

## PART III. INTEGRAL CALCULUS

### CHAPTER XI

#### ON SIMPLE INTEGRALS

**118. Integrals containing a parameter.** Consider

$$\phi(\alpha) = \int_{x_0}^{x_1} f(x, \alpha) dx, \quad (1)$$

a definite integral which contains in the integrand a parameter  $\alpha$ . If the indefinite integral is known, as in the case

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad \int_0^{\frac{\pi}{2}} \cos ax dx = \frac{1}{a} \sin ax \Big|_0^{\frac{\pi}{2}} = \frac{1}{a},$$

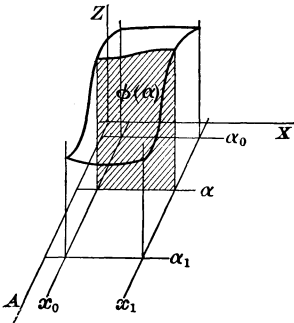
it is seen that the indefinite integral is a function of  $x$  and  $\alpha$ , and that the definite integral is a function of  $\alpha$  alone because the variable  $x$  disappears on the substitution of the limits. If the limits themselves depend on  $\alpha$ , as in the case

$$\int_{\frac{1}{\alpha}}^{\alpha} \cos ax dx = \frac{1}{a} \sin ax \Big|_{\frac{1}{\alpha}}^{\alpha} = \frac{1}{a} (\sin \alpha^2 - \sin 1),$$

the integral is still a function of  $\alpha$ .

In many instances the indefinite integral in (1) cannot be found explicitly and it then becomes necessary to discuss the continuity, differentiation, and integration of the function  $\phi(\alpha)$  defined by the integral without having recourse to the actual evaluation of the integral; in fact these discussions may be required in order to effect that evaluation. Let the limits  $x_0$  and  $x_1$  be taken

as constants independent of  $\alpha$ . Consider the range of values  $x_0 \leq x \leq x_1$  for  $x$ , and let  $\alpha_0 \leq \alpha \leq \alpha_1$  be the range of values over which the function  $\phi(\alpha)$  is to be discussed. The function  $f(x, \alpha)$  may be plotted as the surface  $z = f(x, \alpha)$  over the rectangle of values for  $(x, \alpha)$ . The



value  $\phi(\alpha_i)$  of the function when  $\alpha = \alpha_i$  is then the area of the section of this surface made by the plane  $\alpha = \alpha_i$ . If the surface  $f(x, \alpha)$  is continuous, it is tolerably clear that the area  $\phi(\alpha)$  will be continuous in  $\alpha$ . *The function  $\phi(\alpha)$  is continuous if  $f(x, \alpha)$  is continuous in the two variables  $(x, \alpha)$ .*

To discuss the continuity of  $\phi(\alpha)$  form the difference

$$\phi(\alpha + \Delta\alpha) - \phi(\alpha) = \int_{x_0}^{x_1} [f(x, \alpha + \Delta\alpha) - f(x, \alpha)] dx. \quad (2)$$

Now  $\phi(\alpha)$  will be continuous if the difference  $\phi(\alpha + \Delta\alpha) - \phi(\alpha)$  can be made as small as desired by taking  $\Delta\alpha$  sufficiently small. If  $f(x, y)$  is a continuous function of  $(x, y)$ , it is possible to take  $\Delta x$  and  $\Delta y$  so small that the difference

$$|f(x + \Delta x, y + \Delta y) - f(x, y)| < \epsilon, \quad |\Delta x| < \delta, \quad |\Delta y| < \delta$$

for all points  $(x, y)$  of the region over which  $f(x, y)$  is continuous (Ex. 3, p. 92). Hence in particular if  $f(x, \alpha)$  be continuous in  $(x, \alpha)$  over the rectangle, it is possible to take  $\Delta\alpha$  so small that

$$|f(x, \alpha + \Delta\alpha) - f(x, \alpha)| < \epsilon, \quad |\Delta\alpha| < \delta$$

for all values of  $x$  and  $\alpha$ . Hence, by (65), p. 25,

$$|\phi(\alpha + \Delta\alpha) - \phi(\alpha)| = \left| \int_{x_0}^{x_1} [f(x, \alpha + \Delta\alpha) - f(x, \alpha)] dx \right| < \int_{x_0}^{x_1} \epsilon dx = \epsilon(x_1 - x_0).$$

It is therefore proved that the function  $\phi(\alpha)$  is continuous provided  $f(x, \alpha)$  is continuous in the two variables  $(x, \alpha)$ ; for  $\epsilon(x_1 - x_0)$  may be made as small as desired if  $\epsilon$  may be made as small as desired.

As an illustration of a case where the condition for continuity is violated, take

$$\phi(\alpha) = \int_0^1 \frac{\alpha dx}{\alpha^2 + x^2} = \tan^{-1} \frac{x}{\alpha} \Big|_0^1 = \cot^{-1} \alpha \quad \text{if } \alpha \neq 0, \quad \text{and } \phi(0) = 0.$$

Here the integrand fails to be continuous for  $(0, 0)$ ; it becomes infinite when  $(x, \alpha) \doteq (0, 0)$  along any curve that is not tangent to  $\alpha = 0$ . The function  $\phi(\alpha)$  is defined for all values of  $\alpha \geq 0$ , is equal to  $\cot^{-1} \alpha$  when  $\alpha \neq 0$ , and should therefore be equal to  $\frac{1}{2} \pi$  when  $\alpha = 0$  if it is to be continuous, whereas it is equal to 0. The importance of the imposition of the condition that  $f(x, \alpha)$  be continuous is clear. It should not be inferred, however, that the function  $\phi(\alpha)$  will *necessarily* be discontinuous when  $f(x, \alpha)$  fails of continuity. For instance

$$\phi(\alpha) = \int_0^1 \frac{dx}{\sqrt{\alpha + x}} = \frac{1}{2} (\sqrt{\alpha + 1} - \sqrt{\alpha}), \quad \phi(0) = \frac{1}{2}.$$

This function is continuous in  $\alpha$  for all values  $\alpha \geq 0$ ; yet the integrand is discontinuous and indeed becomes infinite at  $(0, 0)$ . The condition of continuity imposed on  $f(x, \alpha)$  in the theorem is *sufficient* to insure the continuity of  $\phi(\alpha)$  but *by no means necessary*; when the condition is not satisfied some closer examination of the problem will sometimes disclose the fact that  $\phi(\alpha)$  is still continuous.

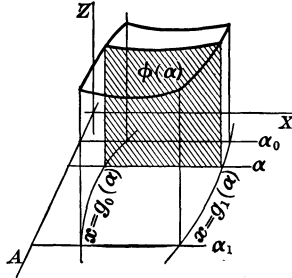
In case the limits of the integral are functions of  $\alpha$ , as

$$\phi(\alpha) = \int_{x_0 = g_0(\alpha)}^{x_1 = g_1(\alpha)} f(x, \alpha) dx, \quad \alpha_0 \leq \alpha \leq \alpha_1, \quad (3)$$

the function  $\phi(\alpha)$  will surely be continuous if  $f(x, \alpha)$  is continuous over the region bounded by the lines  $\alpha = \alpha_0$ ,  $\alpha = \alpha_1$  and the curves  $x_0 = g_0(\alpha)$ ,  $x_1 = g_1(\alpha)$ , and if the functions  $g_0(\alpha)$  and  $g_1(\alpha)$  are continuous.

For in this case

$$\begin{aligned} \phi(\alpha + \Delta\alpha) - \phi(\alpha) &= \int_{g_0(\alpha + \Delta\alpha)}^{g_1(\alpha + \Delta\alpha)} f(x, \alpha + \Delta\alpha) dx \\ &- \int_{g_0(\alpha)}^{g_1(\alpha)} f(x, \alpha) dx = \int_{g_0(\alpha + \Delta\alpha)}^{g_0(\alpha)} f(x, \alpha + \Delta\alpha) dx \\ &+ \int_{g_1(\alpha)}^{g_1(\alpha + \Delta\alpha)} f(x, \alpha + \Delta\alpha) dx \\ &+ \int_{g_0(\alpha)}^{g_1(\alpha)} [f(x, \alpha + \Delta\alpha) - f(x, \alpha)] dx. \end{aligned}$$



The absolute values may be taken and the integrals reduced by (65), (65'), p. 25.

$|\phi(\alpha + \Delta\alpha) - \phi(\alpha)| < \epsilon |g_1(\alpha) - g_0(\alpha)| + |f(\xi_1, \alpha + \Delta\alpha)| |\Delta g_1| + |f(\xi_0, \alpha + \Delta\alpha)| |\Delta g_0|$ , where  $\xi_0$  and  $\xi_1$  are values of  $x$  between  $g_0$  and  $g_0 + \Delta g_0$ , and  $g_1$  and  $g_1 + \Delta g_1$ . By taking  $\Delta\alpha$  small enough,  $g_1(\alpha + \Delta\alpha) - g_1(\alpha)$  and  $g_0(\alpha + \Delta\alpha) - g_0(\alpha)$  may be made as small as desired, and hence  $\Delta\phi$  may be made as small as desired.

**119.** To find the derivative of a function  $\phi(\alpha)$  defined by an integral containing a parameter, form the quotient

$$\begin{aligned} \frac{\Delta\phi}{\Delta\alpha} &= \frac{\phi(\alpha + \Delta\alpha) - \phi(\alpha)}{\Delta\alpha} \\ &= \frac{1}{\Delta\alpha} \left[ \int_{g_0(\alpha + \Delta\alpha)}^{g_1(\alpha + \Delta\alpha)} f(x, \alpha + \Delta\alpha) dx - \int_{g_0(\alpha)}^{g_1(\alpha)} f(x, \alpha) dx \right], \\ \frac{\Delta\phi}{\Delta\alpha} &= \int_{g_0(\alpha)}^{g_1(\alpha)} \frac{f(x, \alpha + \Delta\alpha) - f(x, \alpha)}{\Delta\alpha} dx + \int_{g_0 + \Delta g_0}^{g_0} \frac{f(x, \alpha + \Delta\alpha)}{\Delta\alpha} dx \\ &\quad + \int_{g_1}^{g_1 + \Delta g_1} \frac{f(x, \alpha + \Delta\alpha)}{\Delta\alpha} dx. \end{aligned}$$

The transformation is made by (63), p. 25. A further reduction may be made in the last two integrals by (65'), p. 25, which is the Theorem of the Mean for integrals, and the integrand of the first integral may be modified by the Theorem of the Mean for derivatives (p. 7, and Ex. 14, p. 10). Then

$$\begin{aligned} \frac{\Delta\phi}{\Delta\alpha} &= \int_{g_0(\alpha)}^{g_1(\alpha)} f'_\alpha(x, \alpha + \theta\Delta\alpha) dx - f(\xi_0, \alpha + \Delta\alpha) \frac{\Delta g_0}{\Delta\alpha} + f(\xi_1, \alpha + \Delta\alpha) \frac{\Delta g_1}{\Delta\alpha} \\ \text{and} \quad \frac{d\phi}{d\alpha} &= \int_{g_0(\alpha)}^{g_1(\alpha)} \frac{\partial f}{\partial \alpha} dx - f(g_0, \alpha) \frac{dg_0}{d\alpha} + f(g_1, \alpha) \frac{dg_1}{d\alpha}. \end{aligned} \tag{4}$$

A critical examination of this work shows that the derivative  $\phi'(\alpha)$  exists and may be obtained by (4) in case  $f'_\alpha$  exists and is continuous

in  $(x, \alpha)$  and  $g_0(\alpha), g_1(\alpha)$  are differentiable. In the particular case that the limits  $g_0$  and  $g_1$  are constants, (4) reduces to *Leibniz's Rule*

$$\frac{d\phi}{d\alpha} = \frac{d}{d\alpha} \int_{x_0}^{x_1} f(x, \alpha) dx = \int_{x_0}^{x_1} \frac{\partial f}{\partial \alpha} dx, \quad (4')$$

which states that *the derivative of a function defined by an integral with fixed limits may be obtained by differentiating under the sign of integration*. The additional two terms in (4), when the limits are variable, may be considered as arising from (66), p. 27, and Ex. 11, p. 30.

This process of *differentiating under the sign of integration* is of frequent use in evaluating the function  $\phi(\alpha)$  in cases where the indefinite integral of  $f(x, \alpha)$  cannot be found, but the indefinite integral of  $f'_\alpha$  can be found. For if

$$\phi(\alpha) = \int_{x_0}^{x_1} f(x, \alpha) dx, \quad \text{then} \quad \frac{d\phi}{d\alpha} = \int_{x_0}^{x_1} f'_\alpha dx = \psi(\alpha).$$

Now an integration with respect to  $\alpha$  will give  $\phi$  as a function of  $\alpha$  with a constant of integration which may be determined by the usual method of giving  $\alpha$  some special value. Thus

$$\phi(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x} dx, \quad \frac{d\phi}{d\alpha} = \int_0^1 \frac{x^\alpha \log x}{\log x} dx = \int_0^1 x^\alpha dx.$$

Hence 
$$\frac{d\phi}{d\alpha} = \frac{1}{\alpha + 1} x^{\alpha+1} \Big|_0^1 = \frac{1}{\alpha + 1}, \quad \phi(\alpha) = \log(\alpha + 1) + C.$$

But 
$$\phi(0) = \int_0^1 0 dx = 0 \quad \text{and} \quad \phi(0) = \log 1 + C.$$

Hence 
$$\phi(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x} dx = \log(\alpha + 1).$$

In the way of comment upon this evaluation it may be remarked that the functions  $(x^\alpha - 1)/\log x$  and  $x^\alpha$  are continuous functions of  $(x, \alpha)$  for all values of  $x$  in the interval  $0 \leq x \leq 1$  of integration and all positive values of  $\alpha$  less than any assigned value, that is,  $0 \leq \alpha \leq K$ . The conditions which permit the differentiation under the sign of integration are therefore satisfied. This is not true for negative values of  $\alpha$ . When  $\alpha < 0$  the derivative  $x^\alpha$  becomes infinite at  $(0, 0)$ . The method of evaluation cannot therefore be applied without further examination. As a matter of fact  $\phi(\alpha) = \log(\alpha + 1)$  is defined for  $\alpha > -1$ , and it would be natural to think that some method could be found to justify the above formal evaluation of the integral when  $-1 < \alpha \leq K$  (see Chap. XIII).

To illustrate the application of the rule for differentiation when the limits are functions of  $\alpha$ , let it be required to differentiate

$$\phi(\alpha) = \int_\alpha^{x^2} \frac{x^\alpha - 1}{\log x} dx. \quad \frac{d\phi}{d\alpha} = \int_\alpha^{x^2} x^\alpha dx + \frac{\alpha^{2\alpha} - 1}{\log \alpha} \alpha - \frac{\alpha^\alpha - 1}{\log \alpha},$$

or 
$$\frac{d\phi}{d\alpha} = \frac{\alpha^{\alpha+1}[\alpha^{\alpha+1}-1]}{\alpha+1} + \frac{1}{\log \alpha} [\alpha^{2\alpha} - \alpha^{\alpha} - \alpha + 1].$$

This formal result is only good subject to the conditions of continuity. Clearly  $\alpha$  must be greater than zero. This, however, is the only restriction. It might seem at first as though the value  $x = 1$  with  $\log x = 0$  in the denominator of  $(x^{\alpha} - 1)/\log x$  would cause difficulty; but when  $x = 0$ , this fraction is of the form  $0/0$  and has a finite value which pieces on continuously with the neighboring values.

**120.** The next problem would be to find *the integral of a function defined by an integral containing a parameter*. The attention will be restricted to the case where the limits  $x_0$  and  $x_1$  are constants. Consider the integrals

$$\int_{\alpha_0}^{\alpha} \phi(\alpha) d\alpha = \int_{\alpha_0}^{\alpha} \int_{x_0}^{x_1} f(x, \alpha) dx \cdot d\alpha,$$

where  $\alpha$  may be any point of the interval  $\alpha_0 \leq \alpha \leq \alpha_1$  of values over which  $\phi(\alpha)$  is treated. Let

$$\Phi(\alpha) = \int_{x_0}^{x_1} \int_{\alpha_0}^{\alpha} f(x, \alpha) d\alpha \cdot dx.$$

Then 
$$\Phi'(\alpha) = \int_{x_0}^{x_1} \frac{\partial}{\partial \alpha} \int_{\alpha_0}^{\alpha} f(x, \alpha) d\alpha \cdot dx = \int_{x_0}^{x_1} f(x, \alpha) dx = \phi(\alpha)$$

by (4'), and by (66), p. 27; and the differentiation is legitimate if  $f(x, \alpha)$  be assumed continuous in  $(x, \alpha)$ . Now integrate with respect to  $\alpha$ . Then

$$\int_{\alpha_0}^{\alpha} \Phi'(\alpha) d\alpha = \Phi(\alpha) - \Phi(\alpha_0) = \int_{\alpha_0}^{\alpha} \phi(\alpha) d\alpha.$$

But  $\Phi(\alpha_0) = 0$ . Hence, on substitution,

$$\Phi(\alpha) = \int_{x_0}^{x_1} \int_{\alpha_0}^{\alpha} f(x, \alpha) d\alpha \cdot dx = \int_{\alpha_0}^{\alpha} \phi(\alpha) d\alpha = \int_{\alpha_0}^{\alpha} \int_{x_0}^{x_1} f(x, \alpha) dx \cdot d\alpha. \quad (5)$$

Hence appears the rule for integration, namely, *integrate under the sign of integration*. The rule has here been obtained by a trick from the previous rule of differentiation; it could be proved directly by considering the integral as the limit of a sum.

It is interesting to note the interpretation of this integration on the figure, p. 281. As  $\phi(\alpha)$  is the area of a section of the surface, the product  $\phi(\alpha)d\alpha$  is the infinitesimal volume under the surface and included between two neighboring planes. The integral of  $\phi(\alpha)$  is therefore the volume \* under the surface and boxed in by the four

\* For the "volume of a solid with parallel bases and variable cross section" see Ex. 10, p. 10, and § 35 with Exs. 20, 23 thereunder.

planes  $\alpha = \alpha_0$ ,  $\alpha = \alpha$ ,  $x = x_0$ ,  $x = x_1$ . The geometric significance of the reversal of the order of integrations, as

$$V = \int_{x_0}^{x_1} \int_{\alpha_0}^{\alpha_1} f(x, \alpha) d\alpha \cdot dx = \int_{\alpha_0}^{\alpha_1} \int_{x_0}^{x_1} f(x, \alpha) dx \cdot d\alpha,$$

is in this case merely that the volume may be regarded as generated by a cross section moving parallel to the  $z\alpha$ -plane, or by one moving parallel to the  $zx$ -plane, and that the evaluation of the volume may be made by either method. If the limits  $x_0$  and  $x_1$  depend on  $\alpha$ , the integral of  $\phi(\alpha)$  cannot be found by the simple rule of integration under the sign of integration. It should be remarked that integration under the sign may serve to evaluate functions defined by integrals.

As an illustration of integration under the sign in a case where the method leads to a function which may be considered as evaluated by the method, consider

$$\phi(\alpha) = \int_0^1 x^\alpha dx = \frac{1}{\alpha + 1}, \quad \int_a^b \phi(\alpha) d\alpha = \int_a^b \frac{d\alpha}{\alpha + 1} = \log \frac{b + 1}{a + 1}.$$

But 
$$\int_a^b \phi(\alpha) d\alpha = \int_0^1 \int_a^b x^\alpha d\alpha \cdot dx = \int_0^1 \frac{x^\alpha}{\log x} \Big|_{\alpha=a}^{\alpha=b} dx = \int_0^1 \frac{x^b - x^a}{\log x} dx.$$

Hence 
$$\int_0^1 \frac{x^b - x^a}{\log x} dx = \log \frac{b + 1}{a + 1} = \psi(a, b), \quad a \geq 0, \quad b \geq 0.$$

In this case the integrand contains two parameters  $a, b$ , and the function defined is a function of the two. If  $a = 0$ , the function reduces to one previously found. It would be possible to repeat the integration. Thus

$$\int_0^1 \frac{x^\alpha - 1}{\log x} dx = \log(\alpha + 1), \quad \int_0^\alpha \log(\alpha + 1) d\alpha = (\alpha + 1) \log(\alpha + 1) - \alpha.$$

$$\int_0^1 \int_0^\alpha \frac{x^\alpha - 1}{\log x} d\alpha \cdot dx = \int_0^1 \frac{x^\alpha - 1 - \alpha \log x}{(\log x)^2} dx = (\alpha + 1) \log(\alpha + 1) - \alpha.$$

This is a new form. If here  $\alpha$  be set equal to any number, say 1, then

$$\int_0^1 \frac{x - 1 - \log x}{(\log x)^2} dx = 2 \log 2 - 1.$$

In this way there has been evaluated a definite integral which depends on no parameter and which might have been difficult to evaluate directly. *The introduction of a parameter and its subsequent equation to a particular value is of frequent use in evaluating definite integrals.*

### EXERCISES

1. Evaluate directly and discuss for continuity,  $0 \leq \alpha \leq 1$ :

$$(\alpha) \int_0^1 \frac{\alpha^2 dx}{\alpha^2 + x^2}, \quad (\beta) \int_0^1 \frac{dx}{\sqrt{\alpha^2 + x^2}}, \quad (\gamma) \int_0^1 \frac{x dx}{\sqrt{\alpha^2 + x^2}}.$$

2. If  $f(x, \alpha, \beta)$  is a function containing two parameters and is continuous in the three variables  $(x, \alpha, \beta)$  when  $x_0 \leq x \leq x_1$ ,  $\alpha_0 \leq \alpha \leq \alpha_1$ ,  $\beta_0 \leq \beta \leq \beta_1$ , show

$$\int_{x_0}^{x_1} f(x, \alpha, \beta) dx = \phi(\alpha, \beta) \text{ is continuous in } (\alpha, \beta).$$

3. Differentiate and hence evaluate and state the valid range for  $\alpha$  :

$$(\alpha) \int_0^\pi \log(1 + \alpha \cos x) dx = \pi \log \frac{1 + \sqrt{1 - \alpha^2}}{2},$$

$$(\beta) \int_0^\pi \log(1 - 2\alpha \cos x + \alpha^2) dx = \begin{cases} \pi \log \alpha^2, & \alpha^2 \equiv 1 \\ 0, & \alpha^2 \equiv 1 \end{cases}.$$

4. Find the derivatives without previously integrating :

$$(\alpha) \int_{\tan^{-1}\alpha}^{\sin^{-1}\alpha} \frac{1}{x} \tan \alpha x dx, \quad (\beta) \int_0^{\alpha^2} \tan^{-1} \frac{x}{\alpha^2} dx, \quad (\gamma) \int_{-ax}^{\alpha x} \frac{e^{-h^2 x^2}}{e^{\alpha^2}} dx.$$

5. Extend the assumptions and the work of Ex. 2 to find the partial derivatives  $\phi'_\alpha$  and  $\phi'_\beta$  and the total differential  $d\phi$  if  $x_0$  and  $x_1$  are constants.

6. Prove the rule for integrating under the sign of integration by the direct method of treating the integral as the limit of a sum.

7. From Ex. 6 derive the rule for differentiating under the sign. Can the complete rule including the case of variable limits be obtained this way ?

8. Note that the integral  $\int_{x_0}^{g(x, \alpha)} f(x, \alpha) dx$  will be a function of  $(x, \alpha)$ . Derive formulas for the partial derivatives with respect to  $x$  and  $\alpha$ .

$$9. \text{ Differentiate : } (\alpha) \frac{\partial}{\partial \alpha} \int_0^{\alpha x} \sin(x + \alpha) dx, \quad (\beta) \frac{d}{dx} \int_0^{\sqrt[3]{x}} x^2 dx.$$

10. Integrate under the sign and hence evaluate by subsequent differentiation :

$$(\alpha) \int_0^1 x^\alpha \log x dx, \quad (\beta) \int_0^{\frac{\pi}{2}} x \sin \alpha x dx, \quad (\gamma) \int_0^1 x \sec^2 \alpha x dx.$$

11. Integrate or differentiate both sides of these equations :

$$(\alpha) \int_0^1 x^\alpha dx = \frac{1}{\alpha + 1} \quad \text{to show} \quad \int_0^1 x^\alpha (\log x)^n dx = (-1)^n \frac{n!}{(\alpha + 1)^{n+1}},$$

$$(\beta) \int_0^\infty \frac{dx}{x^2 + \alpha} = \frac{\pi}{2\sqrt{\alpha}} \quad \text{to show} \quad \int_0^\infty \frac{dx}{(x^2 + \alpha)^{n+1}} = \frac{\pi}{2} \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 2 \cdot 4 \cdot 6 \cdots 2n \cdot \alpha^{n+\frac{1}{2}}},$$

$$(\gamma) \int_0^\infty e^{-\alpha x} \cos mx dx = \frac{\alpha}{\alpha^2 + m^2} \quad \text{to show} \quad \int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x \sec mx} dx = \frac{1}{2} \log \left( \frac{\beta^2 + m^2}{\alpha^2 + m^2} \right),$$

$$(\delta) \int_0^\infty e^{-\alpha x} \sin mx dx = \frac{m}{\alpha^2 + m^2} \quad \text{to show} \quad \int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x \csc mx} dx = \tan^{-1} \frac{\beta}{m} - \tan^{-1} \frac{\alpha}{m},$$

$$(\epsilon) \int_0^\pi \frac{dx}{\alpha - \cos x} = \frac{\pi}{\sqrt{\alpha^2 - 1}} \quad \text{to find} \quad \int_0^\pi \frac{dx}{(\alpha - \cos x)^2}, \quad \int_0^\pi \log \frac{b - \cos x}{a - \cos x},$$

$$(\zeta) \int_0^\infty \frac{x^{\alpha-1} dx}{1+x} = \frac{\pi}{\sin \pi \alpha} \quad \text{to find} \quad \int_0^\infty \frac{x^{\alpha-1} \log x dx}{1+x}, \quad \int_0^\infty \frac{x^{b-1} - x^{a-1}}{(1+x) \log x} dx.$$

Note that in  $(\beta)$ - $(\delta)$  the integrals extend to infinity and that, as the rules of the text have been proved on the hypothesis that the interval of integration is finite, a further justification for applying the rules is necessary ; this will be treated in Chap. XIII, but at this point the rules may be applied formally without justification.

12. Evaluate by any means these integrals:

$$(\alpha) \int_0^{\alpha} \sqrt{\alpha^2 - x^2} \cos^{-1} \frac{x}{\alpha} dx = \alpha^2 \left( \frac{\pi^2}{16} + \frac{1}{4} \right),$$

$$(\beta) \int_0^{\frac{\pi}{2}} \frac{\log(1 + \cos \alpha \cos x)}{\cos x} dx = \frac{1}{2} \left( \frac{\pi^2}{4} - \alpha^2 \right),$$

$$(\gamma) \int_0^{\frac{\pi}{2}} \log(\alpha^2 \cos^2 x + \beta^2 \sin^2 x) dx = \pi \log \frac{\alpha + \beta}{2},$$

$$(\delta) \int_0^{\infty} x e^{-ax} \cos \beta x dx = \frac{\alpha^2 - \beta^2}{(\alpha^2 + \beta^2)^2},$$

$$(\epsilon) \int_0^{\frac{\pi}{2}} \log \frac{a + b \sin x}{a - b \sin x} \frac{dx}{\sin x} = \pi \sin^{-1} \frac{b}{a}, \quad b < a,$$

$$(\zeta) \int_0^{\pi} \frac{\log(1 + k \cos x)}{\cos x} dx = \pi \sin^{-1} k,$$

$$(\theta) \int_0^1 \log f(a+x) dx = \int_a^{a+1} \log f(x) dx = \int_0^a \log \frac{f(a+1)}{f(a)} da + \int_0^1 \log f(x) dx.$$

121. **Curvilinear or line integrals.** It is familiar that

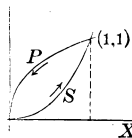
$$A = \int_a^b y dx = \int_a^b f(x) dx$$

is the area between the curve  $y = f(x)$ , the  $x$ -axis, and the ordinates  $x = a$ ,  $x = b$ . The formula may be used to evaluate more complicated areas. For instance, the area between the parabola  $y^2 = x$  and the semi-cubical parabola  $y^2 = x^3$  is

$$A = \int_0^1 x^{\frac{1}{2}} dx - \int_0^1 x^{\frac{3}{2}} dx = \int_0^1 y dx - \int_0^1 y dx,$$

where in the second expression the subscripts  $P$  and  $S$  denote that the integrals are evaluated for the parabola and semicubical parabola. As a change in the order of the limits changes the sign of the integral, the area may be written

$$A = \int_0^1 y dx + \int_1^0 y dx = - \int_1^0 y dx - \int_0^1 y dx,$$



and is the area bounded by the closed curve formed of the portions of the parabola and semicubical parabola from 0 to 1.

In considering the area bounded by a closed curve it is convenient to arrange the limits of the different integrals so that they follow the curve in a definite order. Thus if one advances along  $P$  from 0 to 1 and returns along  $S$  from 1 to 0, the entire closed curve has been described in a uniform direction and the inclosed area has been constantly on the right-hand side; whereas if one advanced along  $S$  from 0 to 1 and

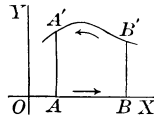


returned from 1 to 0 along  $P$ , the curve would have been described in the opposite direction and the area would have been constantly on the left-hand side. Similar considerations apply to more general closed curves and lead to the definition: If a closed curve which nowhere crosses itself is described in such a direction as to keep the inclosed area always upon the left, the area is considered as positive; whereas if the description were such as to leave the area on the right, it would be taken as negative. It is clear that to a person standing in the inclosure and watching the description of the boundary, the description would appear counterclockwise or positive in the first case (§ 76).

In the case above, the area when positive is

$$A = - \left[ \int_{s_0}^1 y dx + \int_P^0 y dx \right] = - \int_{\bigcirc} y dx, \tag{6}$$

where in the last integral the symbol  $\bigcirc$  denotes that the integral is to be evaluated around the closed curve by describing the curve in the positive direction. That the formula holds for the ordinary case of area under a curve may be verified at once. Here the circuit consists of the contour  $ABB'A'A$ . Then



$$\int_{\bigcirc} y dx = \int_A^B y dx + \int_B^{B'} y dx + \int_{B'}^{A'} y dx + \int_{A'}^A y dx.$$

The first integral vanishes because  $y = 0$ , the second and fourth vanish because  $x$  is constant and  $dx = 0$ . Hence

$$- \int_{\bigcirc} y dx = - \int_{B'}^{A'} y dx = \int_{A'}^{B'} y dx.$$

It is readily seen that the two new formulas

$$A = \int_{\bigcirc} x dy \quad \text{and} \quad A = \frac{1}{2} \int_{\bigcirc} (x dy - y dx) \tag{7}$$

also give the area of the closed curve. The first is proved as (6) was proved and the second arises from the addition of the two. Any one of the three may be used to compute the area of the closed curve; the last has the advantage of symmetry and is particularly useful in finding the area of a sector, because along the lines issuing from the origin  $y : x = dy : dx$  and  $x dy - y dx = 0$ ; the previous form with the integrand  $x dy$  is advantageous when part of the contour consists of lines parallel to the  $x$ -axis so that  $dy = 0$ ; the first form has similar advantages when parts of the contour are parallel to the  $y$ -axis.

The connection of the third formula with the vector expression for the area is noteworthy. For (p. 175)

$$d\mathbf{A} = \frac{1}{2} \mathbf{r} \times d\mathbf{r}, \quad \mathbf{A} = \frac{1}{2} \int_{\circ} \mathbf{r} \times d\mathbf{r},$$

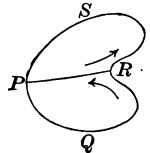
and if  $\mathbf{r} = x\mathbf{i} + y\mathbf{j}, \quad d\mathbf{r} = i dx + j dy,$

then  $\mathbf{A} = \int_{\circ} \mathbf{r} \times d\mathbf{r} = \frac{1}{2} \mathbf{k} \int_{\circ} (x dy - y dx).$

The unit vector  $\mathbf{k}$  merely calls attention to the fact that the area lies in the  $xy$ -plane perpendicular to the  $z$ -axis and is described so as to appear positive.

These formulas for the area as a curvilinear integral taken around the boundary have been derived from a simple figure whose contour was cut in only two points by a line parallel to the axes. The extension to more complicated contours is easy. In the first place note that if two closed areas are contiguous over a part of their contours, the integral around the total area following both contours, but omitting the part in common, is equal to the sum of the integrals. For

$$\int_{PRSP} + \int_{PQRP} = \int_{PR} + \int_{RSP} + \int_{PQR} + \int_{RP} = \int_{QRSP},$$



since the first and last integrals of the four are in opposite directions along the same line and must cancel. But the total area is also the sum of the individual areas and hence the integral around the contour  $PQRSP$  must be the total area. The formulas for determining the area of a closed curve are therefore applicable to such areas as may be composed of a finite number of areas each bounded by an oval curve.

If the contour bounding an area be expressed in parametric form as  $x = f(t), y = \phi(t)$ , the area may be evaluated as

$$\int f(t)\phi'(t) dt = - \int \phi(t)f'(t) dt = \frac{1}{2} \int [f(t)\phi'(t) - \phi(t)f'(t)] dt, \tag{7}$$

where the limits for  $t$  are the value of  $t$  corresponding to any point of the contour and the value of  $t$  corresponding to the same point after the curve has been described once in the positive direction. Thus in the case of the strophoid

$$y^2 = x^2 \frac{a-x}{a+x}, \quad \text{the line } y = tx$$

cuts the curve in the double point at the origin and in only one other point; the coördinates of a point on the curve may be expressed as rational functions

$$x = a(1 - t^2)/(1 + t^2), \quad y = at(1 - t^2)/(1 + t^2)$$

of  $t$  by solving the strophoid with the line; and when  $t$  varies from  $-1$  to  $+1$  the point  $(x, y)$  describes the loop of the strophoid and the limits for  $t$  are  $-1$  and  $+1$ .

122. Consider next the meaning and the evaluation of

$$\int_{a,b}^{x,y} [P(x, y)dx + Q(x, y)dy], \quad \text{where } y = f(x). \quad (8)$$

This is called a *curvilinear or line integral along the curve C* or  $y = f(x)$  from the point  $(a, b)$  to  $(x, y)$ . It is possible to eliminate  $y$  by the relation  $y = f(x)$  and write

$$\int_a^x [P(x, f(x)) + Q(x, f(x))f'(x)]dx. \quad (9)$$

The integral then becomes an ordinary integral in  $x$  alone. If the curve had been given in the form  $x = f(y)$ , it would have been better to convert the line integral into an integral in  $y$  alone. *The method of evaluating the integral is therefore defined.* The differential of the integral may be written as

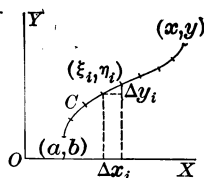
$$d \int_{a,b}^{x,y} (Pdx + Qdy) = Pdx + Qdy, \quad (10)$$

where either  $x$  and  $dx$  or  $y$  and  $dy$  may be eliminated by means of the equation of the curve  $C$ . For further particulars see § 123.

To get at *the meaning of the line integral*, it is necessary to consider it as the limit of a sum (compare § 16). Suppose that the curve  $C$  between  $(a, b)$  and  $(x, y)$  be divided into  $n$  parts, that  $\Delta x_i$  and  $\Delta y_i$  are the increments corresponding to the  $i$ th part, and that  $(\xi_i, \eta_i)$  is any point in that part. Form the sum

$$\sigma = \sum [P(\xi_i, \eta_i) \Delta x_i + Q(\xi_i, \eta_i) \Delta y_i]. \quad (11)$$

If, when  $n$  becomes infinite so that  $\Delta x$  and  $\Delta y$  each approaches 0 as a limit, the sum  $\sigma$  approaches a definite limit independent of how the individual increments  $\Delta x_i$  and  $\Delta y_i$  approach 0, and of how the point  $(\xi_i, \eta_i)$  is chosen in its segment of the curve, then this limit is defined as the line integral



$$\lim \sigma = \int_{a,b}^{x,y} [P(x, y)dx + Q(x, y)dy]. \quad (12)$$

It should be noted that, as in the case of the line integral which gives the area, any line integral which is to be evaluated along two curves which have in common a portion described in opposite directions may be replaced by the integral along so much of the curves as not repeated; for the elements of  $\sigma$  corresponding to the common portion are equal and opposite.

That  $\sigma$  does approach a limit provided  $P$  and  $Q$  are continuous functions of  $(x, y)$  and provided the curve  $C$  is monotonic, that is, that neither  $\Delta x$  nor  $\Delta y$  changes its sign, is easy to prove. For the expression for  $\sigma$  may be written

$$\sigma = \sum [P(\xi_i, f(\xi_i)) \Delta x_i + Q(f^{-1}(\eta_i), \eta_i) \Delta y_i]$$

by using the equation  $y = f(x)$  or  $x = f^{-1}(y)$  of  $C$ . Now as

$$\int_a^x P(x, f(x)) dx \quad \text{and} \quad \int_b^y Q(f^{-1}(y), y) dy$$

are both existent ordinary definite integrals in view of the assumptions as to continuity, the sum  $\sigma$  must approach their sum as a limit. It may be noted that this proof does not require the continuity or existence of  $f'(x)$  as does the formula (9). In practice the added generality is of little use. The restriction to a monotonic curve may be replaced by the assumption of a curve  $C$  which can be regarded as made up of a finite number of monotonic parts including perhaps some portions of lines parallel to the axes. More general varieties of  $C$  are admissible, but are not very useful in practice (§ 127).

Further to examine the line integral and appreciate its utility for mathematics and physics consider some examples. Let

$$F(x, y) = X(x, y) + iY(x, y)$$

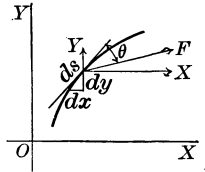
be a complex function (§ 73). Then

$$\begin{aligned} \int_{z=c}^{z=z} F(x, y) dz &= \int_{c(a,b)}^{x,y} [X(x, y) + iY(x, y)] [dx + idy] \\ &= \int_{c(a,b)}^{x,y} (Xd x - Ydy) + i \int_{c(a,b)}^{x,y} (Ydx + Xdy). \end{aligned} \quad (13)$$

It is apparent that *the integral of the complex function is the sum of two line integrals in the complex plane*. The value of the integral can be computed only by the assumption of some definite path  $C$  of integration and will differ for different paths (but see § 124).

By definition *the work done by a constant force  $F$  acting on a particle, which moves a distance  $s$  along a straight line inclined at an angle  $\theta$  to the force, is  $W = Fs \cos \theta$* . If the path were curvilinear and the force were variable, *the differential of work* would be taken as  $dW = F \cos \theta ds$ , where  $ds$  is the infinitesimal arc and  $\theta$  is the angle between the arc and the force. Hence

$$W = \int dW = \int_{a,b}^{x,y} F \cos \theta ds = \int_{r_0}^r \mathbf{F} \cdot d\mathbf{r},$$



where the path must be known to evaluate the integral and where the last expression is merely the equivalent of the others when the

notations of vectors are used (p. 164). These expressions may be converted into the ordinary form of the line integral. For

$$\mathbf{F} = X\mathbf{i} + Y\mathbf{j}, \quad d\mathbf{r} = \mathbf{i}dx + \mathbf{j}dy, \quad \mathbf{F} \cdot d\mathbf{r} = Xdx + Ydy,$$

and 
$$W = \int_{a,b}^{x,y} F \cos \theta ds = \int_{a,b}^{x,y} (Xdx + Ydy),$$

where  $X$  and  $Y$  are the components of the force along the axes. It is readily seen that any line integral may be given this same interpretation. If

$$I = \int_{a,b}^{x,y} Pdx + Qdy, \quad \text{form } \mathbf{F} = P\mathbf{i} + Q\mathbf{j}.$$

Then 
$$I = \int_{a,b}^{x,y} Pdx + Qdy = \int_{a,b}^{x,y} F \cos \theta ds.$$

To the principles of momentum and moment of momentum (§ 80) may now be added the principle of work and energy for mechanics. Consider

$$m \frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} \quad \text{and} \quad m \frac{d^2\mathbf{r}}{dt^2} \cdot d\mathbf{r} = \mathbf{F} \cdot d\mathbf{r} = dW.$$

Then 
$$\frac{d}{dt} \left( \frac{1}{2} \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} \right) = \frac{1}{2} \frac{d^2\mathbf{r}}{dt^2} \cdot \frac{d\mathbf{r}}{dt} + \frac{1}{2} \frac{d\mathbf{r}}{dt} \cdot \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{r}}{dt^2} \cdot \frac{d\mathbf{r}}{dt},$$

or 
$$d \left( \frac{1}{2} v^2 \right) = \frac{d^2\mathbf{r}}{dt^2} \cdot d\mathbf{r} \quad \text{and} \quad d \left( \frac{1}{2} mv^2 \right) = dW.$$

Hence 
$$\frac{1}{2} mv^2 - \frac{1}{2} mv_0^2 = \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = W.$$

In words: *The change of the kinetic energy  $\frac{1}{2} mv^2$  of a particle moving under the action of the resultant force  $\mathbf{F}$  is equal to the work done by the force, that is, to the line integral of the force along the path.* If there were several mutually interacting particles in motion, the results for the energy and work would merely be added as  $\Sigma \frac{1}{2} mv^2 - \Sigma \frac{1}{2} mv_0^2 = \Sigma W$ , and the total change in kinetic energy is the total work done by all the forces. The result gains its significance chiefly by the consideration of what forces may be disregarded in evaluating the work. As  $dW = \mathbf{F} \cdot d\mathbf{r}$ , the work done will be zero if  $d\mathbf{r}$  is zero or if  $\mathbf{F}$  and  $d\mathbf{r}$  are perpendicular. Hence in evaluating  $W$ , forces whose point of application does not move may be omitted (for example, forces of support at pivots), and so may forces whose point of application moves normal to the force (for example, the normal reactions of smooth curves or surfaces). When more than one particle is concerned, the work done by the mutual actions and reactions may be evaluated as follows. Let  $\mathbf{r}_1, \mathbf{r}_2$  be the vectors to the particles and  $\mathbf{r}_1 - \mathbf{r}_2$  the vector joining them. The forces of action and reaction may be written as  $\pm c(\mathbf{r}_1 - \mathbf{r}_2)$ , as they are equal and opposite and in the line joining the particles. Hence

$$\begin{aligned} dW &= dW_1 + dW_2 = c(\mathbf{r}_1 - \mathbf{r}_2) \cdot d\mathbf{r}_1 - c(\mathbf{r}_1 - \mathbf{r}_2) \cdot d\mathbf{r}_2 \\ &= c(\mathbf{r}_1 - \mathbf{r}_2) \cdot d(\mathbf{r}_1 - \mathbf{r}_2) = \frac{1}{2} cd [(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{r}_1 - \mathbf{r}_2)] = \frac{1}{2} cdr_{12}^2, \end{aligned}$$

where  $r_{12}$  is the distance between the particles. Now  $dW$  vanishes when and only when  $dr_{12}$  vanishes, that is, when and only when the distance between the particles

remains constant. Hence when a system of particles is in motion the change in the total kinetic energy in passing from one position to another is equal to the work done by the forces, where, in evaluating the work, forces acting at fixed points or normal to the line of motion of their points of application, and forces due to actions and reactions of particles rigidly connected, may be disregarded.

Another important application is in the theory of thermodynamics. If  $U$ ,  $p$ ,  $v$  are the energy, pressure, volume of a gas inclosed in any receptacle, and if  $dU$  and  $dv$  are the increments of energy and volume when the amount  $dH$  of heat is added to the gas, then

$$dH = dU + p dv, \quad \text{and hence} \quad H = \int dU + p dv$$

is the total amount of heat added. By taking  $p$  and  $v$  as the independent variables,

$$H = \int \left[ \frac{\partial U}{\partial p} dp + \left( \frac{\partial U}{\partial v} + p \right) dv \right] = \int [f(p, v) dp + g(p, v) dv].$$

The amount of heat absorbed by the system will therefore not depend merely on the initial and final values of  $(p, v)$  but on the sequence of these values between those two points, that is, upon the path of integration in the  $pv$ -plane.

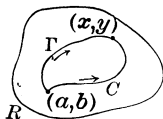
**123.** Let there be given a simply connected region (p. 89) bounded by a closed curve of the type allowed for line integrals, and let  $P(x, y)$  and  $Q(x, y)$  be continuous functions of  $(x, y)$  over this region. Then if the line integrals from  $(a, b)$  to  $(x, y)$  along two paths

$$\int_{a,b}^{x,y} P dx + Q dy = \int_{\Gamma}^{x,y} P dx + Q dy$$

are equal, the line integral taken around the combined path

$$\int_{a,b}^{x,y} + \int_{x,y}^{a,b} = \int_{\circlearrowleft} P dx + Q dy = 0$$

vanishes. This is a corollary of the fact that if the order of description of a curve is reversed, the signs of  $\Delta x_i$  and  $\Delta y_i$  and hence of the line integral are also reversed. Also, conversely, if the integral around the closed circuit is zero, the integrals from any point  $(a, b)$  of the circuit to any other point  $(x, y)$  are equal when evaluated along the two different parts of the circuit leading from  $(a, b)$  to  $(x, y)$ .



The chief value of these observations arises in their application to the case where  $P$  and  $Q$  happen to be such functions that the line integral around any and every closed path lying in the region is zero. In this case if  $(a, b)$  be a fixed point and  $(x, y)$  be any point of the region, the line integral from  $(a, b)$  to  $(x, y)$  along any two paths lying within the region will be the same; for the two paths may be considered as forming one closed path, and the integral around that is zero by hypothesis. The value of the integral will therefore not depend at all on

the path of integration but only on the final point  $(x, y)$  to which the integration is extended. Hence the integral

$$\int_{a,b}^{x,y} [P(x, y) dx + Q(x, y) dy] = F(x, y), \quad (14)$$

extended from a fixed lower limit  $(a, b)$  to a variable upper limit  $(x, y)$ , must be a function of  $(x, y)$ .

This result may be stated as the theorem: *The necessary and sufficient condition that the line integral*

$$\int_{a,b}^{x,y} [P(x, y) dx + Q(x, y) dy]$$

define a single valued function of  $(x, y)$  over a simply connected region is that the circuit integral taken around any and every closed curve in the region shall be zero. This theorem, and in fact all the theorems on line integrals, may be immediately extended to the case of line integrals in space,

$$\int_{a,b,c}^{x,y,z} [P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz]. \quad (15)$$

If the integral about every closed path is zero so that the integral from a fixed lower limit to a variable upper limit

$$F(x, y) = \int_{a,b}^{x,y} P(x, y) dx + Q(x, y) dy$$

defines a function  $F(x, y)$ , that function has continuous first partial derivatives and hence a total differential, namely,

$$\frac{\partial F}{\partial x} = P, \quad \frac{\partial F}{\partial y} = Q, \quad dF = P dx + Q dy. \quad (16)$$

To prove this statement apply the definition of a derivative.

$$\frac{\partial F}{\partial x} = \lim_{\Delta x \neq 0} \frac{\Delta F}{\Delta x} = \lim_{\Delta x \neq 0} \frac{\int_{a,b}^{x+\Delta x,y} P dx + Q dy - \int_{a,b}^{x,y} P dx + Q dy}{\Delta x}.$$

Now as the integral is independent of the path, the integral to  $(x + \Delta x, y)$  may follow the same path as that to  $(x, y)$ , except for the passage from  $(x, y)$  to  $(x + \Delta x, y)$  which may be taken along the straight line joining them. Then  $\Delta y = 0$  and

$$\frac{\Delta F}{\Delta x} = \frac{1}{\Delta x} \int_{x,y}^{x+\Delta x,y} P(x, y) dx = \frac{1}{\Delta x} P(\xi, y) \Delta x = P(\xi, y),$$

by the Theorem of the Mean of (65'), p. 25. Now when  $\Delta x \doteq 0$ , the value  $\xi$  intermediate between  $x$  and  $x + \Delta x$  will approach  $x$  and  $P(\xi, y)$  will approach the limit  $P(x, y)$  by virtue of its continuity. Hence  $\Delta F/\Delta x$  approaches a limit and that limit is  $P(x, y) = \partial F/\partial x$ . The other derivative is treated in the same way.

*If the integrand  $Pdx + Qdy$  of a line integral is the total differential  $dF$  of a single valued function  $F(x, y)$ , then the integral about any closed circuit is zero and*

$$\int_{a,b}^{x,y} Pdx + Qdy = \int_{a,b}^{x,y} dF = F(x, y) - F(a, b). \quad (17)$$

If equation (17) holds, it is clear that the integral around a closed path will be zero provided  $F(x, y)$  is single valued; for  $F(x, y)$  must come back to the value  $F(a, b)$  when  $(x, y)$  returns to  $(a, b)$ . If the function were not single valued, the conclusion might not hold.

To prove the relation (17), note that by definition

$$\int dF = \int Pdx + Qdy = \lim \sum [P(\xi_i, \eta_i) \Delta x_i + Q(\xi_i, \eta_i) \Delta y_i]$$

and

$$\Delta F_i = P(\xi_i, \eta_i) \Delta x_i + Q(\xi_i, \eta_i) \Delta y_i + \epsilon_1 \Delta x_i + \epsilon_2 \Delta y_i,$$

where  $\epsilon_1$  and  $\epsilon_2$  are quantities which by the assumptions of continuity for  $P$  and  $Q$  may be made uniformly (§ 25) less than  $\epsilon$  for all points of the curve provided  $\Delta x_i$  and  $\Delta y_i$  are taken small enough. Then

$$\left| \sum (P_i \Delta x_i + Q_i \Delta y_i) - \sum \Delta F_i \right| < \epsilon \sum (|\Delta x_i| + |\Delta y_i|);$$

and since  $\sum \Delta F_i = F(x, y) - F(a, b)$ , the sum  $\sum P_i \Delta x_i + Q_i \Delta y_i$  approaches a limit, and that limit is

$$\lim \sum [P_i \Delta x_i + Q_i \Delta y_i] = \int_{a,b}^{x,y} Pdx + Qdy = F(x, y) - F(a, b).$$

### EXERCISES

1. Find the area of the loop of the strophoid as indicated above.
2. Find, from (6), (7), the three expressions for the integrand of the line integrals which give the area of a closed curve in polar coordinates.
3. Given the equation of the ellipse  $x = a \cos t$ ,  $y = b \sin t$ . Find the total area, the area of a segment from the end of the major axis to a line parallel to the minor axis and cutting the ellipse at a point whose parameter is  $t$ , also the area of a sector.
4. Find the area of a segment and of a sector for the hyperbola in its parametric form  $x = a \cosh t$ ,  $y = b \sinh t$ .
5. Express the folium  $x^3 + y^3 = 3axy$  in parametric form and find the area of the loop.
6. What area is given by the curvilinear integral around the perimeter of the closed curve  $r = a \sin^3 \frac{1}{3} \phi$ ? What in the case of the lemniscate  $r^2 = a^2 \cos 2\phi$  described as in making the figure 8 or the sign  $\infty$ ?



7. Write for  $y$  the analogous form to (9) for  $x$ . Show that in curvilinear coördinates  $x = \phi(u, v)$ ,  $y = \psi(u, v)$  the area is

$$A = \frac{1}{2} \int \left[ \left| \begin{array}{cc} \phi & \psi \\ \phi'_u & \psi'_u \end{array} \right| du + \left| \begin{array}{cc} \phi & \psi \\ \phi'_v & \psi'_v \end{array} \right| dv \right].$$

8. Compute these line integrals along the paths assigned :

$$(\alpha) \int_{0,0}^{1,1} x^2 y dx + y^3 dy, \quad y^2 = x \quad \text{or} \quad y = x \quad \text{or} \quad y^3 = x^2,$$

$$(\beta) \int_{0,0}^{1,1} (x^2 + y) dx + (x + y^2) dy, \quad y^2 = x \quad \text{or} \quad y = x \quad \text{or} \quad y^3 = x^2,$$

$$(\gamma) \int_{1,0}^{e,1} \frac{y}{x} dx + dy, \quad y = \log x \quad \text{or} \quad y = 0 \quad \text{and} \quad x = e,$$

$$(\delta) \int_{0,0}^{x,y} x \sin y dx + y \cos x dy, \quad y = mx \quad \text{or} \quad x = 0 \quad \text{and} \quad y = y,$$

$$(\epsilon) \int_{z=0}^{1+i} (x - iy) dz, \quad y = x \quad \text{or} \quad x = 0 \quad \text{and} \quad y = 1 \quad \text{or} \quad y = 0 \quad \text{and} \quad x = 1,$$

$$(\zeta) \int_{z=1}^{z=i} (x^2 - (1+i)xy + y^2) dz, \quad \text{quadrant or straight line.}$$

9. Show that  $\int P dx + Q dy = \int \sqrt{P^2 + Q^2} \cos \theta ds$  by working directly with the figure and without the use of vectors.

10. Show that if any circuit is divided into a number of circuits by drawing lines within it, as in a figure on p. 91, the line integral around the original circuit is equal to the sum of the integrals around the subcircuits taken in the proper order.

11. Explain the method of evaluating a line integral in space and evaluate :

$$(\alpha) \int_{0,0,0}^{1,1,1} x dx + 2y dy + z dz, \quad y^2 = x, \quad z^2 = x \quad \text{or} \quad y = z = x,$$

$$(\beta) \int_{1,0,1}^{x,y,z} y \log x dx + y^2 dy + \frac{x}{z} dz, \quad y = x - 1, \quad z = x^2 \quad \text{or} \quad y = \log x, \quad z = x.$$

12. Show that  $\int P dx + Q dy + R dz = \int \sqrt{P^2 + Q^2 + R^2} \cos \theta ds$ .

13. A bead of mass  $m$  strung on a frictionless wire of any shape falls from one point  $(x_0, y_0, z_0)$  to the point  $(x_1, y_1, z_1)$  on the wire under the influence of gravity. Show that  $mg(z_0 - z_1)$  is the work done by all the forces, namely, gravity and the normal reaction of the wire.

14. If  $x = f(t)$ ,  $y = g(t)$ , and  $f'(t)$ ,  $g'(t)$  be assumed continuous, show

$$\int_a^b P(x, y) dx + Q(x, y) dy = \int_{t_0}^{t_1} \left( P \frac{dx}{dt} + Q \frac{dy}{dt} \right) dt,$$

where  $f(t_0) = a$  and  $g(t_0) = b$ . Note that this proves the statement made on page 290 in regard to the possibility of substituting in a line integral. The theorem is also needed for Exs. 1-8.

15. Extend to line integrals (15) in space the results of § 123.

16. *Angle as a line integral.* Show geometrically for a plane curve that  $d\phi = \cos(r, n) ds/r$ , where  $r$  is the radius vector of a curve and  $ds$  the element of

arc and  $(r, n)$  the angle between the radius produced and the normal to the curve, is the angle subtended at  $r = 0$  by the element  $ds$ . Hence show that

$$\phi = \int \frac{\cos(r, n)}{r} ds = \int \frac{1}{r} \frac{dr}{dn} ds = \int \frac{d \log r}{dn} ds,$$

where the integrals are line integrals along the curve and  $dr/dn$  is the normal derivative of  $r$ , is the angle  $\phi$  subtended by the curve at  $r = 0$ . Hence infer that

$$\int_{\circ} \frac{d \log r}{dn} ds = 2\pi \quad \text{or} \quad \int_{\circ} \frac{d \log r}{dn} ds = 0 \quad \text{or} \quad \int_{\circ} \frac{d \log r}{dn} ds = \theta$$

according as the point  $r = 0$  is within the curve or outside the curve or upon the curve at a point where the tangents in the two directions are inclined at the angle  $\theta$  (usually  $\pi$ ). Note that the formula may be applied at any point  $(\xi, \eta)$  if  $r^2 = (\xi - x)^2 + (\eta - y)^2$  where  $(x, y)$  is a point of the curve. What would the integral give if applied to a space curve?

**17.** Are the line integrals of Ex. 16 of the same type  $\int P(x, y) dx + Q(x, y) dy$  as those in the text, or are they more intimately associated with the curve? Cf. § 155.

**18.** Compute  $(\alpha) \int_{1,0}^{0,1} (x - y) ds$ ,  $(\beta) \int_{-1,0}^{0,1} xy ds$  along a right line, along a quadrant, along the axes.

**124. Independency of the path.** It has been seen that in case the integral around every closed path is zero or in case the integrand  $Pdx + Qdy$  is a total differential, the integral is independent of the path, and conversely. Hence if

$$F(x, y) = \int_{a,b}^{x,y} Pdx + Qdy, \quad \text{then} \quad \frac{\partial F}{\partial x} = P, \quad \frac{\partial F}{\partial y} = Q,$$

$$\text{and} \quad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial P}{\partial y}, \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

provided the partial derivatives  $P'_y$  and  $Q'_x$  are continuous functions.\* It remains to prove the converse, namely, that: *If the two partial derivatives  $P'_y$  and  $Q'_x$  are continuous and equal, the integral*

$$\int_{a,b}^{x,y} Pdx + Qdy \quad \text{with} \quad P'_y = Q'_x \quad (18)$$

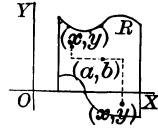
*is independent of the path, is zero around a closed path, and the quantity  $Pdx + Qdy$  is a total differential.*

To show that the integral of  $Pdx + Qdy$  around a closed path is zero if  $P'_y = Q'_x$ , consider first a region  $R$  such that any point  $(x, y)$  of it may

\* See § 52. In particular observe the comments there made relative to differentials which are or which are not exact. This difference corresponds to integrals which are and which are not independent of the path.

be reached from  $(a, b)$  by following the lines  $y = b$  and  $x = x$ . Then define the function  $F(x, y)$  as

$$F(x, y) = \int_a^x P(x, b) dx + \int_b^y Q(x, y) dy \quad (19)$$



for all points of that region  $R$ . Now

$$\frac{\partial F}{\partial y} = Q(x, y), \quad \frac{\partial F}{\partial x} = P(x, b) + \frac{\partial}{\partial x} \int_b^y Q(x, y) dy.$$

But 
$$\frac{\partial}{\partial x} \int_b^y Q(x, y) dy = \int_b^y \frac{\partial Q}{\partial x} dy = \int_b^y \frac{\partial P}{\partial y} dy = P(x, y) \Big|_b^y.$$

This results from Leibniz's rule (4') of § 119, which may be applied since  $Q'_x$  is by hypothesis continuous, and from the assumption  $Q'_x = P'_y$ .

Then

$$\frac{\partial F}{\partial x} = P(x, b) + P(x, y) - P(x, b) = P(x, y).$$

Hence it follows that, within the region specified,  $Pdx + Qdy$  is the total differential of the function  $F(x, y)$  defined by (19). Hence along any closed circuit within that region  $R$  the integral of  $Pdx + Qdy$  is the integral of  $dF$  and vanishes.

It remains to remove the restriction on the type of region within which the integral around a closed path vanishes. Consider any closed path  $C$  which lies within the region over which  $P'_y$  and  $Q'_x$  are equal continuous functions of  $(x, y)$ . As the path lies wholly within  $R$  it is possible to rule  $R$  so finely that any little rectangle which contains a portion of the path shall lie wholly within  $R$ . The reader may construct his own figure, possibly with reference to that of § 128, where a finer ruling would be needed. The path  $C$  may thus be surrounded by a zigzag line which lies within  $R$ . Each of the small rectangles within the zigzag line is a region of the type above considered and, by the proof above given, the integral around any closed curve within the small rectangle must be zero. Now the circuit  $C$  may be replaced by the totality of small circuits consisting either of the perimeters of small rectangles lying wholly within  $C$  or of portions of the curve  $C$  and portions of the perimeters of such rectangles as contain parts of  $C$ . And if  $C$  be so replaced, the integral around  $C$  is resolved into the sum of a large number of integrals about these small circuits; for the integrals along such parts of the small circuits as are portions of the perimeters of the rectangles occur in pairs with opposite signs.\* Hence the integral around  $C$  is zero, where  $C$  is any circuit within  $R$ . Hence the integral of  $Pdx + Qdy$  from  $(a, b)$  to  $(x, y)$  is independent of the path and defines a function  $F(x, y)$  of which  $Pdx + Qdy$  is the total differential. As this function is continuous, its value for points on the boundary of  $R$  may be defined as the limit of  $F(x, y)$  as  $(x, y)$  approaches a point of the boundary, and it may thereby be seen that the line integral of (18) around the boundary is also 0 without any further restriction than that  $P'_y$  and  $Q'_x$  be equal and continuous within the boundary.

\* See Ex. 10 above. It is well, in connection with §§ 123-125, to read carefully the work of §§ 44-45 dealing with varieties of regions, reducibility of circuits, etc.

It should be noticed that *the line integral*

$$\int_{a,b}^{x,y} Pdx + Qdy = \int_a^x P(x, b) dx + \int_b^y Q(x, y) dy, \quad (19)$$

when  $Pdx + Qdy$  is an exact differential, that is, when  $P'_y = Q'_x$ , may be evaluated by the rule given for integrating an exact differential (p. 209), provided the path along  $y = b$  and  $x = x$  does not go outside the region. If that path should cut out of  $R$ , some other method of evaluation would be required. It should, however, be borne in mind that  $Pdx + Qdy$  is best integrated by inspection whenever the function  $F$ , of which  $Pdx + Qdy$  is the differential, can be recognized; if  $F$  is multiple valued, the consideration of the path may be required to pick out the particular value which is needed. It may be added that the work may be extended to line integrals in space without any material modifications.

It was seen (§ 73) that the conditions that the complex function

$$F(x, y) = X(x, y) + iY(x, y), \quad z = x + iy,$$

be a function of the complex variable  $z$  are

$$X'_y = -Y'_x \text{ and } X'_x = Y'_y. \quad (20)$$

If these conditions be applied to the expression (13),

$$\int F(x, y) = \int_{a,b}^{x,y} Xdx - Ydy + i \int_{a,b}^{x,y} Ydx + Xdy,$$

for the line integral of such a function, it is seen that they are precisely the conditions (18) that each of the line integrals entering into the complex line integral shall be independent of the path. Hence *the integral of a function of a complex variable is independent of the path of integration in the complex plane, and the integral around a closed path vanishes.* This applies of course only to simply connected regions of the plane throughout which the derivatives in (20) are equal and continuous.

If the notations of vectors in three dimensions be adopted,

$$\int Xdx + Ydy + Zdz = \int \mathbf{F} \cdot d\mathbf{r},$$

where  $\mathbf{F} = Xi + Yj + Zk, \quad d\mathbf{r} = i dx + j dy + k dz.$

In the particular case where the integrand is an exact differential and the integral around a closed path is zero,

$$Xdx + Ydy + Zdz = \mathbf{F} \cdot d\mathbf{r} = dU = d\mathbf{r} \cdot \nabla U,$$

where  $U$  is the function defined by the integral (for  $\nabla U$  see p. 172). When  $\mathbf{F}$  is interpreted as a force, the function  $V = -U$  such that

$$\mathbf{F} = -\nabla V \quad \text{or} \quad X = -\frac{\partial V}{\partial x}, \quad Y = -\frac{\partial V}{\partial y}, \quad Z = -\frac{\partial V}{\partial z}$$

is called the potential function of the force  $\mathbf{F}$ . *The negative of the slope of the potential function is the force  $\mathbf{F}$  and the negatives of the partial derivatives are the component forces along the axes.*

If the forces are such that they are thus derivable from a potential function, they are said to be *conservative*. In fact if

$$m \frac{d^2 \mathbf{r}}{dt^2} = \mathbf{F} = -\nabla V, \quad m \frac{d^2 \mathbf{r}}{dt^2} \cdot d\mathbf{r} = -d\mathbf{r} \cdot \nabla V = -dV,$$

and 
$$\int_{r_0}^{r_1} m \frac{d^2 \mathbf{r}}{dt^2} \cdot d\mathbf{r} = \frac{m}{2} \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} \Big|_{r_0}^{r_1} = -V \Big|_{r_0}^{r_1},$$

or 
$$\frac{m}{2} (v_1^2 - v_0^2) = V_0 - V_1 \quad \text{or} \quad \frac{m}{2} v_1^2 + V_1 = \frac{m}{2} v_0^2 + V_0.$$

Thus the sum of the kinetic energy  $\frac{1}{2}mv^2$  and the potential energy  $V$  is the same at all times or positions. This is the principle of the *conservation of energy* for the simple case of the motion of a particle when the force is conservative. In case the force is not conservative the integration may still be performed as

$$\frac{m}{2} (v_1^2 - v_0^2) = \int_{r_0}^{r_1} \mathbf{F} \cdot d\mathbf{r} = W,$$

where  $W$  stands for the work done by the force  $\mathbf{F}$  during the motion. The result is that the change in kinetic energy is equal to the work done by the force; but  $dW$  is then not an exact differential and the work must not be regarded as a function of  $(x, y, z)$ ,—it depends on the path. The generalization to any number of particles as in § 123 is immediate.

**125.** The conditions that  $P'_y$  and  $Q'_x$  be continuous and equal, which insures independence of the path for the line integral of  $Pdx + Qdy$ , need to be examined more closely. Consider two examples:

First 
$$\int Pdx + Qdy = \int \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy,$$

where 
$$\frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad \frac{\partial Q}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

It appears formally that  $P'_y = Q'_x$ . If the integral be calculated around a square of side  $2a$  surrounding the origin, the result is

$$\begin{aligned} & \int_{-a}^{+a} \frac{+a}{x^2 + a^2} dx + \int_{-a}^{+a} \frac{ady}{a^2 + y^2} + \int_{+a}^{-a} \frac{-a}{x^2 + a^2} dx + \int_{+a}^{-a} \frac{-ady}{a^2 + y^2} = 2 \int_{-a}^{+a} \frac{adx}{x^2 + a^2} \\ & + 2 \int_{-a}^{+a} \frac{ady}{a^2 + y^2} = 4 \int_{-a}^{+a} \frac{ad\xi}{\xi^2 + a^2} = 4 \frac{\pi}{2} = 2\pi \neq 0. \end{aligned}$$

The integral fails to vanish around the closed path. The reason is not far to seek, the derivatives  $P'_y$  and  $Q'_x$  are not defined for  $(0, 0)$ , and cannot be so defined as to be continuous functions of  $(x, y)$  near the origin. As a matter of fact

$$\int_{a,b}^{x,y} \frac{y dx - x dy}{x^2 + y^2} + \frac{x dy}{x^2 + y^2} = \int_{a,b}^{x,y} d \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{y}{x} \Big|_{a,b}^{x,y},$$

and  $\tan^{-1}(y/x)$  is not a single valued function; it takes on the increment  $2\pi$  when one traces a path surrounding the origin (§ 45).

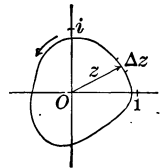
Another illustration may be found in the integral

$$\int \frac{dz}{z} = \int \frac{dx + idy}{x + iy} = \int \frac{x dx + y dy}{x^2 + y^2} + i \int \frac{-y dx + x dy}{x^2 + y^2}$$

taken along a path in the complex plane. At the origin  $z = 0$  the integrand  $1/z$  becomes infinite and so do the partial derivatives of its real and imaginary parts. If the integral be evaluated around a path passing once about the origin, the result is

$$\int_{\circ} \frac{dz}{z} = \left[ \frac{1}{2} \log(x^2 + y^2) + i \tan^{-1} \frac{y}{x} \right]_{a,b}^{x,y} = 2\pi i. \quad (21)$$

In this case, as in the previous, the integral would necessarily be zero about any closed path which did not include the origin; for then the conditions for absolute independence of the path would be satisfied. Moreover the integrals around two different paths each encircling the origin once would be equal; for the paths may be considered as one single closed circuit by joining them with a line as in the device (§ 44) for making a multiply connected region simply connected, the integral around the complete circuit is zero, the parts due to the description of the line in the two directions cancel, and the integrals around the two given circuits taken in opposite directions are therefore equal and opposite. (Compare this work with the multiple valued nature of  $\log z$ , p. 161.)



Suppose in general that  $P(x, y)$  and  $Q(x, y)$  are single valued functions which have the first partial derivatives  $P'_y$  and  $Q'_x$  continuous and equal over a region  $R$  except at certain points  $A, B, \dots$ . Surround these points with small circuits. The remaining portion of  $R$  is such that  $P'_y$  and  $Q'_x$  are everywhere equal and continuous; but the region is not simply connected, that is, it is possible to draw in the region circuits which cannot be shrunk down to a point, owing to the fact that the circuit may surround one or more of the regions which have been cut out. If a circuit can be shrunk down to a point, that is, if it is not inextricably wound about one or more of the deleted portions, the integral around the circuit will vanish; for the previous reasoning will apply. But if the circuit coils about one or more of the deleted regions so that the attempt to shrink it down leads to a circuit which consists of the contours of these regions and of lines joining them, the integral need not vanish; it reduces to the sum of a number of integrals

taken around the contours of the deleted portions. If one circuit can be shrunk into another, the integrals around the two circuits are equal if the direction of description is the same; for a line connecting the two circuits will give a combined circuit which can be shrunk down to a point.

The inference from these various observations is that in a multiply connected region the integral around a circuit need not be zero and the integral from a fixed lower limit  $(a, b)$  to a variable upper limit  $(x, y)$  may not be absolutely independent of the path, but may be different along two paths which are so situated relatively to the excluded regions that the circuit formed of the two paths from  $(a, b)$  to  $(x, y)$  cannot be shrunk down to a point. Hence

$$F(x, y) = \int_{a, b}^{x, y} Pdx + Qdy, \quad P'_y = Q'_x \text{ (generally),}$$

the function defined by the integral, is not necessarily single valued. Nevertheless, any two values of  $F(x, y)$  for the same end point will differ only by a sum of the form

$$F_2(x, y) - F_1(x, y) = m_1 I_1 + m_2 I_2 + \dots$$

where  $I_1, I_2, \dots$  are the values of the integral taken around the contours of the excluded regions and where  $m_1, m_2, \dots$  are positive or negative integers which represent the number of times the combined circuit formed from the two paths will coil around the deleted regions in one direction or the other.

**126.** Suppose that  $f(z) = X(x, y) + iY(x, y)$  is a single valued function of  $z$  over a region  $R$  surrounding the origin (see figure above), and that over this region the derivative  $f'(z)$  is continuous, that is, the relations  $X'_y = -Y'_x$  and  $X'_x = Y'_y$  are fulfilled at every point so that no points of  $R$  need be cut out. Consider the integral

$$\int_0 \frac{f(z)}{z} dz = \int_0 \frac{X + iY}{x + iy} (dx + i dy) \quad (22)$$

over paths lying within  $R$ . The function  $f(z)/z$  will have a continuous derivative at all points of  $R$  except at the origin  $z = 0$ , where the denominator vanishes. If then a small circuit, say a circle, be drawn about the origin, the function  $f(z)/z$  will satisfy the requisite conditions over the region which remains, and the integral (22) taken around a circuit which does not contain the origin will vanish.

The integral (22) taken around a circuit which coils once and only once about the origin will be equal to the integral taken around the

small circle about the origin. Now for the circle,

$$\int_{\odot} \frac{f(z)}{z} dz = \int_{\odot} \frac{f(0) + \eta(z)}{z} dz = f(0) \int_{\odot} \frac{dz}{z} + \int_{\odot} \frac{\eta}{z} dz,$$

where the assumed continuity of  $f(z)$  makes  $|\eta(z)| < \epsilon$  provided the circle about the origin is taken sufficiently small. Hence by (21)

$$\int_{\odot} \frac{f(z)}{z} dz = \int_{\odot} \frac{f(z)}{z} dz = 2\pi i f(0) + \xi$$

with  $|\xi| = \left| \int_{\odot} \frac{\eta}{z} dz \right| \equiv \int_{\odot} \left| \frac{\eta}{z} \right| |dz| \equiv \epsilon \int_{\odot} d\theta = 2\pi\epsilon.$

Hence the difference between (22) and  $2\pi i f(0)$  can be made as small as desired, and as (22) is a certain constant, the result is

$$\int_{\odot} \frac{f(z)}{z} dz = 2\pi i f(0). \quad (23)$$

A function  $f(z)$  which has a continuous derivative  $f'(z)$  at every point of a region is said to be *analytic* over that region. Hence if the region includes the origin, the value of the analytic function at the origin is given by the formula

$$f(0) = \frac{1}{2\pi i} \int_{\odot} \frac{f(z)}{z} dz, \quad (23')$$

where the integral is extended over any circuit lying in the region and passing just once about the origin. It follows likewise that if  $z = \alpha$  is any point within the region, then

$$f(\alpha) = \frac{1}{2\pi i} \int_{\odot} \frac{f(z)}{z - \alpha} dz, \quad (24)$$

where the circuit extends once around the point  $\alpha$  and lies wholly within the region. This important result is due to Cauchy.

A more convenient form of (24) is obtained by letting  $t = z$  represent the value of  $z$  along the circuit of integration and then writing  $\alpha = z$  and regarding  $z$  as variable. Hence Cauchy's Integral:

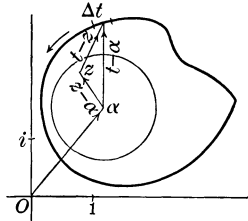
$$f(z) = \frac{1}{2\pi i} \int_{\odot} \frac{f(t)}{t - z} dt. \quad (25)$$

This states that *if any circuit be drawn in the region over which  $f(z)$  is analytic, the value of  $f(z)$  at all points within that circuit may be obtained by evaluating Cauchy's Integral (25)*. Thus  $f(z)$  may be regarded



as defined by an integral containing a parameter  $z$ ; for many purposes this is convenient. It may be remarked that when the values of  $f(z)$  are given along any circuit, the integral may be regarded as defining  $f(z)$  for all points within that circuit.

To find the successive derivatives of  $f(z)$ , it is merely necessary to differentiate with respect to  $z$  under the sign of integration. The conditions of continuity which are required to justify the differentiation are satisfied for all points  $z$  actually within the circuit and not upon it. Then



$$f'(z) = \frac{1}{2\pi i} \int_{\circ} \frac{f(t)}{(t-z)^2} dt, \dots, f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_{\circ} \frac{f(t)}{(t-z)^n} dt.$$

As the differentiations may be performed, these formulas show that an analytic function has continuous derivatives of all orders. The definition of the function only required a continuous first derivative.

Let  $\alpha$  be any particular value of  $z$  (see figure). Then

$$\begin{aligned} \frac{1}{t-z} &= \frac{1}{(t-\alpha) - (z-\alpha)} = \frac{1}{t-\alpha} \frac{1}{1 - \frac{z-\alpha}{t-\alpha}} \\ &= \frac{1}{t-\alpha} \left[ 1 + \frac{z-\alpha}{t-\alpha} + \frac{(z-\alpha)^2}{(t-\alpha)^2} + \dots + \frac{(z-\alpha)^{n-1}}{(t-\alpha)^{n-1}} + \frac{(z-\alpha)^n}{(t-\alpha)^n} \right] \end{aligned}$$

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\circ} \frac{f(t)}{t-z} dt = \frac{1}{2\pi i} \int_{\circ} \frac{f(t)}{t-\alpha} dt + \frac{1}{2\pi i} \int_{\circ} (z-\alpha) \frac{f(t)}{(t-\alpha)^2} dt \\ &+ \frac{1}{2\pi i} \int_{\circ} (z-\alpha)^2 \frac{f(t)}{(t-\alpha)^3} dt + \dots + \frac{1}{2\pi i} \int_{\circ} (z-\alpha)^{n-1} \frac{f(t)}{(t-\alpha)^n} dt + R_n, \end{aligned}$$

with 
$$R_n = \frac{1}{2\pi i} \int_{\circ} \frac{(z-\alpha)^n}{(t-\alpha)^n} \frac{1}{1 - \frac{z-\alpha}{t-\alpha}} \frac{f(t)}{t-\alpha} dt.$$

Now  $t$  is the variable of integration and  $z-\alpha$  is a constant with respect to the integration. Hence

$$\begin{aligned} f(z) &= f(\alpha) + (z-\alpha)f'(\alpha) + \frac{(z-\alpha)^2}{2!} f''(\alpha) \\ &+ \dots + \frac{(z-\alpha)^{n-1}}{(n-1)!} f^{(n-1)}(\alpha) + R_n. \end{aligned} \tag{26}$$

This is Taylor's Formula for a function of a complex variable.

## EXERCISES

1. If  $P'_y = Q'_x$ ,  $Q'_z = R'_y$ ,  $R'_x = P'_z$  and if these derivatives are continuous, show that  $Pdx + Qdy + Rdz$  is a total differential.

2. Show that  $\int_C \int_{a,b}^{x,y} P(x, y, \alpha) dx + Q(x, y, \alpha) dy$ , where  $C$  is a given curve, defines a continuous function of  $\alpha$ , the derivative of which may be found by differentiating under the sign. What assumptions as to the continuity of  $P, Q, P'_\alpha, Q'_\alpha$  do you make?

3. If  $\log z = \int_1^z \frac{dz}{z} = \int_{1,0}^{x,y} \frac{x dx + y dy}{x^2 + y^2} + i \int_{1,0}^{x,y} \frac{-y dx + x dy}{x^2 + y^2}$  be taken as the definition of  $\log z$ , draw paths which make  $\log(\frac{1}{2} + \frac{1}{2}\sqrt{-3}) = \frac{1}{3}\pi i, 2\frac{1}{3}\pi i, -1\frac{2}{3}\pi i$ .

4. Study  $\int_0^z \frac{3z-1}{z^2-1}$  with especial reference to closed paths which surround  $+1, -1$ , or both. Draw a closed path surrounding both and making the integral vanish.

5. If  $f(z)$  is analytic for all values of  $z$  and if  $|f(z)| < K$ , show that

$$f(z) - f(0) = \int_0^z f(t) \left[ \frac{1}{t-z} - \frac{1}{t} \right] dt = \int_0^z \frac{zf(t)}{(t-z)t} dt,$$

taken over a circle of large radius, can be made as small as desired. Hence infer that  $f(z)$  must be the constant  $f(z) = f(0)$ .

6. If  $G(z) = a_0 + a_1z + \dots + a_nz^n$  is a polynomial, show that  $f(z) = 1/G(z)$  must be analytic over any region which does not include a root of  $G(z) = 0$  either within or on its boundary. Show that the assumption that  $G(z) = 0$  has no roots at all leads to the conclusion that  $f(z)$  is constant and equal to zero. Hence infer that an algebraic equation has a root.

7. Show that the absolute value of the remainder in Taylor's Formula is

$$|R_n| = \frac{|z - \alpha|^n}{2\pi} \left| \int_0^1 \frac{f(t) dt}{(t - \alpha)^n (t - z)} \right| \leq \frac{1}{2\pi} \frac{r^n ML}{\rho^n \rho - r}$$

for all points  $z$  within a circle of radius  $r$  about  $\alpha$  as center, when  $\rho$  is the radius of the largest circle concentric with  $\alpha$  which can be drawn within the circuit about which the integral is taken,  $M$  is the maximum value of  $f(t)$  upon the circuit, and  $L$  is the length of the circuit (figure above).

8. Examine for independence of path and in case of independence integrate:

$$(\alpha) \int x^2 y dx + x y^2 dy, \quad (\beta) \int x y^2 dx + x^2 y dy, \quad (\gamma) \int x dy + y dx,$$

$$(\delta) \int (x^2 + xy) dx + (y^2 + xy) dy, \quad (\epsilon) \int y \cos x dy + \frac{1}{2} y^2 \sin x dx.$$

9. Find the conservative forces and the potential:

$$(\alpha) X = \frac{x}{(x^2 + y^2)^{\frac{3}{2}}}, \quad Y = \frac{y}{(x^2 + y^2)^{\frac{3}{2}}}, \quad Z = \frac{z}{(x^2 + y^2)^{\frac{3}{2}}},$$

$$(\beta) X = -nx, \quad Y = -ny, \quad (\gamma) X = 1/x, \quad Y = y/x.$$

10. If  $R(r, \phi)$  and  $\Phi(r, \phi)$  are the component forces resolved along the radius vector and perpendicular to the radius, show that  $dW = Rdr + r\Phi d\phi$  is the differential of work, and express the condition that the forces  $R, \Phi$  be conservative.

11. Show that if a particle is acted on by a force  $R = -f(r)$  directed toward the origin and a function of the distance from the origin, the force is conservative.

12. If a force follows the Law of Nature, that is, acts toward a point and varies inversely as the square  $r^2$  of the distance from the point, show that the potential is  $-k/r$ .

13. From the results  $\mathbf{F} = -\nabla V$  or  $V = -\int \mathbf{F} \cdot d\mathbf{r} = \int Xdx + Ydy + Zdz$  show that if  $V_1$  is the potential of  $\mathbf{F}_1$  and  $V_2$  of  $\mathbf{F}_2$  then  $V = V_1 + V_2$  will be the potential of  $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ , that is, show that for conservative forces the addition of potentials is equivalent to the parallelogram law for adding forces.

14. If a particle is acted on by a retarding force  $-k\mathbf{v}$  proportional to the velocity, show that  $R = \frac{1}{2}kv^2$  is a function such that

$$-\frac{\partial R}{\partial v_x} = -kv_x, \quad \frac{\partial R}{\partial v_y} = -kv_y, \quad \frac{\partial R}{\partial v_z} = -kv_z, \\ dW = -k\mathbf{v} \cdot d\mathbf{r} = -k(v_x dx + v_y dy + v_z dz).$$

Here  $R$  is called the dissipative function; show the force is not conservative.

15. Pick out the integrals independent of the path and integrate:

$$(\alpha) \int yzdx + xzdy + xydz, \quad (\beta) \int ydx/z + xdy/z - xydz/z^2, \\ (\gamma) \int xyz(dx + dy + dz), \quad (\delta) \int \log(xy) dz + xdy + ydz.$$

16. Obtain logarithmic forms for the inverse trigonometric functions, analogous to those for the inverse hyperbolic functions, either algebraically or by considering the inverse trigonometric functions as defined by integrals as

$$\tan^{-1}z = \int_0^z \frac{dz}{1+z^2}, \quad \sin^{-1}z = \int_0^z \frac{dz}{\sqrt{1-z^2}}, \dots$$

17. Integrate these functions of the complex variable directly according to the rules of integration for reals and determine the values of the integrals by substitution:

$$(\alpha) \int_0^{1+i} ze^{2z^2} dz, \quad (\beta) \int_0^{2i} \cos 3z dz, \quad (\gamma) \int_1^{-1+i} (1+z^2)^{-1} dz, \\ (\delta) \int_0^{1+i} \frac{dz}{\sqrt{1-z^2}}, \quad (\epsilon) \int_i^2 \frac{dz}{z\sqrt{z^2-1}}, \quad (\zeta) \int_{-1}^{-2-i} \frac{dz}{\sqrt{1+z^2}}.$$

In the case of multiple valued functions mark two different paths and give two values.

18. Can the algorithm of integration by parts be applied to the definite (or indefinite) integral of a function of a complex variable, it being understood that the integral must be a line integral in the complex plane? Consider the proof of Taylor's Formula by integration by parts, p. 57, to ascertain whether the proof is valid for the complex plane and what the remainder means.

**19.** Suppose that in a plane at  $r = 0$  there is a particle of mass  $m$  which *attracts* according to the law  $F = m/r$ . Show that the potential is  $V = m \log r$ , so that  $\mathbf{F} = -\nabla V$ . The *induction* or *flux* of the force  $\mathbf{F}$  outward across the element  $ds$  of a curve in the plane is by definition  $-F \cos(F, n) ds$ . By reference to Ex. 16, p. 297, show that the total induction or flux of  $\mathbf{F}$  across the curve is the line integral (along the curve)

$$-\int F \cos(F, n) ds = m \int \frac{d \log r}{dn} ds = \int \frac{dV}{dn} ds;$$

and 
$$m = \frac{-1}{2\pi} \int_{\circlearrowleft} F \cos(F, n) ds = \frac{1}{2\pi} \int_{\circlearrowleft} \frac{dV}{dn} ds,$$

where the circuit extends around the point  $r = 0$ , is a formula for obtaining the mass  $m$  within the circuit from the field of force  $\mathbf{F}$  which is set up by the mass.

**20.** Suppose a number of masses  $m_1, m_2, \dots$ , attracting as in Ex. 19, are situated at points  $(\xi_1, \eta_1), (\xi_2, \eta_2), \dots$  in the plane. Let

$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \dots$ ,  $V = V_1 + V_2 + \dots$ ,  $V_i = m_i \log [(\xi_i - x)^2 + (\eta_i - y)^2]^{\frac{1}{2}}$  be the force and potential at  $(x, y)$  due to the masses. Show that

$$\frac{-1}{2\pi} \int_{\circlearrowleft} F \cos(F, n) ds = \frac{1}{2\pi} \sum \int_{\circlearrowleft} \frac{dV}{dn} ds = \sum' m_i = M,$$

where  $\Sigma$  extends over all the masses and  $\Sigma'$  over all the masses within the circuit (none being on the circuit), gives the total mass  $M$  within the circuit.

**127. Some critical comments.** In the discussion of line integrals and in the future discussion of double integrals it is necessary to speak frequently of curves. For the usual problem the intuitive conception of a curve suffices. A curve as ordinarily conceived is continuous, has a continuously turning tangent line except perhaps at a finite number of angular points, and is cut by a line parallel to any given direction in only a finite number of points, except as a portion of the curve may coincide with such a line. The ideas of length and area are also applicable. For those, however, who are interested in more than the intuitive presentation of the idea of a curve and some of the matters therewith connected, the following sections are offered.

If  $\phi(t)$  and  $\psi(t)$  are two single valued real functions of the real variable  $t$  defined for all values in the interval  $t_0 \leq t \leq t_1$ , the pair of equations

$$x = \phi(t), \quad y = \psi(t), \quad t_0 \leq t \leq t_1, \quad (27)$$

will be said to define a *curve*. If  $\phi$  and  $\psi$  are continuous functions of  $t$ , the curve will be called continuous. If  $\phi(t_1) = \phi(t_0)$  and  $\psi(t_1) = \psi(t_0)$ , so that the initial and end points of the curve coincide, the curve will be called a *closed* curve provided it is continuous. If there is no other pair of values  $t$  and  $t'$  which make both  $\phi(t) = \phi(t')$  and  $\psi(t) = \psi(t')$ , the curve will be called *simple*; in ordinary language, the curve does not cut itself. If  $t$  describes the interval from  $t_0$  to  $t_1$  continuously and constantly in the same sense, the point  $(x, y)$  will be said to describe the curve in a given sense; the opposite sense can be had by allowing  $t$  to describe the interval in the opposite direction.

Let the interval  $t_0 \leq t \leq t_1$  be divided into any number  $n$  of subintervals  $\Delta_1 t, \Delta_2 t, \dots, \Delta_n t$ . There will be  $n$  corresponding increments for  $x$  and  $y$ ,

$$\Delta_1 x, \Delta_2 x, \dots, \Delta_n x, \quad \text{and} \quad \Delta_1 y, \Delta_2 y, \dots, \Delta_n y.$$

Then  $\Delta_i c = \sqrt{(\Delta_i x)^2 + (\Delta_i y)^2} \leq |\Delta_i x| + |\Delta_i y|$ ,  $|\Delta_i x| \leq \Delta_i c$ ,  $|\Delta_i y| \leq \Delta_i c$

are obvious inequalities. It will be necessary to consider the three sums

$$\sigma_1 = \sum_1^n |\Delta_i x|, \quad \sigma_2 = \sum_1^n |\Delta_i y|, \quad \sigma_3 = \sum_1^n \Delta_i c = \sum_1^n \sqrt{(\Delta_i x)^2 + (\Delta_i y)^2}.$$

For any division of the interval from  $t_0$  to  $t_1$  each of these sums has a definite positive value. When all possible modes of division are considered for any and every value of  $n$ , the sums  $\sigma_1$  will form an infinite set of numbers which may be either limited or unlimited above (§ 22). In case the set is limited, the upper frontier of the set is called the *variation of  $x$*  over the curve and the curve is said to be of *limited variation in  $x$* ; in case the set is unlimited, the curve is of *unlimited variation in  $x$* . Similar observations for the sums  $\sigma_2$ . It may be remarked that the geometric conception corresponding to the variation in  $x$  is the sum of the projections of the curve on the  $x$ -axis when the sum is evaluated arithmetically and not algebraically. Thus the variation in  $y$  for the curve  $y = \sin x$  from 0 to  $2\pi$  is 4. The curve  $y = \sin(1/x)$  between these same limits is of unlimited variation in  $y$ . In both cases the variation in  $x$  is  $2\pi$ .

If both the sums  $\sigma_1$  and  $\sigma_2$  have upper frontiers  $L_1$  and  $L_2$ , the sum  $\sigma_3$  will have an upper frontier  $L_3 \leq L_1 + L_2$ ; and conversely if  $\sigma_3$  has an upper frontier, both  $\sigma_1$  and  $\sigma_2$  will have upper frontiers. If a new point of division is intercalated in  $\Delta_i t$ , the sum  $\sigma_1$  cannot decrease and, moreover, it cannot increase by more than twice the oscillation of  $x$  in the interval  $\Delta_i t$ . For if  $\Delta_1 i x + \Delta_2 i x = \Delta_i x$ , then

$$|\Delta_1 i x| + |\Delta_2 i x| \geq |\Delta_i x|, \quad |\Delta_1 i x| + |\Delta_2 i x| \leq 2(M_i - m_i).$$

Here  $\Delta_1 i t$  and  $\Delta_2 i t$  are the two intervals into which  $\Delta_i t$  is divided, and  $M_i - m_i$  is the oscillation in the interval  $\Delta_i t$ . A similar theorem is true for  $\sigma_2$ . It now remains to show that if the interval from  $t_0$  to  $t_1$  is divided sufficiently fine, the sums  $\sigma_1$  and  $\sigma_2$  will differ by as little as desired from their frontiers  $L_1$  and  $L_2$ . The proof is like that of the similar problem of § 28. First, the fact that  $L_1$  is the frontier of  $\sigma_1$  shows that some method of division can be found so that  $L_1 - \sigma_1 < \frac{1}{2}\epsilon$ . Suppose the number of points of division is  $n$ . Let it next be assumed that  $\phi(t)$  is continuous; it must then be uniformly continuous (§ 25), and hence it is possible to find a  $\delta$  so small that when  $\Delta_i t < \delta$  the oscillation of  $x$  is  $M_i - m_i < \epsilon/4n$ . Consider then any method of division for which  $\Delta_i t < \delta$ , and its sum  $\sigma'_1$ . The superposition of the former division with  $n$  points upon this gives a sum  $\sigma''_1 \geq \sigma'_1$ . But  $\sigma''_1 - \sigma'_1 < 2n\epsilon/4n = \frac{1}{2}\epsilon$ , and  $\sigma'_1 \geq \sigma_1$ . Hence  $L_1 - \sigma''_1 < \frac{1}{2}\epsilon$  and  $L_1 - \sigma'_1 < \epsilon$ . A similar demonstration may be given for  $\sigma_2$  and  $L_2$ .

To treat the sum  $\sigma_3$  and its upper frontier  $L_3$  note that here, too, the intercalation of an additional point of division cannot decrease  $\sigma_3$  and, as

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} \leq |\Delta x| + |\Delta y|,$$

it cannot increase  $\sigma_3$  by more than twice the sum of the oscillations of  $x$  and  $y$  in the interval  $\Delta t$ . Hence if the curve is continuous, that is, if both  $x$  and  $y$  are continuous, the division of the interval from  $t_0$  to  $t_1$  can be taken so fine that  $\sigma_3$  shall

differ from its upper frontier  $L_3$  by less than any assigned quantity, no matter how small. In this case  $L_3 = s$  is called the *length of the curve*. It is therefore seen that *the necessary and sufficient condition that any continuous curve shall have a length is that its Cartesian coördinates  $x$  and  $y$  shall both be of limited variation*. It is clear that if the frontiers  $L_1(t)$ ,  $L_2(t)$ ,  $L_3(t)$  from  $t_0$  to any value of  $t$  be regarded as functions of  $t$ , they are continuous and nondecreasing functions of  $t$ , and that  $L_3(t)$  is an increasing function of  $t$ ; it would therefore be possible to take  $s$  in place of  $t$  as the parameter for any continuous curve having a length. Moreover if the derivatives  $x'$  and  $y'$  of  $x$  and  $y$  with respect to  $t$  exist and are continuous, the derivative  $s'$  exists, is continuous, and is given by the usual formula  $s' = \sqrt{x'^2 + y'^2}$ . This will be left as an exercise; so will the extension of these considerations to three dimensions or more.

In the sum  $x_1 - x_0 = \Sigma \Delta_i x$  of the actual, not absolute, values of  $\Delta_i x$  there may be both positive and negative terms. Let  $\pi$  be the sum of the positive terms and  $\nu$  be the sum of the negative terms. Then

$$x_1 - x_0 = \pi - \nu, \quad \sigma_1 = \pi + \nu, \quad 2\pi = x_1 - x_0 + \sigma_1, \quad 2\nu = x_0 - x_1 + \sigma_1.$$

As  $\sigma_1$  has an upper frontier  $L_1$  when  $x$  is of limited variation, and as  $x_0$  and  $x_1$  are constants, the sums  $\pi$  and  $\nu$  have upper frontiers. Let these be  $\Pi$  and  $N$ . Considered as functions of  $t$ , neither  $\Pi(t)$  nor  $N(t)$  can decrease. Write  $x(t) = x_0 + \Pi(t) - N(t)$ . Then the function  $x(t)$  of limited variation has been resolved into the difference of two functions each of limited variation and nondecreasing. As a limited nondecreasing function is integrable (Ex. 7, p. 54), this shows that *a function is integrable over any interval over which it is of limited variation*. That the difference  $x = x'' - x'$  of two limited and nondecreasing functions must be a function of limited variation follows from the fact that  $|\Delta x| \leq |\Delta x''| + |\Delta x'|$ . Furthermore if

$$x = x_0 + \Pi - N \quad \text{be written} \quad x = [x_0 + \Pi + |x_0| + t - t_0] - [N + |x_0| + t - t_0],$$

it is seen that *a function of limited variation can be regarded as the difference of two positive functions which are constantly increasing, and that these functions are continuous if the given function  $x(t)$  is continuous*.

Let the curve  $C$  defined by the equations  $x = \phi(t)$ ,  $y = \psi(t)$ ,  $t_0 \leq t \leq t_1$ , be continuous. Let  $P(x, y)$  be a continuous function of  $(x, y)$ . Form the sum

$$\sum P(\xi_i, \eta_i) \Delta_i x = \sum P(\xi_i, \eta_i) \Delta_i x'' - \sum P(\xi_i, \eta_i) \Delta_i x', \quad (28)$$

where  $\Delta_1 x, \Delta_2 x, \dots$  are the increments corresponding to  $\Delta_1 t, \Delta_2 t, \dots$ , where  $(\xi_i, \eta_i)$  is the point on the curve which corresponds to some value of  $t$  in  $\Delta_i t$ , where  $x$  is assumed to be of limited variation, and where  $x''$  and  $x'$  are two continuous increasing functions whose difference is  $x$ . As  $x''$  (or  $x'$ ) is a continuous and constantly increasing function of  $t$ , it is true inversely (Ex. 10, p. 45) that  $t$  is a continuous and constantly increasing function of  $x''$  (or  $x'$ ). As  $P(x, y)$  is continuous in  $(x, y)$ , it is continuous in  $t$  and also in  $x''$  and  $x'$ . Now let  $\Delta_i t \doteq 0$ ; then  $\Delta_i x'' \doteq 0$  and  $\Delta_i x' \doteq 0$ . Also

$$\lim \sum P_i \Delta_i x'' = \int_{x_0''}^{x_1''} P dx'' \quad \text{and} \quad \lim \sum P_i \Delta_i x' = \int_{x_0'}^{x_1'} P dx'.$$

The limits exist and are integrals simply because  $P$  is continuous in  $x''$  or in  $x'$ . Hence the sum on the left of (28) has a limit and

$$\lim \sum P \Delta_i x = \int_C^{x_1} P dx = \int_{x_0''}^{x_1''} P dx'' - \int_{x_0'}^{x_1'} P dx'$$

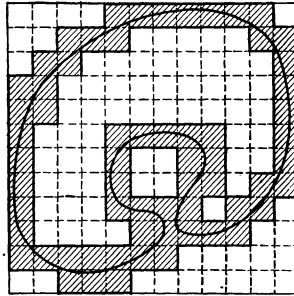
may be defined as the line integral of  $P$  along the curve  $C$  of limited variation in  $x$ . The assumption that  $y$  is of limited variation and that  $Q(x, y)$  is continuous would lead to a corresponding line integral. The assumption that both  $x$  and  $y$  are of limited variation, that is, that the curve is rectifiable, and that  $P$  and  $Q$  are continuous would lead to the existence of the line integral

$$\int_C^{x_1, y_1}_{x_0, y_0} P(x, y) dx + Q(x, y) dy.$$

A considerable theory of line integrals over general rectifiable curves may be constructed. The subject will not be carried further at this point.

**128.** The question of the area of a curve requires careful consideration. In the first place note that the intuitive closed plane curve which does cut itself is intuitively believed to divide the plane into two regions, one interior, one exterior to the curve; and these regions have the property that any two points of the same region may be connected by a continuous curve which does not cut the given curve, whereas any continuous curve which connects any point of one region to a point of the other must cut the given curve. The first question which arises with regard to the general closed simple curve of page 308 is: Does such a curve divide the plane into just two regions with the properties indicated, that is, is there an interior and exterior to the curve? The answer is affirmative, but the proof is somewhat difficult — not because the statement of the problem is involved or the proof replete with advanced mathematics, but rather because the statement is so simple and elementary that there is little to work with and the proof therefore requires the keenest and most tedious logical analysis. The theorem that a closed simple plane curve has an interior and an exterior will therefore be assumed.

As the functions  $x(t), y(t)$  which define the curve are continuous, they are limited, and it is possible to draw a rectangle with sides  $x = a, x = b, y = c, y = d$  so as entirely to surround the curve. This rectangle may next be ruled with a number of lines parallel to its sides, and thus be divided into smaller rectangles. These little rectangles may be divided into three categories, those outside the curve, those inside the curve, and those upon the curve. By one upon the curve is meant one which has so much as a single point of its perimeter or interior upon the curve. Let  $A, A_i, A_u, A_e$  denote the area of the large rectangle, the sum of the areas of the small rectangles, which are interior to the curve, the sum of the areas of those upon the curve, and the sum of those exterior to it. Of course  $A = A_i + A_u + A_e$ .



Now if all methods of ruling be considered, the quantities  $A_i$  will have an upper frontier  $L_i$ , the quantities  $A_e$  will have an upper frontier  $L_e$ , and the quantities  $A_u$  will have a lower frontier  $l_u$ . If to any method of ruling new rulings be added, the quantities  $A_i$  and  $A_e$  become  $A'_i$  and  $A'_e$  with the conditions  $A'_i \supseteq A_i, A'_e \supseteq A_e$ , and hence  $A'_u \supseteq A_u$ . From this it follows that  $A = L_i + l_u + L_e$ . For let there be three modes of ruling which for the respective cases  $A_i, A_e, A_u$  make these three quantities differ from their frontiers  $L_i, L_e, l_u$  by less than  $\frac{1}{3}\epsilon$ . Then the superposition of the three systems of rulings gives rise to a ruling for which  $A'_i, A'_e, A'_u$  must differ from the frontier values by less than

$\frac{1}{3}\epsilon$ , and hence the sum  $L_i + l_u + L_e$ , which is constant, differs from the constant  $A$  by less than  $\epsilon$ , and must therefore be equal to it.

It is now possible to *define as the (qualified) areas of the curve*

$$L_i = \text{inner area}, \quad l_u = \text{area on the curve}, \quad L_i + l_u = \text{total area}.$$

In the case of curves of the sort intuitively familiar, the limit  $l_u$  is zero and  $L_i = A - L_e$  becomes merely the (unqualified) area bounded by the curve. The question arises: Does the same hold for the general curve here under discussion? This time *the answer is negative*; for there are curves which, though closed and simple, are still so sinuous and meandering that a finite area  $l_u$  lies upon the curve, that is, there is a finite area so bestudded with points of the curve that no part of it is free from points of the curve. This fact again will be left as a statement without proof. Two further facts may be mentioned.

In the first place there is applicable a theorem like Theorem 21, p. 51, namely: It is possible to find a number  $\delta$  so small that, when the intervals between the rulings (both sets) are less than  $\delta$ , the sums  $A_u$ ,  $A_i$ ,  $A_e$  differ from their frontiers by less than  $2\epsilon$ . For there is, as seen above, some method of ruling such that these sums differ from their frontiers by less than  $\epsilon$ . Moreover, the adding of a single new ruling cannot change the sums by more than  $\Delta D$ , where  $\Delta$  is the largest interval and  $D$  the largest dimension of the rectangle. Hence if the total number of intervals (both sets) for the given method is  $N$  and if  $\delta$  be taken less than  $\epsilon/N\Delta D$ , the ruling obtained by superposing the given ruling upon a ruling where the intervals are less than  $\delta$  will be such that the sums differ from the given ones by less than  $\epsilon$ , and hence the ruling with intervals less than  $\delta$  can only give rise to sums which differ from their frontiers by less than  $2\epsilon$ .

In the second place it should be observed that the limits  $L_i$ ,  $l_u$  have been obtained by means of all possible modes of ruling where the rules were parallel to the  $x$ - and  $y$ -axes, and that there is no *a priori* assurance that these same limits would have been obtained by rulings parallel to two other lines of the plane or by covering the plane with a network of triangles or hexagons or other figures. In any thorough treatment of the subject of area such matters would have to be discussed. That the discussion is not given here is due entirely to the fact that these critical comments are given not so much with the desire to establish certain theorems as with the aim of showing the reader the sort of questions which come up for consideration in the rigorous treatment of such elementary matters as "the area of a plane curve," which he may have thought he "knew all about."

It is a common intuitive conviction that if a region like that formed by a square be divided into two regions by a continuous curve which runs across the square from one point of the boundary to another, the area of the square and the sum of the areas of the two parts into which it is divided are equal, that is, the curve (counted twice) and the two portions of the perimeter of the square form two simple closed curves, and it is expected that the sum of the areas of the curves is the area of the square. Now in case the curve is such that the frontiers  $l_u$  and  $l'_u$  formed for the two curves are not zero, it is clear that the sum  $L_i + L'_i$  for the two curves will not give the area of the square but a smaller area, whereas the sum  $(L_i + l_u) + (L'_i + l'_u)$  will give a greater area. Moreover in this case, it is not easy to formulate a general definition of area applicable to each of the regions and such that the sum of the areas shall be equal to the area of the combined region. But if  $l_u$  and  $l'_u$  both vanish, then the sum  $L_i + L'_i$  does give the combined area.



It is therefore customary to *restrict the application of the term "area" to such simple closed curves as have  $l_u = 0$ , and to say that the quadrature of such curves is possible*, but that the quadrature of curves for which  $l_u \neq 0$  is impossible.

It may be proved that : *If a curve is rectifiable or even if one of the functions  $x(t)$  or  $y(t)$  is of limited variation, the limit  $l_u$  is zero and the quadrature of the curve is possible.* For let the interval  $t_0 \leqq t \leqq t_1$  be divided into intervals  $\Delta_1 t, \Delta_2 t, \dots$  in which the oscillations of  $x$  and  $y$  are  $\epsilon_1, \epsilon_2, \dots, \eta_1, \eta_2, \dots$ . Then the portion of the curve due to the interval  $\Delta_i t$  may be inscribed in a rectangle  $\epsilon_i \eta_i$ , and that portion of the curve will lie wholly within a rectangle  $2\epsilon_i \cdot 2\eta_i$  concentric with this one. In this way may be obtained a set of rectangles which entirely contain the curve. The total area of these rectangles must exceed  $l_u$ . For if all the sides of all the rectangles be produced so as to rule the plane, the rectangles which go to make up  $A_u$  for this ruling must be contained within the original rectangles, and as  $A_u > l_u$ , the total area of the original rectangles is greater than  $l_u$ . Next suppose  $x(t)$  is of limited variation and is written as  $x_0 + \Pi(t) - N(t)$ , the difference of two nondecreasing functions. Then  $\sum \epsilon_i \leqq \Pi(t_1) + N(t_1)$ , that is, the sum of the oscillations of  $x$  cannot exceed the total variation of  $x$ . On the other hand as  $y(t)$  is continuous, the divisions  $\Delta_i t$  could have been taken so small that  $\eta_i < \eta$ . Hence

$$l_u < A_u \leqq \sum 2\epsilon_i \cdot 2\eta_i < 4\eta \sum \epsilon_i \leqq 4\eta [\Pi(t_1) + N(t_1)].$$

The quantity may be made as small as desired, since it is the product of a finite quantity by  $\eta$ . Hence  $l_u = 0$  and the quadrature is possible.

It may be observed that if  $x(t)$  or  $y(t)$  or both are of limited variation, one or all of the three curvilinear integrals

$$-\int y dx, \quad \int x dy, \quad \frac{1}{2} \int x dy - y dx$$

may be defined, and that it should be expected that in this case the value of the integral or integrals would give the area of the curve. In fact if one desired to deal only with rectifiable curves, it would be possible to take one or all of these integrals as the *definition* of area, and thus to obviate the discussions of the present article. It seems, however, advisable at least to point out the problem of quadrature in all its generality, especially as the treatment of the problem is very similar to that usually adopted for double integrals (§ 132). From the present viewpoint, therefore, it would be a proposition for demonstration that the curvilinear integrals in the cases where they are applicable do give the value of the area as here defined, but the demonstration will not be undertaken.

**EXERCISES**

1. For the continuous curve (27) prove the following properties :

( $\alpha$ ) Lines  $x = a, x = b$  may be drawn such that the curve lies entirely between them, has at least one point on each line, and cuts every line  $x = \xi, a < \xi < b$ , in at least one point ; similarly for  $y$ .

( $\beta$ ) From  $p = x \cos \alpha + y \sin \alpha$ , the normal equation of a line, prove the propositions like those of ( $\alpha$ ) for lines parallel to any direction.

( $\gamma$ ) If  $(\xi, \eta)$  is any point of the  $xy$ -plane, show that the distance of  $(\xi, \eta)$  from the curve has a minimum and a maximum value.

( $\delta$ ) If  $m(\xi, \eta)$  and  $M(\xi, \eta)$  are the minimum and maximum distances of  $(\xi, \eta)$  from the curve, the functions  $m(\xi, \eta)$  and  $M(\xi, \eta)$  are continuous functions of  $(\xi, \eta)$ . Are the coördinates  $x(\xi, \eta)$ ,  $y(\xi, \eta)$  of the points on the curve which are at minimum (or maximum) distance from  $(\xi, \eta)$  continuous functions of  $(\xi, \eta)$ ?

( $\epsilon$ ) If  $t', t'', \dots, t^{(k)}, \dots$  are an infinite set of values of  $t$  in the interval  $t_0 \leq t \leq t_1$  and if  $t^0$  is a point of condensation of the set, then  $x^0 = \phi(t^0)$ ,  $y^0 = \psi(t^0)$  is a point of condensation of the set of points  $(x', y')$ ,  $(x'', y'')$ ,  $\dots$ ,  $(x^{(k)}, y^{(k)})$ ,  $\dots$  corresponding to the set of values  $t', t'', \dots, t^{(k)}, \dots$ .

( $\zeta$ ) Conversely to ( $\epsilon$ ) show that if  $(x', y')$ ,  $(x'', y'')$ ,  $\dots$ ,  $(x^{(k)}, y^{(k)})$ ,  $\dots$  are an infinite set of points on the curve and have a point of condensation  $(x^0, y^0)$ , then the point  $(x^0, y^0)$  is also on the curve.

( $\eta$ ) From ( $\zeta$ ) show that if a line  $x = \xi$  cuts the curve in a set of points  $y', y'', \dots$ , then this suite of  $y$ 's contains its upper and lower frontiers and has a maximum or minimum.

2. Define and discuss rectifiable curves in space.

3. Are  $y = x^2 \sin \frac{1}{x}$  and  $y = \sqrt{x} \sin \frac{1}{x}$  rectifiable between  $x = 0$ ,  $x = 1$ ?

4. If  $x(t)$  in (27) is of total variation  $\Pi(t_1) + N(t_1)$ , show that

$$\int_C^{x_1} P(x, y) dx < M[\Pi(t_1) + N(t_1)],$$

where  $M$  is the maximum value of  $P(x, y)$  on the curve.

5. Consider the function  $\theta(\xi, \eta, t) = \tan^{-1} \frac{\eta - y(t)}{\xi - x(t)}$  which is the inclination of the line joining a point  $(\xi, \eta)$  not on the curve to a point  $(x, y)$  on the curve. With the notations of Ex. 1 ( $\delta$ ) show that

$$|\Delta_t \theta| = |\theta(\xi, \eta, t + \Delta t) - \theta(\xi, \eta, t)| < \frac{2M\delta}{m - 2M\delta},$$

where  $\delta > |\Delta x|$  and  $\delta > |\Delta y|$  may be made as small as desired by taking  $\Delta t$  sufficiently small and where it is assumed that  $m \neq 0$ .

6. From Ex. 5 infer that  $\theta(\xi, \eta, t)$  is of limited variation when  $t$  describes the interval  $t_0 \leq t \leq t_1$  defining the curve. Show that  $\theta(\xi, \eta, t)$  is continuous in  $(\xi, \eta)$  through any region for which  $m > 0$ .

7. Let the parameter  $t$  vary from  $t_0$  to  $t_1$  and suppose the curve (27) is closed so that  $(x, y)$  returns to its initial value. Show that the initial and final values of  $\theta(\xi, \eta, t)$  differ by an integral multiple of  $2\pi$ . Hence infer that this difference is constant over any region for which  $m > 0$ . In particular show that the constant is 0 over all distant regions of the plane. It may be remarked that, by the study of this change of  $\theta$  as  $t$  describes the curve, a proof may be given of the theorem that the closed continuous curve divides the plane into two regions, one interior, one exterior.

8. Extend the last theorem of § 123 to rectifiable curves.