf{i}v_x + \mathbf{j}v_y + \mathbf{k}v_z$ is the velocity, $\partial D/\partial t$ is the rate of change of the density at a point, and dD/dt is the rate of change of density as one moves with the fluid (Ex. 14, p. 101). Explain the meaning of the equation in view of the work of the text. Show that for fluids of constant density $\nabla\cdot\mathbf{v} = 0$.

18. If **f** denotes the acceleration of the particles of a fluid, and if **F** is the external force acting per unit mass upon the elements of fluid, and if p denotes the pressure in the fluid, show that the equation of motion for the fluid within any surface may be written as

$$\sum \mathbf{f}DdV = \sum \mathbf{F}DdV - \sum pd\mathbf{S} \quad \text{or} \quad \int \mathbf{f}DdV = \int \mathbf{F}DdV - \int pd\mathbf{S},$$

where the summations or integrations extend over the volume or its bounding surface and the pressures (except those acting on the bounding surface inward) may be disregarded. (See the first half of § 80.)

19. By the aid of Ex. 6 transform the surface integral in Ex. 18 and find

$$\int D\mathbf{f}dV = \int (D\mathbf{F} - \nabla p)\,dV \quad \text{or} \quad \frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} - \frac{1}{D}\nabla p$$

as the equations of motion for a fluid, where $\mathbf{r}$ is the vector to any particle. Prove

$$(\alpha)\ \frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} - \mathbf{v}\times\nabla\times\mathbf{v} + \frac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v}),$$

$$(\beta)\ \frac{d}{dt}(d\mathbf{r}\cdot\mathbf{v}) = d\mathbf{r}\cdot\frac{d\mathbf{v}}{dt} + d\frac{d\mathbf{r}}{dt}\cdot\mathbf{v} = d\mathbf{r}\cdot\frac{d^2\mathbf{r}}{dt^2} + \frac{1}{2}d(\mathbf{v}\cdot\mathbf{v}).$$

20. If $\mathbf{F}$ is derivable from a potential, so that $\mathbf{F} = -\nabla U$, and if the density is a function of the pressure, so that $dp/D = dP$, show that the equations of motion are

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v}\times\nabla\times\mathbf{v} = -\nabla\left(U + P + \frac{1}{2}v^2\right), \quad \text{or} \quad \frac{d}{dt}(\mathbf{v}\cdot d\mathbf{r}) = -d\left(U + P - \frac{1}{2}v^2\right)$$

after multiplication by $d\mathbf{r}$. The first form is Helmholtz's, the second is Kelvin's. Show

$$\int_{a,\,b,\,c}^{x,\,y,\,z} \frac{d}{dt}(\mathbf{v}\cdot d\mathbf{r}) = \frac{d}{dt}\int_{a,\,b,\,c}^{x,\,y,\,z} \mathbf{v}\cdot d\mathbf{r} = -\left[U + P - \frac{1}{2}v^2\right]_{a,\,b,\,c}^{x,\,y,\,z} \text{ and } \oint \mathbf{v}\cdot d\mathbf{r} = \text{const.}$$

In particular explain that as the differentiation d/dt follows the particles in their motion (in contrast to $\partial/\partial t$, which is executed at a single point of space), the integral must do so if the order of differentiation and integration is to be interchangeable. Interpret the final equation as stating that the circulation in a curve which moves with the fluid is constant.

21. If $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} + \dfrac{\partial^2 U}{\partial z^2} = 0$, show $\displaystyle\int\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2\right]dV = \oint U\frac{dU}{dn}dS.$

22. Show that, apart from the proper restrictions as to continuity and differentiability, the necessary and sufficient condition that the surface integral

$$\iint Pdydz + Qdzdx + Rdxdy = \oint pdx + qdy + rdz$$

depends only on the curve bounding the surface is that $P'_x + Q'_y + R'_z = 0$. Show further that in this case the surface integral reduces to the line integral given above, provided p, q, r are such functions that $r'_y - q'_z = P$, $p'_z - r'_x = Q$, $q'_x - p'_y = R$. Show finally that these differential equations for p, q, r may be satisfied by

$$p = \int_{z_0}^{z} Qdz - \int R(x,\, y,\, z_0)\,dy, \qquad q = -\int_{z_0}^{z} Pdz, \qquad r = 0;$$

and determine by inspection alternative values of p, q, r.