CHAPTER X

DIFFERENTIAL EQUATIONS IN MORE THAN TWO VARIABLES

109. Total differential equations. An equation of the form

$$P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz = 0, \tag{1}$$

involving the differentials of three variables is called a *total differential equation.* A similar equation in any number of variables would also be called total; but the discussion here will be restricted to the case of three. If definite values be assigned to x, y, z, say a, b, c, the equation becomes

$$A\,dx + B\,dy + C\,dz = A(x-a) + B(y-b) + C(z-c) = 0, \tag{2}$$

where x, y, z are supposed to be restricted to values near a, b, c, and represents a small portion of a plane passing through (a, b, c). From the analogy to the lineal element (§ 85), such a portion of a plane may be called a *planar element.* The differential equation therefore represents an infinite number of planar elements, one passing through each point of space.

Now any family of surfaces $F(x, y, z) = C$ also represents an infinity of planar elements, namely, the portions of the tangent planes at every point of all the surfaces in the neighborhood of their respective points of tangency. In fact

$$dF = F'_x dx + F'_y dy + F'_z dz = 0 \tag{3}$$

is an equation similar to (1). If the planar elements represented by (1) and (3) are to be the same, the equations cannot differ by more than a factor $\mu(x, y, z)$. Hence

$$F'_x = \mu P, \qquad F'_y = \mu Q, \qquad F'_z = \mu R.$$

If a function $F(x, y, z) = C$ can be found which satisfies these conditions, it is said to be the integral of (1), and the factor $\mu(x, y, z)$ by which the equations (1) and (3) differ is called an *integrating factor* of (1). Compare § 91.

It may happen that $\mu = 1$ and that (1) is thus an *exact* differential. In this case the conditions

$$P'_y = Q'_x, \qquad Q'_z = R'_y, \qquad R'_x = P'_z, \tag{4}$$

which arise from $F''_{xy} = F''_{yx}$, $F''_{yz} = F''_{zy}$, $F''_{zx} = F''_{xz}$, must be satisfied. Moreover if these conditions are satisfied, the equation (1) will be an exact equation and the integral is given by

$$F(x, y, z) = \int_{x_0}^{x} P(x, y, z)\, dx + \int_{y_0}^{y} Q(x_0, y, z)\, dy + \int R(x_0, y_0, z)\, dz = C,$$

where x_0, y_0, z_0 may be chosen so as to render the integration as simple as possible. The proof of this is so similar to that given in the case of two variables (§ 92) as to be omitted. In many cases which arise in practice the equation, though not exact, may be made so by an obvious integrating factor.

As an example take $zxdy - yzdx + x^2dz = 0$. Here the conditions (4) are not fulfilled but the integrating factor $1/x^2z$ is suggested. Then

$$\frac{xdy - ydx}{x^2} + \frac{dz}{z} = d\left(\frac{y}{x} + \log z\right)$$

is at once perceived to be an exact differential and the integral is $y/x + \log z = C$. It appears therefore that in this simple case neither the renewed application of the conditions (4) nor the general formula for the integral was necessary. It often happens that both the integrating factor and the integral can be recognized at once as above.

If the equation does not suggest an integrating factor, the question arises, Is there any integrating factor? In the case of two variables (§ 94) there always was an integrating factor. In the case of three variables there may be none. For

$$\begin{aligned} F''_{xy} &= P\frac{\partial \mu}{\partial y} + \mu\frac{\partial P}{\partial y} = F''_{yx} = Q\frac{\partial \mu}{\partial x} + \mu\frac{\partial Q}{\partial x}, && \Big| \; R,\\ F''_{yz} &= Q\frac{\partial \mu}{\partial z} + \mu\frac{\partial Q}{\partial z} = F''_{zy} = R\frac{\partial \mu}{\partial y} + \mu\frac{\partial R}{\partial y}, && \Big| \; P,\\ F''_{zx} &= R\frac{\partial \mu}{\partial x} + \mu\frac{\partial R}{\partial x} = F''_{xz} = P\frac{\partial \mu}{\partial z} + \mu\frac{\partial P}{\partial z}, && \Big| \; Q. \end{aligned}$$

If these equations be multiplied by R, P, Q and added and if the result be simplified, the condition

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \qquad (5)$$

is found to be imposed on P, Q, R if there is to be an integrating factor. This is called the *condition of integrability*. For it may be shown conversely that if the condition (5) is satisfied, the equation may be integrated.

Suppose an attempt to integrate (1) be made as follows: First assume that one of the variables is constant (naturally, that one which will make the resulting equation simplest to integrate), say z. Then $Pdx + Qdy = 0$. Now integrate this simplified equation with an integrating factor or otherwise, and let $F(x, y, z) = \phi(z)$ be the integral, where the constant C is taken as a function ϕ of z. Next try to determine ϕ so that the integral $F(x, y, z) = \phi(z)$ will satisfy (1). To do this, differentiate;

$$F'_x dx + F'_y dy + F'_z dz = d\phi.$$

Compare this equation with (1). Then the equations*

$$F'_x = \lambda P, \qquad F'_y = \lambda Q, \qquad (F'_z - \lambda R)\,dz = d\phi$$

must hold. The third equation $(F'_z - \lambda R)\,dz = d\phi$ may be integrated provided the coefficient $S = F'_z - \lambda R$ of dz is a function of z and ϕ, that is, of z and F alone. This is so in case the condition (5) holds. It therefore appears that the integration of the equation (1) for which (5) holds reduces to the succession of two integrations of the type discussed in Chap. VIII.

As an example take $(2x^2 + 2xy + 2xz^2 + 1)\,dx + dy + 2zdz = 0$. The condition

$$(2x^2 + 2xy + 2xz^2 + 1)0 + 1(-4xz) + 2z(2x) = 0$$

of integrability is satisfied. The greatest simplification will be had by making x constant. Then $dy + 2zdz = 0$ and $y + z^2 = \phi(x)$. Compare

$$dy + 2zdz = d\phi \quad \text{and} \quad (2x^2 + 2xy + 2xz^2 + 1)\,dx + dy + 2zdz = 0.$$

Then $\qquad \lambda = 1, \qquad -(2x^2 + 2xy + 2xz^2 + 1)\,dx = d\phi;$

or $\qquad -(2x^2 + 1 + 2x\phi)\,dx = d\phi \quad$ or $\quad d\phi + 2x\phi dx = -(2x^2 + 1)\,dx.$

This is the linear type with the integrating factor e^{x^2}. Then

$$e^{x^2}(d\phi + 2x\phi dx) = -e^{x^2}(2x^2 + 1)\,dx \quad \text{or} \quad e^{x^2}\phi = -\int e^{x^2}(2x^2 + 1)\,dx + C.$$

Hence $y + z^2 + e^{-x^2}\int e^{x^2}(2x^2 + 1)\,dx = Ce^{-x^2}$ or $e^{x^2}(y + z^2) + \int e^{x^2}(2x^2 + 1)\,dx = C$

is the solution. It may be noted that e^{x^2} is the integrating factor for the original equation:

$$e^{x^2}[(2x^2 + 2xy + 2xz^2 + 1)\,dx + dy + 2zdz] = d\left[e^{x^2}(y + z^2) + \int e^{x^2}(2x^2 + 1)\,dx\right].$$

To complete the proof that the equation (1) is integrable if (5) is satisfied, it is necessary to show that when the condition is satisfied the coefficient $S = F'_z - \lambda R$ is a function of z and F alone. Let it be regarded as a function of x, F, z instead of x, y, z. It is necessary to prove that the derivative of S by x when F and z are constant is zero. By the formulas for change of variable

$$\left(\frac{\partial S}{\partial x}\right)_{y,z} = \left(\frac{\partial S}{\partial x}\right)_{F,z} + \left(\frac{\partial S}{\partial F}\right)\frac{\partial F}{\partial x}, \qquad \left(\frac{\partial S}{\partial y}\right)_{x,z} = \left(\frac{\partial S}{\partial F}\right)_{x,z}\frac{\partial F}{\partial y}.$$

* Here the factor λ is not an integrating factor of (1), but only of the reduced equation $Pdx + Qdy = 0$.

But $F'_x = \lambda P$ and $F'_y = \lambda Q$, and hence $Q\left(\frac{\partial S}{\partial x}\right)_{y,z} - P\left(\frac{\partial S}{\partial y}\right)_{x,z} = Q\left(\frac{\partial S}{\partial x}\right)_{F,z}$.

Now $$\left(\frac{\partial S}{\partial x}\right)_{y,z} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z} - \lambda R\right) = \frac{\partial^2 F}{\partial z \partial x} - \frac{\partial \lambda R}{\partial x} = \frac{\partial \lambda P}{\partial z} - \frac{\partial \lambda R}{\partial x}.$$

Hence $$\left(\frac{\partial S}{\partial x}\right)_{y,z} = \lambda\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + P\frac{\partial \lambda}{\partial z} - R\frac{\partial \lambda}{\partial x},$$

and $$\left(\frac{\partial S}{\partial y}\right)_{x,z} = \lambda\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\frac{\partial \lambda}{\partial z} - R\frac{\partial \lambda}{\partial y}.$$

Then $$Q\left(\frac{\partial S}{\partial x}\right)_{y,z} - P\left(\frac{\partial S}{\partial y}\right)_{x,z} = \lambda\left[Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\right] - R\left[Q\frac{\partial \lambda}{\partial x} - P\frac{\partial \lambda}{\partial y}\right]$$

and $$Q\left(\frac{\partial S}{\partial x}\right)_{F,z} = \lambda\left[Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] - R\left[\frac{\partial \lambda Q}{\partial x} - \frac{\partial \lambda P}{\partial y}\right],$$

where a term has been added in the first bracket and subtracted in the second. Now as λ is an integrating factor for $Pdx + Qdy$, it follows that $(\lambda Q)'_x = (\lambda P)'_y$; and only the first bracket remains. By the condition of integrability this, too, vanishes and hence S as a function of x, F, z does not contain x but is a function of F and z alone, as was to be proved.

110. It has been seen that if the equation (1) is integrable, there is an integrating factor and the condition (5) is satisfied; also that conversely if the condition is satisfied the equation may be integrated. Geometrically this means that the infinity of planar elements defined by the equation can be grouped upon a family of surfaces $F(x, y, z) = C$ to which they are tangent. If the condition of integrability is not satisfied, the planar elements cannot be thus grouped into surfaces. Nevertheless if a surface $G(x, y, z) = 0$ be given, the planar element of (1) which passes through any point (x_0, y_0, z_0) of the surface will cut the surface $G = 0$ in a certain lineal element of the surface. Thus upon the surface $G(x, y, z) = 0$ there will be an infinity of lineal elements, one through each point, which satisfy the given equation (1). And these elements may be grouped into curves lying upon the surface. If the equation (1) is integrable, these curves will of course be the intersections of the given surface $G = 0$ with the surfaces $F = C$ defined by the integral of (1).

The method of obtaining the curves upon $G(x, y, z) = 0$ which are the integrals of (1), in case (5) does not possess an integral of the form $F(x, y, z) = C$, is as follows. Consider the two equations

$$Pdx + Qdy + Rdz = 0, \qquad G'_x dx + G'_y dy + G'_z dz = 0,$$

of which the first is the given differential equation and the second is the differential equation of the given surface. From these equations one of the differentials, say dz, may be eliminated, and the corresponding variable z may also be eliminated by substituting its value obtained by solving $G(x, y, z) = 0$. Thus there is obtained a differential equation $Mdx + Ndy = 0$ connecting the other two variables x and y. The integral of this, $F(x, y) = C$, consists of a family of cylinders which cut the given surface $G = 0$ in the curves which satisfy (1).

Consider the equation $ydx + xdy - (x + y + z)dz = 0$. This does not satisfy the condition (5) and hence is not completely integrable; but a set of integral curves may be found on any assigned surface. If the surface be the plane $z = x + y$, then

$$ydx + xdy - (x + y + z)dz = 0 \quad \text{and} \quad dz = dx + dy$$

give $\quad (x + z)dx + (y + z)dy = 0 \quad$ or $\quad (2x + y)dx + (2y + x)dy = 0$

by eliminating dz and z. The resulting equation is exact. Hence

$$x^2 + xy + y^2 = C \quad \text{and} \quad z = x + y$$

give the curves which satisfy the equation and lie in the plane.

If the equation (1) were integrable, the integral curves may be used to obtain the integral surfaces and thus to accomplish the complete integration of the equation by *Mayer's method*. For suppose that $F(x, y, z) = C$ were the integral surfaces and that $F(x, y, z) = F(0, 0, z_0)$ were that particular surface cutting the z-axis at z_0. The family of planes $y = \lambda x$ through the z-axis would cut the surface in a series of curves which would be integral curves, and the surface could be regarded as generated by these curves as the plane turned about the axis. To reverse these considerations let $y = \lambda x$ and $dy = \lambda dx$; by these relations eliminate dy and y from (1) and thus obtain the differential equation $Mdx + Ndz = 0$ of the intersections of the planes with the solutions of (1). Integrate the equation as $f(x, z, \lambda) = C$ and determine the constant so that $f(x, z, \lambda) = f(0, z_0, \lambda)$. For any value of λ this gives the intersection of $F(x, y, z) = F(0, 0, z_0)$ with $y = \lambda x$. Now if λ be eliminated by the relation $\lambda = y/x$, the result will be the surface

$$f\left(x, z, \frac{y}{x}\right) = f\left(0, z_0, \frac{y}{x}\right), \quad \text{equivalent to} \quad F(x, y, z) = F(0, 0, z_0),$$

which is the integral of (1) and passes through $(0, 0, z_0)$. As z_0 is arbitrary, the solution contains an arbitrary constant and is the general solution.

It is clear that instead of using planes through the z-axis, planes through either of the other axes might have been used, or indeed planes or cylinders through any line parallel to any of the axes. Such modifications are frequently necessary owing to the fact that the substitution $f(0, z_0, \lambda)$ introduces a division by 0 or a log 0 or some other impossibility. For instance consider

$$y^2dx + zdy - ydz = 0, \qquad y = \lambda x, \qquad dy = \lambda dx, \qquad \lambda^2x^2dx + \lambda zdx - \lambda xdz = 0.$$

Then $$\lambda dx + \frac{zdx - xdz}{x^2} = 0, \quad \text{and} \quad \lambda x - \frac{z}{x} = f(x, z, \lambda).$$

But here $f(0, z_0, \lambda)$ is impossible and the solution is illusory. If the planes $(y - 1) = \lambda x$ passing through a line parallel to the z-axis and containing the point $(0, 1, 0)$ had been used, the result would be

$$dy = \lambda dx, \qquad (1 + \lambda x)^2dx + \lambda zdx - (1 + \lambda x)dz = 0,$$

or $$dx + \frac{\lambda z dx - (1 + \lambda x)\, dz}{(1 + \lambda x)^2} = 0, \quad \text{and} \quad x - \frac{z}{1 + \lambda x} = f(x, z, \lambda).$$

Hence $$x - \frac{z}{1 + \lambda x} = -z_0 \quad \text{or} \quad x - \frac{z}{y} = -z_0 = C,$$

is the solution. The same result could have been obtained with $x = \lambda z$ or $y = \lambda(x - a)$. In the latter case, however, care should be taken to use $f(x, z, \lambda) = f(a, z_0, \lambda)$.

EXERCISES

1. Test these equations for exactness; if exact, integrate; if not exact, find an integrating factor by inspection and integrate:

(α) $(y + z)\, dx + (z + x)\, dy + (x + y)\, dz = 0$, ($\beta$) $y^2 dx + z dy - y dz = 0$,
(γ) $x dx + y dy - \sqrt{a^2 - x^2 - y^2} dz = 0$, ($\delta$) $2z(dx - dy) + (x - y)\, dz = 0$,
(ϵ) $(2x + y^2 + 2xz)\, dx + 2xy dy + x^2 dz = 0$, ($\zeta$) $zy dx = zx dy + y^2 dx$,
(η) $x(y - 1)(z - 1)\, dx + y(z - 1)(x - 1)\, dy + z(x - 1)(y - 1)\, dz = 0$.

2. Apply the test of integrability and integrate these:

(α) $(x^2 - y^2 - z^2)\, dx + 2xy dy + 2xz dz = 0$,
(β) $(x + y^2 + z^2 + 1)\, dx + 2y dy + 2z dz = 0$,
(γ) $(y + a)^2 dx + z dy = (y + a)\, dz$,
(δ) $(1 - x^2 - 2y^2 z)\, dz = 2xz dx + 2yz^2 dy$,
(ϵ) $x^2 dx^2 + y^2 dy^2 - z^2 dz^2 + 2xy dx dy = 0$,
(ζ) $z(x dx + y dy + z dz)^2 = (z^2 - x^2 - y^2)(x dx + y dy + z dz)\, dz$.

3. If the equation is homogeneous, the substitution $x = uz$, $y = vz$, frequently shortens the work. Show that if the given equation satisfies the condition of integrability, the new equation will satisfy the corresponding condition in the new variables and may be rendered exact by an obvious integrating factor. Integrate:

(α) $(y^2 + yz)\, dx + (xz + z^2)\, dy + (y^2 - xy)\, dz = 0$,
(β) $(x^2 y - y^3 - y^2 z)\, dx + (xy^2 - x^2 z - x^3)\, dy + (xy^2 + x^2 y)\, dz = 0$,
(γ) $(y^2 + yz + z^2)\, dx + (x^2 + xz + z^2)\, dy + (x^2 + xy + y^2)\, dz = 0$.

4. Show that (5) does not hold; integrate subject to the relation imposed:

(α) $y dx + x dy - (x + y + z)\, dz = 0$, $x + y + z = k$ or $y = kx$,
(β) $c(x dy + y dy) + \sqrt{1 - a^2 x^2 - b^2 y^2} dz = 0$, $a^2 x^2 + b^2 y^2 + c^2 z^2 = 1$,
(γ) $dz = ay dx + b dy$, $y = kx$ or $x^2 + y^2 + z^2 = 1$ or $y = f(x)$.

5. Show that if an equation is integrable, it remains integrable after any change of variables from x, y, z to u, v, w.

6. Apply Mayer's method to sundry of Exs. 2 and 3.

7. Find the conditions of exactness for an equation in four variables and write the formula for the integration. Integrate with or without a factor:

(α) $(2x + y^2 + 2xz)\, dx + 2xy dy + x^2 dz + du = 0$,
(β) $yzu dx + xzu dy + xyu dz + xyz du = 0$,
(γ) $(y + z + u)\, dx + (x + z + u)\, dy + (x + y + u)\, dz + (x + y + z)\, du = 0$,
(δ) $u(y + z)\, dx + u(y + z + 1)\, dy + u dz - (y + z) du = 0$.

8. If an equation in four variables is integrable, it must be so when any one of the variables is held constant. Hence the four conditions of integrability obtained by writing (5) for each set of three coefficients must hold. Show that the conditions are satisfied in the following cases. Find the integrals by a generalization of the method in the text by letting one variable be constant and integrating the three remaining terms and determining the constant of integration as a function of the fourth in such a way as to satisfy the equations.

(α) $z(y+z)dx + z(u-x)dy + y(x-u)dz + y(y+z)du = 0,$
(β) $uyzdx + uzx \log xdy + uxy \log xdz - xdu = 0.$

9. Try to extend the method of Mayer to such as the above in Ex. 8.

10. If $G(x, y, z) = a$ and $H(x, y, z) = b$ are two families of surfaces defining a family of curves as their intersections, show that the equation

$$(G'_y H'_z - G'_z H'_y)\,dx + (G'_z H'_x - G'_x H'_z)\,dy + (G'_x H'_y - G'_y H'_x)\,dz = 0$$

is the equation of the planar elements perpendicular to the curves at every point of the curves. Find the conditions on G and H that there shall be a family of surfaces which cut all these curves orthogonally. Determine whether the curves below have orthogonal trajectories (surfaces); and if they have, find the surfaces:

(α) $y = x + a,\ z = x + b,$ (β) $y = ax + 1,\ z = bx,$
(γ) $x^2 + y^2 = a^2,\ z = b,$ (δ) $xy = a,\ xz = b,$
(ϵ) $x^2 + y^2 + z^2 = a^2,\ xy = b,$ (ζ) $x^2 + 2y^2 + 3z^2 = a,\ xy + z = b,$
(η) $\log xy = az,\ x + y + z = b,$ (θ) $y = 2ax + a^2,\ z = 2bx + b^2.$

11. Extend the work of proposition 3, § 94, and Ex. 11, p. 234, to find the normal derivative of the solution of equation (1) and to show that the singular solution may be looked for among the factors of $\mu^{-1} = 0$.

12. If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be formed, show that (1) becomes $\mathbf{F}\cdot d\mathbf{r} = 0$. Show that the condition of exactness is $\nabla\times\mathbf{F} = 0$ by expanding $\nabla\times\mathbf{F}$ as the formal vector product of the operator ∇ and the vector $\mathbf{F}$ (see § 78). Show further that the condition of integrability is $\mathbf{F}\cdot(\nabla\times\mathbf{F}) = 0$ by similar formal expansion.

13. In Ex. 10 consider ∇G and ∇H. Show these vectors are normal to the surfaces $G = a$, $H = b$, and hence infer that $(\nabla G)\times(\nabla H)$ is the direction of the intersection. Finally explain why $d\mathbf{r}\cdot(\nabla G\times\nabla H) = 0$ is the differential equation of the orthogonal family if there be such a family. Show that this vector form of the family reduces to the form above given.

111. Systems of simultaneous equations. The two equations

$$\frac{dy}{dx} = f(x, y, z), \qquad \frac{dz}{dx} = g(x, y, z) \tag{6}$$

in the two dependent variables y and z and the independent variable x constitute a set of simultaneous equations of the first order. It is more customary to write these equations in the form

$$\frac{dx}{X(x, y, z)} = \frac{dy}{Y(x, y, z)} = \frac{dz}{Z(x, y, z)}, \tag{7}$$

which is symmetric in the differentials and where $X : Y : Z = 1 : f : g$. At any assigned point x_0, y_0, z_0 of space the ratios $dx : dy : dz$ of the differentials are determined by substitution in (7). Hence the equations fix a definite direction at each point of space, that is, they determine a lineal element through each point. The problem of integration is to combine these lineal elements into a family of curves $F(x, y, z) = C_1$, $G(x, y, z) = C_2$, depending on two parameters C_1 and C_2, one curve passing through each point of space and having at that point the direction determined by the equations.

For the formal integration there are several allied methods of procedure. In the first place it may happen that two of

$$\frac{dx}{X} = \frac{dy}{Y}, \qquad \frac{dy}{Y} = \frac{dz}{Z}, \qquad \frac{dx}{X} = \frac{dz}{Z}$$

are of such a form as to contain only the variables whose differentials enter. In this case these two may be integrated and the two solutions taken together give the family of curves. Or it may happen that one and only one of these equations can be integrated. Let it be the first and suppose that $F(x, y) = C_1$ is the integral. By means of this integral the variable x may be eliminated from the second of the equations or the variable y from the third. In the respective cases there arises an equation which may be integrated in the form $G(y, z, C_1) = C_2$ or $G(x, z, F) = C_2$, and this result taken with $F(x, y) = C_1$ will determine the family of curves.

Consider the example $\frac{xdx}{yz} = \frac{ydy}{xz} = \frac{dz}{y}$. Here the two equations

$$\frac{xdx}{y} = \frac{ydy}{x} \quad \text{and} \quad \frac{xdx}{z} = dz$$

are integrable with the results $x^3 - y^3 = C_1$, $x^2 - z^2 = C_2$, and these two integrals constitute the solution. The solution might, of course, appear in very different form; for there are an indefinite number of pairs of equations $F(x, y, z, C_1) = 0$, $G(x, y, z, C_2) = 0$ which will intersect in the curves of intersection of $x^3 - y^3 = C_1$, and $x^2 - z^2 = C_2$. In fact $(y^3 + C_1)^2 = (z^2 + C_2)^2$ is clearly a solution and could replace either of those found above.

Consider the example $\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$. Here

$$\frac{dy}{y} = \frac{dz}{z}, \quad \text{with the integral} \quad y = C_1 z,$$

is the only equation the integral of which can be obtained directly. If y be eliminated by means of this first integral, there results the equation

$$\frac{dx}{x^2 - (C_1^2 + 1)z^2} = \frac{dz}{2xz} \quad \text{or} \quad 2xzdx + [(C_1^2 + 1)z^2 - x^2]dz = 0.$$

This is homogeneous and may be integrated with a factor to give

$$x^2 + (C_1^2 + 1)z^2 = C_2 z \quad \text{or} \quad x^2 + y^2 + z^2 = C_2 z.$$

Hence $$y = C_1 z, \qquad x^2 + y^2 + z^2 = C_2 z$$

is the solution, and represents a certain family of circles.

Another method of attack is to use composition and division.

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z}=\frac{\lambda dx+\mu dy+\nu dz}{\lambda X+\mu Y+\nu Z}. \tag{8}$$

Here λ, μ, ν may be chosen as any functions of (x, y, z). It may be possible so to choose them that the last expression, taken with one of the first three, gives an equation which may be integrated. With this first integral a second may be obtained as before. Or it may be that two different choices of λ, μ, ν can be made so as to give the two desired integrals. Or it may be possible so to select two sets of multipliers that the equation obtained by setting the two expressions equal may be solved for a first integral. Or it may be possible to choose λ, μ, ν so that the denominator $\lambda X+\mu Y+\nu Z=0$, and so that the numerator (which must vanish if the denominator does) shall give an equation

$$\lambda dx+\mu dy+\nu dz=0 \tag{9}$$

which satisfies the condition (5) of integrability and may be integrated by the methods of § 109.

Consider the equations $\dfrac{dx}{x^2+y^2+yz}=\dfrac{dy}{x^2+y^2-xz}=\dfrac{dz}{(x+y)z}$. Here take λ, μ, ν as $1, -1, -1$; then $\lambda X+\mu Y+\nu Z=0$ and $dx-dy-dz=0$ is integrable as $x-y-z=C_1$. This may be used to obtain another integral. But another choice of λ, μ, ν as x, y, 0, combined with the last expression, gives

$$\frac{xdx+ydz}{(x^2+y^2)(x+y)}=\frac{dz}{(x+y)z} \quad \text{or} \quad \log(x^2+y^2)=\log z^2+C_2.$$

Hence

$$x-y-z=C_1 \quad \text{and} \quad x^2+y^2=C_2z^2$$

will serve as solutions. This is shorter than the method of elimination.

It will be noted that these equations just solved are homogeneous. The substitution $x=uz$, $y=vz$ might be tried. Then

$$\frac{udz+zdu}{u^2+v^2+v}=\frac{vdz+zdv}{u^2+v^2-u}=\frac{dz}{u+v}=\frac{zdu}{v^2-uv+v}=\frac{zdv}{u^2-uv-u},$$

or

$$\frac{du}{v^2-uv+v}=\frac{dv}{u^2-uv-u}=\frac{dz}{z}.$$

Now the first equations do not contain z and may be solved. This always happens in the homogeneous case and may be employed if no shorter method suggests itself.

It need hardly be mentioned that all these methods apply equally to the case where there are more than three equations. The geometric picture, however, fails, although the geometric language may be continued if one wishes to deal with higher dimensions than three. In some cases the introduction of a fourth variable, as

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z}=\frac{dt}{1} \quad \text{or} \quad =\frac{dt}{t}, \tag{10}$$

is useful in solving a set of equations which originally contained only three variables. This is particularly true when X, Y, Z are linear with constant coefficients, in which case the methods of § 98 may be applied with t as independent variable.

112. Simultaneous differential equations of higher order, as

$$\frac{d^2x}{dt^2} = X\left(x,\, y,\, \frac{dx}{dt}\, \frac{dy}{dt}\right), \qquad \frac{d^2y}{dt^2} = Y\left(x,\, y,\, \frac{dx}{dt}\, \frac{dy}{dt}\right),$$

$$\frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 = R\left(r,\, \phi,\, \frac{dr}{dt},\, \frac{d\phi}{dt}\right), \qquad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right) = \Phi\left(r,\, \phi,\, \frac{dr}{dt},\, \frac{d\phi}{dt}\right),$$

especially those of the second order like these, are of constant occurrence in mechanics; for the acceleration requires second derivatives with respect to the time for its expression, and the forces are expressed in terms of the coördinates and velocities. The complete integration of such equations requires the expression of the dependent variables as functions of the independent variable, generally the time, with a number of constants of integration equal to the sum of the orders of the equations. Frequently even when the complete integrals cannot be found, it is possible to carry out some integrations and replace the given system of equations by fewer equations or equations of lower order containing some constants of integration.

No special or general rules will be laid down for the integration of systems of higher order. In each case some particular combinations of the equations may suggest themselves which will enable an integration to be performed.* In problems in mechanics the principles of energy, momentum, and moment of momentum frequently suggest combinations leading to integrations. Thus if

$$x'' = X, \qquad y'' = Y, \qquad z'' = Z,$$

where accents denote differentiation with respect to the time, be multiplied by dx, dy, dz and added, the result

$$x''dx + y''dy + z''dz = Xdx + Ydy + Zdz \tag{11}$$

contains an exact differential on the left; then if the expression on the right is an exact differential, the integration

$$\tfrac{1}{2}(x'^2 + y'^2 + z'^2) = \int Xdx + Ydy + Zdz + C \tag{11'}$$

* It is possible to differentiate the given equations repeatedly and eliminate all the dependent variables except one. The resulting differential equation, say in x and t, may then be treated by the methods of previous chapters; but this is rarely successful except when the equation is linear.

can be performed. This is *the principle of energy* in its simplest form. If two of the equations are multiplied by the chief variable of the other and subtracted, the result is

$$yx'' - xy'' = yX - xY \tag{12}$$

and the expression on the left is again an exact differential; if the right-hand side reduces to a constant or a function of t, then

$$yx' - xy' = \int f(t) + C \tag{12'}$$

is an integral of the equations. This is *the principle of moment of momentum.* If the equations can be multiplied by constants as

$$lx'' + my'' + nz'' = lX + mY + nZ, \tag{13}$$

so that the expression on the right reduces to a function of t, an integration may be performed. This is *the principle of momentum.* These three are the most commonly usable devices.

As an example: Let a particle move in a plane subject to forces attracting it toward the axes by an amount proportional to the mass and to the distance from the axes; discuss the motion. Here the equations of motion are merely

$$m\frac{d^2x}{dt^2} = -kmx, \qquad m\frac{d^2y}{dt^2} = -kmy \quad \text{or} \quad \frac{d^2x}{dt^2} = -kx, \qquad \frac{d^2y}{dt^2} = -ky.$$

Then $\quad x\dfrac{d^2x}{dt^2} + y\dfrac{d^2y}{dt^2} = -k(xdx + ydy) \quad$ and $\quad \left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = -(x^2 + y^2) + C.$

Also $\quad y\dfrac{d^2x}{dt^2} - x\dfrac{d^2y}{dt^2} = 0 \quad$ and $\quad y\dfrac{dx}{dt} - x\dfrac{dy}{dt} = C'.$

In this case the two principles of energy and moment of momentum give two integrals and the equations are reduced to two of the first order. But as it happens, the original equations could be integrated directly as

$$\frac{d^2x}{dt^2}dx = -kxdx, \qquad \left(\frac{dx}{dt}\right)^2 = -kx^2 + C^2, \qquad \frac{dx}{\sqrt{C^2 - kx^2}} = dt$$

$$\frac{d^2y}{dt^2}dy = -kydy, \qquad \left(\frac{dy}{dt}\right)^2 = -ky^2 + K^2, \qquad \frac{dy}{\sqrt{K^2 - ky^2}} = dt.$$

The constants C^2 and K^2 of integration have been written as squares because they are necessarily positive. The complete integration gives

$$\sqrt{k}x = C\sin(\sqrt{k}t + C_1), \qquad \sqrt{k}y = K\sin(\sqrt{k}t + K_2).$$

As another example: A particle, attracted toward a point by a force equal to $r/m^2 + h^2/r^3$ per unit mass, where m is the mass and h is the double areal velocity and r is the distance from the point, is projected perpendicularly to the radius vector at the distance $\sqrt{mh}$; discuss the motion. In polar coördinates the equations of motion are

$$m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2\right] = R = -\frac{mr}{m^2} - \frac{mh^2}{r^3}, \qquad \frac{m}{r}\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right) = \Phi = 0.$$

The second integrates directly as $r^2 d\phi/dt = h$ where the constant of integration h is twice the areal velocity. Now substitute in the first to eliminate ϕ.

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\frac{r}{m^2} - \frac{h^2}{r^3} \quad \text{or} \quad \frac{d^2r}{dt^2} = -\frac{r}{m^2} \quad \text{or} \quad \left(\frac{dr}{dt}\right)^2 = -\frac{r^2}{m^2} + C.$$

Now as the particle is projected perpendicularly to the radius, $dr/dt = 0$ at the start when $r = \sqrt{mh}$. Hence the constant C is h/m. Then

$$\frac{dr}{\sqrt{\frac{h}{m} - \frac{r^2}{m^2}}} = dt \quad \text{and} \quad \frac{r^2 d\phi}{h} = dt \quad \text{give} \quad \frac{\sqrt{mh}dr}{r^2\sqrt{1 - \frac{r^2}{hm}}} = d\phi.$$

Hence $$\sqrt{mh}\sqrt{\frac{1}{r^2} - \frac{1}{h}} = \phi + C \quad \text{or} \quad \frac{1}{r^2} - \frac{1}{hm} = \frac{(\phi + C)^2}{mh}.$$

Now if it be assumed that $\phi = 0$ at the start when $r = \sqrt{mh}$, we find $C = 0$.

Hence $$r^2 = \frac{mh}{1 + \phi^2} \quad \text{is the orbit.}$$

To find the relation between ϕ and the time,

$$r^2 d\phi = hdt \quad \text{or} \quad \frac{m d\phi}{1 + \phi^2} = dt \quad \text{or} \quad t = m \tan^{-1}\phi,$$

if the time be taken as $t = 0$ when $\phi = 0$. Thus the orbit is found, the expression of ϕ as a function of the time is found, and the expression of r as a function of the time is obtainable. The problem is completely solved. It will be noted that the constants of integration have been determined after each integration by the initial conditions. This simplifies the subsequent integrations which might in fact be impossible in terms of elementary functions without this simplification.

EXERCISES

1. Integrate these equations:

(α) $\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$, (β) $\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2y^2z^2}$,

(γ) $\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$, (δ) $\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{x + y}$,

(ϵ) $-\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{1 + z^2}$, (ζ) $\frac{dx}{-1} = \frac{dy}{3y + 4z} = \frac{dz}{2y + 5z}$.

2. Integrate the equations:

(α) $\frac{dx}{bz - cy} = \frac{dy}{cx - az} = \frac{dz}{ay - bx}$,

(β) $\frac{dx}{x^2 + y^2} = \frac{dy}{2xy} = \frac{dz}{xz + yz}$, (γ) $\frac{dx}{y + z} = \frac{dy}{x + z} = \frac{dz}{x + y}$,

(δ) $\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{yz(x^3 - y^3)}$, (ϵ) $\frac{dx}{x(y - z)} = \frac{dy}{y(z - x)} = \frac{dz}{z(x - y)}$,

(ζ) $\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}$, (η) $\frac{dx}{x(y^2 - z^2)} = \frac{-dy}{y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}$,

(θ) $\frac{dx}{y - z} = \frac{dy}{x + y} = \frac{dz}{x + z} = dt$, (ι) $\frac{dx}{y - z} = \frac{dy}{x + y + t} = \frac{dz}{x + z + t} = dt$.

3. Show that the differential equations of the orthogonal trajectories (curves of the family of surfaces $F(x, y, z) = C$ are $dx : dy : dz = F'_x : F'_y : F'_z$. Find the curves which cut the following families of surfaces orthogonally:

(α) $a^2x^2 + b^2y^2 + c^2z^2 = C$, (β) $xyz = C$, (γ) $y^2 = Cxz$,
(δ) $y = x \tan(z + C)$, (ϵ) $y = x \tan Cz$, (ζ) $z = Cxy$.

4. Show that the solution of $dx : dy : dz = X : Y : Z$, where X, Y, Z are linear expressions in x, y, z, can always be found provided a certain cubic equation can be solved.

5. Show that the solutions of the two equations

$$\frac{dx}{dt} + T(ax + by) = T_1, \qquad \frac{dy}{dt} + T(a'x + b'y) = T_2,$$

where T, T_1, T_2 are functions of t, may be obtained by adding the equation as

$$\frac{d}{dt}(x + ly) + \lambda T(x + ly) = T_1 + lT_2$$

after multiplying one by l, and by determining λ as a root of

$$\lambda^2 - (a + b')\lambda + ab' - a'b = 0.$$

6. Solve:

(α) $t\dfrac{dx}{dt} + 2(x - y) = t, \qquad t\dfrac{dy}{dt} + x + 5y = t^2,$

(β) $tdx = (t - 2x)\,dt, \qquad tdy = (tx + ty + 2x - t)\,dt,$

(γ) $\dfrac{ldx}{mn(y - z)} = \dfrac{mdy}{nl(z - x)} = \dfrac{ndz}{lm(x - y)} = \dfrac{dt}{t}.$

7. A particle moves in vacuo in a vertical plane under the force of gravity alone. Integrate. Determine the constants if the particle starts from the origin with a velocity V and at an angle of α degrees with the horizontal and at the time $t = 0$.

8. Same problem as in Ex. 7 except that the particle moves in a medium which resists proportionately to the velocity of the particle.

9. A particle moves in a plane about a center of force which attracts proportionally to the distance from the center and to the mass of the particle.

10. Same as Ex. 9 but with a repulsive force instead of an attracting force.

11. A particle is projected parallel to a line toward which it is attracted with a force proportional to the distance from the line.

12. Same as Ex. 11 except that the force is inversely proportional to the square of the distance and only the path of the particle is wanted.

13. A particle is attracted toward a center by a force proportional to the square of the distance. Find the orbit.

14. A particle is placed at a point which repels with a constant force under which the particle moves away to a distance a where it strikes a peg and is deflected off at a right angle with undiminished velocity. Find the orbit of the subsequent motion.

15. Show that equations (7) may be written in the form $d\mathbf{r} \times \mathbf{F} = 0$. Find the condition on $\mathbf{F}$ or on X, Y, Z that the integral curves have orthogonal surfaces.

113. Introduction to partial differential equations. An equation which contains a dependent variable, two or more independent variables, and one or more partial derivatives of the dependent variable with respect to the independent variables is called a *partial differential equation.* The equation

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} = R(x, y, z), \qquad p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}, \tag{14}$$

is clearly a linear partial differential equation of the first order in one dependent and two independent variables. The discussion of this equation preliminary to its integration may be carried on by means of the concept of *planar elements,* and the discussion will immediately suggest the method of integration.

When any point (x_0, y_0, z_0) of space is given, the coefficients P, Q, R in the equation take on definite values and the derivatives p and q are connected by a linear relation. Now any planar element through (x_0, y_0, z_0) may be considered as specified by the two slopes p and q; for it is an infinitesimal portion of the plane $z - z_0 = p(x - x_0) + q(y - y_0)$ in the neighborhood of the point. This plane contains the line or lineal element whose direction is

$$dx : dy : dz = P : Q : R, \tag{15}$$

because the substitution of P, Q, R for $dx = x - x_0$, $dy = y - y_0$, $dz = z - z_0$ in the plane gives the original equation $Pp + Qq = R$. Hence it appears that the planar elements defined by (14), of which there are an infinity through each point of space, are so related that all which pass through a given point of space pass through a certain line through that point, namely the line (15).

Now the problem of integrating the equation (14) is that of grouping the planar elements which satisfy it into surfaces. As at each point they are already grouped in a certain way by the lineal elements through which they pass, it is first advisable to group these lineal elements into curves by integrating the simultaneous equations (15). The integrals of these equations are the curves defined by two families of surfaces $F(x, y, z) = C_1$ and $G(x, y, z) = C_2$. These curves are called the *characteristic curves* or merely the *characteristics* of the equation (14). Through each lineal element of these curves there pass an infinity of the planar elements which satisfy (14). It is therefore clear that if these curves be in any wise grouped into surfaces, the planar elements of the surfaces must satisfy (14); for through each point of the surfaces will pass one of the curves, and the planar element of the surface at that point must therefore pass through the lineal element of the curve and hence satisfy (14).

To group the curves $F(x, y, z) = C_1$, $G(x, y, z) = C_2$ which depend on two parameters C_1, C_2 into a surface, it is merely necessary to introduce some functional relation $C_2 = f(C_1)$ between the parameters so that when one of them, as C_1, is given, the other is determined, and thus a particular curve of the family is fixed by one parameter alone and will sweep out a surface as the parameter varies. Hence *to integrate* (14), *first integrate* (15) *and then write*

$$G(x, y, z) = \Phi[F(x, y, z)] \quad \text{or} \quad \Phi(F, G) = 0, \tag{16}$$

where Φ denotes any arbitrary function. This will be the integral of (14) and will contain an arbitrary function Φ.

As an example, integrate $(y-z)p + (z-x)q = x - y$. Here the equations

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} \quad \text{give} \quad x^2 + y^2 + z^2 = C_1, \quad x + y + z = C_2$$

as the two integrals. Hence the solution of the given equation is

$$x + y + z = \Phi(x^2 + y^2 + z^2) \quad \text{or} \quad \Phi(x^2 + y^2 + z^2, x + y + z) = 0,$$

where Φ denotes an arbitrary function. The arbitrary function allows a solution to be determined which shall pass through any desired curve; for if the curve be $f(x, y, z) = 0$, $g(x, y, z) = 0$, the elimination of x, y, z from the four simultaneous equations

$$F(x, y, z) = C_1, \qquad G(x, y, z) = C_2, \qquad f(x, y, z) = 0, \qquad g(x, y, z) = 0$$

will express the condition that the four surfaces meet in a point, that is, that the curve given by the first two will cut that given by the second two; and this elimination will determine a relation between the two parameters C_1 and C_2 which will be precisely the relation to express the fact that the integral curves cut the given curve and that consequently the surface of integral curves passes through the given curve. Thus in the particular case here considered, suppose the solution were to pass through the curve $y = x^2$, $z = x$; then

$$x^2 + y^2 + z^2 = C_1, \qquad x + y + z = C_2, \qquad y = x^2, \qquad z = x$$

give $$2x^2 + x^4 = C_1, \qquad x^2 + 2x = C_2,$$

whence $$(C_2^2 + 2C_2 - C_1)^2 + 8C_2^2 - 24C_1 - 16C_1C_2 = 0.$$

The substitution of $C_1 = x^2 + y^2 + z^2$ and $C_2 = x + y + z$ in this equation will give the solution of $(y-z)p + (z-x)q = x - y$ which passes through the parabola $y = x^2$, $z = x$.

114. It will be recalled that the integral of an ordinary differential equation $f(x, y, y', \cdots, y^{(n)}) = 0$ of the nth order contains n constants, and that conversely if a system of curves in the plane, say $F(x, y, C_1, \cdots, C_n) = 0$, contains n constants, the constants may be eliminated from the equation and its first n derivatives with respect to x. It has now been seen that the integral of a certain partial differential equation contains an arbitrary function, and it might be inferred that the elimination of an arbitrary function would give rise to a partial differential equation of the first order. To show this, suppose $F(x, y, z) = \Phi[G(x, y, z)]$. Then

$$F'_x + F'_z p = \Phi' \cdot (G'_x + G'_z p), \qquad F'_y + F'_z q = \Phi' \cdot (G'_y + G'_z q)$$

follow from partial differentiation with respect to x and y; and

$$(F'_z G'_y - F'_y G'_z)p + (F'_x G'_z - F'_z G'_x)q = F'_y G'_x - F'_x G'_y$$

is a partial differential equation arising from the elimination of Φ'. More generally, the elimination of n arbitrary functions will give rise to an equation of the nth order; conversely it may be believed that the integration of such an equation would introduce n arbitrary functions in the general solution.

As an example, eliminate from $z = \Phi(xy) + \Psi(x + y)$ the two arbitrary functions Φ and Ψ. The first differentiation gives

$$p = \Phi' \cdot y + \Psi', \qquad q = \Phi' \cdot x + \Psi', \qquad p - q = (y - x)\Phi'.$$

Now differentiate again and let $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$. Then

$$r - s = -\Phi' + (y - x)\Phi'' \cdot y, \qquad s - t = \Phi' + (y - x)\Phi'' \cdot x.$$

These two equations with $p - q = (y - x)\Phi'$ make three from which

$$xr - (x + y)s + yt = \frac{x + y}{x - y}(p - q) \quad \text{or} \quad x\frac{\partial^2 z}{\partial x^2} - (x + y)\frac{\partial^2 z}{\partial x \partial y} + y\frac{\partial^2 z}{\partial y^2} = \frac{x + y}{x - y}\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

may be obtained as a partial differential equation of the second order free from Φ and Ψ. The general integral of this equation would be $z = \Phi(xy) + \Psi(x + y)$.

A partial differential equation may represent a certain definite type of surface. For instance by definition a conoidal surface is a surface generated by a line which moves parallel to a given plane, the director plane, and cuts a given line, the directrix. If the director plane be taken as $z = 0$ and the directrix be the z-axis, the equations of any line of the surface are

$$z = C_1, \qquad y = C_2 x, \qquad \text{with} \quad C_1 = \Phi(C_2)$$

as the relation which picks out a definite family of the lines to form a particular conoidal surface. Hence $z = \Phi(y/x)$ may be regarded as the general equation of a conoidal surface of which $z = 0$ is the director plane and the z-axis the directrix. The elimination of Φ gives $px + qy = 0$ as the differential equation of any such conoidal surface.

Partial differentiation may be used not only to eliminate arbitrary functions, but to eliminate constants. For if an equation $f(x, y, z, C_1, C_2) = 0$ contained two constants, the equation and its first derivatives with respect to x and y would yield three equations from which the constants could be eliminated, leaving a partial differential equation $F(x, y, z, p, q) = 0$ of the first order. If there had been five constants, the equation with its two first derivatives and its three second derivatives with respect to x and y would give a set of six equations from which the constants could be eliminated, leaving a differential equation of the second order. And so on. As the differential equation is obtained by eliminating the constants, the original equation will be a solution of the resulting differential equation.

For example, eliminate from $z = Ax^2 + 2Bxy + Cy^2 + Dx + Ey$ the five constants. The two first and three second derivatives are

$$p = 2Ax + 2By + D, \quad q = 2Bx + 2Cy + E, \quad r = 2A, \quad s = 2B, \quad t = 2C.$$

Hence
$$z = -\tfrac{1}{2}rx^2 - \tfrac{1}{2}ty^2 - sxy + px + qy$$

is the differential equation of the family of surfaces. The family of surfaces do not constitute the general solution of the equation, for that would contain two arbitrary functions, but they give what is called a *complete solution*. If there had been only three or four constants, the elimination would have led to a differential equation of the second order which need have contained only one or two of the second derivatives instead of all three; it would also have been possible to find three or two simultaneous partial differential equations by differentiating in different ways.

115. If
$$f(x, y, z, C_1, C_2) = 0 \quad \text{and} \quad F(x, y, z, p, q) = 0 \tag{17}$$

are two equations of which the second is obtained by the elimination of the two constants from the first, the first is said to be the *complete solution* of the second. That is, any equation which contains two distinct arbitrary constants and which satisfies a partial differential equation of the first order is said to be a complete solution of the differential equation. A complete solution has an interesting geometric interpretation. The differential equation $F = 0$ defines a series of planar elements through each point of space. So does $f(x, y, z, C_1, C_2) = 0$. For the tangent plane is given by

$$\left.\frac{\partial f}{\partial x}\right|_0 (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_0 (y - y_0) + \left.\frac{\partial f}{\partial z}\right|_0 (z - z_0) = 0$$

with
$$f(x_0, y_0, z_0, C_1, C_2) = 0$$

as the condition that C_1 and C_2 shall be so related that the surface passes through (x_0, y_0, z_0). As there is only this one relation between the two arbitrary constants, there is a whole series of planar elements through the point. As $f(x, y, z, C_1, C_2) = 0$ satisfies the differential equation, the planar elements defined by it are those defined by the differential equation. Thus a complete solution establishes an arrangement of the planar elements defined by the differential equation upon a family of surfaces dependent upon two arbitrary constants of integration.

From the idea of a solution of a partial differential equation of the first order as a surface pieced together from planar elements which satisfy the equation, it appears that the envelope (p. 140) of any family of solutions will itself be a solution; for each point of the envelope is a point of tangency with some one of the solutions of the family, and the planar element of the envelope at that point is identical with the planar element of the solution and hence satisfies the differential equation. *This observation allows the general solution to be determined from any complete solution.* For if in $f(x, y, z, C_1, C_2) = 0$ any relation $C_2 = \Phi(C_1)$ is introduced between the two arbitrary constants, there arises a family depending on one parameter, and the envelope of the family is found by eliminating C_1 from the three equations

$$C_2 = \Phi(C_1), \qquad \frac{\partial f}{\partial C_1} + \frac{d\Phi}{dC_1}\frac{\partial f}{\partial C_2} = 0, \qquad f = 0. \tag{18}$$

As the relation $C_2 = \Phi(C_1)$ contains an arbitrary function Φ, the result of the elimination may be considered as containing an arbitrary function even though it is generally impossible to carry out the elimination except in the case where Φ has been assigned and is therefore no longer arbitrary.

A family of surfaces $f(x, y, z, C_1, C_2) = 0$ depending on two parameters may also have an envelope (p. 139). This is found by eliminating C_1 and C_2 from the three equations

$$f(x, y, z, C_1, C_2) = 0, \qquad \frac{\partial f}{\partial C_1} = 0, \qquad \frac{\partial f}{\partial C_2} = 0.$$

This surface is tangent to all the surfaces in the complete solution. This envelope is called the ***singular solution*** of the partial differential equation. As in the case of ordinary differential equations (§ 101), the singular solution may be obtained directly from the equation; * it is merely necessary to eliminate p and q from the three equations

$$F(x, y, z, p, q) = 0, \qquad \frac{\partial F}{\partial p} = 0, \qquad \frac{\partial F}{\partial q} = 0.$$

The last two equations express the fact that $F(p, q) = 0$ regarded as a function of p and q should have a double point (§ 57). A reference to § 67 will bring out another point, namely, that not only are all the surfaces represented by the complete solution tangent to the singular solution, but so is any surface which is represented by the general solution.

* It is hardly necessary to point out the fact that, as in the case of ordinary equations, extraneous factors may arise in the elimination, whether of C_1, C_2 or of p, q.

EXERCISES

1. Integrate these linear equations:

(α) $xzp + yzq = xy$, (β) $a(p + q) = z$, (γ) $x^2p + y^2q = z^2$,
(δ) $-yp + xq + 1 + z^2 = 0$, (ϵ) $yp - xq = x^2 - y^2$, (ζ) $(x + z)p = y$,
(η) $x^2p - xyq + y^2 = 0$, (θ) $(a - x)p + (b - y)q = c - z$,
(ι) $p \tan x + q \tan y = \tan z$, (κ) $(y^2 + z^2 - x^2)p - 2xyq + 2xz = 0$.

2. Determine the integrals of the preceding equations to pass through the curves:

for (α) $x^2 + y^2 = 1$, $z = 0$, for (β) $y = 0$, $x = z$,
for (γ) $y = 2x$, $z = 1$, for (ϵ) $x = z$, $y = z$.

3. Show analytically that if $F(x, y, z) = C_1$ is a solution of (15), it is a solution of (14). State precisely what is meant by a solution of a partial differential equation, that is, by the statement that $F(x, y, z) = C_1$ satisfies the equation. Show that the equations

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R \quad \text{and} \quad P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} + R\frac{\partial F}{\partial z} = 0$$

are equivalent and state what this means. Show that if $F = C_1$ and $G = C_2$ are two solutions, then $F = \Phi(G)$ is a solution, and show conversely that a functional relation must exist between any two solutions (see § 62).

4. Generalize the work in the text along the analytic lines of Ex. 3 to establish the rules for integrating a linear equation in one dependent and four or n independent variables. In particular show that the integral of

$$P_1\frac{\partial z}{\partial x_1} + \cdots + P_n\frac{\partial z}{\partial x_n} = P_{n+1} \quad \text{depends on} \quad \frac{dx_1}{P_1} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{P_{n+1}},$$

and that if $F_1 = C_1, \cdots, F_n = C_n$ are n integrals of the simultaneous system, the integral of the partial differential equation is $\Phi(F_1, \cdots, F_n) = 0$.

5. Integrate: (α) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = xyz$,

$$(\beta)\quad (y + z + u)\frac{\partial u}{\partial x} + (z + u + x)\frac{\partial u}{\partial y} + (u + x + y)\frac{\partial u}{\partial z} = x + y + z.$$

6. Interpret the general equation of the first order $F(x, y, z, p, q) = 0$ as determining at each point (x_0, y_0, z_0) of space a series of planar elements tangent to a certain cone, namely, the cone found by eliminating p and q from the three simultaneous equations

$$F(x_0, y_0, z_0, p, q) = 0, \qquad (x - x_0)p + (y - y_0)q = z - z_0,$$
$$(x - x_0)\frac{\partial F}{\partial q} - (y - y_0)\frac{\partial F}{\partial p} = 0.$$

7. Eliminate the arbitrary functions:

(α) $x + y + z = \Phi(x^2 + y^2 + z^2)$, (β) $\Phi(x^2 + y^2, z - xy) = 0$,
(γ) $z = \Phi(x + y) + \Psi(x - y)$, (δ) $z = e^{ay}\Phi(x - y)$,
(ϵ) $z = y^2 + 2\Phi(x^{-1} + \log y)$, (ζ) $\Phi\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right) = 0$.

8. Find the differential equations of these types of surfaces:

(α) cylinders with generators parallel to the line $x = az$, $y = bz$,
(β) conical surfaces with vertex at (a, b, c),
(γ) surfaces of revolution about the line $x : y : z = a : b : c$.

9. Eliminate the constants from these equations:

(α) $z = (x + a)(y + b)$, (β) $a(x^2 + y^2) + bz^2 = 1$,
(γ) $(x - a)^2 + (y - b)^2 + (z - c)^2 = 1$, ($\delta$) $(x - a)^2 + (y - b)^2 + (z - c)^2 = d^2$,
(ϵ) $Ax^2 + Bxy + Cy^2 + Dxz + Eyz = z^2$.

10. Show geometrically and analytically that $F(x, y, z) + aG(x, y, z) = b$ is a complete solution of the linear equation.

11. How many constants occur in the complete solution of the equation of the third, fourth, or nth order?

12. Discuss the complete, general, and singular solutions of an equation of the first order $F(x, y, z, u, u'_x, u'_y, u'_z) = 0$ with three independent variables.

13. Show that the planes $z = ax + by + C$, where a and b are connected by the relation $F(a, b) = 0$, are complete solutions of the equation $F(p, q) = 0$. Integrate:

(α) $pq = 1$, (β) $q = p^2 + 1$, (γ) $p^2 + q^2 = m^2$,
(δ) $pq = k$, (ϵ) $k \log q + p = 0$, (ζ) $3p^2 - 2q^2 = 4pq$,

and determine also the singular solutions.

14. Note that a simple change of variable will often reduce an equation to the type of Ex. 13. Thus the equations

$$F\left(\frac{p}{z}, \frac{q}{z}\right) = 0, \quad F(xp, q) = 0, \quad F\left(\frac{xp}{z}, \frac{yq}{z}\right) = 0,$$

with $\qquad z = e^{z'}, \qquad x = e^{x'}, \qquad z = e^{z'}, x = e^{x'}, y = e^{y'},$

take a simpler form. Integrate and determine the singular solutions:

(α) $q = z + px$, (β) $x^2p^2 + y^2q^2 = z^2$, (γ) $z = pq$,
(δ) $q = 2yp^2$, (ϵ) $(p - y)^2 + (q - x)^2 = 1$, ($\zeta$) $z = p^m q^m$.

15. What is the obvious complete solution of the extended Clairaut equation $z = xp + yq + f(p, q)$? Discuss the singular solution. Integrate the equations:

(α) $z = xp + yq + \sqrt{p^2 + q^2 + 1}$, ($\beta$) $z = xp + yq + (p + q)^2$,
(γ) $z = xp + yq + pq$, (δ) $z = xp + yq - 2\sqrt{pq}$.

116. Types of partial differential equations. In addition to the linear equation and the types of Exs. 13–15 above, there are several types which should be mentioned. Of these the first is *the general equation of the first order*. If $F(x, y, z, p, q) = 0$ is the given equation and if a second equation $\Phi(x, y, z, p, q, a) = 0$, which holds simultaneously with the first and contains an arbitrary constant can be found, the two equations may be solved together for the values of p and q, and the results may be substituted in the relation $dz = pdx + qdy$ to give a total differential equation of which the integral will contain the constant a and a second constant of integration b. This integral will then be a complete integral of the given equation; the general integral may then be obtained by (18) of § 115. This is known as *Charpit's method.*

To find a relation $\Phi = 0$ differentiate the two equations

$$F(x, y, z, p, q) = 0, \qquad \Phi(x, y, z, p, q, a) = 0 \tag{19}$$

with respect to x and y and use the relation that dz be exact.

$$\begin{array}{r|c}
F'_x + F'_z p + F'_p \dfrac{dp}{dx} + F'_q \dfrac{dq}{dx} = 0, & \Phi'_p, \\
\Phi'_x + \Phi'_z p + \Phi'_p \dfrac{dp}{dx} + \Phi'_q \dfrac{dq}{dx} = 0, & -F'_p, \\
F'_y + F'_z q + F'_p \dfrac{dp}{dy} + F'_q \dfrac{dq}{dy} = 0, & \Phi'_q, \\
\Phi'_y + \Phi'_z q + \Phi'_p \dfrac{dp}{dy} + \Phi'_q \dfrac{dq}{dy} = 0, & -F'_q, \\
\dfrac{dp}{dy} - \dfrac{dq}{dx} = 0, & F'_q\Phi'_p - \Phi'_q F'_p.
\end{array}$$

Multiply by the quantities on the right and add. Then

$$(F'_x + pF'_z)\frac{\partial \Phi}{\partial p} + (F'_y + qF'_z)\frac{\partial \Phi}{\partial q} - F'_p\frac{\partial \Phi}{\partial x} - F'_q\frac{\partial \Phi}{\partial y} - (pF'_p + qF'_q)\frac{\partial \Phi}{\partial z} = 0. \tag{20}$$

Now this is a linear equation for Φ and is equivalent to

$$\frac{dp}{F'_x + pF'_z} = \frac{dq}{F'_y + qF'_z} = \frac{dx}{-F'_p} = \frac{dy}{-F'_q} = \frac{dz}{-(pF'_p + qF'_q)} = \frac{d\Phi}{0}. \tag{21}$$

Any integral of this system containing p or q and a will do for Φ, and the simplest integral will naturally be chosen.

As an example take $zp(x+y) + p(q-p) - z^2 = 0$. Then Charpit's equations are

$$\frac{dp}{-zp + p^2(x+y)} = \frac{dq}{zp - 2zq + pq(x+y)} = \frac{dx}{2p - q - z(x+y)}$$
$$= \frac{dy}{-p} = \frac{dz}{2p^2 - 2pq - pz(x+y)}.$$

How to combine these so as to get a solution is not very clear. Suppose the substitution $z = e^{z'}$, $p = e^{z'}p'$, $q = e^{z'}q'$ be made in the equation. Then

$$p'(x+y) + p'(q'-p') - 1 = 0$$

is the new equation. For this Charpit's simultaneous system is

$$\frac{dp'}{p'} = \frac{dq'}{p'} = \frac{dx}{2p' - q' - (x+y)} = \frac{dy}{-p'} = \frac{dz}{2p'^2 - 2pq - p'(x+y)}.$$

The first two equations give at once the solution $dp' = dq'$ or $q' = p' + a$. Solving

$$p'(x+y) + p'(q'-p') - 1 = 0 \quad \text{and} \quad q' = p' + a,$$

$$p' = \frac{1}{a+x+y}, \qquad q' = \frac{1}{a+x+y} + a, \qquad dz' = \frac{dx+dy}{a+x+y} + a\,dy.$$

Then $z = \log(a + x + y) + ay + b$ or $\log z = \log(a + x + y) + ay + b$

is a complete solution of the given equation. This will determine the general integral by eliminating a between the three equations

$$z = e^{ay+b}(a + x + y), \qquad b = f(a), \qquad 0 = (y + f'(a))(a + x + y) + 1,$$

where $f(a)$ denotes an arbitrary function. The rules for determining the singular solution give $z = 0$; but it is clear that the surfaces in the complete solution cannot be tangent to the plane $z = 0$ and hence the result $z = 0$ must be not a singular solution but an extraneous factor. There is no singular solution.

The method of solving a partial differential equation of higher order than the first is to reduce it first to an equation of the first order and then to complete the integration. Frequently the form of the equation will suggest some method easily applied. For instance, if the derivatives of lower order corresponding to one of the independent variables are absent, an integration may be performed as if the equation were an ordinary equation with that variable constant, and the constant of integration may be taken as a function of that variable. Sometimes a change of variable or an interchange of one of the independent variables with the dependent variable will simplify the equation. In general the solver is left mainly to his own devices. Two special methods will be mentioned below.

117. If the equation is *linear with constant coefficients* and all the derivatives are of the same order, the equation is

$$(a_0D_x^n + a_1D_x^{n-1}D_y + \cdots + a_{n-1}D_xD_y^{n-1} + a_nD_y^n)z = R(x, y). \qquad (22)$$

Methods like those of § 95 may be applied. Factor the equation.

$$a_0(D_x - \alpha_1D_y)(D_x - \alpha_2D_y)\cdots(D_x - \alpha_nD_y)z = R(x, y). \qquad (22')$$

Then the equation is reduced to a succession of equations

$$D_xz - \alpha D_yz = R(x, y),$$

each of which is linear of the first order (and with constant coefficients). Short cuts analogous to those previously given may be developed, but will not be given. If the derivatives are not all of the same order but the polynomial can be factored into linear factors, the same method will apply. For those interested, the several exercises given below will serve as a synopsis for dealing with these types of equation.

There is one equation of the second order,* namely

$$\frac{1}{V^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \qquad (23)$$

* This is one of the important differential equations of physics; other important equations and methods of treating them are discussed in Chap. XX.

which occurs constantly in the discussion of waves and which has therefore the name of the *wave equation.* The solution may be written down by inspection. For try the form

$$u(x, y, z, t) = F(ax + by + cz - Vt) + G(ax + by + cz + Vt). \quad (24)$$

Substitution in the equation shows that this is a solution if the relation $a^2 + b^2 + c^2 = 1$ holds, no matter what functions F and G may be. Note that the equation

$$ax + by + cz - Vt = 0, \qquad a^2 + b^2 + c^2 = 1,$$

is the equation of a plane at a perpendicular distance Vt from the origin along the direction whose cosines are a, b, c. If t denotes the time and if the plane moves away from the origin with a velocity V, the function $F(ax + by + cz - Vt) = F(0)$ remains constant; and if $G = 0$, the value of u will remain constant. Thus $u = F$ represents a phenomenon which is constant over a plane and retreats with a velocity V, that is, a plane wave. In a similar manner $u = G$ represents a plane wave approaching the origin. The general solution of (23) therefore represents the superposition of an advancing and a retreating plane wave.

To Monge is due a method sometimes useful in treating differential equations of the second order linear in the derivatives r, s, t; it is known as *Monge's method.* Let

$$Rr + Ss + Tt = V \quad (25)$$

be the equation, where R, S, T, V are functions of the variables and the derivatives p and q. From the given equation and

$$dp = rdx + sdy, \qquad dq = sdx + tdy,$$

the elimination of r and t gives the equation

$$s(Rdy^2 - Sdxdy + Tdx^2) - (Rdydp + Tdxdq - Vdxdy) = 0,$$

and this will surely be satisfied if the two equations

$$Rdy^2 - Sdxdy + Tdx^2 = 0, \qquad Rdydp + Tdxdq - Vdxdy = 0 \quad (25')$$

can be satisfied simultaneously. The first may be factored as

$$dy - f_1(x, y, z, p, q)\,dx = 0, \qquad dy - f_2(x, y, z, p, q)\,dx = 0. \quad (26)$$

The problem then is reduced to integrating the system consisting of one of these factors with (25′) and $dz = pdx + qdy$, that is, a system of three total differential equations.

If two independent solutions of this system can be found, as

$$u_1(x, y, z, p, q) = C_1, \qquad u_2(x, y, z, p, q) = C_2,$$

then $u_1 = \Phi(u_2)$ is a first or intermediary integral of the given equation, the general integral of which may be found by integrating this equation of the first order. If the two factors are distinct, it may happen that the two systems which arise may both be integrated. Then two first integrals $u_1 = \Phi(u_2)$ and $v_1 = \Psi(v_2)$ will be found, and instead of integrating one of these equations it may be better to solve both for p and q and to substitute in the expression $dz = pdx + qdy$ and integrate. When, however, it is not possible to find even one first integral, Monge's method fails.

As an example take $(x+y)(r-t)=-4p$. The equations are

$$(x+y)\,dy^2-(x+y)\,dx^2=0 \quad\text{or}\quad dy-dx=0, \qquad dy+dx=0$$

and
$$(x+y)\,dydp-(x+y)\,dxdq+4\,pdxdy=0. \tag{A}$$

Now the equation $dy-dx=0$ may be integrated at once to give $y=x+C_1$. The second equation (A) then takes the form

$$2\,xdp+4\,pdx-2\,xdq+C_1(dp-dq)=0\,;$$

but as $dz=pdx+qdy=(p+q)\,dx$ in this case, we have by combination

$$2\,(xdp+pdx)-2\,(xdq+qdx)+C_1(dp-dq)+2\,dz=0$$

or
$$(2\,x+C_1)(p-q)+2\,z=C_2 \quad\text{or}\quad (x+y)(p-q)+2\,z=C_2.$$

Hence
$$(x+y)(p-q)+2\,z=\Phi(y-x) \tag{27}$$

is a first integral. This is linear and may be integrated by

$$\frac{dx}{x+y}=-\frac{dy}{x+y}=\frac{dz}{\Phi(y-x)-2\,z} \quad\text{or}\quad x+y=K_1, \qquad \frac{dx}{K_1}=\frac{dz}{\Phi(K_1-2\,x)-2\,z}.$$

This equation is an ordinary linear equation in z and x. The integration gives

$$K_1ze^{\frac{2x}{K_1}}=\int e^{\frac{2x}{K_1}}\Phi(K_1-2\,x)\,dx+K_2.$$

Hence
$$(x+y)\,ze^{\frac{2x}{x+y}}-\int e^{\frac{2x}{K_1}}\Phi(K_1-2\,x)\,dx=K_2=\Psi(K_1)=\Psi(x+y)$$

is the general integral of the given equation when K_1 has been replaced by $x+y$ after integration, — an integration which cannot be performed until Φ is given.

The other method of solution would be to use also the second system containing $dy+dx=0$ instead of $dy-dx=0$. Thus in addition to the first integral (27) a second intermediary integral might be sought. The substitution of $dy+dx=0$, $y+x=C_1$ in (A) gives $C_1(dp+dq)+4\,pdx=0$. This equation is not integrable, because $dp+dq$ is a perfect differential and pdx is not. The combination with $dz=pdx+qdy=(p-q)\,dx$ does not improve matters. Hence it is impossible to determine a second intermediary integral, and the method of completing the solution by integrating (27) is the only available method.

Take the equation $ps-qr=0$. Here $S=p$, $R=-q$, $T=V=0$. Then

$$-\,qdy^2-pdxdy=0 \quad\text{or}\quad dy=0, \quad pdx+qdy=0 \quad\text{and}\quad -\,qdydp=0$$

are the equations to work with. The system $dy=0$, $qdydp=0$, $dz=pdx+qdy$, and the system $pdx+qdy=0$, $qdydp=0$, $dz=pdx+qdy$ are not very satisfactory for obtaining an intermediary integral $u_1=\Phi(u_2)$, although $p=\Phi(z)$ is an obvious solution of the first set. It is better to use a method adapted to this special equation. Note that

$$\frac{\partial}{\partial x}\left(\frac{q}{p}\right)=\frac{ps-qr}{p^2}, \quad\text{and}\quad \frac{\partial}{\partial x}\left(\frac{q}{p}\right)=0 \quad\text{gives}\quad \frac{q}{p}=f(y).$$

By (11), p. 124,
$$\frac{q}{p}=-\left(\frac{\partial x}{\partial y}\right)_z; \quad\text{then}\quad \frac{\partial x}{\partial y}=-f(y)$$

and
$$x=-\int f(y)\,dy+\Psi(z)=\Phi(y)+\Psi(z).$$

EXERCISES

1. Integrate these equations and discuss the singular solution:

(α) $p^{\frac{1}{2}} + q^{\frac{1}{2}} = 2x$, (β) $(p^2 + q^2)x = pz$, (γ) $(p + q)(px + qy) = 1$,
(δ) $pq = px + qy$, (ϵ) $p^2 + q^2 = x + y$, (ζ) $xp^2 - 2zp + xy = 0$,
(η) $q^2 = z^2(p - q)$, (θ) $q(p^2z + q^2) = 1$, (ι) $p(1 + q^2) = q(z - c)$,
(κ) $xp(1 + q) = qz$, (λ) $y^2(p^2 - 1) = x^2p^2$, (μ) $z^2(p^2 + q^2 + 1) = c^2$,
(ν) $p = (z + yq)^2$, (o) $pz = 1 + q^2$, (π) $z - pq = 0$, (ρ) $q = xp + p^2$.

2. Show that the rule for the type of Ex. 13, p. 273, can be deduced by Charpit's method. How about the generalized Clairaut form of Ex. 15?

3. (α) For the solution of the type $f_1(x, p) = f_2(y, q)$, the rule is: Set

$$f_1(x, p) = f_2(y, q) = a,$$

and solve for p and q as $p = g_1(x, a)$, $q = g_2(y, a)$; the complete solution is

$$z = \int g_1(x, a)\,dx + \int g_2(y, a)\,dy + b.$$

(β) For the type $F(z, p, q) = 0$ the rule is: Set $X = x + ay$, solve

$$F\left(z, \frac{dz}{dX}, a\frac{dz}{dX}\right) \quad \text{for} \quad \frac{dz}{dX} = \phi(z, a), \quad \text{and let} \quad \int\frac{dz}{\phi(z, a)} = f(z, a);$$

the complete solution is $x + ay + b = f(z, a)$. Discuss these rules in the light of Charpit's method. Establish a rule for the type $F(x + y, p, q) = 0$. Is there any advantage in using the rules over the use of the general method? Assort the examples of Ex. 1 according to these rules as far as possible.

4. What is obtainable for partial differential equations out of any characteristics of homogeneity that may be present?

5. By differentiating $p = f(x, y, z, q)$ successively with respect to x and y show that the expansion of the solution by Taylor's Formula about the point (x_0, y_0, z_0) may be found if the successive derivatives with respect to y alone,

$$\frac{\partial z}{\partial y}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^3 z}{\partial y^3}, \quad \cdots, \quad \frac{\partial^n z}{\partial y^n}, \quad \cdots,$$

are assigned arbitrary values at that point. Note that this arbitrariness allows the solution to be passed through any curve through (x_0, y_0, z_0) in the plane $x = x_0$.

6. Show that $F(x, y, z, p, q) = 0$ satisfies Charpit's equations

$$du = \frac{dx}{-F'_p} = \frac{dy}{-F'_q} = \frac{dz}{-(pF'_p + qF'_q)} = \frac{dp}{F'_x + pF'_z} = \frac{dq}{F'_y + qF'_z}, \tag{28}$$

where u is an auxiliary variable introduced for symmetry. Show that the first three equations are the differential equations of the lineal elements of the cones of Ex. 6, p. 272. The integrals of (28) therefore define a system of curves which have a planar element of the equation $F = 0$ passing through each of their lineal tangential elements. If the equations be integrated and the results be solved for the variables, and if the constants be so determined as to specify one particular curve with the initial conditions x_0, y_0, z_0, p_0, q_0, then

$$x = x(u, x_0, y_0, z_0, p_0, q_0), \quad y = y(\cdots), \; z = z(\cdots), \quad p = p(\cdots), \quad q = q(\cdots).$$

Note that, along the curve, $q = f(p)$ and that consequently the planar elements just mentioned must lie upon a developable surface containing the curve (§ 67). The curve and the planar elements along it are called a characteristic and a *characteristic strip* of the given differential equation. In the case of the linear equation the characteristic curves afforded the integration and any planar element through their lineal tangential elements satisfied the equation; but here it is only those planar elements which constitute the characteristic strip that satisfy the equation. What the complete integral does is to piece the characteristic strips into a family of surfaces dependent on two parameters.

7. By simple devices integrate the equations. Check the answers:

$(\alpha)\ \dfrac{\partial^2 z}{\partial x^2} = f(x),\qquad (\beta)\ \dfrac{\partial^n z}{\partial y^n} = 0,\qquad (\gamma)\ \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a,$

$(\delta)\ s + pf(x) = g(y),\qquad (\epsilon)\ ar = xy,\qquad (\zeta)\ xr = (n-1)p.$

8. Integrate these equations by the method of factoring:

$(\alpha)\ (D_x^2 - a^2 D_y^2)z = 0,\qquad (\beta)\ (D_x - D_y)^3 z = 0,\qquad (\gamma)\ (D_x D_y^2 - D_y^3)z = 0,$

$(\delta)\ (D_x^2 + 3D_xD_y + 2D_y^2)z = x + y,\qquad (\epsilon)\ (D_x^2 - D_xD_y - 6D_y^2)z = xy,$

$(\zeta)\ (D_x^2 - D_y^2 - 3D_x + 3D_y)z = 0,\qquad (\eta)\ (D_x^2 - D_y^2 + 2D_x + 1)z = e^{-x}.$

9. Prove the operational equations:

$(\alpha)\ e^{axD_y}\phi(y) = (1 + axD_y + \tfrac{1}{2}a^2x^2D_y^2 + \cdots)\phi(y) = \phi(y + ax),$

$(\beta)\ \dfrac{1}{D_x - aD_y}0 = e^{axD_y}\dfrac{1}{D_x}0 = e^{axD_y}\phi(y) = \phi(y + ax),$

$(\gamma)\ \dfrac{1}{D_x - aD_y}R(x, y) = e^{axD_y}\displaystyle\int^x e^{-a\xi D_y}R(\xi, y)\,d\xi = \int^x R(\xi, y + ax - a\xi)\,d\xi.$

10. Prove that if $[(D_x - a_1D_y)^{m_1}\cdots(D_x - a_kD_y)^{m_k}]z = 0$, then

$$z = \Phi_{11}(y + a_1x) + x\Phi_{12}(y + a_1x) + \cdots + x^{m_1-1}\Phi_{1m_1}(y + a_1x) + \cdots$$
$$+ \Phi_{k1}(y + a_kx) + x\Phi_{k2}(y + a_kx) + \cdots + x^{m_k-1}\Phi_{km_k}(y + a_kx),$$

where the Φ's are all arbitrary functions. This gives the solution of the reduced equation in the simplest case. What terms would correspond to $(D_x - aD_y - \beta)^m z = 0$?

11. Write the solutions of the equations (or equations reduced) of Ex. 8.

12. State the rule of Ex. 9 (γ) as: Integrate $R(x, y - ax)$ with respect to x and in the result change y to $y + ax$. Apply this to obtaining particular solutions of Ex. 8 (δ), (ϵ), (η) with the aid of any short cuts that are analogous to those of Chap. VIII.

13. Integrate the following equations:

$(\alpha)\ (D_x^2 - D_{xy}^2 + D_y - 1)z = \cos(x + 2y) + e^y,\qquad (\beta)\ x^2r^2 + 2xys + y^2t^2 = x^2 + y^2,$

$(\gamma)\ (D_x^2 + D_{xy} + D_y - 1)z = \sin(x + 2y),\qquad (\delta)\ r - t - 3p + 3q = e^{x+2y},$

$(\epsilon)\ (D_x^3 - 2D_xD_y^2 + D_y^3)z = x^{-2},\qquad (\zeta)\ r - t + p + 3q - 2z = e^{x-y} - x^2y,$

$(\eta)\ (D_x^2 - D_xD_y - 2D_y^2 + 2D_x + 2D_y)z = e^{2x+3y} + \sin(2x + y) + xy.$

14. Try Monge's method on these equations of the second order:

$(\alpha)\ q^2r - 2pqs + p^2t = 0,\qquad (\beta)\ r - a^2t = 0,\qquad (\gamma)\ r + s = -p,$

$(\delta)\ q(1 + q)r - (p + q + 2pq)s + p(1 + p)t = 0,\qquad (\epsilon)\ x^2r + 2xys + y^2t = 0,$

$(\zeta)\ (b + cq)^2r - 2(b + cq)(a + cp)s + (a + cp)^2t = 0,\qquad (\eta)\ r + ka^2t = 2as.$

If any simpler method is available, state what it is and apply it also.

15. Show that an equation of the form $Rr + Ss + Tt + U(rt - s^2) = V$ necessarily arises from the elimination of the arbitrary function from

$$u_1(x, y, z, p, q) = f[u_2(x, y, z, p, q)].$$

Note that only such an equation can have an intermediary integral.

16. Treat the more general equation of Ex. 15 by the methods of the text and thus show that an intermediary integral may be sought by solving one of the systems

$$\begin{array}{ll} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0, & Udx + \lambda_1 Rdy + \lambda_1 Udq = 0, \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0, & Udy + \lambda_2 Tdx + \lambda_2 Udp = 0, \\ dz = pdx + qdy, & dz = pdx + qdy, \end{array}$$

where λ_1 and λ_2 are roots of the equation $\lambda^2(RT + UV) + \lambda US + U^2 = 0$.

17. Solve the equations: (α) $s^2 - rt = 0$, (β) $s^2 - rt = a^2$,
(γ) $ar + bs + ct + e(rt - s^2) = h$, ($\delta$) $xqr + ypt + xy(s^2 - rt) = pq$.