## PARTIAL DIFFERENTIATION; IMPLICIT FUNCTIONS

56. The simplest case; $F(x, y) = 0$. The total differential

$$dF = F_x' dx + F_y' dy = d0 = 0$$

indicates
$$\frac{dy}{dx} = -\frac{F_x'}{F_y'}, \qquad \frac{dx}{dy} = -\frac{F_y'}{F_x'} \tag{1}$$

as the derivative of y by x, or of x by y, where y is defined as a function of x, or x as a function of y, by the relation $F(x, y) = 0$; and this method of obtaining a derivative of an *implicit function* without solving explicitly for the function has probably been familiar long before the notion of a partial derivative was obtained. The relation $F(x, y) = 0$ is pictured as a curve, and the function $y = \phi(x)$, which would be obtained by solution, is considered as multiple valued or as restricted to some definite portion or branch of the curve $F(x, y) = 0$. If the results (1) are to be applied to find the derivative at some point (x_0, y_0) of the curve $F(x, y) = 0$, it is necessary that at that point the denominator F_y' or F_x' should not vanish.
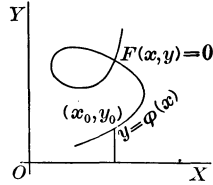
These pictorial and somewhat vague notions may be stated precisely as a *theorem* susceptible of proof, namely: Let x_0 be any real value of x such that 1°, the equation $F(x_0, y) = 0$ has a real solution y_0; and 2°, the function $F(x, y)$ regarded as a function of two independent variables (x, y) is continuous and has continuous first partial derivatives F_x', F_y' in the neighborhood of (x_0, y_0); and 3°, the derivative $F_y'(x_0, y_0) \neq 0$ does not vanish for (x_0, y_0); then $F(x, y) = 0$ may be solved (theoretically) as $y = \phi(x)$ in the vicinity of $x = x_0$ and in such a manner that $y_0 = \phi(x_0)$, that $\phi(x)$ is continuous in x, and that $\phi(x)$ has a derivative $\phi'(x) = -F_x'/F_y'$; and the solution is unique. This is the fundamental theorem on implicit functions for the simple case, and the proof follows.

By the conditions on F_x', F_y', the Theorem of the Mean is applicable. Hence

$$F(x, y) - F(x_0, y_0) = F(x, y) = (hF_x' + kF_y')_{x_0 + \theta h,\, y_0 + \theta k}. \tag{2}$$

Furthermore, in any square $|h| < \delta$, $|k| < \delta$ surrounding (x_0, y_0) and sufficiently small, the continuity of F_x' insures $|F_x'| < M$ and the continuity of F_y' taken with

the fact that $F_y'(x_0, y_0) \neq 0$ insures $|F_y'| > m$. Consider the range of x as further restricted to values such that $|x - x_0| < m\delta/M$ if $m < M$. Now consider the value of $F(x, y)$ for any x in the permissible interval and for $y = y_0 + \delta$ or $y = y_0 - \delta$. As $|kF_y'| > m\delta$ but $|(x - x_0) F_x'| < m\delta$, it follows from (2) that $F(x, y_0 + \delta)$ has the sign of $\delta F_y'$ and $F(x, y_0 - \delta)$ has the sign of $- \delta F_y'$; and as the sign of F_y' does not change, $F(x, y_0 + \delta)$ and $F(x, y_0 - \delta)$ have opposite signs. Hence by Ex. 10, p. 45, there is one and only one value of y between $y_0 - \delta$ and $y_0 + \delta$ such that $F(x, y) = 0$. Thus for each x in the interval there is one and only one y such that $F(x, y) = 0$. The equation $F(x, y) = 0$ has a unique solution near (x_0, y_0). Let $y = \phi(x)$ denote the solution. The solution is continuous at $x = x_0$ because $|y - y_0| < \delta$. If (x, y) are restricted to values $y = \phi(x)$ such that $F(x, y) = 0$, equation (2) gives at once
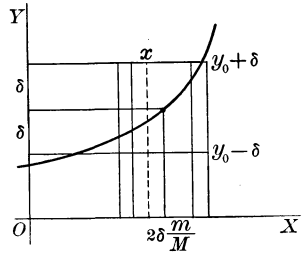
$$\frac{k}{h} = \frac{y - y_0}{x - x_0} = \frac{\Delta y}{\Delta x} = - \frac{F_x'(x + \theta h, y + \theta k)}{F_y'(x + \theta h, y + \theta k)}, \qquad \frac{dy}{dx} = - \frac{F_x'(x_0, y_0)}{F_y'(x_0, y_0)}.$$

As F_x', F_y' are continuous and $F_y' \neq 0$, the fraction k/h approaches a limit and the derivative $\phi'(x_0)$ exists and is given by (1). The same reasoning would apply to any point x in the interval. The theorem is completely proved. It may be added that the expression for $\phi'(x)$ is such as to show that $\phi'(x)$ itself is continuous.

The values of higher derivatives of implicit functions are obtainable by successive total differentiation as

$$F_x' + F_y'y' = 0,$$
$$F_{xx}'' + 2 F_{xy}''y' + F_{yy}''y'^2 + F_y'y'' = 0, \qquad (3)$$

etc. It is noteworthy that these successive equations may be solved for the derivative of highest order by dividing by F_y' which has been assumed not to vanish. The question of whether the function $y = \phi(x)$ defined implicitly by $F(x, y) = 0$ has derivatives of order higher than the first may be seen by these equations to depend on whether $F(x, y)$ has higher partial derivatives which are continuous in (x, y).

57. To find the *maxima and minima* of $y = \phi(x)$, that is, to find the points where the tangent to $F(x, y) = 0$ is parallel to the x-axis, observe that at such points $y' = 0$. Equations (3) give

$$F_x' = 0, \qquad F_{xx}'' + F_y'y'' = 0. \qquad (4)$$

Hence always under the assumption that $F_y' \neq 0$, *there are maxima at the intersections of $F = 0$ and $F_x' = 0$ if F_{xx}'' and F_y' have the same sign, and minima at the intersections for which F_{xx}'' and F_y' have opposite signs;* the case $F_{xx}'' = 0$ still remains undecided.

For example if $F(x, y) = x^3 + y^3 - 3\,axy = 0$, the derivatives are

$$3\,(x^2 - ay) + 3\,(y^2 - ax)\,y' = 0, \qquad \frac{dy}{dx} = -\frac{x^2 - ay}{y^2 - ax},$$

$$6\,x - 6\,ay' + 6\,yy'^2 + 3\,(y^2 - ax)\,y'' = 0, \qquad \frac{d^2y}{dx^2} = -\frac{2\,a^3xy}{(y^2 - ax)^3}.$$

To find the maxima or minima of y as a function of x, solve

$$F_x' = 0 = x^2 - ay, \qquad F = 0 = x^3 + y^3 - 3\,axy, \qquad F_y' \neq 0.$$

The real solutions of $F_x' = 0$ and $F = 0$ are $(0, 0)$ and $\left(\sqrt[3]{2}\,a,\ \sqrt[3]{4}\,a\right)$ of which the first must be discarded because $F_y'(0, 0) = 0$. At $\left(\sqrt[3]{2}\,a,\ \sqrt[3]{4}\,a\right)$ the derivatives F_y' and F_{xx}'' are positive; and the point is a maximum. The curve $F = 0$ is the folium of Descartes.

The rôle of the variables x and y may be interchanged if $F_x' \neq 0$ and the equation $F(x, y) = 0$ may be solved for $x = \psi(y)$, the functions ϕ and ψ being inverse. In this way the vertical tangents to the curve $F = 0$ may be discussed. For the points of $F = 0$ at which both $F_x' = 0$ and $F_y' = 0$, the equation cannot be solved in the sense here defined. Such points are called *singular points* of the curve. The questions of the singular points of $F = 0$ and of maxima, minima, or minimax (§ 55) of the surface $z = F(x, y)$ are related. For if $F_x' = F_y' = 0$, the surface has a tangent plane parallel to $z = 0$, and if the condition $z = F = 0$ is also satisfied, the surface is tangent to the xy-plane. Now if $z = F(x, y)$ has a maximum or minimum at its point of tangency with $z = 0$, the surface lies entirely on one side of the plane and the point of tangency is an isolated point of $F(x, y) = 0$; whereas if the surface has a minimax it cuts through the plane $z = 0$ and the point of tangency is not an isolated point of $F(x, y) = 0$. The shape of the curve $F = 0$ in the neighborhood of a singular point is discussed by developing $F(x, y)$ about that point by Taylor's Formula.

For example, consider the curve $F(x, y) = x^3 + y^3 - x^2y^2 - \frac{1}{2}(x^2 + y^2) = 0$ and the surface $z = F(x, y)$. The common real solutions of

$$F_x' = 3\,x^2 - 2\,xy^2 - x = 0, \qquad F_y' = 3\,y^2 - 2\,x^2y - y = 0, \qquad F(x, y) = 0$$

are the singular points. The real solutions of $F_x' = 0$, $F_y' = 0$ are $(0, 0)$, $(1, 1)$, $(\frac{1}{2}, \frac{1}{2})$ and of these the first two satisfy $F(x, y) = 0$ but the last does not. The singular points of the curve are therefore $(0, 0)$ and $(1, 1)$. The test (34) of § 55 shows that $(0, 0)$ is a maximum for $z = F(x, y)$ and hence an isolated point of $F(x, y) = 0$. The test also shows that $(1, 1)$ is a minimax. To discuss the curve $F(x, y) = 0$ near $(1, 1)$ apply Taylor's Formula.

$$\begin{aligned}
0 = F(x, y) &= \tfrac{1}{2}(3\,h^2 - 8\,hk + 3\,k^2) + \tfrac{1}{6}(6\,h^3 - 12\,h^2k - 12\,hk^2 + 6\,k^3) + \text{remainder} \\
&= \tfrac{1}{2}(3\cos^2\phi - 8\sin\phi\cos\phi + 3\sin^2\phi) \\
&\qquad + r(\cos^3\phi - 2\cos^2\phi\sin\phi - 2\cos\phi\sin^2\phi + \sin^3\phi) + \cdots.
\end{aligned}$$

if polar coördinates $h = r \cos \phi$, $k = r \sin \phi$ be introduced at (1, 1) and r^2 be canceled. Now for very small values of r, the equation can be satisfied only when the first parenthesis is very small. Hence the solutions of

$$3 - 4 \sin 2 \phi = 0, \qquad \sin 2 \phi = \tfrac{3}{4}, \qquad \text{or} \qquad \phi = 24^\circ 17\tfrac{1}{2}', \ 65^\circ 42\tfrac{1}{2}',$$

and $\phi + \pi$, are the directions of the tangents to $F(x, y) = 0$. The equation $F = 0$ is

$$0 = (1\tfrac{1}{2} - 2 \sin 2 \phi) + r (\cos \phi + \sin \phi) (1 - 1\tfrac{1}{2} \sin 2 \phi)$$

if only the first two terms are kept, and this will serve to sketch $F(x, y) = 0$ for very small values of r, that is, for ϕ very near to the tangent directions.

58. It is important to obtain conditions for the maximum or minimum of a function $z = f(x, y)$ where the variables x, y are connected by a relation $F(x, y) = 0$ so that z really becomes a function of x alone or y alone. For it is not always possible, and frequently it is inconvenient, to solve $F(x, y) = 0$ for either variable and thus eliminate that variable from $z = f(x, y)$ by substitution. When the variables x, y in $z = f(x, y)$ are thus connected, the minimum or maximum is called a *constrained minimum* or *maximum ;* when there is no equation $F(x, y) = 0$ between them the minimum or maximum is called *free* if any designation is needed.* The conditions are obtained by differentiating $z = f(x, y)$ and $F(x, y) = 0$ totally with respect to x. Thus

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0, \qquad \frac{d0}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0,$$

and
$$\frac{\partial f}{\partial x} \frac{\partial F}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial F}{\partial x} = 0, \qquad \frac{d^2 z}{dx^2} \gtreqless 0, \qquad F = 0, \tag{5}$$

where the first equation arises from the two above by eliminating dy/dx and the second is added to insure a minimum or maximum, are the conditions desired. Note that all singular points of $F(x, y) = 0$ satisfy the first condition identically, but that the process by means of which it was obtained excludes such points, and that the rule cannot be expected to apply to them.

Another method of treating the problem of constrained maxima and minima is to introduce *a multiplier* and form the function

$$z = \Phi(x, y) = f(x, y) + \lambda F(x, y), \qquad \lambda \text{ a multiplier.} \tag{6}$$

Now if this function z is to have a free maximum or minimum, then

$$\Phi'_x = f'_x + \lambda F'_x = 0, \qquad \Phi'_y = f'_y + \lambda F'_y = 0. \tag{7}$$

These two equations taken with $F = 0$ constitute a set of three from which the three values x, y, λ may be obtained by solution. Note that

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* The adjective " relative " is sometimes used for constrained, and "absolute" for free; but the term "absolute" is best kept for the greatest of the maxima or least of the minima, and the term " relative " for the other maxima and minima.

λ cannot be obtained from (7) if both F'_x and F'_y vanish; and hence this method also rejects the singular points. That this method really determines the constrained maxima and minima of $f(x, y)$ subject to the constraint $F(x, y) = 0$ is seen from the fact that if λ be eliminated from (7) the condition $f''_x F'_y - f''_y F'_x = 0$ of (5) is obtained. The new method is therefore identical with the former, and its introduction is more a matter of convenience than necessity. It is possible to show directly that the new method gives the constrained maxima and minima. For the conditions (7) are those of a free extreme for the function $\Phi(x, y)$ which depends on two independent variables (x, y). Now if the equations (7) be solved for (x, y), it appears that the position of the maximum or minimum will be expressed in terms of λ as a parameter and that consequently the point $(x(\lambda), y(\lambda))$ cannot in general lie on the curve $F(x, y) = 0$; but if λ be so determined that the point shall lie on this curve, the function $\Phi(x, y)$ has a free extreme at a point for which $F = 0$ and hence in particular must have a constrained extreme for the particular values for which $F(x, y) = 0$. In speaking of (7) as the conditions for an extreme, the conditions which should be imposed on the second derivative have been disregarded.

For example, suppose the maximum radius vector from the origin to the folium of Descartes were desired. The problem is to render $f(x, y) = x^2 + y^2$ maximum subject to the condition $F(x, y) = x^3 + y^3 - 3\,axy = 0$. Hence

$$2\,x + 3\,\lambda\,(x^2 - ay) = 0, \qquad 2\,y + 3\,\lambda\,(y^2 - ax) = 0, \qquad x^3 + y^3 - 3\,axy = 0$$

or $$2\,x \cdot 3\,(y^2 - ax) - 2\,y \cdot 3\,(x^2 - ay) = 0, \qquad x^3 + y^3 - 3\,axy = 0$$

are the conditions in the two cases. These equations may be solved for $(0, 0)$, $(1\frac{1}{2}\,a, 1\frac{1}{2}\,a)$, and some imaginary values. The value $(0, 0)$ is singular and λ cannot be determined, but the point is evidently a minimum of $x^2 + y^2$ by inspection. The point $(1\frac{1}{2}\,a, 1\frac{1}{2}\,a)$ gives $\lambda = -1\frac{1}{3}\,a$. That the point is a (relative constrained) maximum of $x^2 + y^2$ is also seen by inspection. There is no need to examine d^2f. In most practical problems the examination of the conditions of the second order may be waived. This example is one which may be treated in polar coördinates by the ordinary methods; but it is noteworthy that if it could not be treated that way, the method of solution by eliminating one of the variables by solving the cubic $F(x, y) = 0$ would be unavailable and the methods of constrained maxima would be required.

EXERCISES

1. By total differentiation and division obtain dy/dx in these cases. Do not substitute in (1), but use the method by which it was derived.

(α) $ax^2 + 2\,bxy + cy^2 - 1 = 0$,   ($\beta$) $x^4 + y^4 = 4\,a^2xy$,   (γ) $(\cos x)^y - (\sin y)^x = 0$,
(δ) $(x^2 + y^2)^2 = a^2(x^2 - y^2)$,   ($\epsilon$) $e^x + e^y = 2\,xy$,   (ζ) $x^{-2}y^{-2} = \tan^{-1} xy$.

2. Obtain the second derivative d^2y/dx^2 in Ex. 1 (α), (β), (ϵ), (ζ) by differentiating the value of dy/dx obtained above. Compare with use of (3).

3. Prove $\dfrac{d^2y}{dx^2} = -\dfrac{F_y'^2 F_{xx}'' - 2\,F_x'F_y'F_{xy}'' + F_x'^2 F_{yy}''}{F_y'^3}$.

4. Find the radius of curvature of these curves :

(α) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},\ R = 3\,(axy)^{\frac{1}{3}}$,　　($\beta$) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}},\ R = 2\,\sqrt{(x+y)^3/a}$,

(γ) $b^2x^2 + a^2y^2 = a^2b^2$,　　(δ) $xy^2 = a^2\,(a-x)$,　　(ϵ) $(ax)^2 + (by)^{\frac{2}{3}} = 1$.

5. Find y', y'', y''' in case $x^3 + y^3 - 3\,axy = 0$.

6. Extend equations (3) to obtain y''' and reduce by Ex. 3.

7. Find tangents parallel to the x-axis for $(x^2 + y^2)^2 = 2\,a^2\,(x^2 - y^2)$.

8. Find tangents parallel to the y-axis for $(x^2 + y^2 + ax)^2 = a^2\,(x^2 + y^2)$.

9. If $b^2 < ac$ in $ax^2 + 2\,bxy + cy^2 + fx + gy + h = 0$, circumscribe about the curve a rectangle parallel to the axes. Check algebraically.

10. Sketch $x^3 + y^3 = x^2y^2 + \frac{1}{2}\,(x^2 + y^2)$ near the singular point (1, 1).

11. Find the singular points and discuss the curves near them :

(α) $x^3 + y^3 = 3\,axy$,　　　　(β) $(x^2 + y^2)^2 = 2\,a^2\,(x^2 - y^2)$,

(γ) $x^4 + y^4 = 2\,(x - y)^2$,　　　(δ) $y^5 + 2\,xy^2 = x^2 + y^4$.

12. Make these functions maxima or minima subject to the given conditions. Discuss the work both with and without a multiplier :

(α) $\dfrac{a}{u\cos x} + \dfrac{b}{v\cos y}$,　$a\tan x + b\tan y = c$.　　　　　　*Ans.* $\dfrac{\sin x}{\sin y} = \dfrac{u}{v}$.

(β) $x^2 + y^2$,　$ax^2 + 2\,bxy + cy^2 = f$.　　　　　　Find axes of conic.

(γ) Find the shortest distance from a point to a line (in a plane).

13. Write the second and third total differentials of $F(x, y) = 0$ and compare with (3) and Ex. 5. Try this method of calculating in Ex. 2.

14. Show that $F_x'dx + F_y'dy = 0$ does and should give the tangent line to $F(x, y) = 0$ at the points (x, y) if $dx = \xi - x$ and $dy = \eta - y$, where ξ, η are the coördinates of points other than (x, y) on the tangent line. Why is the equation inapplicable at singular points of the curve ?

59. More general cases of implicit functions.
The problem of implicit functions may be generalized in two ways. In the first place a greater number of variables may occur in the function, as

$$F(x, y, z) = 0, \qquad F(x, y, z, \cdots, u) = 0\,;$$

and the question may be to solve the equation for one of the variables in terms of the others and to determine the partial derivatives of the chosen dependent variable. In the second place there may be several equations connecting the variables and it may be required to solve the equations for some of the variables in terms of the others and to determine the partial derivatives of the chosen dependent variables

with respect to the independent variables. In both cases the formal differentiation and attempted formal solution of the equations for the derivatives will indicate the results and the theorem under which the solution is proper.

Consider the case $F(x, y, z) = 0$ and form the differential.

$$dF(x, y, z) = F'_x dx + F'_y dy + F'_z dz = 0. \tag{8}$$

If z is to be the dependent variable, the partial derivative of z by x is found by setting $dy = 0$ so that y is constant. Thus

$$\frac{\partial z}{\partial x} = \left(\frac{dz}{dx}\right)_y = -\frac{F'_x}{F'_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = \left(\frac{dz}{dy}\right)_x = -\frac{F'_y}{F'_z} \tag{9}$$

are obtained by ordinary division after setting $dy = 0$ and $dx = 0$ respectively. If this division is to be legitimate, F'_z must not vanish at the point considered. The immediate suggestion is the theorem : If, when real values (x_0, y_0) are chosen and a real value z_0 is obtained from $F(z, x_0, y_0) = 0$ by solution, the function $F(x, y, z)$ regarded as a function of three independent variables (x, y, z) is continuous at and near (x_0, y_0, z_0) and has continuous first partial derivatives and $F'_z(x_0, y_0, z_0) \neq 0$, then $F(x, y, z) = 0$ may be solved uniquely for $z = \phi(x, y)$ and $\phi(x, y)$ will be continuous and have partial derivatives (9) for values of (x, y) sufficiently near to (x_0, y_0).

The theorem is again proved by the Law of the Mean, and in a similar manner.

$$F(x, y, z) - F(x_0, y_0, z_0) = F(x, y, z) = \left(hF'_x + kF'_y + lF'_z\right)_{x_0 + \theta h, y_0 + \theta k, z_0 + \theta l}.$$

As F'_x, F'_y, F'_z are continuous and $F'_z(x_0, y_0, z_0) \neq 0$, it is possible to take δ so small that, when $|h| < \delta, |k| < \delta, |l| < \delta$, the derivative $|F'_z| > m$ and $|F'_x| < \mu, |F'_y| < \mu$. Now it is desired so to restrict h, k that $\pm \delta F'_z$ shall determine the sign of the parenthesis. Let

$$|x - x_0| < \tfrac{1}{2} m\delta/\mu, \qquad |y - y_0| < \tfrac{1}{2} m\delta/\mu, \quad \text{then} \quad |hF'_x + kF'_y| < m\delta$$

and the signs of the parenthesis for $(x, y, z_0 + \delta)$ and $(x, y, z_0 - \delta)$ will be opposite since $|F'_z| > m$. Hence if (x, y) be held fixed, there is one and only one value of z for which the parenthesis vanishes between $z_0 + \delta$ and $z_0 - \delta$. Thus z is defined as a single valued function of (x, y) for sufficiently small values of $h = x - x_0, k = y - y_0$.

Also $\qquad \dfrac{l}{h} = -\dfrac{F'_x(x_0 + \theta h, y_0 + \theta k, z_0 + \theta l)}{F'_z(x_0 + \theta h, y_0 + \theta k, z_0 + \theta l)}, \qquad \dfrac{l}{k} = -\dfrac{F'_y(\cdots)}{F'_z(\cdots)}$

when k and h respectively are assigned the values 0. The limits exist when $h \doteq 0$ or $k \doteq 0$. But in the first case $l = \Delta z = \Delta_x z$ is the increment of z when x alone varies, and in the second case $l = \Delta z = \Delta_y z$. The limits are therefore the desired partial derivatives of z by x and y. The proof for any number of variables would be similar.

If none of the derivatives F_x', F_y', F_z' vanish, the equation $F(x, y, z) = 0$ may be solved for any one of the variables, and formulas like (9) will express the partial derivatives. It then appears that

$$\left(\frac{dz}{dx}\right)_y\left(\frac{dx}{dz}\right)_y = \frac{\partial z}{\partial x}\frac{\partial x}{\partial z} = \frac{F_x'}{F_z'}\frac{F_z'}{F_x'} = 1, \tag{10}$$

and

$$\left(\frac{dz}{dx}\right)_y\left(\frac{dx}{dy}\right)_z\left(\frac{dy}{dz}\right)_x = \frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = -1 \tag{11}$$

in like manner. The first equation is in this case identical with (4) of § 2 because if y is constant the relation $F(x, y, z) = 0$ reduces to $G(x, z) = 0$. The second equation is new. By virtue of (10) and similar relations, the derivatives in (11) may be inverted and transformed to the right side of the equation. As it is assumed in thermodynamics that the pressure, volume, and temperature of a given simple substance are connected by an equation $F(p, v, T) = 0$, called the characteristic equation of the substance, a relation between different thermodynamic magnitudes is furnished by (11).

60. In the next place suppose there are two equations

$$F(x, y, u, v) = 0, \qquad G(x, y, u, v) = 0 \tag{12}$$

between four variables. Let each equation be differentiated.

$$dF = 0 = F_x'dx + F_y'dy + F_u'du + F_v'dv,$$

$$dG = 0 = G_x'dx + G_y'dy + G_u'du + G_v'dv. \tag{13}$$

If it be desired to consider u, v as the dependent variables and x, y as independent, it would be natural to solve these equations for the differentials du and dv in terms of dx and dy; for example,

$$du = \frac{(F_x'G_v' - F_v'G_x')dx + (F_y'G_v' - F_v'G_y')dy}{F_u'G_v' - F_v'G_u'}. \tag{13'}$$

The differential dv would have a different numerator but the same denominator. The solution requires $F_u'G_v' - F_v'G_u' \neq 0$. This suggests the desired theorem: If (u_0, v_0) are solutions of $F = 0$, $G = 0$ corresponding to (x_0, y_0) and if $F_u'G_v' - F_v'G_u'$ does not vanish for the values (x_0, y_0, u_0, v_0), the equations $F = 0$, $G = 0$ may be solved for $u = \phi(x, y)$, $v = \psi(x, y)$ and the solution is unique and valid for (x, y) sufficiently near (x_0, y_0) — it being assumed that F and G regarded as functions in four variables are continuous and have continuous first partial derivatives at and near (x_0, y_0, u_0, v_0); moreover, the total differentials du, dv are given by (13') and a similar equation.

The proof of this theorem may be deferred (§ 64). Some observations should be made. The equations (13) may be solved for any two variables in terms of the other two. The partial derivatives

$$\frac{\partial u\,(x,\ y)}{\partial x}\,, \qquad \frac{\partial u\,(x,\ v)}{\partial x}\,, \qquad \frac{\partial x\,(u,\ v)}{\partial u}\,, \qquad \frac{\partial x\,(u,\ y)}{\partial u} \qquad (14)$$

of u by x or of x by u will naturally depend on whether the solution for u is in terms of $(x,\ y)$ or of $(x,\ v)$, and the solution for x is in $(u,\ v)$ or $(u,\ y)$. Moreover, it must not be assumed that $\partial u/\partial x$ and $\partial x/\partial u$ are reciprocals no matter which meaning is attached to each. In obtaining relations between the derivatives analogous to (10), (11), the values of the derivatives in terms of the derivatives of F and G may be found or the equations (12) may first be considered as solved.

Thus if
$$u = \phi\,(x,\ y), \qquad\qquad du = \phi'_x dx + \phi'_y dy,$$
$$v = \psi\,(x,\ y), \qquad\qquad dv = \psi'_x dx + \psi'_y dy.$$

Then
$$dx = \frac{\psi'_y du - \phi'_y dv}{\phi'_x \psi'_y - \phi'_y \psi'_x}, \qquad dy = \frac{-\psi'_x du + \phi'_x dv}{\phi'_x \psi'_y - \phi'_y \psi'_x}.$$

and
$$\frac{\partial x}{\partial u} = \frac{\psi'_y}{\phi'_x \psi'_y - \phi'_y \psi'_x}, \qquad \frac{\partial x}{\partial v} = \frac{-\phi'_y}{\phi'_x \psi'_y - \phi'_y \psi'_x}, \text{ etc.}$$

Hence
$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} = 1, \qquad (15)$$

as may be seen by direct substitution. Here u, v are expressed in terms of x, y for the derivatives u'_x, v'_x; and x, y are considered as expressed in terms of u, v for the derivatives x'_u, x'_v.

61. The questions of free or constrained maxima and minima, at any rate in so far as the determination of the conditions of the first order is concerned, may now be treated. If $F(x,\ y,\ z) = 0$ is given and the maxima and minima of z as a function of $(x,\ y)$ are wanted,

$$F'_x(x,\ y,\ z) = 0, \qquad F'_y(x,\ y,\ z) = 0, \qquad F(x,\ y,\ z) = 0 \qquad (16)$$

are three equations which may be solved for x, y, z. If for any of these solutions the derivative F'_z does not vanish, the surface $z = \phi(x,\ y)$ has at that point a tangent plane parallel to $z = 0$ and there is a maximum, minimum, or minimax. To distinguish between the possibilities further investigation must be made if necessary; the details of such an investigation will not be outlined for the reason that special methods are usually available. The conditions for an extreme of u as a function of $(x,\ y)$ defined implicitly by the equations (13') are seen to be

$$F'_x G'_v - F'_v G'_x = 0, \quad F'_y G'_v - F'_v G'_y = 0, \quad F = 0, \quad G = 0. \qquad (17)$$

The four equations may be solved for x, y, u, v or merely for x, y.

Suppose that the maxima, minima, and minimax of $u = f(x, y, z)$ subject either to one equation $F(x, y, z) = 0$ or two equations $F(x, y, z) = 0$, $G(x, y, z) = 0$ of constraint are desired. Note that if only one equation of constraint is imposed, the function $u = f(x, y, z)$ becomes a function of two variables; whereas if two equations are imposed, the function u really contains only one variable and the question of a minimax does not arise. The *method of multipliers* is again employed. Consider

$$\Phi(x, y, z) = f + \lambda F \quad \text{or} \quad \Phi = f + \lambda F + \mu G \tag{18}$$

as the case may be. The conditions for a free extreme of Φ are

$$\Phi_x' = 0, \qquad \Phi_y' = 0, \qquad \Phi_z' = 0. \tag{19}$$

These three equations may be solved for the coördinates x, y, z which will then be expressed as functions of λ or of λ and μ according to the case. If then λ or λ and μ be determined so that (x, y, z) satisfy $F = 0$ or $F = 0$ and $G = 0$, the constrained extremes of $u = f(x, y, z)$ will be found except for the examination of the conditions of higher order.

As a problem in constrained maxima and minima let the axes of the section of an ellipsoid by a plane through the origin be determined. Form the function

$$\Phi = x^2 + y^2 + z^2 + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) + \mu(lx + my + nz)$$

by adding to $x^2 + y^2 + z^2$, which is to be made extreme, the equations of the ellipsoid and plane, which are the equations of constraint. Then apply (19). Hence

$$x + \lambda\frac{x}{a^2} + \frac{\mu}{2}l = 0, \qquad y + \lambda\frac{y}{b^2} + \frac{\mu}{2}m = 0, \qquad z + \lambda\frac{z}{c^2} + \frac{\mu}{2}n = 0$$

taken with the equations of ellipsoid and plane will determine x, y, z, λ, μ. If the equations are multiplied by x, y, z and reduced by the equations of plane and ellipsoid, the solution for λ is $\lambda = -r^2 = -(x^2 + y^2 + z^2)$. The three equations then become

$$x = \frac{1}{2}\frac{\mu la^2}{r^2 - a^2}, \qquad y = \frac{1}{2}\frac{\mu mb^2}{r^2 - b^2}, \qquad z = \frac{1}{2}\frac{\mu nc^2}{r^2 - c^2}, \qquad \text{with} \quad lx + my + nz = 0.$$

Hence $\qquad\qquad \dfrac{l^2a^2}{r^2 - a^2} + \dfrac{m^2b^2}{r^2 - b^2} + \dfrac{n^2c^2}{r^2 - c^2} = 0 \quad$ determines r^2. $\qquad$ (20)

The two roots for r are the major and minor axes of the ellipse in which the plane cuts the ellipsoid. The substitution of x, y, z above in the ellipsoid determines

$$\frac{\mu^2}{4} = \left(\frac{al}{r^2 - a^2}\right)^2 + \left(\frac{bm}{r^2 - b^2}\right)^2 + \left(\frac{cn}{r^2 - c^2}\right)^2 \quad \text{since} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{21}$$

Now when (20) is solved for any particular root r and the value of μ is found by (21), the actual coördinates x, y, z of the extremities of the axes may be found.

EXERCISES

1. Obtain the partial derivatives of z by x and y directly from (8) and not by substitution in (9). Where does the solution fail?

$$(\alpha)\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad\qquad (\beta)\ x + y + z = \frac{1}{xyz},$$

$$(\gamma)\ (x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2, \qquad (\delta)\ xyz = c.$$

2. Find the second derivatives in Ex. 1 (α), (β), (δ) by repeated differentiation.

3. State and prove the theorem on the solution of $F(x, y, z, u) = 0$.

4. Show that the product $\alpha_p E_T$ of the coefficient of expansion by the modulus of elasticity (§ 52) is equal to the rate of rise of pressure with the temperature if the volume is constant.

5. Establish the proportion $E_S : E_T = C_p : C_v$ (see § 52).

6. If $F(x, y, z, u) = 0$, show $\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial y}\dfrac{\partial y}{\partial z}\dfrac{\partial z}{\partial u} = 1$, $\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial u} = 1$.

7. Write the equations of tangent plane and normal line to $F(x, y, z) = 0$ and find the tangent planes and normal lines to Ex. 1 (β), (δ) at $x = 1$, $y = 1$.

8. Find, by using (13), the indicated derivatives on the assumption that either x, y or u, v are dependent and the other pair independent:

$(\alpha)\ u^5 + v^5 + x^5 - 3y = 0, \qquad u^3 + v^3 + y^3 + 3x = 0, \qquad u'_x,\ u'_y,\ u''_{xy},\ v''_{xx}$

$(\beta)\ x + y + u + v = a, \qquad x^2 + y^2 + u^2 + v^2 = b, \qquad x'_u,\ u'_x,\ v'_y,\ v''_{xy}$

(γ) Find dy in both cases if x, v are independent variables.

9. Prove $\dfrac{\partial u}{\partial x}\dfrac{\partial y}{\partial u} + \dfrac{\partial v}{\partial x}\dfrac{\partial y}{\partial v} = 0$ if $F(x, y, u, v) = 0$, $G(x, y, u, v) = 0$.

10. Find du and the derivatives $u'_x,\ u'_y,\ u'_z$ in case

$$x^2 + y^2 + z^2 = uv, \qquad xy = u^2 + v^2 + w^2, \qquad \cdot xyz = uvw.$$

11. If $F(x, y, z) = 0$, $G(x, y, z) = 0$ define a curve, show that

$$\frac{x - x_0}{\left(F'_y G'_z - F'_z G'_y\right)_0} = \frac{y - y_0}{\left(F'_z G'_x - F'_x G'_z\right)_0} = \frac{z - z_0}{\left(F'_x G'_y - F'_y G'_x\right)_0}$$

is the tangent line to the curve at (x_0, y_0, z_0). Write the normal plane.

12. Formulate the problem of implicit functions occurring in Ex. 11.

13. Find the perpendicular distance from a point to a plane.

14. The sum of three positive numbers is $x + y + z = N$, where N is given. Determine x, y, z so that the product $x^p y^q z^r$ shall be maximum if p, q, r are given.
$$Ans.\ x : y : z : N = p : q : r : (p + q + r).$$

15. The sum of three positive numbers and the sum of their squares are both given. Make the product a maximum or minimum.

16. The surface $(x^2 + y^2 + z^2)^2 = ax^2 + by^2 + cz^2$ is cut by the plane $lx + my + nz = 0$.

Find the maximum or minimum radius of the section. $Ans.\ \displaystyle\sum \frac{l^2}{r^2 - a} = 0.$

17. In case $F(x, y, u, v) = 0$, $G(x, y, u, v) = 0$ consider the differentials

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy, \qquad dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv, \qquad dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv.$$

Substitute in the first from the last two and obtain relations like (15) and Ex. 9.

18. If $f(x, y, z)$ is to be maximum or minimum subject to the constraint $F(x, y, z) = 0$, show that the conditions are that $dx : dy : dz = 0 : 0 : 0$ are indeterminate when their solution is attempted from

$$f_x' dx + f_y' dy + f_z' dz = 0 \quad \text{and} \quad F_x' dx + F_y' dy + F_z' dz = 0.$$

From what geometrical considerations should this be obvious? Discuss in connection with the problem of inscribing the maximum rectangular parallelepiped in the ellipsoid. These equations,

$$dx : dy : dz = f_y' F_z' - f_z' F_y' : f_z' F_x' - f_x' F_z' : f_x' F_y' - f_y' F_x' = 0 : 0 : 0,$$

may sometimes be used to advantage for such problems.

19. Given the curve $F(x, y, z) = 0$, $G(x, y, z) = 0$. Discuss the conditions for the highest or lowest points, or more generally the points where the tangent is parallel to $z = 0$, by treating $u = f(x, y, z) = z$ as a maximum or minimum subject to the two constraining equations $F = 0$, $G = 0$. Show that the condition $F_x' G_y' = F_y' G_x'$ which is thus obtained is equivalent to setting $dz = 0$ in

$$F_x' dx + F_y' dy + F_z' dz = 0 \quad \text{and} \quad G_x' dx + G_y' dy + G_z' dz = 0.$$

20. Find the highest and lowest points of these curves :

(α) $x^2 + y^2 = z^2 + 1$, $x + y + 2z = 0$,   (β) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, $lx + my + nz = 0$.

21. Show that $F_x' dx + F_y' dy + F_z' dz = 0$, with $dx = \xi - x$, $dy = \eta - y$, $dz = \zeta - z$, is the tangent plane to the surface $F(x, y, z) = 0$ at (x, y, z). Apply to Ex. 1.

22. Given $F(x, y, u, v) = 0$, $G(x, y, u, v) = 0$. Obtain the equations

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = 0, \qquad \frac{\partial F}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial x} = 0, \qquad \frac{\partial G}{\partial y} + \frac{\partial G}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial y} = 0,$$

and explain their significance as a sort of partial-total differentiation of $F = 0$ and $G = 0$. Find u_x' from them and compare with (13'). Write similar equations where x, y are considered as functions of (u, v). Hence prove, and compare with (15) and Ex. 9,

$$\frac{\partial u}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} = 1, \qquad \frac{\partial u}{\partial y}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial x}{\partial v} = 0.$$

23. Show that the differentiation with respect to x and y of the four equations under Ex. 22 leads to eight equations from which the eight derivatives

$$\frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial^2 u}{\partial x \partial y}, \qquad \frac{\partial^2 u}{\partial y \partial x}, \qquad \frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial^2 v}{\partial x^2}, \qquad \cdots, \qquad \frac{\partial^2 v}{\partial y^2}$$

may be obtained. Show thus that formally $u_{xy}'' = u_{yx}''$.

62. Functional determinants or Jacobians. Let two functions

$$u = \phi(x, y), \qquad v = \psi(x, y) \tag{22}$$

of two independent variables be given. The continuity of the functions and of their first derivatives is assumed throughout this discussion and will not be mentioned again. Suppose that there were a relation $F(u, v) = 0$ or $F(\phi, \psi) = 0$ between the functions. Then

$$F(\phi, \psi) = 0, \qquad F'_u \phi'_x + F'_v \psi'_x = 0, \qquad F'_u \phi'_y + F'_v \psi'_y = 0. \tag{23}$$

The last two equations arise on differentiating the first with respect to x and y. The elimination of F'_u and F'_v from these gives

$$\phi'_x \psi'_y - \phi'_y \psi'_x = \begin{vmatrix} \phi'_x & \psi'_x \\ \phi'_y & \psi'_y \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)} = J\left(\frac{u, v}{x, y}\right) = 0. \tag{24}$$

The determinant is merely another way of writing the first expression; the next form is the customary short way of writing the determinant and denotes that the elements of the determinant are the first derivatives of u and v with respect to x and y. This determinant is called the *functional determinant* or *Jacobian* of the functions u, v or ϕ, ψ with respect to the variables x, y and is denoted by J. It is seen that: *If there is a functional relation $F(\phi, \psi) = 0$ between two functions, the Jacobian of the functions vanishes identically*, that is, vanishes for all values of the variables (x, y) under consideration.

Conversely, *if the Jacobian vanishes identically over a two-dimensional region for (x, y), the functions are connected by a functional relation*. For, the functions u, v may be assumed not to reduce to mere constants and hence there may be assumed to be points for which at least one of the partial derivatives ϕ'_x, ϕ'_y, ψ'_x, ψ'_y does not vanish. Let ϕ'_x be the derivative which does not vanish at some particular point of the region. Then $u = \phi(x, y)$ may be solved as $x = \chi(u, y)$ in the vicinity of that point and the result may be substituted in v.

$$v = \psi(\chi, y), \qquad \frac{\partial v}{\partial y} = \psi'_x \frac{\partial \chi}{\partial y} + \psi'_y = \psi'_x \frac{\partial x}{\partial y} + \psi'_y.$$

But $\qquad \dfrac{\partial x}{\partial y} = -\dfrac{\partial u}{\partial y}\dfrac{\partial x}{\partial u} \quad$ and $\quad \dfrac{\partial v}{\partial y} = \dfrac{1}{\phi'_x}(\phi'_x \psi'_y - \psi'_x \phi'_y) \tag{24'}$

by (11) and substitution. Thus $\partial v/\partial y = J/\phi'_x$; and if $J = 0$, then $\partial v/\partial y = 0$. This relation holds at least throughout the region for which $\phi'_x \neq 0$, and for points in this region $\partial v/\partial y$ vanishes identically. Hence v does not depend on y but becomes a function of u alone. This establishes the fact that v and u are functionally connected.

These considerations may be extended to other cases. Let

$$u = \phi(x, y, z), \qquad v = \psi(x, y, z), \qquad w = \chi(x, y, z). \qquad (25)$$

If there is a functional relation $F(u, v, w) = 0$, differentiate it.

$$
\begin{aligned}
F'_u\phi'_x + F'_v\psi'_x + F'_w\chi'_x &= 0, \\
F'_u\phi'_y + F'_v\psi'_y + F'_w\chi'_y &= 0, \\
F'_u\phi'_z + F'_v\psi'_z + F'_w\chi'_z &= 0,
\end{aligned}
\qquad
\begin{vmatrix}
\phi'_x & \psi'_x & \chi'_x \\
\phi'_y & \psi'_y & \chi'_y \\
\phi'_z & \psi'_z & \chi'_z
\end{vmatrix} = 0,
\qquad (26)
$$

or
$$\frac{\partial(\phi, \psi, \chi)}{\partial(x, y, z)} = \frac{\partial(u, v, w)}{\partial(x, y, z)} = J = 0.$$

The result is obtained by eliminating F'_u, F'_v, F'_w from the three equations. The assumption is made, here as above, that F'_u, F'_v, F'_w do not all vanish; for if they did, the three equations would not imply $J = 0$. On the other hand their vanishing would imply that F did not contain u, v, w, — as it must if there is really a relation between them. And now conversely it may be shown that if J vanishes identically, there is a functional relation between u, v, w. Hence again *the necessary and sufficient conditions that the three functions* (25) *be functionally connected is that their Jacobian vanish.*

The proof of the converse part is about as before. It may be assumed that at least one of the derivatives of u, v, w or ϕ, ψ, χ by x, y, z does not vanish. Let $\phi'_x \neq 0$ be that derivative. Then $u = \phi(x, y, z)$ may be solved as $x = \omega(u, y, z)$ and the result may be substituted in v and w as

$$v = \psi(x, y, z) = \psi(\omega, y, z), \qquad w = \chi(x, y, z) = \chi(\omega, y, z).$$

Next the Jacobian of v and w relative to y and z may be written as

$$
\begin{vmatrix}
\dfrac{\partial v}{\partial y} & \dfrac{\partial w}{\partial y} \\[2mm]
\dfrac{\partial v}{\partial z} & \dfrac{\partial w}{\partial z}
\end{vmatrix}
=
\begin{vmatrix}
\psi'_x \dfrac{\partial x}{\partial y} + \psi'_y & \chi'_x \dfrac{\partial x}{\partial y} + \chi'_y \\[2mm]
\psi'_x \dfrac{\partial x}{\partial z} + \psi'_z & \chi'_x \dfrac{\partial x}{\partial z} + \chi'_z
\end{vmatrix}
$$

$$
=
\begin{vmatrix} \psi'_y & \chi'_y \\ \psi'_z & \chi'_z \end{vmatrix}
+ \psi'_x
\begin{vmatrix} -\phi'_y/\phi'_x & \chi'_y \\ -\phi'_z/\phi'_x & \chi'_z \end{vmatrix}
+ \chi'_x
\begin{vmatrix} \psi'_y & -\phi'_y/\phi'_x \\ \psi'_z & -\phi'_z/\phi'_x \end{vmatrix}
$$

$$
= \frac{1}{\phi'_x}\left[\phi'_x
\begin{vmatrix} \psi'_y & \chi'_y \\ \psi'_z & \chi'_z \end{vmatrix}
+ \psi'_x
\begin{vmatrix} \chi'_y & \phi'_y \\ \chi'_z & \phi'_z \end{vmatrix}
+ \chi'_x
\begin{vmatrix} \phi'_y & \psi'_y \\ \phi'_z & \psi'_z \end{vmatrix}\right] = \frac{J}{\phi'_x}.
$$

As J vanishes identically, the Jacobian of v and w expressed as functions of y, z, and u vanishes. Hence by the case previously discussed there is a functional relation $F(v, w) = 0$ independent of y, z; and as v, w now contain u, this relation may be considered as a functional relation between u, v, w.

63. If in (22) the variables u, v be assigned constant values, the equations define two curves, and if u, v be assigned a series of such values, the equations (22) define a network of curves in some part of the

xy-plane. If there is a functional relation $u = F(v)$, that is, if the Jacobian vanishes identically, a constant value of v implies a constant value of u and hence the locus for which v is constant is also a locus for which u is constant; the set of v-curves coincides with the set of u-curves and no true network is formed. This case is uninteresting. Let it be assumed that the Jacobian does not vanish identically and even that it does not vanish for any point (x, y) of a certain region of the xy-plane. The indications of § 60 are that the equations (22) may then be solved for x, y in terms of u, v at any point of the region and that there is a pair of the curves through each point. It is then proper to consider (u, v) as the coördinates of the points in the region. To any point there correspond not only the rectangular coördinates (x, y) but also the *curvilinear coördinates* (u, v).
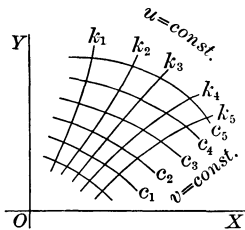
The equations connecting the rectangular and curvilinear coördinates may be taken in either of the two forms

$$u = \phi(x, y), \quad v = \psi(x, y) \quad \text{or} \quad x = f(u, v), \quad y = g(u, v), \quad (22')$$

each of which are the solutions of the other. The Jacobians

$$J\left(\frac{u, v}{x, y}\right) \cdot J\left(\frac{x, y}{u, v}\right) = 1 \tag{27}$$

are reciprocal each to each; and this relation may be regarded as the analogy of the relation (4) of § 2 for the case of the function $y = \phi(x)$ and the solution $x = f(y) = \phi^{-1}(y)$ in the case of a single variable. The *differential of arc* is
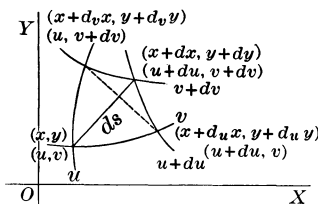
$$ds^2 = dx^2 + dy^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2, \tag{28}$$

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2, \qquad F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v}, \qquad G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2.$$

The *differential of area* included between two neighboring u-curves and two neighboring v-curves may be written in the form

$$dA = J\left(\frac{x, y}{u, v}\right) du\,dv = du\,dv \div J\left(\frac{u, v}{x, y}\right). \tag{29}$$

These statements will now be proved in detail.

To prove (27) write out the Jacobians at length and reduce the result.

$$J\left(\frac{u,\ v}{x,\ y}\right)J\left(\frac{x,\ y}{u,\ v}\right)=\begin{vmatrix}\dfrac{\partial u}{\partial x}&\dfrac{\partial v}{\partial x}\\[2mm]\dfrac{\partial u}{\partial y}&\dfrac{\partial v}{\partial y}\end{vmatrix}\cdot\begin{vmatrix}\dfrac{\partial x}{\partial u}&\dfrac{\partial y}{\partial u}\\[2mm]\dfrac{\partial x}{\partial v}&\dfrac{\partial y}{\partial v}\end{vmatrix}$$

$$=\begin{vmatrix}\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial u}+\dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial v}&\dfrac{\partial u}{\partial x}\dfrac{\partial y}{\partial u}+\dfrac{\partial v}{\partial x}\dfrac{\partial y}{\partial v}\\[2mm]\dfrac{\partial u}{\partial y}\dfrac{\partial x}{\partial u}+\dfrac{\partial v}{\partial y}\dfrac{\partial x}{\partial v}&\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial u}+\dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1&0\\0&1\end{vmatrix}=1,$$

where the rule for multiplying determinants has been applied and the reduction has been made by (15), Ex. 9 above, and similar formulas. If the rule for multiplying determinants is unfamiliar, the Jacobians may be written and multiplied without that notation and the reduction may be made by the same formulas as before.

To establish the formula for the differential of arc it is only necessary to write the total differentials of dx and dy, to square and add, and then collect. To obtain the differential area between four adjacent curves consider the triangle determined by (u, v), $(u + du, v)$, $(u, v + dv)$, which is half that area, and double the result. The determinantal form of the area of a triangle is the best to use.

$$dA=2\cdot\frac{1}{2}\begin{vmatrix}d_u x&d_u y\\d_v x&d_v y\end{vmatrix}=\begin{vmatrix}\dfrac{\partial x}{\partial u}du&\dfrac{\partial y}{\partial u}du\\[2mm]\dfrac{\partial x}{\partial v}dv&\dfrac{\partial y}{\partial v}dv\end{vmatrix}=\begin{vmatrix}\dfrac{\partial x}{\partial u}&\dfrac{\partial y}{\partial u}\\[2mm]\dfrac{\partial x}{\partial v}&\dfrac{\partial y}{\partial v}\end{vmatrix}dudv.$$

The subscripts on the differentials indicate which variable changes; thus $d_u x$, $d_u y$ are the coördinates of $(u + du, v)$ relative to (u, v). This method is easily extended to determine the analogous quantities in three dimensions or more. It may be noticed that the triangle does not look as if it were half the area (except for infinitesimals of higher order) in the figure; but see Ex. 12 below.

It should be remarked that as the differential of area dA is usually considered positive when du and dv are positive, it is usually better to replace J in (29) by its absolute value. Instead of regarding (u, v) as curvilinear coördinates in the xy-plane, it is possible to plot them in their own uv-plane and thus to establish by (22') a *transformation* of the xy-plane over onto the uv-plane. A small area in the xy-plane then becomes a small area in the uv-plane. If $J > 0$, the transformation is called direct; but if $J < 0$, the transformation is called perverted. The significance of the distinction can be made clear only when the question of the signs of areas has been treated. The transformation is called *conformal* when elements of arc in the neighborhood of a point in the xy-plane are proportional to the elements of arc in the neighborhood of the corresponding point in the uv-plane, that is, when

$$ds^2 = dx^2 + dy^2 = k\,(du^2 + dv^2) = k\,d\sigma^2. \qquad (30)$$

For in this case any little triangle will be transformed into a little triangle similar to it, and hence angles will be unchanged by the transformation. That the transformation be conformal requires that $F = 0$ and $E = G$. It is not necessary that $E = G = k$ be constants; the ratio of similitude may be different for different points.

64. There remains outstanding the proof that equations may be solved in the neighborhood of a point at which the Jacobian does not vanish. The fact was indicated in § 60 and used in § 63.

THEOREM. Let p equations in $n + p$ variables be given, say,

$$F_1(x_1, x_2, \cdots, x_{n+p}) = 0, \qquad F_2 = 0, \cdots, F_p = 0. \tag{31}$$

Let the p functions be soluble for $x_{10}, x_{20}, \cdots, x_{p0}$ when a particular set $x_{(p+1)0}, \cdots, x_{(n+p)0}$ of the other n variables are given. Let the functions and their first derivatives be continuous in all the $n + p$ variables in the neighborhood of $(x_{10}, x_{20}, \cdots, x_{(n+p)0})$. Let the Jacobian of the functions with respect to $x_1, x_2, \cdots, x_p$,

$$J\left(\frac{F_1, \cdots, F_p}{x_1, \cdots, x_p}\right) = \begin{vmatrix} \dfrac{\partial F_1}{\partial x_1} \cdots \dfrac{\partial F_p}{\partial x_1} \\ \vdots \qquad \vdots \\ \dfrac{\partial F_1}{\partial x_p} \cdots \dfrac{\partial F_p}{\partial x_p} \end{vmatrix}_{x_{10}, \cdots, x_{(n+p)0}} \neq 0, \tag{32}$$

fail to vanish for the particular set mentioned. Then the p equations may be solved for the p variables $x_1, x_2, \cdots, x_p$, and the solutions will be continuous, unique, and differentiable with continuous first partial derivatives for all values of $x_{p+1}, \cdots, x_{n+p}$ sufficiently near to the values $x_{(p+1)0}, \cdots, x_{(n+p)0}$.

THEOREM. The necessary and sufficient condition that a functional relation exist between p functions of p variables is that the Jacobian of the functions with respect to the variables shall vanish identically, that is, for all values of the variables.

The proofs of these theorems will naturally be given by mathematical induction. Each of the theorems has been proved in the simplest cases and it remains only to show that the theorems are true for p functions in case they are for $p - 1$. Expand the determinant J.

$$J = J_1 \frac{\partial F_1}{\partial x_1} + J_2 \frac{\partial F_1}{\partial x_2} + \cdots + J_p \frac{\partial F_1}{\partial x_p}, \qquad J_1, \cdots, J_p, \text{ minors.}$$

For the first theorem $J \neq 0$ and hence at least one of the minors $J_1, \cdots, J_p$ must fail to vanish. Let that one be J_1, which is the Jacobian of $F_2, \cdots, F_p$ with respect to $x_2, \cdots, x_p$. By the assumption that the theorem holds for the case $p - 1$, these $p - 1$ equations may be solved for $x_2, \cdots, x_p$ in terms of the $n + 1$ variables x_1,

$x_{p+1}, \cdots, x_{n+p}$, and the results may be substituted in F_1. It remains to show that $F_1 = 0$ is soluble for x_1. Now

$$\frac{dF_1}{dx_1} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial x_2}\frac{\partial x_2}{\partial x_1} + \cdots + \frac{\partial F_1}{\partial x_p}\frac{\partial x_p}{\partial x_1} = J/J_1 \neq 0. \tag{32'}$$

For the derivatives of $x_2, \cdots, x_p$ with respect to x_1 are obtained from the equations

$$0 = \frac{\partial F_2}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\frac{\partial x_2}{\partial x_1} + \cdots + \frac{\partial F_2}{\partial x_p}\frac{\partial x_p}{\partial x_1}, \quad \cdots, \quad 0 = \frac{\partial F_p}{\partial x_1} + \frac{\partial F_p}{\partial x_2}\frac{\partial x_2}{\partial x_1} + \cdots + \frac{\partial F_p}{\partial x_p}\frac{\partial x_p}{\partial x_1}$$

resulting from the differentiation of $F_2 = 0, \cdots, F_p = 0$ with respect to x_1. The derivative $\partial x_i/\partial x_1$ is therefore merely J_i/J_1, and hence $dF_1/dx_1 = J/J_1$ and does not vanish. The equation therefore may be solved for x_1 in terms of $x_{p+1}, \cdots, x_{n+p}$, and this result may be substituted in the solutions above found for $x_2, \cdots, x_p$. Hence the equations have been solved for $x_1, x_2, \cdots, x_p$ in terms of $x_{p+1}, \cdots, x_{n+p}$ and the theorem is proved.

For the second theorem the procedure is analogous to that previously followed. If there is a relation $F(u_1, \cdots, u_p) = 0$ between the p functions

$$u_1 = \phi_1(x_1, \cdots, x_p), \cdots, \qquad u_p = \phi_p(x_1, \cdots, x_p),$$

differentiation with respect to $x_1, \cdots, x_p$ gives p equations from which the derivatives of F by $u_1, \cdots, u_p$ may be eliminated and $J\left(\dfrac{u_1, \cdots, u_p}{x_1, \cdots, x_p}\right) = 0$ becomes the condition desired. If conversely this Jacobian vanishes identically and it be assumed that one of the derivatives of u_i by x_j, say $\partial u_1/\partial x_1$, does not vanish, then the solution $x_1 = \omega(u_1, x_2, \cdots, x_p)$ may be effected and the result may be substituted in $u_2, \cdots, u_p$. The Jacobian of $u_2, \cdots, u_p$ with respect to $x_2, \cdots, x_p$ will then turn out to be $J \div \partial u_1/\partial x_1$ and will vanish because J vanishes. Now, however, only $p - 1$ functions are involved, and hence if the theorem is true for $p - 1$ functions it must be true for p functions.

EXERCISES

1. If $u = ax + by + c$ and $v = a'x + b'y + c'$ are functionally dependent, the lines $u = 0$ and $v = 0$ are parallel; and conversely.

2. Prove $x + y + z$, $xy + yz + zx$, $x^2 + y^2 + z^2$ functionally dependent.

3. If $u = ax + by + cz + d$, $v = a'x + b'y + c'z + d'$, $w = a''x + b''y + c''z + d''$ are functionally dependent, the planes $u = 0$, $v = 0$, $w = 0$ are parallel to a line.

4. In what senses are $\dfrac{\partial v}{\partial y}$ and ψ_y' of (24') and $\dfrac{dF_1}{dx_1}$ and $\dfrac{\partial F_1}{\partial x_1}$ of (32') partial or total derivatives? Are not the two sets completely analogous?

5. Given (25), suppose $\begin{vmatrix} \psi_y' & \chi_y' \\ \psi_z' & \chi_z' \end{vmatrix} \neq 0$. Solve $v = \psi$ and $w = \chi$ for y and z, substitute in $u = \phi$, and prove $\partial u/\partial x = J \div \begin{vmatrix} \psi_y' & \chi_y' \\ \psi_z' & \chi_z' \end{vmatrix}$.

6. If $u = u(x, y)$, $v = v(x, y)$, and $x = x(\xi, \eta)$, $y = y(\xi, \eta)$, prove

$$J\left(\frac{u, v}{x, y}\right) J\left(\frac{x, y}{\xi, \eta}\right) = J\left(\frac{u, v}{\xi, \eta}\right). \tag{27'}$$

State the extension to any number of variables. How may (27') be used to prove (27)? Again state the extension to any number of variables.

7. Prove $dV = J\left(\dfrac{x,\,y,\,z}{u,\,v,\,w}\right) du\,dv\,dw = du\,dv\,dw \div J\left(\dfrac{u,\,v,\,w}{x,\,y,\,z}\right)$ is the element of volume in space with curvilinear coördinates u, v, $w =$ consts.

8. In what parts of the plane can $u = x^2 + y^2$, $v = xy$ not be used as curvilinear coördinates ? Express ds^2 for these coördinates.

9. Prove that $2\,u = x^2 - y^2$, $v = xy$ is a conformal transformation.

10. Prove that $x = \dfrac{u}{u^2 + v^2}$, $y = \dfrac{v}{u^2 + v^2}$ is a conformal transformation.

11. Define conformal transformation in space. If the transformation

$$x = au + bv + cw, \qquad y = a'u + b'v + c'w, \qquad z = a''u + b''v + c''w$$

is conformal, is it orthogonal ? See Ex. 10 (ζ), p. 100.

12. Show that the areas of the triangles whose vertices are

$$(u,\,v),\ (u + du,\,v),\ (u,\,v + dv) \quad \text{and} \quad (u + du,\,v + dv),\ (u + du,\,v),\ (u,\,v + dv)$$

are infinitesimals of the same order, as suggested in § 63.

13. Would the condition $F = 0$ in (28) mean that the set of curves $u =$ const. were perpendicular to the set $v =$ const. ?

**14.** Express E, F, G in (28) in terms of the derivatives of u, v by x, y.

15. If $x = \phi\,(s,\,t)$, $y = \psi\,(s,\,t)$, $z = \chi\,(s,\,t)$ are the parametric equations of a surface (from which s, t could be eliminated to obtain the equation between x, y, z), show

$$\frac{\partial z}{\partial x} = J\left(\frac{\chi,\,\psi}{s,\,t}\right) \div J\left(\frac{\phi,\,\psi}{s,\,t}\right) \quad \text{and find} \quad \frac{\partial z}{\partial y}.$$

65. Envelopes of curves and surfaces. Let the equation $F(x,\,y,\,\alpha) = 0$ be considered as representing a family of curves where the different curves of the family are obtained by assigning different values to the parameter α. Such families are illustrated by

$$(x - \alpha)^2 + y^2 = 1 \quad \text{and} \quad \alpha x + y/\alpha = 1, \tag{33}$$

which are circles of unit radius centered on the x-axis and lines which cut off the area $\frac{1}{2}\,\alpha^2$ from the first quadrant. As α changes, the circles remain always tangent to the two lines $y = \pm\,1$ and the point of tangency traces those lines. Again, as α changes, the lines (33) remain tangent to the hyperbola $xy = k$, owing to the property of the hyperbola that a tangent forms a triangle of constant area with the asymptotes. The lines $y = \pm\,1$ are called the *envelope* of the system of circles and the hyperbola $xy = k$ the envelope of the set of lines. In general, *if there is a curve to which the curves of a family $F(x,\,y,\,\alpha) = 0$ are tangent and if the point of tangency describes that curve as α varies, the curve is called*
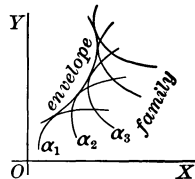
the envelope (or part of the envelope if there are several such curves) *of the family* $F(x, y, \alpha) = 0$. Thus any curve may be regarded as the envelope of its tangents or as the envelope of its circles of curvature.

To find the equations of the envelope note that by definition the enveloping curves of the family $F(x, y, \alpha) = 0$ are tangent to the envelope and that the point of tangency moves along the envelope as α varies. The equation of the envelope may therefore be written

$$x = \phi(\alpha), \qquad y = \psi(\alpha) \quad \text{with} \quad F(\phi, \psi, \alpha) = 0, \tag{34}$$

where the first equations express the dependence of the points on the envelope upon the parameter α and the last equation states that each point of the envelope lies also on some curve of the family $F(x, y, \alpha) = 0$. Differentiate (34) with respect to α. Then

$$F_x' \phi'(\alpha) + F_y' \psi'(\alpha) + F_\alpha' = 0. \tag{35}$$

Now if the point of contact of the envelope with the curve $F = 0$ is an ordinary point of that curve, the tangent to the curve is

$$F_x'(x - x_0) + F_y'(y - y_0) = 0; \quad \text{and} \quad F_x'\phi' + F_y'\psi' = 0,$$

since the tangent direction $dy : dx = \psi' : \phi'$ along the envelope is by definition identical with that along the enveloping curve; and if the point of contact is a singular point for the enveloping curve, $F_x' = F_y' = 0$. Hence in either case $F_\alpha' = 0$.

Thus *for points on the envelope the two equations*

$$F(x, y, \alpha) = 0, \qquad F_\alpha'(x, y, \alpha) = 0 \tag{36}$$

are satisfied and the equation of the envelope of the family $F = 0$ may be found by solving (36) *to find the parametric equations* $x = \phi(\alpha)$, $y = \psi(\alpha)$ *of the envelope or by eliminating α between* (36) *to find the equation of the envelope in the form* $\Phi(x, y) = 0$. It should be remarked that the locus found by this process may contain other curves than the envelope. For instance if the curves of the family $F = 0$ have singular points and if $x = \phi(\alpha)$, $y = \psi(\alpha)$ be the locus of the singular points as α varies, equations (34), (35) still hold and hence (36) also. The rule for finding the envelope therefore finds also the locus of singular points. Other extraneous factors may also be introduced in performing the elimination. It is therefore important to test graphically or analytically the solution obtained by applying the rule.

As a first example let the envelope of $(x - \alpha)^2 + y^2 = 1$ be found.

$$F(x, y, \alpha) = (x - \alpha)^2 + y^2 - 1 = 0, \qquad F_\alpha' = -2(x - \alpha) = 0.$$

The elimination of α from these equations gives $y^2 - 1 = 0$ and the solution for α gives $x = \alpha$, $y = \pm 1$. The loci indicated as envelopes are $y = \pm 1$. It is

geometrically evident that these are really envelopes and not extraneous factors. But as a second example consider $\alpha x + y/\alpha = 1$. Here

$$F(x,\, y,\, \alpha) = \alpha x + y/\alpha - 1 = 0, \qquad F'_\alpha = x - y/\alpha^2 = 0.$$

The solution is $y = \alpha/2$, $x = 1/2\alpha$, which gives $xy = \tfrac{1}{4}$. This is the envelope; it could not be a locus of singular points of $F = 0$ as there are none. Suppose the elimination of α be made by Sylvester's method as

$$\begin{array}{l}
-y/\alpha^2 + 0/\alpha + x + 0\alpha = 0 \\
0/\alpha^2 - y/\alpha + 0 + x\alpha = 0 \\
y/\alpha^2 - 1/\alpha + x + 0\alpha = 0 \\
0/\alpha^2 + y/\alpha - 1 + x\alpha = 0
\end{array}
\quad \text{and} \quad
\begin{vmatrix}
-y & 0 & x & 0 \\
0 & -y & 0 & x \\
y & -1 & x & 0 \\
0 & y & -1 & x
\end{vmatrix} = 0\,;$$

the reduction of the determinant gives $xy\,(4\,xy - 1) = 0$ as the eliminant, and contains not only the envelope $4\,xy = 1$, but the factors $x = 0$ and $y = 0$ which are obviously extraneous.

As a third problem find the envelope of a line of which the length intercepted between the axes is constant. The necessary equations are

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1, \qquad \alpha^2 + \beta^2 = K^2, \qquad \frac{x}{\alpha^2}\,d\alpha + \frac{y}{\beta^2}\,d\beta = 0, \qquad \alpha\,d\alpha + \beta\,d\beta = 0.$$

Two parameters α, β connected by a relation have been introduced; both equations have been differentiated totally with respect to the parameters; and the problem is to eliminate α, β, $d\alpha$, $d\beta$ from the equations. In this case it is simpler to carry both parameters than to introduce the radicals which would be required if only one parameter were used. The elimination of $d\alpha$, $d\beta$ from the last two equations gives $x : y = \alpha^3 : \beta^3$ or $\sqrt[3]{x} : \sqrt[3]{y} = \alpha : \beta$. From this and the first equation,

$$\frac{1}{\alpha} = \frac{1}{x^{\frac{1}{3}}\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)}, \quad \frac{1}{\beta} = \frac{1}{y^{\frac{1}{3}}\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)}, \quad \text{and hence} \quad x^{\frac{2}{3}} + y^{\frac{2}{3}} = K^{\frac{2}{3}}.$$

66. Consider two neighboring curves of $F(x,\, y,\, \alpha) = 0$. Let $(x_0,\, y_0)$ be an ordinary point of $\alpha = \alpha_0$ and $(x_0 + dx,\, y_0 + dy)$ of $\alpha_0 + d\alpha$. Then

$$F(x_0 + dx,\, y_0 + dy,\, \alpha_0 + d\alpha) - F(x_0,\, y_0,\, \alpha_0)$$
$$= F'_x dx + F'_y dy + F'_\alpha d\alpha = 0 \qquad (37)$$

holds except for infinitesimals of higher order. The distance from the point on $\alpha_0 + d\alpha$ to the tangent to α_0 at $(x_0,\, y_0)$ is

$$\frac{F'_x dx + F'_y dy}{\pm\sqrt{F'^2_x + F'^2_y}} = \frac{\pm F'_\alpha d\alpha}{\sqrt{F'^2_x + F'^2_y}} = dn \qquad (38)$$

except for infinitesimals of higher order. This distance is of the first order with $d\alpha$, and the normal derivative $d\alpha/dn$ of § 48 is finite except when $F'_\alpha = 0$. The distance is of higher order than $d\alpha$, and $d\alpha/dn$ is infinite or $dn/d\alpha$ is zero when $F'_\alpha = 0$. It appears therefore that *the envelope is the locus of points at which the distance between two neighboring curves is of higher order than $d\alpha$.* This is also apparent geometrically from the fact that the distance from a point on a curve to the

tangent to the curve at a neighboring point is of higher order (§ 36). Singular points have been ruled out because (38) becomes indeterminate. In general the locus of singular points is not tangent to the curves of the family and is not an envelope but an extraneous factor; in exceptional cases this locus is an envelope.

If two neighboring curves $F(x, y, \alpha) = 0$, $F(x, y, \alpha + \Delta\alpha) = 0$ intersect, their point of intersection satisfies both of the equations, and hence also the equation

$$\frac{1}{\Delta\alpha}\left[F(x, y, \alpha + \Delta\alpha) - F(x, y, \alpha)\right] = F'_\alpha(x, y, \alpha + \theta\Delta\alpha) = 0.$$

If the limit be taken for $\Delta\alpha \doteq 0$, the limiting position of the intersection satisfies $F'_\alpha = 0$ and hence may lie on the envelope, and will lie on the envelope if the common point of intersection is remote from singular points of the curves $F(x, y, \alpha) = 0$. This idea of an *envelope as the limit of points in which neighboring curves of the family intersect* is valuable. It is sometimes taken as the definition of the envelope. But, unless imaginary points of intersection are considered, it is an inadequate definition; for otherwise $y = (x - \alpha)^3$ would have no envelope according to the definition (whereas $y = 0$ is obviously an envelope) and a curve could not be regarded as the envelope of its osculating circles.

Care must be used in applying the rule for finding an envelope. Otherwise not only may extraneous solutions be mistaken for the envelope, but the envelope may be missed entirely. Consider

$$y - \sin \alpha x = 0 \quad \text{or} \quad \alpha - x^{-1}\sin^{-1}y = 0, \tag{39}$$

where the second form is obtained by solution and contains a multiple valued function. These two families of curves are identical, and it is geometrically clear that they have an envelope, namely $y = \pm 1$. This is precisely what would be found on applying the rule to the first of (39) ; but if the rule be applied to the second of (39), it is seen that $F'_\alpha = 1$, which does not vanish and hence indicates no envelope. The whole matter should be examined carefully in the light of implicit functions.

Hence let $F(x, y, \alpha) = 0$ be a continuous single valued function of the three variables (x, y, α) and let its derivatives F'_x, F'_y, F'_α exist and be continuous. Consider the behavior of the curves of the family near a point (x_0, y_0) of the curve for $\alpha = \alpha_0$ provided that (x_0, y_0) is an ordinary (nonsingular) point of the curve and that the derivative $F'_\alpha(x_0, y_0, \alpha_0)$ does not vanish. As $F'_\alpha \neq 0$ and either $F'_x \neq 0$ or $F'_y \neq 0$ for (x_0, y_0, α_0), it is possible to surround (x_0, y_0) with a region so small that $F(x, y, \alpha) = 0$ may be solved for $\alpha = f(x, y)$ which will be single valued and differentiable; and the region may further be taken so small that F'_x or F'_y remains different from 0 throughout the region. Then through every point of the region there is one and only one curve $\alpha = f(x, y)$ and the curves have no singular points within the region. In particular no two curves of the family can be tangent to each other within the region.

Furthermore, in such a region there is no envelope. For let any curve which traverses the region be $x = \phi(t)$, $y = \psi(t)$. Then

$$\alpha(t) = f(\phi(t),\, \psi(t)), \qquad \alpha'(t) = f'_x\phi'(t) + f'_y\psi'(t).$$

Along any curve $\alpha = f(x,\, y)$ the equation $f'_x dx + f'_y dy = 0$ holds, and if $x = \phi(t)$, $y = \psi(t)$ be tangent to this curve, $dy = dx = \psi' : \phi'$ and $\alpha'(t) = 0$ or $\alpha = $ const. Hence the only curve which has at each point the direction of the curve of the family through that point is a curve which coincides throughout with some curve of the family and is tangent to no other member of the family. Hence there is no envelope. The result is that an envelope can be present only when $F'_\alpha = 0$ or when $F'_x = F'_y = 0$, and this latter case has been seen to be included in the condition $F'_\alpha = 0$. If $F(x,\, y,\, \alpha)$ were not single valued but the branches were separable, the same conclusion would hold. Hence in case $F(x,\, y,\, \alpha)$ is not single valued the loci over which two or more values become inseparable must be added to those over which $F'_\alpha = 0$ in order to insure that all the loci which may be envelopes are taken into account.

67. The preceding considerations apply with so little change to other cases of envelopes that the facts will merely be stated without proof. Consider a family of surfaces $F(x,\, y,\, z,\, \alpha,\, \beta) = 0$ depending on two parameters. The envelope may be defined by the property of tangency as in § 65; and *the conditions for an envelope would be*

$$F(x,\, y,\, z,\, \alpha,\, \beta) = 0, \qquad F'_\alpha = 0, \qquad F'_\beta = 0. \tag{40}$$

These three equations may be solved to express the envelope as

$$x = \phi(\alpha,\, \beta), \qquad y = \psi(\alpha,\, \beta), \qquad z = \chi(\alpha,\, \beta)$$

parametrically in terms of α, β; or the two parameters may be eliminated and the envelope may be found as $\Phi(x,\, y,\, z) = 0$. In any case extraneous loci may be introduced and the results of the work should therefore be tested, which generally may be done at sight.

It is also possible to determine the distance from the tangent plane of one surface to the neighboring surfaces as

$$\frac{F'_x dx + F'_y dy + F'_z dz}{\sqrt{F'^2_x + F'^2_y + F'^2_z}} = \frac{F'_\alpha d\alpha + F'_\beta d\beta}{\sqrt{F'^2_x + F'^2_y + F'^2_z}} = dn, \tag{41}$$

and to define the envelope as the locus of points such that this distance is of higher order than $|d\alpha| + |d\beta|$. The equations (40) would then also follow. This definition would apply only to ordinary points of the surfaces of the family, that is, to points for which not all the derivatives F'_x, F'_y, F'_z vanish. But as the elimination of α, β from (40) would give an equation which included the loci of these singular points, there would be no danger of losing such loci in the rare instances where they, too, happened to be tangent to the surfaces of the family.

The application of implicit functions as in § 66 could also be made in this case and would show that no envelope could exist in regions where no singular points occurred and where either F'_α or F'_β failed to vanish. This work could be based either on the first definition involving tangency directly or on the second definition which involves tangency indirectly in the statements concerning infinitesimals of higher order. It may be added that if $F(x, y, z, \alpha, \beta) = 0$ were not single valued, the surfaces over which two values of the function become inseparable should be added as possible envelopes.

A family of surfaces $F(x, y, z, \alpha) = 0$ depending on a single parameter may have an envelope, and *the envelope is found from*

$$F(x, y, z, \alpha) = 0, \qquad F'_\alpha(x, y, z, \alpha) = 0 \qquad (42)$$

by the elimination of the single parameter. The details of the deduction of the rule will be omitted. If two neighboring surfaces intersect, the limiting position of the curve of intersection lies on the envelope and the envelope is the surface generated by this curve as α varies. The surfaces of the family touch the envelope not at a point merely but along these curves. The curves are called *characteristics* of the family. In the case where consecutive surfaces of the family do not intersect in a real curve it is necessary to fall back on the conception of imaginaries or on the definition of an envelope in terms of tangency or infinitesimals; the characteristic curves are still the curves along which the surfaces of the family are in contact with the envelope and along which two consecutive surfaces of the family are distant from each other by an infinitesimal of higher order than $d\alpha$.

A particular case of importance is the envelope of a plane which depends on one parameter. The equations (42) are then

$$Ax + By + Cz + D = 0, \qquad A'x + B'y + C'z + D' = 0, \qquad (43)$$

where A, B, C, D are functions of the parameter and differentiation with respect to it is denoted by accents. The case where the plane moves parallel to itself or turns about a line may be excluded as trivial. As the intersection of two planes is a line, the characteristics of the system are straight lines, the envelope is a *ruled surface*, and *a plane tangent to the surface at one point of the lines is tangent to the surface throughout the whole extent of the line*. Cones and cylinders are examples of this sort of surface. Another example is the surface enveloped by the osculating planes of a curve in space; for the osculating plane depends on only one parameter. As the osculating plane (§ 41) may be regarded as passing through three consecutive points of the curve, two consecutive osculating planes may be considered as having two consecutive points of the curve in common and hence the characteristics are

the tangent lines to the curve. Surfaces which are the envelopes of a plane which depends on a single parameter are called *developable surfaces*.

A family of curves dependent on two parameters as

$$F(x, y, z, \alpha, \beta) = 0, \quad G(x, y, z, \alpha, \beta) = 0 \qquad (44)$$

is called a *congruence of curves*. The curves may have an envelope, that is, there may be a surface to which the curves are tangent and which may be regarded as the locus of their points of tangency. The envelope is obtained by eliminating α, β from the equations

$$F = 0, \qquad G = 0, \qquad F_\alpha' G_\beta' - F_\beta' G_\alpha' = 0. \qquad (45)$$

To see this, suppose that the third condition is not fulfilled. The equations (44) may then be solved as $\alpha = f(x, y, z)$, $\beta = g(x, y, z)$. Reasoning like that of § 66 now shows that there cannot possibly be an envelope in the region for which the solution is valid. It may therefore be inferred that the only possibilities for an envelope are contained in the equations (45). As various extraneous loci might be introduced in the elimination of α, β from (45) and as the solutions should therefore be tested individually, it is hardly necessary to examine the general question further. The envelope of a congruence of curves is called the *focal surface* of the congruence and the points of contact of the curves with the envelope are called the *focal points* on the curves.

EXERCISES

1. Find the envelopes of these families of curves. In each case test the answer or its individual factors and check the results by a sketch :

$\quad$ (α) $y = 2\alpha x + \alpha^4$, $\quad$ (β) $y^2 = \alpha(x - \alpha)$, $\quad$ (γ) $y = \alpha x + k/\alpha$,

$\quad$ (δ) $\alpha(y + \alpha)^2 = x^3$, $\quad$ (ϵ) $y = \alpha(x + \alpha)^2$, $\quad$ (ζ) $y^2 = \alpha(x - \alpha)^3$.

2. Find the envelope of the ellipses $x^2/a^2 + y^2/b^2 = 1$ under the condition that (α) the sum of the axes is constant or (β) the area is constant.

3. Find the envelope of the circles whose center is on a given parabola and which pass through the vertex of the parabola.

4. Circles pass through the origin and have their centers on $x^2 - y^2 = c^2$. Find their envelope. $\qquad\qquad\qquad\qquad\qquad$ *Ans.* A lemniscate.

5. Find the envelopes in these cases :

$\quad$ (α) $x + xy\alpha = \sin^{-1} xy$, $\quad$ (β) $x + \alpha = \mathrm{vers}^{-1} y + \sqrt{2y - y^2}$,

$\qquad\qquad$ (γ) $y + \alpha = \sqrt{1 - 1/x}$.

6. Find the envelopes in these cases :

$\quad$ (α) $\alpha x + \beta y + \alpha\beta z = 1$, $\quad$ (β) $\dfrac{x}{\alpha} + \dfrac{y}{\beta} + \dfrac{z}{1 - \alpha - \beta} = 1$,

$\qquad\qquad$ (γ) $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} + \dfrac{z^2}{\gamma^2} = 1$ with $\alpha\beta\gamma = k^3$.

7. Find the envelopes in Ex. 6 (α), (β) if $\alpha = \beta$ or if $\alpha = -\beta$.

8. Prove that the envelope of $F(x, y, z, \alpha) = 0$ is tangent to the surface along the whole characteristic by showing that the normal to $F(x, y, z, \alpha) = 0$ and to the eliminant of $F = 0$, $F'_\alpha = 0$ are the same, namely

$$F'_x : F'_y : F'_z \quad \text{and} \quad F'_x + F'_\alpha \frac{\partial \alpha}{\partial x} : F'_y + F'_\alpha \frac{\partial \alpha}{\partial y} : F'_z + F'_\alpha \frac{\partial \alpha}{\partial z},$$

where $\alpha(x, y, z)$ is the function obtained by solving $F'_\alpha = 0$. Consider the problem also from the point of view of infinitesimals and the normal derivative.

9. If there is a curve $x = \phi(\alpha)$, $y = \psi(\alpha)$, $z = \chi(\alpha)$ tangent to the curves of the family defined by $F(x, y, z, \alpha) = 0$, $G(x, y, z, \alpha) = 0$ in space, then that curve is called the envelope of the family. Show, by the same reasoning as in § 65 for the case of the plane, that the four conditions $F = 0$, $G = 0$, $F'_\alpha = 0$, $G'_\alpha = 0$ must be satisfied for an envelope ; and hence infer that ordinarily a family of curves in space dependent on a single parameter has no envelope.

10. Show that the family $F(x, y, z, \alpha) = 0$, $F'_\alpha(x, y, z, \alpha) = 0$ of curves which are the characteristics of a family of surfaces has in general an envelope given by the three equations $F = 0$, $F'_\alpha = 0$, $F''_{\alpha\alpha} = 0$.

11. Derive the condition (45) for the envelope of a two-parametered family of curves from the idea of tangency, as in the case of one parameter.

12. Find the envelope of the normals to a plane curve $y = f(x)$ and show that the envelope is the locus of the center of curvature.

13. The locus of Ex. 12 is called the *evolute* of the curve $y = f(x)$. In these cases find the evolute as an envelope :

　(α) $y = x^2$, 　　　　(β) $x = a \sin t$, $y = b \cos t$, 　　　　(γ) $2xy = a^2$,
　(δ) $y^2 = 2mx$, 　　(ϵ) $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, 　(ζ) $y = \cosh x$.

14. Given a surface $z = f(x, y)$. Construct the family of normal lines and find their envelope.

15. If rays of light issuing from a point in a plane are reflected from a curve in the plane, the angle of reflection being equal to the angle of incidence, the envelope of the reflected rays is called the *caustic* of the curve with respect to the point. Show that the caustic of a circle with respect to a point on its circumference is a cardioid.

16. The curve which is the envelope of the characteristic lines, that is, of the rulings, on the developable surface (43) is called the *cuspidal edge* of the surface. Show that the equations of this curve may be found parametrically in terms of the parameter of (43) by solving simultaneously

$$Ax + By + Cz + D = 0, \ A'x + B'y + C'z + D' = 0, \ A''x + B''y + C''z + D'' = 0$$

for x, y, z. Consider the exceptional cases of cones and cylinders.

17. The term "developable" signifies that a *developable surface may be developed or mapped on a plane in such a way that lengths of arcs on the surface become equal lengths in the plane*, that is, the map may be made without distortion of size or shape. In the case of cones or cylinders this map may be made by slitting the cone or cylinder along an element and rolling it out upon a plane. What is the analytic statement in this case ? In the case of any developable surface with a cuspidal edge, the developable surface being the locus of all tangents to the cuspidal edge,

the length of arc upon the surface may be written as $d\sigma^2 = (dt + ds)^2 + t^2 ds^2/R^2$, where s denotes arc measured along the cuspidal edge and t denotes distance along the tangent line. This form of $d\sigma^2$ may be obtained geometrically by infinitesimal analysis or analytically from the equations

$$x = f(s) + tf'(s), \quad y = g(s) + tg'(s), \quad z = h(s) + th'(s)$$

of the developable surface of which $x = f(s)$, $y = g(s)$, $z = h(s)$ is the cuspidal edge. It is thus seen that $d\sigma^2$ is the same at corresponding points of all developable surfaces for which the radius of curvature R of the cuspidal edge is the same function of s without regard to the torsion; in particular the torsion may be zero and the developable may reduce to a plane.

18. Let the line $x = az + b$, $y = cz + d$ depend on one parameter so as to generate a ruled surface. By identifying this form of the line with (43) obtain by substitution the conditions

$$\begin{aligned}&Aa + Bc + C = 0, \quad A'a + B'c + C' = 0 \\ &Ab + Bd + D = 0, \quad A'b + B'd + D' = 0\end{aligned} \quad \text{or} \quad \begin{aligned}&Aa' + Bc' = 0 \\ &Ab' + Bd' = 0\end{aligned} \quad \text{or} \quad \begin{vmatrix} a' & c' \\ b' & d' \end{vmatrix} = 0$$

as the condition that the line generates a developable surface.

68. More differential geometry. The representation

$$F(x, y, z) = 0, \quad \text{or} \quad z = f(x, y) \tag{46}$$

or $\quad x = \phi(u, v), \quad y = \psi(u, v), \quad z = \chi(u, v)$

of a surface may be taken in the unsolved, the solved, or the parametric form. The parametric form is equivalent to the solved form provided u, v be taken as x, y. The notation

$$p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}, \qquad r = \frac{\partial^2 z}{\partial x^2}, \qquad s = \frac{\partial^2 z}{\partial x \partial y}, \qquad t = \frac{\partial^2 z}{\partial y^2}$$

is adopted for the derivatives of z with respect to x and y. The application of Taylor's Formula to the solved form gives

$$\Delta z = ph + qk + \tfrac{1}{2}(rh^2 + 2shk + tk^2) + \cdots \tag{47}$$

with $h = \Delta x$, $k = \Delta y$. The linear terms $ph + qk$ constitute the differential dz and represent that part of the increment of z which would be obtained by replacing the surface by its tangent plane. Apart from infinitesimals of the third order, the distance from the tangent plane up or down to the surface along a parallel to the z-axis is given by the quadratic terms $\tfrac{1}{2}(rh^2 + 2shk + tk^2)$.

Hence if the quadratic terms at any point are a positive definite form (§ 55), the surface lies above its tangent plane and is concave up; but if the form is negative definite, the surface lies below its tangent plane and is concave down or convex up. If the form is indefinite but not singular, the surface lies partly above and partly below its tangent plane and may be called concavo-convex, that is, it is saddle-shaped. If the form is singular nothing can be definitely stated. These statements

are merely generalizations of those of § 55 made for the case where the tangent plane is parallel to the xy-plane. It will be assumed in the further work of these articles that at least one of the derivatives r, s, t is not 0.

To examine more closely the behavior of a surface in the vicinity of a particular point upon it, let the xy-plane be taken in coincidence with the tangent plane at the point and let the point be taken as origin. Then Maclaurin's Formula is available.

$$z = \tfrac{1}{2}(rx^2 + 2\,sxy + ty^2) + \text{terms of higher order}$$
$$= \tfrac{1}{2}\rho^2(r\cos^2\theta + 2\,s\sin\theta\cos\theta + t\sin^2\theta) + \text{higher terms,} \quad (48)$$

where (ρ, θ) are polar coördinates in the xy-plane. Then

$$\frac{1}{R} = r\cos^2\theta + 2\,s\sin\theta\cos\theta + t\sin^2\theta = \frac{d^2z}{d\rho^2} \div \left[1 + \left(\frac{dz}{d\rho}\right)^2\right]^{\frac{3}{2}} \quad (49)$$

is the curvature of a normal section of the surface. The sum of the curvatures in two normal sections which are in perpendicular planes may be obtained by giving θ the values θ and $\theta + \tfrac{1}{2}\pi$. This sum reduces to $r + t$ and is therefore independent of θ.

As the sum of the curvatures in two perpendicular normal planes is constant, the maximum and minimum values of the curvature will be found in perpendicular planes. These values of the curvature are called the *principal values* and their reciprocals are the *principal radii of curvature* and the sections in which they lie are the *principal sections*. If $s = 0$, the principal sections are $\theta = 0$ and $\theta = \tfrac{1}{2}\pi$; and conversely if the axes of x and y had been chosen in the tangent plane so as to be tangent to the principal sections, the derivative s would have vanished. The equation of the surface would then have taken the simple form

$$z = \tfrac{1}{2}(rx^2 + ty^2) + \text{higher terms.} \quad (50)$$

The principal curvatures would be merely r and t, and the curvature in any normal section would have had the form

$$\frac{1}{R} = \frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2} = r\cos^2\theta + t\sin^2\theta.$$

If the two principal curvatures have opposite signs, that is, if the signs of r and t in (50) are opposite, the surface is saddle-shaped. There are then two directions for which the curvature of a normal section vanishes, namely the directions of the lines

$$\theta = \pm \tan^{-1}\sqrt{-R_2/R_1} \quad \text{or} \quad \sqrt{|r|}\,x = \pm\sqrt{|t|}\,y.$$

These are called the *asymptotic directions*. Along these directions the surface departs from its tangent plane by infinitesimals of the third

order, or higher order. If a curve is drawn on a surface so that at each point of the curve the tangent to the curve is along one of the asymptotic directions, the curve is called an *asymptotic curve or line* of the surface. As the surface departs from its tangent plane by infinitesimals of higher order than the second along an asymptotic line, the tangent plane to a surface at any point of an asymptotic line must be the osculating plane of the asymptotic line.

The character of a point upon a surface is indicated by the *Dupin indicatrix* of the point. The indicatrix is the conic

$$\frac{x^2}{R_1} + \frac{y^2}{R_2} = 1, \qquad \text{cf. } z = \tfrac{1}{2}(rx^2 + ty^2),\tag{51}$$

which has the principal directions as the directions of its axes and the square roots of the absolute values of the principal radii of curvature as the magnitudes of its axes. The conic may be regarded as similar to the conic in which a plane infinitely near the tangent plane cuts the surface when infinitesimals of order higher than the second are neglected. In case the surface is concavo-convex the indicatrix is a hyperbola and should be considered as either or both of the two conjugate hyperbolas that would arise from giving z positive or negative values in (51). The point on the surface is called elliptic, hyperbolic, or parabolic according as the indicatrix is an ellipse, a hyperbola, or a pair of lines, as happens when one of the principal curvatures vanishes. These classes of points correspond to the distinctions definite, indefinite, and singular applied to the quadratic form $rh^2 + 2shk + tk^2$.

Two further results are noteworthy. Any curve drawn on the surface differs from the section of its osculating plane with the surface by infinitesimals of higher order than the second. For as the osculating plane passes through three consecutive points of the curve, its intersection with the surface passes through the same three consecutive points and the two curves have contact of the second order. It follows that the radius of curvature of any curve on the surface is identical with that of the curve in which its osculating plane cuts the surface. The other result is *Meusnier's Theorem:* The radius of curvature of an oblique section of the surface at any point is the projection upon the plane of that section of the radius of curvature of the normal section which passes through the same tangent line. In other words, if the radius of curvature of a normal section is known, that of the oblique sections through the same tangent line may be obtained by multiplying it by the cosine of the angle between the plane normal to the surface and the plane of the oblique section.

The proof of Meusnier's Theorem may be given by reference to (48). Let the x-axis in the tangent plane be taken along the intersection with the oblique plane. Neglect infinitesimals of higher order than the second. Then

$$y = \phi(x) = \tfrac{1}{2} ax^2, \qquad z = \tfrac{1}{2}(rx^2 + 2\,sxy + ty^2) = \tfrac{1}{2}\,rx^2 \tag{48'}$$

will be the equations of the curve. The plane of the section is $az - ry = 0$, as may be seen by inspection. The radius of curvature of the curve in this plane may be found at once. For if u denote distance in the plane and perpendicular to the x-axis and if ν be the angle between the normal plane and the oblique plane $az - ry = 0$,

$$u = z \sec \nu = y \csc \nu = \tfrac{1}{2}\, r \sec \nu \cdot x^2 = \tfrac{1}{2}\, a \csc \nu \cdot x^2.$$

The form $u = \tfrac{1}{2}\, r \sec \nu \cdot x^2$ gives the curvature as $r \sec \nu$. But the curvature in the normal section is r by (48'). As the curvature in the oblique section is $\sec \nu$ times that in the normal section, the radius of curvature in the oblique section is $\cos \nu$ times that of the normal section. Meusnier's Theorem is thus proved.

69. These investigations with a special choice of axes give geometric properties of the surface, but do not express those properties in a convenient analytic form ; for if a surface $z = f(x, y)$ is given, the transformation to the special axes is difficult. The idea of the indicatrix or its similar conic as the section of the surface by a plane near the tangent plane and parallel to it will, however, determine the general conditions readily. If in the expansion

$$\Delta z - dz = \tfrac{1}{2}(rh^2 + 2\,shk + tk^2) = \text{const.} \tag{52}$$

the quadratic terms be set equal to a constant, the conic obtained is the projection of the indicatrix on the xy-plane, or if (52) be regarded as a cylinder upon the xy-plane, the indicatrix (or similar conic) is the intersection of the cylinder with the tangent plane. As the character of the conic is unchanged by the projection, *the point on the surface is elliptic if $s^2 < rt$, hyperbolic if $s^2 > rt$, and parabolic if $s^2 = rt$.* Moreover if the indicatrix is hyperbolic, its asymptotes must project into the asymptotes of the conic (52), and hence if dx and dy replace h and k, the equation

$$r\,dx^2 + 2\,s\,dx\,dy + t\,dy^2 = 0 \tag{53}$$

may be regarded as *the differential equation of the projection of the asymptotic lines on the xy-plane.* If r, s, t be expressed as functions $f''_{xx}, f''_{xy}, f''_{yy}$ of (x, y) and (53) be factored, the integration of the two equations $M(x, y)\,dx + N(x, y)\,dy$ thus found will give the finite equations of the projections of the asymptotic lines and, taken with the equation $z = f(x, y)$, will give the curves on the surface.

To find the lines of curvature is not quite so simple ; for it is necessary to determine the directions which are the projections of the axes of the indicatrix, and these are not the axes of the projected conic. Any radius of the indicatrix may be regarded as the intersection of the tangent plane and a plane perpendicular to the xy-plane through the radius of the projected conic. Hence

$$z - z_0 = p(x - x_0) + q(y - y_0), \qquad (x - x_0)\,k = (y - y_0)\,h$$

are the two planes which intersect in the radius that projects along the direction determined by h, k. The direction cosines

$$\frac{h : k : ph + qk}{\sqrt{h^2 + k^2 + (ph + qk)^2}} \quad \text{and} \quad h : k : 0 \tag{54}$$

are therefore those of the radius in the indicatrix and of its projection and they determine the cosine of the angle ϕ between the radius and its projection. The square of the radius in (52) is

$$h^2 + k^2, \quad \text{and} \quad (h^2 + k^2)\sec^2 \phi = h^2 + k^2 + (ph + qk)^2$$

is therefore the square of the corresponding radius in the indicatrix. To determine the axes of the indicatrix, this radius is to be made a maximum or minimum subject to (52). With a multiplier λ,

$$h + ph + qk + \lambda(rh + sk) = 0, \qquad k + ph + qk + \lambda(sh + tk) = 0$$

are the conditions required, and the elimination of λ gives

$$h^2[s(1 + p^2) - pqr] + hk[t(1 + p^2) - r(1 + q^2)] - k^2[t(1 + q^2) - pqt] = 0$$

as the equation that determines the projection of the axes. Or

$$\frac{(1 + p^2)\,dx + pq\,dy}{r\,dx + s\,dy} = \frac{pq\,dx + (1 + q^2)\,dy}{s\,dx + t\,dy} \tag{55}$$

is the differential equation of the projected lines of curvature.

In addition to the asymptotic lines and lines of curvature the *geodesic or shortest lines* on the surface are important. These, however, are better left for the methods of the calculus of variations (§ 159). The attention may therefore be turned to finding the value of the radius of curvature in any normal section of the surface.

A reference to (48) and (49) shows that the curvature is

$$\frac{1}{R} = \frac{2\,z}{\rho^2} = \frac{rh^2 + 2\,shk + tk^2}{\rho^2} = \frac{rh^2 + 2\,shk + tk^2}{h^2 + k^2}$$

in the special case. But in the general case the normal distance to the surface is $(\Delta z - dz)\cos \gamma$, with $\sec \gamma = \sqrt{1 + p^2 + q^2}$, instead of the $2\,z$ of the special case, and the radius ρ^2 of the special case becomes $\rho^2 \sec^2 \phi = h^2 + k^2 + (ph + qk)^2$ in the tangent plane. Hence

$$\frac{1}{R} = \frac{2(\Delta z - dz)\cos \gamma}{h^2 + k^2 + (ph + qk)^2} = \frac{rl^2 + 2\,slm + tm^2}{\sqrt{1 + p^2 + q^2}}, \tag{56}$$

where the direction cosines l, m of a radius in the tangent plane have been introduced from (54), is the general expression for the curvature of a normal section. The form

$$\frac{1}{R} = \frac{rh^2 + 2\,shk + tk^2}{h^2 + k^2 + (ph + qk)^2} \frac{1}{\sqrt{1 + p^2 + q^2}}, \tag{56$'$}$$

where the direction h, k of the projected radius remains, is frequently more convenient than (56) which contains the direction cosines l, m of the original direction in the tangent plane. Meusnier's Theorem may now be written in the form

$$\frac{\cos \nu}{R} = \frac{rl^2 + 2\,slm + tm^2}{\sqrt{1 + p^2 + q^2}}, \tag{57}$$

where ν is the angle between an oblique section and the tangent plane and where l, m are the direction cosines of the intersection of the planes.

The work here given has depended for its relative simplicity of statement upon the assumption of the surface (46) in solved form. It is merely a problem in implicit partial differentiation to pass from p, q, r, s, t to their equivalents in terms of F'_x, F'_y, F'_z or the derivatives of ϕ, ψ, χ by α, β.

EXERCISES

• **1.** In (49) show $\dfrac{1}{R} = \dfrac{r+t}{2} + \dfrac{r-t}{2}\cos 2\theta + s\sin 2\theta$ and find the directions of maximum and minimum R. If R_1 and R_2 are the maximum and minimum values of R, show

$$\frac{1}{R_1} + \frac{1}{R_2} = r + t \quad\text{and}\quad \frac{1}{R_1}\frac{1}{R_2} = rt - s^2.$$

Half of the sum of the curvatures is called the *mean curvature*; the product of the curvatures is called the *total curvature*.

2. Find the mean curvature, the total curvature, and therefrom (by constructing and solving a quadratic equation) the principal radii of curvature at the origin:

$$(\alpha)\ z = xy, \qquad (\beta)\ z = x^2 + xy + y^2, \qquad (\gamma)\ z = x(x+y).$$

3. In the surfaces (α) $z = xy$ and (β) $z = 2x^2 + y^2$ find at $(0, 0)$ the radius of curvature in the sections made by the planes

$$(\alpha)\ x + y = 0, \qquad (\beta)\ x + y + z = 0, \qquad (\gamma)\ x + y + 2z = 0,$$
$$(\delta)\ x - 2y = 0, \qquad (\epsilon)\ x - 2y + z = 0, \qquad (\zeta)\ x + 2y + \tfrac{1}{2}z = 0.$$

The oblique sections are to be treated by applying Meusnier's Theorem.

4. Find the asymptotic directions at $(0, 0)$ in Exs. 2 and 3.

5. Show that *a developable surface is everywhere parabolic*, that is, that $rt - s^2 = 0$ at every point; and conversely. To do this consider the surface as the envelope of its tangent plane $z - p_0 x - q_0 y = z_0 - p_0 x_0 - q_0 y_0$, where p_0, q_0, x_0, y_0, z_0 are functions of a single parameter α. Hence show

$$J\!\left(\frac{p_0, q_0}{x_0, y_0}\right) = 0 = (rt - s^2)_0 \quad\text{and}\quad J\!\left(\frac{p_0,\ z_0 - p_0 x_0 - q_0 y_0}{x_0,\ y_0}\right) = y_0\,(s^2 - rt)_0.$$

The first result proves the statement; the second, its converse.

6. Find the differential equations of the asymptotic lines and lines of curvature on these surfaces:

$$(\alpha)\ z = xy, \qquad (\beta)\ z = \tan^{-1}(y/x), \qquad (\gamma)\ z^2 + y^2 = \cosh x, \qquad (\delta)\ xyz = 1.$$

7. Show that the mean curvature and total curvature are

$$\frac{1}{2}\!\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{r(1 + q^2) + t(1 + p^2) - 2pqs}{2(1 + p^2 + q^2)^{\frac{3}{2}}}, \qquad \frac{1}{R_1 R_2} = \frac{rt - s^2}{(1 + p^2 + q^2)^2}.$$

8. Find the principal radii of curvature at $(1, 1)$ in Ex. 6.

9. An umbilic is a point of a surface at which the principal radii of curvature (and hence all radii of curvature for normal sections) are equal. Show that the conditions are $\dfrac{r}{1 + p^2} = \dfrac{s}{pq} = \dfrac{t}{1 + q^2}$ for an umbilic, and determine the umbilics of the ellipsoid with semiaxes a, b, c.