PART II. DIFFERENTIAL EQUATIONS

CHAPTER VII

GENERAL INTRODUCTION TO DIFFERENTIAL EQUATIONS

81. Some geometric problems. The application of the differential calculus to plane curves has given a means of determining some geometric properties of the curves. For instance, the length of the subnormal of a curve (§ 7) is ydy/dx, which in the case of the parabola $y^2 = 4px$ is $2p$, that is, the subnormal is constant. Suppose now it were desired conversely to find all curves for which the subnormal is a given constant m. The statement of this problem is evidently contained in the equation

$$y\frac{dy}{dx} = m \quad \text{or} \quad yy' = m \quad \text{or} \quad ydy = mdx.$$

Again, the radius of curvature of the lemniscate $r^2 = a^2 \cos 2\phi$ is found to be $R = a^2/3r$, that is, the radius of curvature varies inversely as the radius. If conversely it were desired to find all curves for which the radius of curvature varies inversely as the radius of the curve, the statement of the problem would be the equation

$$\frac{\left[r^2 + \left(\dfrac{dr}{d\phi}\right)^2\right]^{\frac{3}{2}}}{r^2 - r\dfrac{d^2r}{d\phi^2} + 2\left(\dfrac{dr}{d\phi}\right)^2} = \frac{k}{r},$$

where k is a constant called a factor of proportionality.*

Equations like these are unlike ordinary algebraic equations because, in addition to the variables x, y or r, ϕ and certain constants m or k, they contain also derivatives, as dy/dx or $dr/d\phi$ and $d^2r/d\phi^2$, of one of the variables with respect to the other. An equation which contains

* Many problems in geometry, mechanics, and physics are stated in terms of variation. For purposes of analysis the statement x varies as y, or $x \propto y$, is written as $x = ky$, introducing a constant k called a factor of proportionality to convert the variation into an equation. In like manner the statement x varies inversely as y, or $x \propto 1/y$, becomes $x = k/y$, and x varies jointly with y and z becomes $x = kyz$.

derivatives is called a *differential equation.* The *order* of the differential equation is the order of the highest derivative it contains. The equations above are respectively of the first and second orders. A differential equation of the first order may be symbolized as $\Phi(x, y, y') = 0$, and one of the second order as $\Phi(x, y, y', y'') = 0$. A function $y = f(x)$ given explicitly or defined implicitly by the relation $F(x, y) = 0$ is said to be a *solution* of a given differential equation if the equation is true for all values of the independent variable x when the expressions for y and its derivatives are substituted in the equation.

Thus to show that (no matter what the value of a is) the relation

$$4ay - x^2 + 2a^2 \log x = 0$$

gives a solution of the differential equation of the second order

$$1 + \left(\frac{dy}{dx}\right)^2 - x^2\left(\frac{d^2y}{dx^2}\right)^2 = 0,$$

it is merely necessary to form the derivatives

$$2a\frac{dy}{dx} = x - \frac{a^2}{x}, \qquad 2a\frac{d^2y}{dx^2} = 1 + \frac{a^2}{x^2}$$

and substitute them in the given equation together with y to see that

$$1 + \left(\frac{dy}{dx}\right)^2 - x^2\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \frac{1}{4a^2}\left(x^2 - 2a^2 + \frac{a^4}{x^2}\right) - \frac{x^2}{4a^2}\left(1 + \frac{2a^2}{x^2} + \frac{a^4}{x^4}\right) = 0$$

is clearly satisfied for all values of x. It appears therefore that the given relation for y is a solution of the given equation.

To *integrate* or *solve* a differential equation is to find all the functions which satisfy the equation. Geometrically speaking, it is to find all the curves which have the property expressed by the equation. In mechanics it is to find all possible motions arising from the given forces. The method of integrating or solving a differential equation depends largely upon the *ingenuity* of the solver. In many cases, however, some method is immediately obvious. For instance if it be possible to *separate the variables,* so that the differential dy is multiplied by a function of y alone and dx by a function of x alone, as in the equation

$$\phi(y)\,dy = \psi(x)\,dx, \quad \text{then} \quad \int \phi(y)\,dy = \int \psi(x)\,dx + C \tag{1}$$

will clearly be the integral or solution of the differential equation.

As an example, let the curves of constant subnormal be determined. Here

$$ydy = mdx \quad \text{and} \quad y^2 = 2mx + C.$$

The variables are already separated and the integration is immediate. The curves are parabolas with semi-latus rectum equal to the constant and with the axis coincident with the axis of x. If in particular it were desired to determine that curve whose subnormal was m and which passed through the origin, it would merely be necessary to substitute (0, 0) in the equation $y^2 = 2mx + C$ to ascertain what particular value must be assigned to C in order that the curve pass through (0, 0). The value is $C = 0$.

Another example might be to determine the curves for which the x-intercept varies as the abscissa of the point of tangency. As the expression (§ 7) for the x-intercept is $x - ydx/dy$, the statement is

$$x - y\frac{dx}{dy} = kx \quad \text{or} \quad (1-k)x = y\frac{dx}{dy}.$$

Hence
$$(1-k)\frac{dy}{y} = \frac{dx}{x} \quad \text{and} \quad (1-k)\log y = \log x + C.$$

If desired, this expression may be changed to another form by using each side of the equality as an exponent with the base e. Then

$$e^{(1-k)\log y} = e^{\log x + C} \quad \text{or} \quad y^{1-k} = e^C x = C'x.$$

As C is an arbitrary constant, the constant $C' = e^C$ is also arbitrary and the solution may simply be written as $y^{1-k} = Cx$, where the accent has been omitted from the constant. If it were desired to pick out that particular curve which passed through the point (1, 1), it would merely be necessary to determine C from the equation

$$1^{1-k} = C\,1, \quad \text{and hence} \quad C = 1.$$

As a third example let the curves whose tangent is constant and equal to a be determined. The length of the tangent is $y\sqrt{1+y'^2}/y'$ and hence the equation is

$$y\frac{\sqrt{1+y'^2}}{y'} = a \quad \text{or} \quad y^2\frac{1+y'^2}{y'^2} = a \quad \text{or} \quad 1 = \frac{\sqrt{a^2-y^2}}{y}y'.$$

The variables are therefore separable and the results are

$$dx = \frac{\sqrt{a^2-y^2}}{y}dy \quad \text{and} \quad x + C = \sqrt{a^2-y^2} - a\log\frac{a+\sqrt{a^2-y^2}}{y}.$$

If it be desired that the tangent at the origin be vertical so that the curve passes through (0, a), the constant C is 0. The curve is the tractrix or "curve of pursuit" as described by a calf dragged at the end of a rope by a person walking along a straight line.

82. Problems which involve the radius of curvature will lead to differential equations of the second order. The method of solving such problems is to *reduce the equation, if possible, to one of the first order.* For the second derivative may be written as

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dy}y', \tag{2}$$

and
$$R = \frac{(1+y'^2)^{\frac{3}{2}}}{y''} = \frac{(1+y'^2)^{\frac{3}{2}}}{\dfrac{dy'}{dx}} = \frac{(1+y'^2)^{\frac{3}{2}}}{y'\dfrac{dy'}{dy}} \tag{2'}$$

is the expression for the radius of curvature. If it be given that the radius of curvature is of the form $f(x)\,\phi(y')$ or $f(y)\,\phi(y')$,

$$\frac{(1+y'^2)^{\frac{3}{2}}}{\dfrac{dy'}{dx}} = f(x)\,\phi(y') \quad \text{or} \quad \frac{(1+y'^2)^{\frac{3}{2}}}{y'\dfrac{dy'}{dy}} = f(y)\,\phi(y'), \tag{3}$$

the variables x and y' or y and y' are immediately separable, and an integration may be performed. This will lead to an equation of the first order; and if the variables are again separable, the solution may be completed by the methods of the above examples.

In the first place consider curves whose radius of curvature is constant. Then

$$\frac{(1+y'^2)^{\frac{3}{2}}}{\dfrac{dy'}{dx}} = a \quad \text{or} \quad \frac{dy'}{(1+y'^2)^{\frac{3}{2}}} = \frac{dx}{a} \quad \text{and} \quad \frac{y'}{\sqrt{1+y'^2}} = \frac{x-C}{a},$$

where the constant of integration has been written as $-C/a$ for future convenience. The equation may now be solved for y' and the variables become separated with the results

$$y' = \frac{x-C}{\sqrt{a^2-(x-C)^2}} \quad \text{or} \quad dy = \frac{(x-C)}{\sqrt{a^2-(x-C)^2}}\,dx.$$

Hence $\quad y - C' = -\sqrt{a^2-(x-C)^2} \quad$ or $\quad (x-C)^2 + (y-C')^2 = a^2.$

The curves, as should be anticipated, are circles of radius a and with any arbitrary point (C, C') as center. It should be noted that, as the solution has required two successive integrations, there are two arbitrary constants C and C' of integration in the result.

As a second example consider the curves whose radius of curvature is double the normal. As the length of the normal is $y\sqrt{1+y'^2}$, the equation becomes

$$\frac{(1+y'^2)^{\frac{3}{2}}}{y'\dfrac{dy'}{dy}} = 2y\sqrt{1+y'^2} \quad \text{or} \quad \frac{1+y'^2}{y'\dfrac{dy'}{dy}} = \pm\, 2y,$$

where the double sign has been introduced when the radical is removed by cancellation. This is necessary; for before the cancellation the signs were ambiguous and there is no reason to assume that the ambiguity disappears. In fact, if the curve is concave up, the second derivative is positive and the radius of curvature is reckoned as positive, whereas the normal is positive or negative according as the curve is above or below the axis of x; similarly, if the curve is concave down. Let the negative sign be chosen. This corresponds to a curve above the axis and concave down, or below the axis and concave up, that is, the normal and the radius of curvature have the same direction. Then

$$\frac{dy}{y} = -\frac{2\,y'dy'}{1+y'^2} \quad \text{and} \quad \log y = -\log(1+y'^2) + \log 2\,C,$$

where the constant has been given the form $\log 2\,C$ for convenience. This expression may be thrown into algebraic form by exponentiation, solved for y', and then

$$y(1+y'^2)=2C \quad \text{or} \quad y'^2=\frac{2C-y}{y} \quad \text{or} \quad \frac{ydy}{\sqrt{2Cy-y^2}}=dx.$$

Hence $$x-C'=C\,\text{vers}^{-1}\frac{y}{C}-\sqrt{2Cy-y^2}.$$

The curves are cycloids of which the generating circle has an arbitrary radius C and of which the cusps are upon the x-axis at the points $C'\pm 2k\pi C$. If the positive sign had been taken in the equation, the curves would have been entirely different; see Ex. 5 (α).

The number of arbitrary constants of integration which enter into the solution of a differential equation depends on the number of integrations which are performed and is equal to the order of the equation. This results in giving a family of curves, dependent on one or more parameters, as the solution of the equation. To pick out any particular member of the family, additional conditions must be given. Thus, if there is only one constant of integration, the curve may be required to pass through a given point; if there are two constants, the curve may be required to pass through a given point and have a given slope at that point, or to pass through two given points. These additional conditions are called *initial conditions*. In mechanics the initial conditions are very important; for the point reached by a particle describing a curve under the action of assigned forces depends not only on the forces, but on the point at which the particle started and the velocity with which it started. In all cases the distinction between the *constants of integration* and the *given constants of the problem* (in the foregoing examples, the distinction between C, C' and m, k, a) should be kept clearly in mind

EXERCISES

1. Verify the solutions of the differential equations:

(α) $xy+\frac{1}{2}x^2=C,\ y+x+xy'=0,$ (β) $x^3y^3(3e^x+C)=1,\ xy'+y+x^4y^4e^x=0,$
(γ) $(1+x^2)y'^2=1,\ 2x=Ce^y-C^{-1}e^{-y},$ (δ) $y+xy'=x^4y'^2,\ xy=C^2x+C,$
(ϵ) $y''+y'/x=0,\ y=C\log x+C_1,$ (ζ) $y=Ce^x+C_1e^{2x},\ y''+2y=3y',$
(η) $y'''-y=x^2,\ y=Ce^x+e^{-\frac{1}{2}x}\left(C_1\cos\frac{x\sqrt{3}}{2}+C_2\sin\frac{x\sqrt{3}}{2}\right)-x^2.$

2. Determine the curves which have the following properties:

(α) The subtangent is constant; $y^m=Ce^x$. If through (2, 2), $y^m=2^me^{x-2}$.

(β) The right triangle formed by the tangent, subtangent, and ordinate has the constant area $k/2$; the hyperbolas $xy+Cy+k=0$. Show that if the curve passes through (1, 2) and (2, 1), the arbitrary constant C is 0 and the given k is -2.

(γ) The normal is constant in length; the circles $(x-C)^2+y^2=k^2$.

(δ) The normal varies as the square of the ordinate; catenaries $ky=\cosh k(x-C)$. If in particular the curve is perpendicular to the y-axis, $C=0$.

(ϵ) The area of the right triangle formed by the tangent, normal, and x-axis is inversely proportional to the slope; the circles $(x-C)^2+y^2=k$.

3. Determine the curves which have the following properties:

(α) The angle between the radius vector and tangent is constant; spirals $r = Ce^{k\phi}$.

(β) The angle between the radius vector and tangent is half that between the radius and initial line; cardioids $r = C(1 - \cos\phi)$.

(γ) The perpendicular from the pole to a tangent is constant; $r\cos(\phi - C) = k$.

(δ) The tangent is equally inclined to the radius vector and to the initial line; the two sets of parabolas $r = C/(1 \pm \cos\phi)$.

(ϵ) The radius is equally inclined to the normal and to the initial line; circles $r = C\cos\phi$ or lines $r\cos\phi = C$.

4. The arc s of a curve is proportional to the area A, where in rectangular coördinates A is the area under the curve and in polar coördinates it is the area included by the curve and the radius vectors. From the equation $ds = dA$ show that the curves which satisfy the condition are catenaries for rectangular coördinates and lines for polar coördinates.

5. Determine the curves for which the radius of curvature

(α) is twice the normal and oppositely directed; parabolas $(x - C)^2 = C'(2y - C')$.

(β) is equal to the normal and in same direction; circles $(x - C)^2 + y^2 = C'^2$.

(γ) is equal to the normal and in opposite direction; catenaries.

(δ) varies as the cube of the normal; conics $kCy^2 - C^2(x + C')^2 = k$.

(ϵ) projected on the x-axis equals the abscissa; catenaries.

(ζ) projected on the x-axis is the negative of the abscissa; circles.

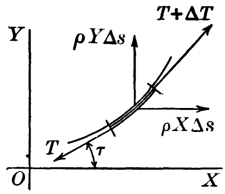
(η) projected on the x-axis is twice the abscissa.

(θ) is proportional to the slope of the tangent or of the normal.

83. Problems in mechanics and physics. In many physical problems the statement involves an equation between the *rate of change* of some quantity and the value of that quantity. In this way the solution of the problem is made to depend on the integration of a differential equation of the first order. If x denotes any quantity, the rate of increase in x is dx/dt and the rate of decrease in x is $-dx/dt$; and consequently when the rate of change of x is a function of x, the variables are immediately separated and the integration may be performed. The constant of integration has to be determined from the initial conditions; the constants inherent in the problem may be given in advance or their values may be determined by comparing x and t at some subsequent time. The exercises offered below will exemplify the treatment of such problems.

In other physical problems the statement of the question as a differential equation is not so direct and is carried out by an examination of the problem with a view to stating a relation between the increments or differentials of the dependent and independent variables, as in some geometric relations already discussed (§ 40), and in the problem of the tension in a rope wrapped around a cylindrical post discussed below.

The method may be further illustrated by the derivation of the differential equations of the curve of equilibrium of a flexible string or chain. Let ρ be the density of the chain so that $\rho\Delta s$ is the mass of the length Δs; let X and Y be the components of the force (estimated per unit mass) acting on the elements of the chain. Let T denote the tension in the chain, and τ the inclination of the element of chain. From the figure it then appears that the components of all the forces acting on Δs are



$$(T + \Delta T)\cos(\tau + \Delta\tau) - T\cos\tau + X\rho\Delta s = 0,$$
$$(T + \Delta T)\sin(\tau + \Delta\tau) - T\sin\tau + Y\rho\Delta s = 0;$$

for these must be zero if the element is to be in a position of equilibrium. The equations may be written in the form

$$\Delta(T\cos\tau) + X\rho\Delta s = 0, \qquad \Delta(T\sin\tau) + Y\rho\Delta s = 0;$$

and if they now be divided by Δs and if Δs be allowed to approach zero, the result is the two equations of equilibrium

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + \rho X = 0, \qquad \frac{d}{ds}\left(T\frac{dy}{ds}\right) + \rho Y = 0, \tag{4}$$

where $\cos\tau$ and $\sin\tau$ are replaced by their values dx/ds and dy/ds.

If the string is acted on only by forces parallel to a given direction, let the y-axis be taken as parallel to that direction. Then the component X will be zero and the first equation may be integrated. The result is

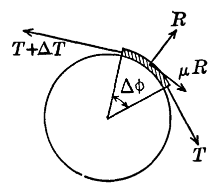
$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) = 0, \qquad T\frac{dx}{ds} = C, \qquad T = C\frac{ds}{dx}.$$

This value of T may be substituted in the second equation. There is thus obtained a differential equation of the second order

$$\frac{d}{ds}\left(C\frac{dy}{dx}\right) + \rho Y = 0 \quad \text{or} \quad C\frac{y''}{\sqrt{1 + y'^2}} + \rho Y = 0. \tag{4'}$$

If this equation can be integrated, the form of the curve of equilibrium may be found.

Another problem of a different nature in strings is to consider the variation of the tension in a rope wound around a cylinder without overlapping. The forces acting on the element Δs of the rope are the tensions T and $T + \Delta T$, the normal pressure or reaction R of the cylinder, and the force of friction which is proportional to the pressure. It will be assumed that the normal reaction lies in the angle $\Delta\phi$ and that the coefficient of friction is μ so that the force of friction is μR. The components along the radius and along the tangent are



$$(T + \Delta T)\sin \Delta\phi - R\cos(\theta\Delta\phi) - \mu R\sin(\theta\Delta\phi) = 0, \qquad 0<\theta<1,$$
$$(T + \Delta T)\cos \Delta\phi + R\sin(\theta\Delta\phi) - \mu R\cos(\theta\Delta\phi) - T = 0.$$

Now discard all infinitesimals except those of the first order. It must be borne in mind that the pressure R is the reaction on the infinitesimal arc Δs and hence is itself infinitesimal. The substitutions are therefore $Td\phi$ for $(T + \Delta T)\sin\Delta\phi$, R for $R\cos\theta\Delta\phi$, 0 for $R\sin\theta\Delta\phi$, and $T + dT$ for $(T+\Delta T)\cos\Delta\phi$. The equations therefore reduce to two simple equations

$$Td\phi - R = 0, \qquad dT - \mu R = 0,$$

from which the unknown R may be eliminated with the result

$$dT = \mu Td\phi \quad \text{or} \quad T = Ce^{\mu\phi} \quad \text{or} \quad T = T_0e^{\mu\phi},$$

where T_0 is the tension when ϕ is 0. The tension therefore runs up exponentially and affords ample explanation of why a man, by winding a rope about a post, can readily hold a ship or other object exerting a great force at the other end of the rope. If μ is 1/3, three turns about the post will hold a force 535 T_0, or over 25 tons, if the man exerts a force of a hundredweight.

84. If a constant mass m is moving along a line under the influence of a force F acting along the line, Newton's Second Law of Motion (p. 13) states the problem of the motion as the differential equation

$$mf = F \quad \text{or} \quad m\frac{d^2x}{dt^2} = F \tag{5}$$

of the second order; and it therefore appears that the complete solution of a problem in rectilinear motion requires the integration of this equation. The acceleration may be written as

$$f = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx};$$

and hence the equation of motion takes either of the forms

$$m\frac{dv}{dt} = F \quad \text{or} \quad mv\frac{dv}{dx} = F. \tag{5'}$$

It now appears that there are several cases in which the first integration may be performed. For if the force is a function of the velocity or of the time or a product of two such functions, the variables are separated in the first form of the equation; whereas if the force is a function of the velocity or of the coördinate x or a product of two such functions, the variables are separated in the second form of the equation.

When the first integration is performed according to either of these methods, there will arise an equation between the velocity and either the time t or the coördinate x. In this equation will be contained a constant of integration which may be determined by the initial conditions, that is, by the knowledge of the velocity at the start, whether in time or in position. Finally it will be possible (at least theoretically) to solve the equation and express the velocity as a function of the time t or of the position x, as the case may be, and integrate a second time. The carrying through in practice of this sketch of the work will be exemplified in the following two examples.

Suppose a particle of mass m is projected vertically upward with the velocity V. Solve the problem of the motion under the assumption that the resistance of the air varies as the velocity of the particle. Let the distance be measured vertically upward. The forces acting on the particle are two, — the force of gravity which is the weight $W = mg$, and the resistance of the air which is kv. Both these forces are negative because they are directed toward diminishing values of x. Hence

$$mf = -mg - kv \quad \text{or} \quad m\frac{dv}{dt} = -mg - kv,$$

where the first form of the equation of motion has been chosen, although in this case the second form would be equally available. Then integrate.

$$\frac{dv}{g + \frac{k}{m}v} = -dt \quad \text{and} \quad \log\left(g + \frac{k}{m}v\right) = -\frac{k}{m}t + C.$$

As by the initial conditions $v = V$ when $t = 0$, the constant C is found from

$$\log\left(g + \frac{k}{m}V\right) = -\frac{k}{m}0 + C; \quad \text{hence} \quad \frac{g + \frac{k}{m}v}{g + \frac{k}{m}V} = e^{-\frac{k}{m}t}$$

is the relation between v and t found by substituting the value of C. The solution for v gives

$$v = \frac{dx}{dt} = \left(\frac{m}{k}g + V\right)e^{-\frac{k}{m}t} - \frac{m}{k}g.$$

Hence
$$x = -\frac{m}{k}\left(\frac{m}{k}g + V\right)e^{-\frac{k}{m}t} - \frac{m}{k}gt + C.$$

If the particle starts from the origin $x = 0$, the constant C is found to be

$$C = \frac{m}{k}\left(\frac{m}{k}g + V\right) \quad \text{and} \quad x = \frac{m}{k}\left(\frac{m}{k}g + V\right)\left(1 - e^{-\frac{k}{m}t}\right) - \frac{m}{k}gt.$$

Hence the position of the particle is expressed in terms of the time and the problem is solved. If it be desired to find the time which elapses before the particle comes to rest and starts to drop back, it is merely necessary to substitute $v = 0$ in the relation connecting the velocity and the time, and solve for the time $t = T$; and if this value of t be substituted in the expression for x, the total distance X covered in the ascent will be found. The results are

$$T = \frac{m}{k}\log\left(1 + \frac{k}{mg}V\right), \qquad X = \left(\frac{m}{k}\right)^2\left[\frac{k}{m}V - g\log\left(1 + \frac{k}{mg}V\right)\right].$$

As a second example consider the motion of a particle vibrating up and down at the end of an elastic string held in the field of gravity. By Hooke's Law for elastic strings the force exerted by the string is proportional to the extension of the string over its natural length, that is, $F = k\Delta l$. Let l be the length of the string, $\Delta_0 l$ the extension of the string just sufficient to hold the weight $W = mg$ at rest so that $k\Delta_0 l = mg$, and let x measured downward be the additional extension of the string at any instant of the motion. The force of gravity mg is positive and the force of elasticity $-k(\Delta_0 l + x)$ is negative. The second form of the equation of motion is to be chosen. Hence

$$mv\frac{dv}{dx} = mg - k(\Delta_0 l + x) \quad \text{or} \quad mv\frac{dv}{dx} = -kx, \quad \text{since} \quad mg = k\Delta_0 l.$$

Then $$mvdv = -kxdx \quad \text{or} \quad mv^2 = -kx^2 + C.$$

Suppose that $x = a$ is the amplitude of the motion, so that when $x = a$ the velocity $v = 0$ and the particle stops and starts back. Then $C = ka^2$. Hence

$$v = \frac{dx}{dt} = \sqrt{\frac{k}{m}}\sqrt{a^2 - x^2} \quad \text{or} \quad \frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\frac{k}{m}}\,dt,$$

and $$\sin^{-1}\frac{x}{a} = \sqrt{\frac{k}{m}}\,t + C \quad \text{or} \quad x = a\sin\left(\sqrt{\frac{k}{m}}\,t + C\right).$$

Now let the time be measured from the instant when the particle passes through the position $x = 0$. Then C satisfies the equation $0 = a\sin C$ and may be taken as zero. The motion is therefore given by the equation $x = a\sin\sqrt{k/m}t$ and is periodic. While t changes by $2\pi\sqrt{m/k}$ the particle completes an entire oscillation. The time $T = 2\pi\sqrt{m/k}$ is called the *periodic time*. The motion considered in this example is characterized by the fact that the total force $-kx$ is proportional to the displacement from a certain origin and is directed toward the origin. Motion of this sort is called *simple harmonic motion* (briefly S. H. M.) and is of great importance in mechanics and physics.

EXERCISES

1. The sum of $100 is put at interest at 4 per cent per annum under the condition that the interest shall be compounded at each instant. Show that the sum will amount to $200 in 17 yr. 4 mo., and to $1000 in 57½ yr.

2. Given that the rate of decomposition of an amount x of a given substance is proportional to the amount of the substance remaining undecomposed. Solve the problem of the decomposition and determine the constant of integration and the physical constant of proportionality if $x = 5.11$ when $t = 0$ and $x = 1.48$ when $t = 40$ min. *Ans.* $k = .0309$.

3. A substance is undergoing transformation into another at a rate which is assumed to be proportional to the amount of the substance still remaining untransformed. If that amount is 35.6 when $t = 1$ hr. and 13.8 when $t = 4$ hr., determine the amount at the start when $t = 0$ and the constant of proportionality and find how many hours will elapse before only one-thousandth of the original amount will remain.

4. If the activity A of a radioactive deposit is proportional to its rate of diminution and is found to decrease to $\frac{1}{2}$ its initial value in 4 days, show that A satisfies the equation $A/A_0 = e^{-0.173t}$.

5. Suppose that amounts a and b respectively of two substances are involved in a reaction in which the velocity of transformation dx/dt is proportional to the product $(a-x)(b-x)$ of the amounts remaining untransformed. Integrate on the supposition that $a \neq b$.

$$\log \frac{b(a-x)}{a(b-x)} = (a-b)kt; \quad \text{and if} \quad \begin{array}{c|c|c} t & a-x & b-x \\ \hline 393 & 0.4866 & 0.2342 \\ 1265 & 0.3879 & 0.1354 \end{array}$$

determine the product $k(a-b)$.

6. Integrate the equation of Ex. 5 if $a = b$, and determine a and k if $x = 9.87$ when $t = 15$ and $x = 13.69$ when $t = 55$.

7. If the velocity of a chemical reaction in which three substances are involved is proportional to the continued product of the amounts of the substances remaining, show that the equation between x and the time is

$$\frac{\log\left(\dfrac{a}{a-x}\right)^{b-c}\left(\dfrac{b}{b-x}\right)^{c-a}\left(\dfrac{c}{c-x}\right)^{a-b}}{(a-b)(b-c)(c-a)} = -kt, \quad \text{where} \quad \begin{cases} x = 0 \\ t = 0. \end{cases}$$

8. Solve Ex. 7 if $a = b \neq c$; also when $a = b = c$. Note the very different forms of the solution in the three cases.

9. The rate at which water runs out of a tank through a small pipe issuing horizontally near the bottom of the tank is proportional to the square root of the height of the surface of the water above the pipe. If the tank is cylindrical and half empties in 30 min., show that it will completely empty in about 100 min.

10. Discuss Ex. 9 in case the tank were a right cone or frustum of a cone.

11. Consider a vertical column of air and assume that the pressure at any level is due to the weight of the air above. Show that $p = p_0 e^{-kh}$ gives the pressure at any height h, if Boyle's Law that the density of a gas varies as the pressure be used.

12. Work Ex. 11 under the assumption that the adiabatic law $p \propto \rho^{1.4}$ represents the conditions in the atmosphere. Show that in this case the pressure would become zero at a finite height. (If the proper numerical data are inserted, the height turns out to be about 20 miles. The adiabatic law seems to correspond better to the facts than Boyle's Law.)

13. Let l be the natural length of an elastic string, let Δl be the extension, and assume Hooke's Law that the force is proportional to the extension in the form $\Delta l = klF$. Let the string be held in a vertical position so as to elongate under its own weight W. Show that the elongation is $\frac{1}{2}kWl$.

14. The density of water under a pressure of p atmospheres is $\rho = 1 + 0.00004p$. Show that the surface of an ocean six miles deep is about 600 ft. below the position it would have if water were incompressible.

15. Show that the equations of the curve of equilibrium of a string or chain are

$$\frac{d}{ds}\left(T\frac{dr}{ds}\right) + \rho R = 0, \qquad \frac{d}{ds}\left(T\frac{rd\phi}{ds}\right) + \rho\Phi = 0$$

in polar coördinates, where R and Φ are the components of the force along the radius vector and perpendicular to it.

16. Show that $dT + \rho S ds = 0$ and $T + \rho RN = 0$ are the equations of equilibrium of a string if R is the radius of curvature and S and N are the tangential and normal components of the forces.

17.* Show that when a uniform chain is supported at two points and hangs down between the points under its own weight, the curve of equilibrium is the catenary.

18. Suppose the mass dm of the element ds of a chain is proportional to the projection dx of ds on the x-axis, and that the chain hangs in the field of gravity. Show that the curve is a parabola. (This is essentially the problem of the shape of the cables in a suspension bridge when the roadbed is of uniform linear density; for the weight of the cables is negligible compared to that of the roadbed.)

19. It is desired to string upon a cord a great many uniform heavy rods of varying lengths so that when the chord is hung up with the rods dangling from it the rods will be equally spaced along the horizontal and have their lower ends on the same level. Required the shape the chord will take. (It should be noted that the shape must be known before the rods can be cut in the proper lengths to hang as desired.) The weight of the chord may be neglected.

20. A masonry arch carries a horizontal roadbed. On the assumption that the material between the arch and the roadbed is of uniform density and that each element of the arch supports the weight of the material above it, find the shape of the arch.

21. In equations (4′) the integration may be carried through in terms of quadratures if ρY is a function of y alone; and similarly in Ex. 15 the integration may be carried through if $\Phi = 0$ and ρR is a function of r alone so that the field is central. Show that the results of thus carrying through the integration are the formulas

$$x + C' = \int \frac{C dy}{\sqrt{(\int \rho Y dy)^2 - C^2}}, \qquad \phi + C' = \int \frac{C dr/r}{\sqrt{(\int \rho R dr)^2 - C^2}}.$$

22. A particle falls from rest through the air, which is assumed to offer a resistance proportional to the velocity. Solve the problem with the initial conditions $v = 0$, $x = 0$, $t = 0$. Show that as the particle falls, the velocity does not increase indefinitely, but approaches a definite limit $V = mg/k$.

23. Solve Ex. 22 with the initial conditions $v = v_0$, $x = 0$, $t = 0$, where v_0 is greater than the limiting velocity V. Show that the particle slows down as it falls.

24. A particle rises through the air, which is assumed to resist proportionally to the square of the velocity. Solve the motion.

25. Solve the problem analogous to Ex. 24 for a falling particle. Show that there is a limiting velocity $V = \sqrt{mg/k}$. If the particle were projected down with an initial velocity greater than V, it would slow down as in Ex. 23.

26. A particle falls towards a point which attracts it inversely as the square of the distance and directly as its mass. Find the relation between x and t and determine the total time T taken to reach the center. Initial conditions $v = 0$, $x = a$, $t = 0$.

$$\sqrt{\frac{2k}{a}}\, t = \frac{a}{2} \cos^{-1} \frac{2x - a}{a} + \sqrt{ax - x^2}, \qquad T = \pi k^{-\frac{1}{2}} \left(\frac{a}{2}\right)^{\frac{3}{2}}.$$

* Exercises 17–20 should be worked *ab initio* by the method by which (4) were derived, not by applying (4) directly.

27. A particle starts from the origin with a velocity V and moves in a medium which resists proportionally to the velocity. Find the relations between velocity and distance, velocity and time, and distance and time; also the limiting distance traversed.

$$v = V - kx/m, \qquad v = Ve^{-\frac{k}{m}t}, \qquad kx = mV(1 - e^{-\frac{k}{m}t}), \qquad mV/k.$$

28. Solve Ex. 27 under the assumption that the resistance varies as $\sqrt{v}$.

29. A particle falls toward a point which attracts inversely as the cube of the distance and directly as the mass. The initial conditions are $x = a$, $v = 0$, $t = 0$. Show that $x^2 = a^2 - kt^2/a^2$ and the total time of descent is $T = a^2/\sqrt{k}$.

30. A cylindrical spar buoy stands vertically in the water. The buoy is pressed down a little and released. Show that, if the resistance of the water and air be neglected, the motion is simple harmonic. Integrate and determine the constants from the initial conditions $x = 0$, $v = V$, $t = 0$, where x measures the displaçement from the position of equilibrium.

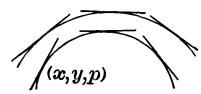
31. A particle slides down a rough inclined plane. Determine the motion. Note that of the force of gravity only the component $mg \sin i$ acts down the plane, whereas the component $mg \cos i$ acts perpendicularly to the plane and develops the force $\mu mg \cos i$ of friction. Here i is the inclination of the plane and μ is the coefficient of friction.

32. A bead is free to move upon a frictionless wire in the form of an inverted cycloid (vertex down). Show that the component of the weight along the tangent to the cycloid is proportional to the distance of the particle from the vertex. Hence determine the motion as simple harmonic and fix the constants of integration by the initial conditions that the particle starts from rest at the top of the cycloid.

33. Two equal weights are hanging at the end of an elastic string. One drops off. Determine completely the motion of the particle remaining.

34. One end of an elastic spring (such as is used in a spring balance) is attached rigidly to a point on a horizontal table. To the other end a particle is attached. If the particle be held at such a point that the spring is elongated by the amount a and then released, determine the motion on the assumption that the coefficient of friction between the particle and the table is μ; and discuss the possibility of different cases according as the force of friction is small or large relative to the force exerted by the spring.

85. Lineal element and differential equation. The idea of a curve as made up of the points upon it is familiar. Points, however, have no extension and therefore must be regarded not as pieces of a curve but merely as positions on it. Strictly speaking, the pieces of a curve are the elements Δs of arc; but for many purposes it is convenient to replace the complicated element Δs by a piece of the tangent to the curve at some point of the arc Δs, and from this point of view a curve is made up of an infinite number of infinitesimal elements tangent to it. This is analogous to the point of view by which a curve is regarded as made up of an infinite number of infinitesimal chords and is intimately related to the conception of the curve as the envelope of its tangents (§ 65). A point on a curve taken with an infinitesimal portion of the tangent to the curve at that point is called a *lineal element* of the curve. These concepts and definitions are clearly equally available in two or three dimensions. For the present the curves under discussion will be plane curves and the lineal elements will therefore all lie in a plane.



To specify any particular lineal element *three coördinates* x, y, p will be used, of which the two (x, y) determine the point through which the element passes and of which the third p is the slope of the element. If a curve $f(x, y) = 0$ is given, the slope at any point may be found by differentiation,

$$p = \frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}, \tag{6}$$

and hence the third coördinate p of the lineal elements of this particular curve is expressed in terms of the other two. If in place of one curve $f(x, y) = 0$ the whole family of curves $f(x, y) = C$, where C is an arbitrary constant, had been given, the slope p would still be found from (6), and it therefore appears that the third coördinate of the lineal elements of such a family of curves is expressible in terms of x and y.

In the more general case where the family of curves is given in the unsolved form $F(x, y, C) = 0$, the slope p is found by the same formula but it now depends apparently on C in addition to on x and y. If, however, the constant C be eliminated from the two equations

$$F(x, y, C) = 0 \quad \text{and} \quad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} p = 0, \tag{7}$$

there will arise an equation $\Phi(x, y, p) = 0$ which connects the slope p of any curve of the family with the coördinates (x, y) of any point through which a curve of the family passes and at which the slope of that curve is p. Hence it appears that the three coördinates (x, y, p) of the lineal elements of all the curves of a family are connected by an equation $\Phi(x, y, p) = 0$, just as the coördinates (x, y, z) of the points of a surface are connected by an equation $\Phi(x, y, z) = 0$. As the equation $\Phi(x, y, z) = 0$ is called the equation of the surface, so the equation $\Phi(x, y, p) = 0$ is called the equation of the family of curves; it is, however, not the finite equation $F(x, y, C) = 0$ but the differential equation of the family, because it involves the derivative $p = dy/dx$ of y by x instead of the parameter C.

As an example of the elimination of a constant, consider the case of the parabolas

$$y^2 = Cx \quad \text{or} \quad y^2/x = C.$$

The differentiation of the equation in the second form gives at once

$$-\,y^2/x^2 + 2\,yp/x = 0 \quad \text{or} \quad y = 2\,xp$$

as the differential equation of the family. In the unsolved form the work is

$$2\,yp = C, \qquad y^2 = 2\,ypx, \qquad y = 2\,xp.$$

The result is, of course, the same in either case. For the family here treated it makes little difference which method is followed. As a general rule it is perhaps best to solve for the constant if the solution is simple and leads to a simple form of the function $f(x, y)$; whereas if the solution is not simple or the form of the function is complicated, it is best to differentiate first because the differentiated equation may be simpler to solve for the constant than the original equation, or because the elimination of the constant between the two equations can be conducted advantageously.

If an equation $\Phi(x, y, p) = 0$ connecting the three coördinates of the lineal element be given, the elements which satisfy the equation may be plotted much as a surface is plotted; that is, a pair of values (x, y) may be assumed and substituted in the equation, the equation may then be solved for one or more values of p, and lineal elements with these values of p may be drawn through the point (x, y). In this manner the elements through as many points as desired may be found. The detached elements are of interest and significance chiefly from the fact that they can be *assembled into curves*, — in fact, into the curves of a family $F(x, y, C) = 0$ of which the equation $\Phi(x, y, p) = 0$ is the differential equation. This is the converse of the problem treated above and requires the integration of the differential equation $\Phi(x, y, p) = 0$ for its solution. In some simple cases the assembling may be accomplished intuitively from the geometric properties implied in the equation, in other cases it follows from the integration of the equation by analytic means, in other cases it can be done only approximately and by methods of computation.

As an example of intuitively assembling the lineal elements into curves, take

$$\Phi(x, y, p) = y^2p^2 + y^2 - r^2 = 0 \quad \text{or} \quad p = \pm\frac{\sqrt{r^2 - y^2}}{y}.$$

The quantity $\sqrt{r^2 - y^2}$ may be interpreted as one leg of a right triangle of which y is the other leg and r the hypotenuse. The slope of the hypotenuse is then $\pm\, y/\sqrt{r^2 - y^2}$ according to the position of the figure, and the differential equation $\Phi(x, y, p) = 0$ states that the coördinate p of the lineal element which satisfies it is the negative reciprocal of this slope. Hence the lineal element is perpendicular to the hypotenuse. It therefore appears that the lineal elements are tangent to circles of radius r described about points of the x-axis. The equation of these circles is

$(x-C)^2+y^2=r^2$, and this is therefore the integral of the differential equation. The correctness of this integral may be checked by direct integration. For

$$p=\frac{dy}{dx}=\pm\frac{\sqrt{r^2-y^2}}{y}\quad\text{or}\quad\frac{ydy}{\sqrt{r^2-y^2}}=dx\quad\text{or}\quad\sqrt{r^2-y^2}=x-C.$$

86. In geometric problems which relate the slope of the tangent of a curve to other lines in the figure, it is clear that not the tangent but the lineal element is the vital thing. Among such problems that of the *orthogonal trajectories* (or trajectories under any angle) of a given family of curves is of especial importance. If two families of curves are so related that the angle at which any curve of one of the families cuts any curve of the other family is a right angle, then the curves of either family are said to be the orthogonal trajectories of the curves of the other family. Hence at any point (x, y) at which two curves belonging to the different families intersect, there are two lineal elements, one belonging to each curve, which are perpendicular. As the slopes of two perpendicular lines are the negative reciprocals of each other, it follows that if the coördinates of one lineal element are (x, y, p) the coördinates of the other are $(x, y, -1/p)$; and if the coördinates of the lineal element (x, y, p) satisfy the equation $\Phi(x, y, p)=0$, the coördinates of the orthogonal lineal element must satisfy $\Phi(x, y, -1/p)=0$. Therefore the *rule for finding the orthogonal trajectories of the curves $F(x, y, C)=0$ is to find first the differential equation $\Phi(x, y, p)=0$ of the family, then to replace p by $-1/p$ to find the differential equation of the orthogonal family, and finally to integrate this equation to find the family.* It may be noted that if $F(z)=X(x, y)+iY(x, y)$ is a function of $z=x+iy$ (§ 73), the families $X(x, y)=C$ and $Y(x, y)=K$ are orthogonal.

As a problem in orthogonal trajectories find the trajectories of the semicubical parabolas $(x-C)^3=y^2$. The differential equation of this family is found as

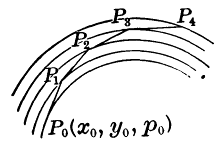
$$3(x-C)^2=2yp,\qquad x-C=(\tfrac{2}{3}yp)^{\frac{1}{2}},\qquad (\tfrac{2}{3}yp)^{\frac{3}{2}}=y^2\quad\text{or}\quad \tfrac{2}{3}p=y^{\frac{1}{3}}.$$

This is the differential equation of the given family. Replace p by $-1/p$ and integrate:

$$-\frac{2}{3p}=y^{\frac{1}{3}}\quad\text{or}\quad 1+\frac{3}{2}py^{\frac{1}{3}}=0\quad\text{or}\quad dx+\frac{3}{2}y^{\frac{1}{3}}dy=0,\quad\text{and}\quad x+\frac{9}{8}y^{\frac{4}{3}}=C.$$

Thus the differential equation and finite equation of the orthogonal family are found. The curves look something like parabolas with axis horizontal and vertex toward the right.

Given a differential equation $\Phi(x, y, p)=0$ or, in solved form, $p=\phi(x, y)$; the *lineal element affords a means for obtaining graphically and numerically an approximation to the solution which passes through* *an assigned point* $P_0(x_0, y_0)$. For the value p_0 of p at this point may be computed from the equation and a lineal element P_0P_1 may be drawn, the length being taken small. As the lineal element is tangent to the curve, its end point will not lie upon the curve but will depart from it by an infinitesimal of higher order. Next the slope p_1 of the lineal element which satisfies the equation and passes through P_1 may be found and the element P_1P_2 may be drawn. This element will not be tangent to the desired solution but to a solution lying near that one. Next the element P_2P_3 may be drawn, and so on. The broken line $P_0P_1P_2P_3\cdots$ is clearly an approximation to the solution and will be a better approximation the shorter the elements P_iP_{i+1} are taken. If the radius of curvature of the solution at P_0 is not great, the curve will be bending rapidly and the elements must be taken fairly short in order to get a fair approximation; but if the radius of curvature is great, the elements need not be taken so small. (This method of approximate graphical solution indicates a method which is of value in proving by the method of limits that the equation $p = \phi(x, y)$ actually has a solution; but that matter will not be treated here.)



Let it be required to plot approximately that solution of $yp + x = 0$ which passes through (0, 1) and thus to find the ordinate for $x = 0.5$, and the area under the curve and the length of the curve to this point. Instead of assuming the lengths of the successive lineal elements, let the lengths of successive increments δx of x be taken as $\delta x = 0.1$. At the start $x_0 = 0$, $y_0 = 1$, and from $p = -x/y$ it follows that $p_0 = 0$. The increment δy of y acquired in moving along the tangent is $\delta y = p\delta x = 0$. Hence the new point of departure (x_1, y_1) is (0.1, 1) and the new slope is $p_1 = -x_1/y_1 = -0.1$. The results of the work, as it is continued, may be grouped in the table. Hence it appears that the final ordinate is $y = 0.90$. By adding up the trapezoids the area is computed as 0.48, and by finding the elements $\delta s = \sqrt{\delta x^2 + \delta y^2}$ the length is found as 0.51. Now the particular equation here treated can be integrated.

i	δx	δy	x_i	y_i	p_i
0	...	...	0.	1.00	0.
1	0.1	0.	0.1	1.00	− 0.1
2	0.1	− 0.01	0.2	0.99	− 0.2
3	0.1	− 0.02	0.3	0.97	− 0.31
4	0.1	− 0.03	0.4	0.94	− 0.43
5	0.1	− 0.04	0.5	0.90	...

$$yp + x = 0, \qquad ydy + xdx = 0, \qquad x^2 + y^2 = C, \quad \text{and hence} \quad x^2 + y^2 = 1$$

is the solution which passes through (0, 1). The ordinate, area, and length found from the curve are therefore 0.87, 0.48, 0.52 respectively. The errors in the approximate results to two places are therefore respectively 3, 0, 2 per cent. If δx had been chosen as 0.01 and four places had been kept in the computations, the errors would have been smaller.

EXERCISES

1. In the following cases eliminate the constant C to find the differential equation of the family given:

(α) $x^2 = 2\,Cy + C^2$, (β) $y = Cx + \sqrt{1 - C^2}$,

(γ) $x^2 - y^2 = Cx$, (δ) $y = x \tan (x + C)$,

(ϵ) $\dfrac{x^2}{a^2 - C} + \dfrac{y^2}{b^2 - C} = 1$, *Ans.* $\left(\dfrac{dy}{dx}\right)^2 + \dfrac{(x^2 - y^2) - (a^2 - b^2)}{xy}\dfrac{dy}{dx} - 1 = 0.$

2. Plot the lineal elements and intuitively assemble them into the solution:

$$(\alpha)\ yp + x = 0, \qquad (\beta)\ xp - y = 0, \qquad (\gamma)\ r\frac{d\phi}{dr} = 1.$$

Check the results by direct integration of the differential equations.

3. Lines drawn from the points $(\pm\, c, 0)$ to the lineal element are equally inclined to it. Show that the differential equation is that of Ex. 1 (ϵ). What are the curves?

4. The trapezoidal area under the lineal element equals the sectorial area formed by joining the origin to the extremities of the element (disregarding infinitesimals of higher order). (α) Find the differential equation and integrate. (β) Solve the same problem where the areas are equal in magnitude but opposite in sign. What are the curves?

5. Find the orthogonal trajectories of the following families. Sketch the curves.

(α) parabolas $y^2 = 2\,Cx$, *Ans.* ellipses $2\,x^2 + y^2 = C$.

(β) exponentials $y = Ce^{kx}$, *Ans.* parabolas $\frac{1}{2}\,ky^2 + x = C$.

(γ) circles $(x - C)^2 + y^2 = a^2$, *Ans.* tractrices.

(δ) $x^2 - y^2 = C^2$, (ϵ) $Cy^2 = x^3$, (ζ) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = C^{\frac{2}{3}}$.

6. Show from the answer to Ex. 1 (ϵ) that the family is self-orthogonal and illustrate with a sketch. From the fact that the lineal element of a parabola makes equal angles with the axis and with the line drawn to the focus, derive the differential equation of all coaxial confocal parabolas and show that the family is self-orthogonal.

7. If $\Phi(x, y, p) = 0$ is the differential equation of a family, show

$$\Phi\left(x, y, \frac{p - m}{1 + mp}\right) = 0 \quad \text{and} \quad \Phi\left(x, y, \frac{p + m}{1 - mp}\right) = 0$$

are the differential equations of the family whose curves cut those of the given family at $\tan^{-1} m$. What is the difference between these two cases?

8. Show that the differential equations

$$\Phi\left(\frac{dr}{d\phi}, r, \phi\right) = 0 \quad \text{and} \quad \Phi\left(-r^2\frac{d\phi}{dr}, r, \phi\right) = 0$$

define orthogonal families in polar coördinates, and write the equation of the family which cuts the first of these at the constant angle $\tan^{-1} m$.

9. Find the orthogonal trajectories of the following families. Sketch.

(α) $r = e^{C\phi}$, (β) $r = C(1 - \cos\phi)$, (γ) $r = C\phi$, (δ) $r^2 = C^2 \cos 2\,\phi$.

10. Recompute the approximate solution of $yp + x = 0$ under the conditions of the text but with $\delta x = 0.05$, and carry the work to three decimals.

11. Plot the approximate solution of $p = xy$ between $(1, 1)$ and the y-axis. Take $\delta x = -0.2$. Find the ordinate, area, and length. Check by integration and comparison.

12. Plot the approximate solution of $p = -x$ through $(1, 1)$, taking $\delta x = 0.1$ and following the curve to its intersection with the x-axis. Find also the area and the length.

13. Plot the solution of $p = \sqrt{x^2 + y^2}$ from the point $(0, 1)$ to its intersection with the x-axis. Take $\delta x = -0.2$ and find the area and length.

14. Plot the solution of $p = s$ which starts from the origin into the first quadrant (s is the length of the arc). Take $\delta x = 0.1$ and carry the work for five steps to find the final ordinate, the area, and the length. Compare with the true integral.

87. The higher derivatives; analytic approximations. Although a differential equation $\Phi(x, y, y') = 0$ does not determine the relation between x and y without the application of some process equivalent to integration, it does afford a means of computing the higher derivatives simply by differentiation. Thus

$$\frac{d\Phi}{dx} = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} y' + \frac{\partial \Phi}{\partial y'} y'' = 0$$

is an equation which may be solved for y'' as a function of x, y, y'; and y'' may therefore be expressed in terms of x and y by means of $\Phi(x, y, y') = 0$. A further differentiation gives the equation

$$\frac{d^2\Phi}{dx^2} = \frac{\partial^2 \Phi}{\partial x^2} + 2\frac{\partial^2 \Phi}{\partial x \partial y} y' + 2\frac{\partial^2 \Phi}{\partial x \partial y'} y'' + \frac{\partial^2 \Phi}{\partial y^2} y'^2 + 2\frac{\partial^2 \Phi}{\partial y \partial y'} y'y''$$
$$+ \frac{\partial^2 \Phi}{\partial y'^2} y''^2 + \frac{\partial \Phi}{\partial y} y'' + \frac{\partial \Phi}{\partial y'} y''' = 0,$$

which may be solved for y''' in terms of x, y, y', y''; and hence, by the preceding results, y''' is expressible as a function of x and y; and so on to all the higher derivatives. In this way any property of the integrals of $\Phi(x, y, y') = 0$ which, like the radius of curvature, is expressible in terms of the derivatives, may be found as a function of x and y.

As the differential equation $\Phi(x, y, y') = 0$ defines y' and all the higher derivatives as functions of x, y, it is clear that the values of the derivatives may be found as $y_0', y_0'', y_0''', \cdots$ at any given point (x_0, y_0). Hence it is possible to write the series

$$y = y_0 + y_0'(x - x_0) + \tfrac{1}{2} y_0''(x - x_0)^2 + \tfrac{1}{6} y_0'''(x - x_0)^3 + \cdots. \qquad (8)$$

If this power series in $x - x_0$ converges, it defines y as a function of x for values of x near x_0; it is indeed *the Taylor development of the* *function* y (§ 167). The convergence is assumed. Then

$$y' = y_0' + y_0''(x - x_0) + \tfrac{1}{2} y_0'''(x - x_0)^2 + \cdots.$$

It may be shown that the function y defined by the series actually satisfies the differential equation $\Phi(x, y, y') = 0$, that is, that

$$\Omega(x) = \Phi[x, y_0 + y_0'(x - x_0) + \tfrac{1}{2} y_0''(x - x_0)^2 + \cdots, y_0' + y_0''(x - x_0) + \cdots] = 0$$

for all values of x near x_0. To prove this accurately, however, is beyond the scope of the present discussion; the fact may be taken for granted. Hence an analytic expansion for the integral of a differential equation has been found.

As an example of computation with higher derivatives let it be required to determine the radius of curvature of that solution of $y' = \tan(y/x)$ which passes through $(1, 1)$. Here the slope $y'_{(1,1)}$ at $(1, 1)$ is $\tan 1 = 1.557$. The second derivative is

$$y'' = \frac{dy'}{dx} = \frac{d}{dx} \tan \frac{y}{x} = \sec^2 \frac{y}{x} \frac{xy' - y}{x^2}.$$

From these data the radius of curvature is found to be

$$R = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''} = \sec \frac{y}{x} \frac{x^2}{xy' - y}, \qquad R_{(1,1)} = \sec 1 \frac{1}{\tan 1 - 1} = 3.250.$$

The equation of the circle of curvature may also be found. For as $y''_{(1,1)}$ is positive, the curve is concave up. Hence $(1 - 3.250 \sin 1,\ 1 + 3.250 \cos 1)$ is the center of curvature; and the circle is

$$(x + 1.735)^2 + (y - 2.757)^2 = (3.250)^2.$$

As a second example let four terms of the expansion of that integral of $x \tan y' = y$ which passes through $(2, 1)$ be found. The differential equation may be solved; then

$$\frac{dy}{dx} = \tan^{-1}\left(\frac{y}{x}\right), \qquad \frac{d^2y}{dx^2} = \frac{xy' - y}{x^2 + y^2},$$

$$\frac{d^3y}{dx^3} = \frac{(x^2 + y^2)(x - 1)y'' + (3y^2 - x^2)y' - 2xyy'^2 + 2xy}{(x^2 + y^2)^2}.$$

Now it must be noted that the problem is not wholly determinate; for y' is multiple valued and any one of the values for $\tan^{-1}\frac{1}{2}$ may be taken as the slope of a solution through $(2, 1)$. Suppose that the angle be taken in the first quadrant; then $\tan^{-1}\frac{1}{2} = 0.462$. Substituting this in y'', we find $y''_{(2,1)} = -0.0152$; and hence may be found $y'''_{(2,1)} = 0.110$. The series for y to four terms is therefore

$$y = 1 + 0.462(x - 2) - 0.0076(x - 2)^2 + 0.018(x - 3)^3.$$

It may be noted that it is generally simpler not to express the higher derivatives in terms of x and y, but to compute each one successively from the preceding ones.

88. Picard has given a method for the integration of the equation $y' = \phi(x, y)$ by *successive approximations* which, although of the highest theoretic value and importance, is not particularly suitable to analytic uses in finding an approximate solution. The method is this. Let the equation $y' = \phi(x, y)$ be given in solved form, and suppose (x_0, y_0) is the point through which the solution is to pass. To find the first approximation let y be held constant and equal to y_0, and integrate the equation $y' = \phi(x, y_0)$. Thus

$$dy = \phi(x, y_0)\,dx; \qquad y = y_0 + \int_{x_0}^{x} \phi(x, y_0)\,dx = f_1(x), \tag{9}$$

where it will be noticed that the constant of integration has been chosen so that the curve passes through (x_0, y_0). For the second approximation let y have the value just found, substitute this in $\phi(x, y)$, and integrate again. Then

$$y = y_0 + \int_{x_0}^{x} \phi\left[x,\, y_0 + \int_{x_0}^{x} \phi(x, y_0)\,dx\right] dx = f_2(x). \tag{9'}$$

With this new value for y continue as before. The successive determinations of y as a function of x actually converge toward a limiting function which is a solution of the equation and which passes through (x_0, y_0). It may be noted that at each step of the work an integration is required. The difficulty of actually performing this integration in formal practice limits the usefulness of the method in such cases. It is clear, however, that with an integrating machine such as the integraph the method could be applied as rapidly as the curves $\phi(x, f_i(x))$ could be plotted.

To see how the method works, consider the integration of $y' = x + y$ to find the integral through (1, 1). For the first approximation $y = 1$. Then

$$dy = (x + 1)\,dx, \qquad y = \tfrac{1}{2}x^2 + x + C, \qquad y = \tfrac{1}{2}x^2 + x - \tfrac{1}{2} = f_1(x).$$

From this value of y the next approximation may be found, and then still another:

$$dy = [x + (\tfrac{1}{2}x^2 + x - \tfrac{1}{2})]\,dx, \qquad y = \tfrac{1}{6}x^3 + x^2 - \tfrac{1}{2}x + \tfrac{1}{3} = f_2(x),$$
$$dy = [x + f_2(x)]\,dx, \qquad y = \tfrac{1}{24}x^4 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^2 + \tfrac{1}{3}x + \tfrac{1}{24}.$$

In this case there are no difficulties which would prevent any number of applications of the method. In fact it is evident that if y' is a polynomial in x and y, the result of any number of applications of the method will be a polynomial in x.

The method of *undetermined coefficients* may often be employed to advantage to develop the solution of a differential equation into a series. The result is of course identical with that obtained by the application of successive differentiation and Taylor's series as above; the work is sometimes shorter. Let the equation be in the form $y' = \phi(x, y)$ and assume an integral in the form

$$y = y_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots. \tag{10}$$

Then $\phi(x, y)$ may also be expanded into a series, say,

$$\phi(x, y) = A_0 + A_1(x - x_0) + A_2(x - x_0)^2 + A_3(x - x_0)^3 + \cdots.$$

But by differentiating the assumed form for y we have

$$y' = a_1 + 2\,a_2(x - x_0) + 3\,a_3(x - x_0)^2 + 4\,a_4(x - x_0)^3 + \cdots.$$

Thus there arise two different expressions as series in $x - x_0$ for the function y', and therefore the corresponding coefficients must be equal. The resulting set of equations

$$a_1 = A_0, \quad 2\,a_2 = A_1, \quad 3\,a_3 = A_2, \quad 4\,a_4 = A_3, \quad \cdots \tag{11}$$

may be solved successively for the undetermined coefficients a_1, a_2, a_3, a_4, $\cdots$ which enter into the assumed expansion. This method is particularly useful when the form of the differential equation is such that some of the terms may be omitted from the assumed expansion (see Ex. 14).

As an example in the use of undetermined coefficients consider that solution of the equation $y' = \sqrt{x^2 + 3\,y^2}$ which passes through (1, 1). The expansion will proceed according to powers of $x - 1$, and for convenience the variable may be changed to $t = x - 1$ so that

$$\frac{dy}{dt} = \sqrt{(t+1)^2 + 3\,y^2}, \qquad y = 1 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + \cdots$$

are the equation and the assumed expansion. One expression for y' is

$$y' = a_1 + 2\,a_2t + 3\,a_3t^2 + 4\,a_4t^3 + \cdots.$$

To find the other it is necessary to expand into a series in t the expression

$$y' = \sqrt{(1+t)^2 + 3\,(1 + a_1t + a_2t^2 + a_3t^3)^2}.$$

If this had to be done by Maclaurin's series, nothing would be gained over the method of § 87; but in this and many other cases algebraic methods and known expansions may be applied (§ 32). First square y and retain only terms up to the third power. Hence

$$y' = 2\sqrt{1 + \tfrac{1}{2}(1 + 3\,a_1)\,t + \tfrac{1}{4}\left(1 + 6\,a_2 + 3\,a_1^2\right)t^2 + \tfrac{3}{2}\,(a_1a_2 + a_3)\,t^3}.$$

Now let the quantity under the radical be called $1 + h$ and expand so that

$$y' = 2\sqrt{1+h} = 2\,(1 + \tfrac{1}{2}h - \tfrac{1}{8}h^2 + \tfrac{1}{16}h^3).$$

Finally raise h to the indicated powers and collect in powers of t. Then

$$y' = 2 + \tfrac{1}{2}(1 + 3\,a_1)\left|\begin{array}{l} \hfill t \\ {} \end{array}\right. \left.\begin{array}{l} \hfill t^2 \\ +\,\tfrac{1}{4}\left(1 + 6\,a_2 + 3\,a_1^2\right) \\ -\,\tfrac{1}{16}(1 + 3\,a_1)^2 \\ {} \end{array}\right| \left.\begin{array}{l} \hfill t^3 \\ +\,\tfrac{3}{2}\,(a_1a_2 + a_3) \\ -\,\tfrac{1}{16}\,(1 + 3\,a_1)\left(1 + 6\,a_2 + 3\,a_1^2\right) \\ +\,\tfrac{1}{64}\,(1 + 3\,a_1)^3 \end{array}\right|.$$

Hence the successive equations for determining the coefficients are $a_1 = 2$ and

$$2a_2 = \tfrac{1}{2}(1 + 3a_1) \text{ or } a_2 = \tfrac{7}{4},$$
$$3a_3 = \tfrac{1}{4}(1 + 6a_2 + 3a_1^2) - \tfrac{1}{16}(1 + 3a_1)^2 \text{ or } a_3 = \tfrac{15}{16},$$
$$4a_4 = \tfrac{3}{2}(a_1a_2 + a_3) - \tfrac{1}{16}(1 + 3a_1)(1 + 6a_2 + 3a_1^2) + \tfrac{1}{64}(1 + 3a_1)^3 \text{ or } a_4 = \tfrac{111}{256}.$$

Therefore to five terms the expansion desired is

$$y = 1 + 2(x-1) + \tfrac{7}{4}(x-1)^2 + \tfrac{15}{16}(x-1)^3 + \tfrac{111}{256}(x-1)^4.$$

The methods of developing a solution by Taylor's series or by undetermined coefficients apply equally well to equations of higher order. For example consider an equation of the second order in solved form $y'' = \phi(x, y, y')$ and its derivatives

$$y''' = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}y' + \frac{\partial\phi}{\partial y'}y''$$

$$y^{\text{iv}} = \frac{\partial^2\phi}{\partial x^2} + 2\frac{\partial^2\phi}{\partial x\partial y}y' + 2\frac{\partial^2\phi}{\partial x\partial y'}y'' + \frac{\partial^2\phi}{\partial y^2}y'^2 + 2\frac{\partial^2\phi}{\partial y\partial y'}y'y'' + \frac{\partial^2\phi}{\partial y'^2}y''^2 + \frac{\partial\phi}{\partial y}y'' + \frac{\partial\phi}{\partial y'}y'''.$$

Evidently the higher derivatives of y may be obtained in terms of x, y, y'; and y itself may be written in the expanded form

$$y = y_0 + y_0'(x - x_0) + \tfrac{1}{2}y_0''(x - x_0)^2 + \tfrac{1}{6}y_0'''(x - x_0)^3 + \tfrac{1}{24}y_0^{\text{iv}}(x - x_0)^4 + \cdots, \tag{12}$$

where any desired values may be attributed to the ordinate y_0 at which the curve cuts the line $x = x_0$, and to the slope y_0' of the curve at that point. Moreover the coefficients y_0'', y_0''', $\cdots$ are determined in such a way that they depend on the assumed values of y_0 and y_0'. It therefore is seen that the solution (12) of the differential equation of the second order really involves two arbitrary constants, and the justification of writing it as $F(x, y, C_1, C_2) = 0$ is clear.

In following out the method of undetermined coefficients a solution of the equation would be assumed in the form

$$y = y_0 + y_0'(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + a_4(x - x_0)^4 + \cdots, \tag{13}$$

from which y' and y'' would be obtained by differentiation. Then if the series for y and y' be substituted in $y'' = \phi(x, y, y')$ and the result arranged as a series, a second expression for y'' is obtained and the comparison of the coefficients in the two series will afford a set of equations from which the successive coefficients may be found in terms of y_0 and y_0' by solution. These results may clearly be generalized to the case of differential equations of the nth order, whereof the solutions will depend on n arbitrary constants, namely, the values assumed for y and its first $n - 1$ derivatives when $x = x_0$.

EXERCISES

1. Find the radii and circles of curvature of the solutions of the following equations at the points indicated:

(α) $y' = \sqrt{x^2 + y^2}$ at $(0, 1)$, (β) $yy' + x = 0$ at (x_0, y_0).

2. Find $y'''_{(1,1)} = (5\sqrt{2} - 2)/4$ if $y' = \sqrt{x^2 + y^2}$.

3. Given the equation $y^2y'^3 + xyy'^2 - yy' + x^2 = 0$ of the third degree in y' so that there will be three solutions with different slopes through any ordinary point (x, y). Find the radii of curvature of the three solutions through $(0, 1)$.

4. Find three terms in the expansion of the solution of $y' = e^{xy}$ about $(2, \frac{1}{2})$.

5. Find four terms in the expansion of the solution of $y = \log \sin xy$ about $(\frac{1}{2}\pi, 1)$.

6. Expand the solution of $y' = xy$ about $(1, y_0)$ to five terms.

7. Expand the solution of $y' = \tan(y/x)$ about $(1, 0)$ to four terms. Note that here x should be expanded in terms of y, not y in terms of x.

8. Expand two of the solutions of $y^2y'^3 + xyy'^2 - yy' + x^2 = 0$ about $(-2, 1)$ to four terms.

9. Obtain four successive approximations to the integral of $y' = xy$ through $(1, 1)$.

10. Find four successive approximations to the integral of $y' = x + y$ through $(0, y_0)$.

11. Show by successive approximations that the integral of $y' = y$ through $(0, y_0)$ is the well-known $y = y_0e^x$.

12. Carry the approximations to the solution of $y' = -x/y$ through $(0, 1)$ as far as you can integrate, and plot each approximation on the same figure with the exact integral.

13. Find by the method of undetermined coefficients the number of terms indicated in the expansions of the solutions of these differential equations about the points given:

(α) $y' = \sqrt{x + y}$, five terms, $(0, 1)$, (β) $y' = \sqrt{x + y}$, four terms, $(1, 3)$,
(γ) $y' = x + y$, n terms, $(0, y_0)$, (δ) $y' = \sqrt{x^2 + y^2}$, four terms, $(\frac{3}{8}, \frac{1}{4})$.

14. If the solution of an equation is to be expanded about $(0, y_0)$ and if the change of x into $-x$ and y' into $-y'$ does not alter the equation, the solution is necessarily symmetric with respect to the y-axis and the expansion may be assumed to contain only even powers of x. If the solution is to be expanded about $(0, 0)$ and a change of x into $-x$ and y into $-y$ does not alter the equation, the solution is symmetric with respect to the origin and the expansion may be assumed in odd powers. Obtain the expansions to four terms in the following cases and compare the labor involved in the method of undetermined coefficients with that which would be involved in performing the requisite six or seven differentiations for the application of Maclaurin's series:

(α) $y' = \dfrac{x}{\sqrt{x^2 + y^2}}$ about $(0, 2)$, (β) $y' = \sin xy$ about $(0, 1)$,

(γ) $y' = e^{xy}$ about $(0, 0)$, (δ) $y' = x^3y + xy^3$ about $(0, 0)$.

15. Expand to and including the term x^4:

(α) $y'' = y'^2 + xy$ about $x_0 = 0$, $y_0 = a_0$, $y'_0 = a_1$ (by both methods),
(β) $xy'' + y' + y = 0$ about $x_0 = 0$, $y_0 = a_0$, $y'_0 = -a_0$ (by und. coeffs.).