CHAPTER II

REVIEW OF FUNDAMENTAL THEORY*

18. Numbers and limits. The concept and theory of *real number*, integral, rational, and irrational, will not be set forth in detail here. Some matters, however, which are necessary to the proper understanding of rigorous methods in analysis must be mentioned; and numerous points of view which are adopted in the study of irrational number will be suggested in the text or exercises.

It is taken for granted that by his earlier work the reader has become familiar with the use of real numbers. In particular it is assumed that he is accustomed to represent numbers as a *scale*, that is, by points on a straight line, and that he knows that when a line is given and an origin chosen upon it and a unit of measure and a positive direction have been chosen, then to each point of the line corresponds one and only one real number, and conversely. Owing to this correspondence, that is, owing to the conception of a scale, it is possible to interchange statements about numbers with statements about points and hence to obtain a more vivid and graphic or a more abstract and arithmetic phraseology as may be desired. Thus instead of saying that the numbers $x_1, x_2, \cdots$ are increasing algebraically, one may say that the points (whose coördinates are) $x_1, x_2, \cdots$ are moving in the positive direction or to the right; with a similar correlation of a decreasing suite of numbers with points moving in the negative direction or to the left. It should be remembered, however, that whether a statement is couched in geometric or algebraic terms, it is always a statement concerning numbers when one has in mind the point of view of pure analysis.†

It may be recalled that arithmetic begins with the integers, including 0, and with addition and multiplication. That second, the rational numbers of the form p/q are introduced with the operation of division and the negative rational numbers with the operation of subtraction. Finally, the irrational numbers are introduced by various processes. Thus $\sqrt{2}$ occurs in geometry through the necessity of expressing the length of the diagonal of a square, and $\sqrt{3}$ for the diagonal of a cube. Again, π is needed for the ratio of circumference to diameter in a circle. In algebra any equation of odd degree has at least one real root and hence may be regarded as defining a number. But there is an essential difference between rational and irrational numbers in that any rational number is of the

* The object of this chapter is to set forth systematically, with attention to precision of statement and accuracy of proof, those fundamental definitions and theorems which lie at the basis of calculus and which have been given in the previous chapter from an intuitive rather than a critical point of view.

† Some illustrative graphs will be given; the student should make many others.

form $\pm p/q$ with $q \neq 0$ and can therefore be written down explicitly; whereas the irrational numbers arise by a variety of processes and, although they may be represented to any desired accuracy by a decimal, they cannot all be written down explicitly. It is therefore necessary to have some definite axioms regulating the essential properties of irrational numbers. The particular axiom upon which stress will here be laid is the axiom of continuity, the use of which is essential to the proof of elementary theorems on limits.

19. Axiom of Continuity. *If all the points of a line are divided into two classes such that every point of the first class precedes every point of the second class, there must be a point C such that any point preceding C is in the first class and any point succeeding C is in the second class.* This principle may be stated in terms of numbers, as: *If all real numbers be assorted into two classes such that every number of the first class is algebraically less than every number of the second class, there must be a number N such that any number less than N is in the first class and any number greater than N is in the second.* The number N (or point C) is called the frontier number (or point), or simply the *frontier* of the two classes, and in particular it is the *upper frontier* for the first class and the *lower frontier* for the second.

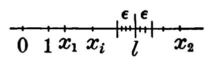
To consider a particular case, let all the negative numbers and zero constitute the first class and all the positive numbers the second, or let the negative numbers alone be the first class and the positive numbers with zero the second. In either case it is clear that the classes satisfy the conditions of the axiom and that zero is the frontier number such that any lesser number is in the first class and any greater in the second. If, however, one were to consider the system of all positive and negative numbers but without zero, it is clear that there would be no number N which would satisfy the conditions demanded by the axiom when the two classes were the negative and positive numbers; for no matter how small a positive number were taken as N, there would be smaller numbers which would also be positive and would not belong to the first class; and similarly in case it were attempted to find a negative N. Thus the axiom insures the presence of zero in the system, and in like manner insures the presence of every other number — a matter which is of importance because there is no way of writing all (irrational) numbers in explicit form.

Further to appreciate the continuity of the number scale, consider the four significations attributable to the phrase "*the interval from a to b.*" They are

$$a \leqq x \leqq b, \qquad a < x \leqq b, \qquad a \leqq x < b, \qquad a < x < b.$$

That is to say, both end points or either or neither may belong to the interval. In the case a is absent, the interval has no first point; and if b is absent, there is no last point. Thus if zero is not counted as a positive number, there is no least positive number; for if any least number were named, half of it would surely be less, and hence the absurdity. The axiom of continuity shows that if all numbers be divided into two classes as required, there must be either a greatest in the first class or a least in the second — the frontier — but not both unless the frontier is counted twice, once in each class.

20. Definition of a Limit. *If x is a variable which takes on successive values $x_1, x_2, \cdots, x_i, x_j, \cdots$, the variable x is said to approach the constant l as a limit if the numerical difference between x and l ultimately becomes, and for all succeeding values of x remains, less than any preassigned number no matter how small.* The numerical difference between x and l is denoted by $|x-l|$ or $|l-x|$ and is called the *absolute value* of the difference. The fact of the approach to a limit may be stated as



$$|x-l| < \epsilon \quad \text{for all } x\text{'s subsequent to some } x$$

or $$x = l + \eta, \quad |\eta| < \epsilon \quad \text{for all } x\text{'s subsequent to some } x,$$

where ϵ is a positive number which may be assigned at pleasure and must be assigned before the attempt be made to find an x such that for all subsequent x's the relation $|x-l| < \epsilon$ holds.

So long as the conditions required in the definition of a limit are satisfied there is no need of bothering about how the variable approaches its limit, whether from one side or alternately from one side and the other, whether discontinuously as in the case of the area of the polygons used for computing the area of a circle or continuously as in the case of a train brought to rest by its brakes. To speak geometrically, a point x which changes its position upon a line approaches the point l as a limit if the point x ultimately comes into and remains in an assigned interval, no matter how small, surrounding l.

A variable is said to *become infinite* if the numerical value of the variable ultimately becomes and remains greater than any preassigned number K, no matter how large.* The notation is $x = \infty$, but had best be read "x becomes infinite," not "x equals infinity."

Theorem 1. If a variable is always increasing, it either becomes infinite or approaches a limit.

That the variable *may* increase indefinitely is apparent. But if it does not become infinite, there must be numbers K which are greater than any value of the variable. Then any number must satisfy one of two conditions: either there are values of the variable which are greater than it or there are no values of the variable greater than it. Moreover all numbers that satisfy the first condition are less than any number which satisfies the second. All numbers are therefore divided into two classes fulfilling the requirements of the axiom of continuity, and there must be a number N such that there are values of the variable greater than any number $N-\epsilon$ which is less than N. Hence if ϵ be assigned, there is a value of the variable which lies in the interval $N-\epsilon < x \leqq N$, and as the variable is always increasing, all subsequent values must lie in this interval. Therefore the variable approaches N as a limit.

* This definition means what it says, and no more. Later, additional or different meanings may be assigned to infinity, but not now. Loose and extraneous concepts in this connection are almost certain to introduce errors and confusion.

EXERCISES

1. If $x_1, x_2, \cdots, x_n, \cdots, x_{n+p}, \cdots$ is a suite approaching a limit, apply the definition of a limit to show that when ϵ is given it must be possible to find a value of n so great that $|x_{n+p} - x_n| < \epsilon$ for all values of p.

2. If $x_1, x_2, \cdots$ is a suite approaching a limit and if $y_1, y_2, \cdots$ is any suite such that $|y_n - x_n|$ approaches zero when n becomes infinite, show that the y's approach a limit which is identical with the limit of the x's.

3. As the definition of a limit is phrased in terms of inequalities and absolute values, note the following rules of operation:

$$(\alpha)\ \text{If}\quad a > 0 \quad\text{and}\quad c > b, \quad\text{then}\quad \frac{c}{a} > \frac{b}{a} \quad\text{and}\quad \frac{a}{c} < \frac{a}{b},$$

$$(\beta)\ |a + b + c + \cdots| \leqq |a| + |b| + |c| + \cdots, \qquad (\gamma)\ |abc\cdots| = |a|\cdot|b|\cdot|c|\cdots,$$

where the equality sign in (β) holds only if the numbers $a, b, c, \cdots$ have the same sign. By these relations and the definition of a limit prove the fundamental theorems:

If $\lim x = X$ and $\lim y = Y$, then $\lim (x \pm y) = X \pm Y$ and $\lim xy = XY$.

4. Prove Theorem 1 when restated in the slightly changed form: If a variable x never decreases and never exceeds K, then x approaches a limit N and $N \leqq K$. Illustrate fully. State and prove the corresponding theorem for the case of a variable never increasing.

5. If $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$ are two suites of which the first never decreases and the second never increases, all the y's being greater than any of the x's, and if when ϵ is assigned an n can be found such that $y_n - x_n < \epsilon$, show that the limits of the suites are identical.

6. If $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$ are two suites which never decrease, show by Ex. 4 (not by Ex. 3) that the suites $x_1 + y_1, x_2 + y_2, \cdots$ and $x_1y_1, x_2y_2, \cdots$ approach limits. Note that two infinite decimals are precisely two suites which never decrease as more and more figures are taken. They do not always increase, for some of the figures may be 0.

7. If the word "all" in the hypothesis of the axiom of continuity be assumed to refer only to rational numbers so that the statement becomes: If all rational numbers be divided into two classes $\cdots$, there shall be a number N (not necessarily rational) such that $\cdots$; then the conclusion may be taken as defining a number as the frontier of a sequence of rational numbers. Show that if two numbers X, Y be defined by two such sequences, and if the sum of the numbers be *defined* as the number defined by the sequence of the sums of corresponding terms as in Ex. 6, and if the product of the numbers be *defined* as the number defined by the sequence of the products as in Ex. 6, then the fundamental rules

$$X + Y = Y + X, \qquad XY = YX, \qquad (X + Y)Z = XZ + YZ$$

of arithmetic hold for the numbers X, Y, Z defined by sequences. In this way a complete theory of irrationals may be built up from the properties of rationals combined with the principle of continuity, namely, 1° by defining irrationals as frontiers of sequences of rationals, 2° by defining the operations of addition, multiplication, $\cdots$ as operations upon the rational numbers in the sequences, 3° by showing that the fundamental rules of arithmetic still hold for the irrationals.

8. Apply the principle of continuity to show that there is a positive number x such that $x^2 = 2$. To do this it should be shown that the rationals are divisible into two classes, those whose square is less than 2 and those whose square is not less than 2; and that these classes satisfy the requirements of the axiom of continuity. In like manner if a is any positive number and n is any positive integer, show that there is an x such that $x^n = a$.

21. Theorems on limits and on sets of points.

The theorem on limits which is of fundamental algebraic importance is

THEOREM 2. If $R(x, y, z, \cdots)$ be any rational function of the variables $x, y, z, \cdots$, and if these variables are approaching limits $X, Y, Z, \cdots$, then the value of R approaches a limit and the limit is $R(X, Y, Z, \cdots)$, provided there is no division by zero.

As any rational expression is made up from its elements by combinations of addition, subtraction, multiplication, and division, it is sufficient to prove the theorem for these four operations. All except the last have been indicated in the above Ex. 3. As multiplication has been cared for, division need be considered only in the simple case of a reciprocal $1/x$. It must be proved that if $\lim x = X$, then $\lim (1/x) = 1/X$. Now

$$\left|\frac{1}{x} - \frac{1}{X}\right| = \frac{|x - X|}{|x||X|}, \qquad \text{by Ex. 3 } (\gamma) \text{ above.}$$

This quantity must be shown to be less than any assigned ϵ. As the quantity is complicated it will be replaced by a simpler one which is greater, owing to an increase in the denominator. Since $x \doteq X$, $x - X$ may be made numerically as small as desired, say less than ϵ', for all x's subsequent to some particular x. Hence if ϵ' be taken at least as small as $\frac{1}{2}|X|$, it appears that $|x|$ must be greater than $\frac{1}{2}|X|$. Then

$$\frac{|x - X|}{|x||X|} < \frac{|x - X|}{\frac{1}{2}|X|^2} = \frac{\epsilon'}{\frac{1}{2}|X|^2}, \qquad \text{by Ex. 3 } (\alpha) \text{ above,}$$

and if ϵ' be restricted to being less than $\frac{1}{2}|X|^2\epsilon$, the difference is less than ϵ and the theorem that $\lim (1/x) = 1/X$ is proved, and also Theorem 2. The necessity for the restriction $X \neq 0$ and the corresponding restriction in the statement of the theorem is obvious.

THEOREM 3. If when ϵ is given, no matter how small, it is possible to find a value of n so great that the difference $|x_{n+p} - x_n|$ between x_n and every subsequent term x_{n+p} in the suite $x_1, x_2, \cdots, x_n, \cdots$ is less than ϵ, the suite approaches a limit, and conversely.

The converse part has already been given as Ex. 1 above. The theorem itself is a consequence of the axiom of continuity. First note that as $|x_{n+p} - x_n| < \epsilon$ for all x's subsequent to x_n, the x's cannot become infinite. Suppose 1° that there is some number l such that no matter how remote x_n is in the suite, there are always subsequent values of x which are greater than l and others which are less than l. As all the x's after x_n lie in the interval 2ϵ and as l is less than some x's and greater than others, l must lie in that interval. Hence $|l - x_{n+p}| < 2\epsilon$ for all

x's subsequent to x_n. But now 2ϵ can be made as small as desired because ϵ can be taken as small as desired. Hence the definition of a limit applies and the x's approach l as a limit.

Suppose 2° that there is no such number l. Then every number k is such that either it is possible to go so far in the suite that all subsequent numbers x are as great as k or it is possible to go so far that all subsequent x's are less than k. Hence all numbers k are divided into two classes which satisfy the requirements of the axiom of continuity, and there must be a number N such that the x's ultimately come to lie between $N - \epsilon'$ and $N + \epsilon'$, no matter how small ϵ' is. Hence the x's approach N as a limit. Thus under either supposition the suite approaches a limit and the theorem is proved. It may be noted that under the second supposition the x's ultimately lie entirely upon one side of the point N and that the condition $|x_{n+p} - x_n| < \epsilon$ is not used except to show that the x's remain finite.

22. Consider next a set of points (or their correlative numbers) without any implication that they form a suite, that is, that one may be said to be subsequent to another. If there is only a finite number of points in the set, there is a point farthest to the right and one farthest to the left. If there is an infinity of points in the set, two possibilities arise. Either 1° it is not possible to assign a point K so far to the right that no point of the set is farther to the right — in which case the set is said to be *unlimited above* — or 2° there is a point K such that no point of the set is beyond K — and the set is said to be *limited above*. Similarly, a set may be *limited below* or *unlimited below*. If a set is limited above and below so that it is entirely contained in a finite interval, it is said merely to be *limited*. If there is a point C such that in any interval, no matter how small, surrounding C there are points of the set, then C is called a *point of condensation* of the set (C itself may or may not belong to the set).

Theorem 4. Any infinite set of points which is limited has an upper frontier (maximum?), a lower frontier (minimum?), and at least one point of condensation.

Before proving this theorem, consider three infinite sets as illustrations:

$$(\alpha)\ 1, 1.9, 1.99, 1.999, \cdots, \qquad (\beta)\ -2, \cdots, -1.99, -1.9, -1,$$

$$(\gamma)\ -1, -\tfrac{1}{2}, -\tfrac{1}{4}, \cdots, \tfrac{1}{4}, \tfrac{1}{2}, 1.$$

In (α) the element 1 is the minimum and serves also as the lower frontier; it is clearly not a point of condensation, but is isolated. There is no maximum; but 2 is the upper frontier and also a point of condensation. In (β) there is a maximum -1 and a minimum -2 (for -2 has been incorporated with the set). In (γ) there is a maximum and minimum; the point of condensation is 0. If one could be sure that an infinite set had a maximum and minimum, as is the case with finite sets, there would be no need of considering upper and lower frontiers. It is clear that if the upper or lower frontier belongs to the set, there is a maximum or minimum and the frontier is not necessarily a point of condensation; whereas *if the frontier does not belong to the set, it is necessarily a point of condensation and the corresponding extreme point is missing.*

To prove that there is an upper frontier, divide the points of the line into two classes, one consisting of points which are to the left of some point of the set, the other of points which are not to the left of any point of the set — then apply the axiom. Similarly for the lower frontier. To show the existence of a point of condensation, note that as there is an infinity of elements in the set, any point p is such that either there is an infinity of points of the set to the right of it or there is not. Hence the two classes into which all points are to be assorted are suggested, and the application of the axiom offers no difficulty.

EXERCISES

1. In a manner analogous to the proof of Theorem 2, show that

$$(\alpha)\ \lim_{x \doteq 0} \frac{x-1}{x-2} = \frac{1}{2}, \qquad (\beta)\ \lim_{x \doteq 2} \frac{3x-1}{x+5} = \frac{5}{7}, \qquad (\gamma)\ \lim_{x \doteq -1} \frac{x^2+1}{x^3-1} = -1.$$

2. Given an infinite series $S = u_1 + u_2 + u_3 + \cdots$. Construct the suite

$$S_1 = u_1,\ S_2 = u_1 + u_2,\ S_3 = u_1 + u_2 + u_3,\ \cdots,\ S_i = u_1 + u_2 + \cdots + u_i,\ \cdots,$$

where S_i is the sum of the first i terms. Show that Theorem 3 gives: The necessary and sufficient condition that the series S converge is that it is possible to find an n so large that $|S_{n+p} - S_n|$ shall be less than an assigned ϵ for all values of p. It is to be understood that a series *converges* when the suite of S's approaches a limit, and conversely.

3. If in a series $u_1 - u_2 + u_3 - u_4 + \cdots$ the terms approach the limit 0, are alternately positive and negative, and each term is less than the preceding, the series converges. Consider the suites $S_1, S_3, S_5, \cdots$ and $S_2, S_4, S_6, \cdots$.

4. Given three infinite suites of numbers

$$x_1, x_2, \cdots, x_n, \cdots; \quad y_1, y_2, \cdots, y_n, \cdots; \quad z_1, z_2, \cdots, z_n, \cdots$$

of which the first never decreases, the second never increases, and the terms of the third lie between corresponding terms of the first two, $x_n \leqq z_n \leqq y_n$. Show that the suite of z's has a point of condensation at or between the limits approached by the x's and by the y's; and that if $\lim x = \lim y = l$, then the z's approach l as a limit.

5. Restate the definitions and theorems on sets of points in arithmetic terms.

6. Give the details of the proof of Theorem 4. Show that the proof as outlined gives the least point of condensation. How would the proof be worded so as to give the greatest point of condensation? Show that if a set is limited above, it has an upper frontier but need not have a lower frontier.

7. If a set of points is such that between any two there is a third, the set is said to be *dense*. Show that the rationals form a dense set; also the irrationals. Show that any point of a dense set is a point of condensation for the set.

8. Show that the rationals p/q where $q < K$ do not form a dense set — in fact are a finite set in any limited interval. Hence in regarding any irrational as the limit of a set of rationals it is necessary that the denominators and also the numerators should become infinite.

9. Show that if an infinite set of points lies in a limited region of the plane, say in the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, there must be at least one point of condensation of the set. Give the necessary definitions and apply the axiom of continuity successively to the abscissas and ordinates.

23. Real functions of a real variable. *If x be a variable which takes on a certain set of values of which the totality may be denoted by $[x]$ and if y is a second variable the value of which is uniquely determined for each x of the set $[x]$, then y is said to be a function of x defined over the set $[x]$.* The terms "limited," "unlimited," "limited above," "unlimited below," $\cdots$ are applied to a function if they are applicable to the set $[y]$ of values of the function. Hence Theorem 4 has the corollary:

THEOREM 5. If a function is limited over the set $[x]$, it has an upper frontier M and a lower frontier m for that set.

If the function takes on its upper frontier M, that is, if there is a value x_0 in the set $[x]$ such that $f(x_0) = M$, the function has the absolute *maximum* M at x_0; and similarly with respect to the lower frontier. In any case, the difference $M - m$ between the upper and lower frontiers is called the *oscillation* of the function for the set $[x]$. The set $[x]$ is generally an interval.

Consider some illustrations of functions and sets over which they are defined. The reciprocal $1/x$ is defined for all values of x save 0. In the neighborhood of 0 the function is unlimited above for positive x's and unlimited below for negative x's. It should be noted that the function is not limited in the interval $0 < x \leqq a$ but is limited in the interval $\epsilon \leqq x \leqq a$ where ϵ is any assigned positive number. The function $+\sqrt{x}$ is defined for all positive x's including 0 and is limited below. It is not limited above for the totality of all positive numbers; but if K is assigned, the function is limited in the interval $0 \leqq x \leqq K$. The factorial function $x!$ is defined only for positive integers, is limited below by the value 1, but is not limited above unless the set $[x]$ is limited above. The function $E(x)$ denoting the integer not greater than x or "the integral part of x" is defined for all positive numbers — for instance $E(3) = E(\pi) = 3$. This function is not expressed, like the elementary functions of calculus, as a "formula"; it is defined by a definite law, however, and is just as much of a function as $x^2 + 3x + 2$ or $\frac{1}{2}\sin^2 2x + \log x$. Indeed it should be noted that the elementary functions themselves are in the first instance defined by definite laws and that it is not until after they have been made the subject of considerable study and have been largely developed along analytic lines that they appear as formulas. The ideas of function and formula are essentially distinct and the latter is essentially secondary to the former.

The definition of function as given above excludes the so-called *multiple-valued* functions such as $\sqrt{x}$ and $\sin^{-1} x$ where to a given value of x correspond more than one value of the function. It is usual, however, in treating multiple-valued functions to resolve the functions into different parts or *branches* so that each branch is a single-valued function. Thus $+\sqrt{x}$ is one branch and $-\sqrt{x}$ the other branch of $\sqrt{x}$; in fact when x is positive the symbol $\sqrt{x}$ is usually restricted to mean merely $+\sqrt{x}$ and thus becomes a single-valued symbol. One branch of $\sin^{-1} x$ consists of the values between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, other branches give values between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$ or $-\frac{1}{2}\pi$ and $-\frac{3}{2}\pi$, and so on. Hence the term "function" will be restricted in this chapter to the single-valued functions allowed by the definition.

24. *If $x = a$ is any point of an interval over which $f(x)$ is defined, the function $f(x)$ is said to be continuous at the point $x = a$ if*

$$\lim_{x \doteq a} f(x) = f(a), \qquad \textit{no matter how } x \doteq a.$$

The function is said to be continuous in the interval if it is continuous at every point of the interval. If the function is not continuous at the point a, it is said to be *discontinuous* at a; and if it fails to be continuous at any one point of an interval, it is said to be discontinuous in the interval.

THEOREM 6. If any finite number of functions are continuous (at a point or over an interval), any rational expression formed of those functions is continuous (at the point or over the interval) provided no division by zero is called for.

THEOREM 7. If $y = f(x)$ is continuous at x_0 and takes the value $y_0 = f(x_0)$ and if $z = \phi(y)$ is a continuous function of y at $y = y_0$, then $z = \phi[f(x)]$ will be a continuous function of x at x_0.

In regard to the definition of continuity note that a function cannot be continuous at a point unless it is defined at that point. Thus e^{-1/x^2} is not continuous at $x = 0$ because division by 0 is impossible and the function is undefined. If, however, the function be defined at 0 as $f(0) = 0$, the function becomes continuous at $x = 0$. In like manner the function $1/x$ is not continuous at the origin, and in this case it is impossible to assign to $f(0)$ any value which will render the function continuous; the function becomes infinite at the origin and the very idea of becoming infinite precludes the possibility of approach to a definite limit. Again, the function $E(x)$ is in general continuous, but is discontinuous for integral values of x. When a function is discontinuous at $x = a$, the *amount of the discontinuity* is the limit of the oscillation $M - m$ of the function in the interval $a - \delta < x < a + \delta$ surrounding the point a when δ approaches zero as its limit. The discontinuity of $E(x)$ at each integral value of x is clearly 1; that of $1/x$ at the origin is infinite no matter what value is assigned to $f(0)$.

In case the interval over which $f(x)$ is defined has end points, say $a \leqq x \leqq b$, the question of continuity at $x = a$ must of course be decided by allowing x to approach a from the right-hand side only; and similarly it is a question of left-handed approach to b. In general, if for any reason it is desired to restrict the approach of a variable to its limit to being one-sided, the notations $x \doteq a^+$ and $x \doteq b^-$ respectively are used to denote approach through greater values (right-handed) and through lesser values (left-handed). It is not necessary to make this specification in the case of the ends of an interval; for it is understood that x shall take on only values in the interval in question. It should be noted that

$\lim f(x) = f(x_0)$ when $x \doteq x_0^+$ in no wise implies the continuity of $f(x)$ at x_0; a simple example is that of $E(x)$ at the positive integral points.

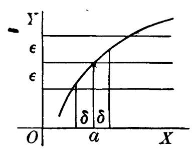
The proof of Theorem 6 is an immediate corollary application of Theorem 2. For

$$\lim R[f(x), \phi(x) \cdots] = R[\lim f(x), \lim \phi(x), \cdots] = R[f(\lim x), \phi(\lim x), \cdots],$$

and the proof of Theorem 7 is equally simple.

Theorem 8. If $f(x)$ is continuous at $x = a$, then for any positive ϵ which has been assigned, no matter how small, there may be found a number δ such that $|f(x) - f(a)| < \epsilon$ in the interval $|x - a| < \delta$, and hence in this interval the oscillation of $f(x)$ is less than 2ϵ. And conversely, if these conditions hold, the function is continuous.

This theorem is in reality nothing but a restatement of the definition of continuity combined with the definition of a limit. For "$\lim f(x) = f(a)$ when $x \doteq a$, no matter how" means that the difference between $f(x)$ and $f(a)$ can be made as small as desired by taking x sufficiently near to a; and conversely. The reason for this restatement is that the present form is more amenable to analytic operations. It also suggests the geometric picture which corresponds to the usual idea of continuity in graphs. For the theorem states that if the two lines $y = f(a) \pm \epsilon$ be drawn, the graph of the function remains between them for at least the short distance δ on each side of $x = a$; and as ϵ may be assigned a value as small as desired, the graph cannot exhibit breaks. On the other hand it should be noted that the actual physical graph is not a curve but a band, a two-dimensional region of greater or less breadth, and that a function could be discontinuous at every point of an interval and yet lie entirely within the limits of any given physical graph.



It is clear that δ, which has to be determined *subsequently* to ϵ, is in general more and more restricted as ϵ is taken smaller and that for different points it is more restricted as the graph rises more rapidly. Thus if $f(x) = 1/x$ and $\epsilon = 1/1000$, δ can be nearly 1/10 if $x_0 = 100$, but must be slightly less than 1/1000 if $x_0 = 1$, and something less than 10^{-6} if x is 10^{-3}. Indeed, if x be allowed to approach zero, the value δ for any assigned ϵ also approaches zero; and although the function $f(x) = 1/x$ is continuous in the interval $0 < x \leqq 1$ and for any given x_0 and ϵ a number δ may be found such that $|f(x) - f(x_0)| < \epsilon$ when $|x - x_0| < \delta$, yet it is not possible to assign a number δ which shall serve *uniformly* for all values of x_0.

25. Theorem 9. If a function $f(x)$ is continuous in an interval $a \leqq x \leqq b$ with end points, it is possible to find a δ such that $|f(x) - f(x_0)| < \epsilon$ when $|x - x_0| < \delta$ for all points x_0; and the function is said to be *uniformly continuous.*

The proof is conducted by the method of reductio ad absurdum. Suppose ϵ is assigned. Consider the suite of values $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\cdots$, or any other suite which approaches zero as a limit. Suppose that no one of these values will serve as a δ for all points of the interval. Then there must be at least one point for which $\frac{1}{2}$ will not serve, at least one for which $\frac{1}{4}$ will not serve, at least one for which $\frac{1}{8}$ will not serve, and so on indefinitely. This infinite set of points must have at least one point of condensation C such that in any interval surrounding C there are points for which 2^{-k} will not serve as δ, no matter how large k. But now by hypothesis $f(x)$ is continuous at C and hence a number δ can be found such that $|f(x)-f(C)|<\frac{1}{2}\epsilon$ when $|x-x_0|<2\delta$. The oscillation of $f(x)$ in the whole interval 4δ is less than ϵ. Now if x_0 be any point in the middle half of this interval, $|x_0-C|<\delta$; and if x satisfies the relation $|x-x_0|<\delta$, it must still lie in the interval 4δ and the difference $|f(x)-f(x_0)|<\epsilon$, being surely not greater than the oscillation of f in the whole interval. Hence it is possible to surround C with an interval so small that the same δ will serve for any point of the interval. This contradicts the former conclusion, and hence the hypothesis upon which that conclusion was based must have been false and it must have been possible to find a δ which would serve for all points of the interval. The reason why the proof would not apply to a function like $1/x$ defined in the interval $0<x\leqq 1$ lacking an end point is precisely that the point of condensation C would be 0, and at 0 the function is not continuous and $|f(x)-f(C)|<\frac{1}{2}\epsilon$, $|x-C|<2\delta$ could not be satisfied.

THEOREM 10. If a function is continuous in a region which includes its end points, the function is limited.

THEOREM 11. If a function is continuous in an interval which includes its end points, the function takes on its upper frontier and has a maximum M; similarly it has a minimum m.

These are successive corollaries of Theorem 9. For let ϵ be assigned and let δ be determined so as to serve uniformly for all points of the interval. Divide the interval $b-a$ into n successive intervals of length δ or less. Then in each such interval f cannot increase by more than ϵ nor decrease by more than ϵ. Hence f will be contained between the values $f(a)+n\epsilon$ and $f(a)-n\epsilon$, and is limited. And $f(x)$ has an upper and a lower frontier in the interval. Next consider the rational function $1/(M-f)$ of f. By Theorem 6 this is continuous in the interval unless the denominator vanishes, and if continuous it is limited. This, however, is impossible for the reason that, as M is a frontier of values of f, the difference $M-f$ may be made as small as desired. Hence $1/(M-f)$ is not continuous and there must be some value of x for which $f=M$.

THEOREM 12. If $f(x)$ is continuous in the interval $a\leqq x\leqq b$ with end points and if $f(a)$ and $f(b)$ have opposite signs, there is at least one point ξ, $a<\xi<b$, in the interval for which the function vanishes. And whether $f(a)$ and $f(b)$ have opposite signs or not, there is a point ξ, $a<\xi<b$, such that $f(\xi)=\mu$, where μ is any value intermediate between the maximum and minimum of f in the interval.

For convenience suppose that $f(a)<0$. Then in the neighborhood of $x=a$ the function will remain negative on account of its continuity; and in the neighborhood of b it will remain positive. Let ξ be the lower frontier of values of x which make $f(x)$ positive. Suppose that $f(\xi)$ were either positive or negative. Then as f is continuous, an interval could be chosen surrounding ξ and so small that f remained positive or negative in that interval. In neither case could ξ be the lower frontier of positive values. Hence the contradiction, and $f(\xi)$ must be zero. To prove the second part of the theorem, let c and d be the values of x which make f a minimum and maximum. Then the function $f - \mu$ has opposite signs at c and d, and must vanish at some point of the interval between c and d; and hence a fortiori at some point of the interval from a to b.

EXERCISES

1. Note that x is a continuous function of x, and that consequently it follows from Theorem 6 that any rational fraction $P(x)/Q(x)$, where P and Q are polynomials in x, must be continuous for all x's except roots of $Q(x) = 0$.

2. Graph the function $x - E(x)$ for $x \geqq 0$ and show that it is continuous except for integral values of x. Show that it is limited, has a minimum 0, an upper frontier 1, but no maximum.

3. Suppose that $f(x)$ is defined for an infinite set $[x]$ of which $x = a$ is a point of condensation (not necessarily itself a point of the set). Suppose

$$\lim_{x', x'' \doteq a} [f(x') - f(x'')] = 0 \quad \text{or} \quad |f(x') - f(x'')| < \epsilon, \ |x' - a| < \delta, \ |x'' - a| < \delta,$$

when x' and x'' regarded as *independent* variables approach a as a limit (passing only over values of the set $[x]$, of course). Show that $f(x)$ approaches a limit as $x \doteq a$. By considering the set of values of $f(x)$, the method of Theorem 3 applies almost verbatim. Show that there is no essential change in the proof if it be assumed that x' and x'' become infinite, the set $[x]$ being unlimited instead of having a point of condensation a.

4. From the formula $\sin x < x$ and the formulas for $\sin u - \sin v$ and $\cos u - \cos v$ show that $\Delta \sin x$ and $\Delta \cos x$ are numerically less than $2|\Delta x|$; hence infer that $\sin x$ and $\cos x$ are continuous functions of x for all values of x.

5. What are the intervals of continuity for $\tan x$ and $\csc x$? If $\epsilon = 10^{-4}$, what are approximately the largest available values of δ that will make $|f(x) - f(x_0)| < \epsilon$ when $x_0 = 1°, 30°, 60°, 89°$ for each? Use a four-place table.

6. Let $f(x)$ be defined in the interval from 0 to 1 as equal to 0 when x is irrational and equal to $1/q$ when x is rational and expressed as a fraction p/q in lowest terms. Show that f is continuous for irrational values and discontinuous for rational values. Ex. 8, p. 39, will be of assistance in treating the irrational values.

7. Note that in the definition of continuity a generalization may be introduced by allowing the set $[x]$ over which f is defined to be any set each point of which is a point of condensation of the set, and that hence continuity over a dense set (Ex. 7 above), say the rationals or irrationals, may be defined. This is important because many functions are in the first instance defined only for rationals and are subsequently defined for irrationals by interpolation. Note that if a function is continuous over a dense set (say, the rationals), it does not follow that it is uniformly continuous over the set. For the point of condensation C which was used in the proof of Theorem 9 may not be a point of the set (may be irrational), and the proof would fall through for the same reason that it would in the case of $1/x$ in the interval $0 < x \leqq 1$, namely, because it could not be affirmed that the function was continuous at C. Show that if a function is defined and is uniformly continuous over a dense set, the value $f(x)$ will approach a limit when x approaches any value a (not necessarily of the set, but situated between the upper and lower frontiers of the set), and that if this limit be defined as the value of $f(a)$, the function will remain continuous. Ex. 3 may be used to advantage.

8. By factoring $(x + \Delta x)^n - x^n$, show for integral values of n that when $0 \leqq x \leqq K$, then $\Delta (x^n) < nK^{n-1} \Delta x$ for small Δx's and consequently x^n is uniformly continuous in the interval $0 \leqq x \leqq K$. If it be assumed that x^n has been defined only for rational x's, it follows from Ex. 7 that the definition may be extended to all x's and that the resulting x^n will be continuous.

9. Suppose (α) that $f(x) + f(y) = f(x + y)$ for any numbers x and y. Show that $f(n) = nf(1)$ and $nf(1/n) = f(1)$, and hence infer that $f(x) = xf(1) = Cx$, where $C = f(1)$, for all rational x's. From Ex. 7 it follows that if $f(x)$ is continuous, $f(x) = Cx$ for all x's. Consider (β) the function $f(x)$ such that $f(x) f(y) = f(x + y)$. Show that it is $Ce^x = a^x$.

10. Show by Theorem 12 that if $y = f(x)$ is a continuous constantly increasing function in the interval $a \leqq x \leqq b$, then to each value of y corresponds a single value of x so that the function $x = f^{-1}(y)$ exists and is single-valued; show also that it is continuous and constantly increasing. State the corresponding theorem if $f(x)$ is constantly decreasing. The function $f^{-1}(y)$ is called the *inverse* function to $f(x)$.

11. Apply Ex. 10 to discuss $y = \sqrt[n]{x}$, where n is integral, x is positive, and only positive roots are taken into consideration.

12. In arithmetic it may readily be shown that the equations

$$a^m a^n = a^{m+n}, \qquad (a^m)^n = a^{mn}, \qquad a^n b^n = (ab)^n,$$

are true when a and b are rational and positive and when m and n are any positive and negative integers or zero. (α) Can it be inferred that they hold when a and b are positive irrationals? (β) How about the extension of the fundamental inequalities

$$x^n > 1, \quad \text{when} \quad x > 1, \qquad x^n < 1, \quad \text{when} \quad 0 \leqq x < 1$$

to all rational values of n and the proof of the inequalities

$$x^m > x^n \quad \text{if} \quad m > n \quad \text{and} \quad x > 1, \qquad x^m < x^n \quad \text{if} \quad m > n \quad \text{and} \quad 0 < x < 1.$$

(γ) Next consider x as held constant and the exponent n as variable. Discuss the exponential function a^x from this relation, and Exs. 10, 11, and other theorems that may seem necessary. Treat the logarithm as the inverse of the exponential.

26. The derivative. *If $x = a$ is a point of an interval over which $f(x)$ is defined and if the quotient*

$$\frac{\Delta f}{\Delta x} = \frac{f(a+h) - f(a)}{h}, \qquad h = \Delta x,$$

approaches a limit when h approaches zero, no matter how, the function $f(x)$ is said to be differentiable at $x = a$ and the value of the limit of the quotient is the derivative $f'(a)$ of f at $x = a$. In the case of differentiability, the definition of a limit gives

$$\frac{f(a+h) - f(a)}{h} = f'(a) + \eta \quad \text{or} \quad f(a+h) - f(a) = hf'(a) + \eta h, \quad (1)$$

where $\lim \eta = 0$ when $\lim h = 0$, no matter how.

In other words if ϵ is given, a δ can be found so that $|\eta|<\epsilon$ when $|h|<\delta$. This shows that a function differentiable at a as in (1) is continuous at a. For

$$|f(a+h)-f(a)| \leqq |f'(a)|\delta + \epsilon\delta, \qquad |h|<\delta.$$

If the limit of the quotient exists when $h \doteq 0$ through positive values only, the function has a right-hand derivative which may be denoted by $f'(a^+)$ and similarly for the left-hand derivative $f'(a^-)$. At the end points of an interval the derivative is always considered as one-handed; but for interior points the right-hand and left-hand derivatives must be equal if the function is to have a derivative (unqualified). The function is said to have an *infinite derivative* at a if the quotient becomes infinite as $h \doteq 0$; but if a is an interior point, the quotient must become positively infinite or negatively infinite for all manners of approach and not positively infinite for some and negatively infinite for others. Geometrically this allows a vertical tangent with an inflection point, but not with a cusp as in Fig. 3, p. 8. If infinite derivatives are allowed, the function may have a derivative and yet be discontinuous, as is suggested by any figure where $f(a)$ is any value between $\lim f(x)$ when $x \doteq a^+$ and $\lim f(x)$ when $x \doteq a^-$.

THEOREM 13. If a function takes on its maximum (or minimum) at an interior point of the interval of definition and if it is differentiable at that point, the derivative is zero.

THEOREM 14. *Rolle's Theorem.* If a function $f(x)$ is continuous over an interval $a \leqq x \leqq b$ with end points and vanishes at the ends and has a derivative at each interior point $a < x < b$, there is some point ξ, $a < \xi < b$, such that $f'(\xi) = 0$.

THEOREM 15. *Theorem of the Mean.* If a function is continuous over an interval $a \leqq x \leqq b$ and has a derivative at each interior point, there is some point ξ such that

$$\frac{f(b)-f(a)}{b-a} = f'(\xi) \quad \text{or} \quad \frac{f(a+h)-f(a)}{h} = f'(a+\theta h),$$

where $h \leqq b - a$* and θ is a proper fraction, $0 < \theta < 1$.

To prove the first theorem, note that if $f(a) = M$, the difference $f(a+h) - f(a)$ cannot be negative for any value of h and the quotient $\Delta f/h$ cannot be positive when $h > 0$ and cannot be negative when $h < 0$. Hence the right-hand derivative cannot be positive and the left-hand derivative cannot be negative. As these two must be equal if the function has a derivative, it follows that they must be zero, and the derivative is zero. The second theorem is an immediate corollary. For as the function is continuous it must have a maximum and a minimum (Theorem 11) both of which cannot be zero unless the function is always zero in the interval. Now if the function is identically zero, the derivative is identically zero and the theorem is true; whereas if the function is not identically zero, either the maximum or minimum must be at an interior point, and at that point the derivative will vanish.

* That the theorem is true for any part of the interval from a to b if it is true for the whole interval follows from the fact that the conditions, namely, that f be continuous and that f' exist, hold for any part of the interval if they hold for the whole.

To prove the last theorem construct the auxiliary function

$$\psi(x) = f(x) - f(a) - (x - a)\frac{f(b) - f(a)}{b - a}, \qquad \psi'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

As $\psi(a) = \psi(b) = 0$, Rolle's Theorem shows that there is some point for which $\psi'(\xi) = 0$, and if this value be substituted in the expression for $\psi'(x)$ the solution for $f'(\xi)$ gives the result demanded by the theorem. The proof, however, requires the use of the function $\psi(x)$ and its derivative and is not complete until it is shown that $\psi(x)$ really satisfies the conditions of Rolle's Theorem, namely, is continuous in the interval $a \leqq x \leqq b$ and has a derivative for every point $a < x < b$. The continuity is a consequence of Theorem 6; that the derivative exists follows from the direct application of the definition combined with the assumption that the derivative of f exists.

27. Theorem 16. If a function has a derivative which is identically zero in the interval $a \leqq x \leqq b$, the function is constant; and if two functions have derivatives equal throughout the interval, the functions differ by a constant.

Theorem 17. If $f(x)$ is differentiable and becomes infinite when $x \doteq a$, the derivative cannot remain finite as $x \doteq a$.

Theorem 18. If the derivative $f'(x)$ of a function exists and is a continuous function of x in the interval $a \leqq x \leqq b$, the quotient $\Delta f/h$ converges uniformly toward its limit $f'(x)$.

These theorems are consequences of the Theorem of the Mean. For the first,

$$f(a + h) - f(a) = hf'(a + \theta h) = 0, \quad \text{if} \quad h \leqq b - a, \quad \text{or} \quad f(a + h) = f(a).$$

Hence $f(x)$ is constant. And in case of two functions f and ϕ with equal derivatives, the difference $\psi(x) = f(x) - \phi(x)$ will have a derivative that is zero and the difference will be constant. For the second, let x_0 be a fixed value near a and suppose that in the interval from x_0 to a the derivative remained finite, say less than K. Then

$$|f(x_0 + h) - f(x_0)| = |hf'(x_0 + \theta h)| \leqq |h| K.$$

Now let $x_0 + h$ approach a and note that the left-hand term becomes infinite and the supposition that f' remained finite is contradicted. For the third, note that f', being continuous, must be uniformly continuous (Theorem 9), and hence that if ϵ is given, a δ may be found such that

$$\left|\frac{f(x + h) - f(x)}{h} - f'(x)\right| \leqq |f'(x + \theta h) - f'(x)| < \epsilon$$

when $|h| < \delta$ and for all x's in the interval; and the theorem is proved.

Concerning derivatives of higher order no special remarks are necessary. Each is the derivative of a definite function — the previous derivative. If the derivatives of the first n orders exist and are continuous, the derivative of order $n + 1$ may or may not exist. In practical applications, however, the functions are generally indefinitely differentiable except at certain isolated points. The proof of Leibniz's Theorem (§ 8) may be revised so as to depend on elementary processes. Let the formula be assumed for a given value of n. The only terms which can contribute to the term $D^iuD^{n+1-i}v$ in the formula for the $(n+1)$st derivative of uv are the terms

$$\frac{n(n-1)\cdots(n-i+2)}{1\cdot 2\cdots(i-1)}D^{i-1}u\,D^{n+1-i}v, \qquad \frac{n(n-1)\cdots(n-i+1)}{1\cdot 2\cdots i}D^iu\,D^{n-i}v,$$

in which the first factor is to be differentiated in the first and the second in the second. The sum of the coefficients obtained by differentiating is

$$\frac{n(n-1)\cdots(n-i+2)}{1\cdot 2\cdots(i-1)}+\frac{n(n-1)\cdots(n-i+1)}{1\cdot 2\cdots i}=\frac{(n+1)n\cdots(n-i+2)}{1\cdot 2\cdots i},$$

which is precisely the proper coefficient for the term $D^iuD^{n+1-i}v$ in the expansion of the $(n+1)$st derivative of uv by Leibniz's Theorem.

With regard to this rule and the other elementary rules of operation (4)–(7) of the previous chapter it should be remarked that *a theorem* as well as a rule is involved — thus: If two functions u and v are differentiable at x_0, then the product uv is differentiable at x_0, and the value of the derivative is $u(x_0)v'(x_0)+u'(x_0)v(x_0)$. And similar theorems arise in connection with the other rules. As a matter of fact the ordinary proof needs only to be gone over with care in order to convert it into a rigorous demonstration. But care does need to be exercised both in stating the theorem and in looking to the proof. For instance, the above theorem concerning a product is not true if infinite derivatives are allowed. For let u be -1, 0, or $+1$ according as x is negative, 0, or positive, and let $v=x$. Now v has always a derivative which is 1 and u has always a derivative which is 0, $+\infty$, or 0 according as x is negative, 0, or positive. The product uv is $|x|$, of which the derivative is -1 for negative x's, $+1$ for positive x's, and *nonexistent* for 0. Here the product has no derivative at 0, although each factor has a derivative, and it would be useless to have a formula for attempting to evaluate something that did not exist.

EXERCISES

1. Show that if at a point the derivative of a function exists and is positive, the function must be increasing at that point.

2. Suppose that the derivatives $f'(a)$ and $f'(b)$ exist and are not zero. Show that $f(a)$ and $f(b)$ are relative maxima or minima of f in the interval $a\leqq x\leqq b$, and determine the precise criteria in terms of the signs of the derivatives $f'(a)$ and $f'(b)$.

3. Show that if a continuous function has a positive right-hand derivative at every point of the interval $a\leqq x\leqq b$, then $f(b)$ is the maximum value of f. Similarly, if the right-hand derivative is negative, show that $f(b)$ is the minimum of f.

4. Apply the Theorem of the Mean to show that if $f'(x)$ is continuous at a, then

$$\lim_{x',\,x''\doteq a}\frac{f(x')-f(x'')}{x'-x''}=f'(a),$$

x' and x'' being regarded as independent.

5. Form the increments of a function f for *equicrescent* values of the variable:

$$\Delta_1f=f(a+h)-f(a), \qquad \Delta_2f=f(a+2h)-f(a+h),$$
$$\Delta_3f=f(a+3h)-f(a+2h),\ \cdots.$$

These are called first differences; the differences of these differences are

$$\Delta_1^2 f = f(a+2h) - 2f(a+h) + f(a),$$
$$\Delta_2^2 f = f(a+3h) - 2f(a+2h) + f(a+h), \cdots$$

which are called the second differences; in like manner there are third differences

$$\Delta_1^3 f = f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a), \cdots$$

and so on. Apply the Law of the Mean to all the differences and show that

$$\Delta_1^2 f = h^2 f''(a+\theta_1 h+\theta_2 h), \qquad \Delta_1^3 f = h^3 f'''(a+\theta_1 h+\theta_2 h+\theta_3 h), \cdots.$$

Hence show that if the first n derivatives of f are continuous at a, then

$$f''(a) = \lim_{h \doteq 0} \frac{\Delta^2 f}{h^2}, \qquad f'''(a) = \lim_{h \doteq 0} \frac{\Delta^3 f}{h^3}, \qquad \cdots, \qquad f^{(n)}(a) = \lim_{h \doteq 0} \frac{\Delta^n f}{h^n}.$$

6. *Cauchy's Theorem.* If $f(x)$ and $\phi(x)$ are continuous over $a \leqq x \leqq b$, have derivatives at each interior point, and if $\phi'(x)$ does not vanish in the interval,

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(\xi)}{\phi'(\xi)} \quad \text{or} \quad \frac{f(a+h)-f(a)}{\phi(a+h)-\phi(a)} = \frac{f'(a+\theta h)}{\phi'(a+\theta h)}.$$

Prove that this follows from the application of Rolle's Theorem to the function

$$\psi(x) = f(x) - f(a) - [\phi(x) - \phi(a)] \frac{f(b)-f(a)}{\phi(b)-\phi(a)}.$$

7. One application of Ex. 6 is to the theory of indeterminate forms. Show that if $f(a) = \phi(a) = 0$ and if $f'(x)/\phi'(x)$ approaches a limit when $x \doteq a$, then $f(x)/\phi(x)$ will approach the same limit.

8. *Taylor's Theorem.* Note that the form $f(b) = f(a) + (b-a)f'(\xi)$ is one way of writing the Theorem of the Mean. By the application of Rolle's Theorem to

$$\psi(x) = f(b) - f(x) - (b-x)f'(x) - (b-x)^2 \frac{f(b)-f(a)-(b-a)f'(a)}{(b-a)^2},$$

show
$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2} f''(\xi),$$

and to
$$\psi(x) = f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2}f''(x) - \cdots - \frac{(b-x)^{n-1}}{(n-1)!}f^{(n-1)}(x)$$
$$- \frac{(b-x)^n}{(b-a)^n}\left[f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{2}f''(a) - \cdots - \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)\right],$$

show
$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2}f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^n}{n!}f^{(n)}(\xi).$$

What are the restrictions that must be imposed on the function and its derivatives?

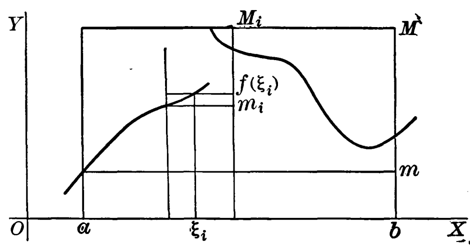
9. If a continuous function over $a \leqq x \leqq b$ has a right-hand derivative at each point of the interval which is zero, show that the function is constant. Apply Ex. 2 to the functions $f(x) + \epsilon(x-a)$ and $f(x) - \epsilon(x-a)$ to show that the maximum difference between the functions is $2\epsilon(b-a)$ and that f must therefore be constant.

10. State and prove the theorems implied in the formulas (4)–(6), p. 2.

11. Consider the extension of Ex. 7, p. 44, to derivatives of functions defined over a dense set. If the derivative exists and is uniformly continuous over the dense set, what of the existence and continuity of the derivative of the function when its definition is extended as there indicated ?

12. If $f(x)$ has a finite derivative at each point of the interval $a \leqq x \leqq b$, the derivative $f'(x)$ must take on every value intermediate between any two of its values. To show this, take first the case where $f'(a)$ and $f'(b)$ have opposite signs and show, by the continuity of f and by Theorem 13 and Ex. 2, that $f'(\xi) = 0$. Next if $f'(a) < \mu < f'(b)$ without any restrictions on $f'(a)$ and $f'(b)$, consider the function $f(x) - \mu x$ and its derivative $f'(x) - \mu$. Finally, prove the complete theorem. It should be noted that the continuity of $f'(x)$ is not assumed, nor is it proved; for there are functions which take every value intermediate between two given values and yet are not continuous.

28. Summation and integration. Let $f(x)$ be defined and limited over the interval $a \leqq x \leqq b$ and let M, m, and $O = M - m$ be the upper frontier, lower frontier, and oscillation of $f(x)$ in the interval. Let $n - 1$ points of division be introduced in the interval dividing it into n consecutive intervals $\delta_1, \delta_2, \cdots, \delta_n$ of which the largest has the length Δ and let M_i, m_i, O_i, and $f(\xi_i)$ be the upper and lower frontiers, the oscillation, and any value of the function in the interval δ_i. Then the inequalities



$$m\delta_i \leqq m_i\delta_i \leqq f(\xi_i)\,\delta_i \leqq M_i\delta_i \leqq M\delta_i$$

will hold, and if these terms be summed up for all n intervals,

$$m(b-a) \leqq \sum m_i\delta_i \leqq \sum f(\xi_i)\,\delta_i \leqq \sum M_i\delta_i \leqq M(b-a) \qquad (A)$$

will also hold. Let $s = \Sigma m_i\delta_i$, $\sigma = \Sigma f(\xi_i)\,\delta_i$, and $S = \Sigma M_i\delta_i$. From (A) it is clear that the difference $S - s$ does not exceed

$$(M - m)(b - a) = O(b - a),$$

the product of the length of the interval by the oscillation in it. The values of the sums S, s, σ will evidently depend on the number of parts into which the interval is divided and on the way in which it is divided into that number of parts.

THEOREM 19. If n' additional points of division be introduced into the interval, the sum S' constructed for the $n + n' - 1$ points of division cannot be greater than S and cannot be less than S by more than $n'O\Delta$. Similarly, s' cannot be less than s and cannot exceed s by more than $n'O\Delta$.

THEOREM 20. There exists a lower frontier L for all possible methods of constructing the sum S and an upper frontier l for s.

THEOREM 21. *Darboux's Theorem.* When ϵ is assigned it is possible to find a Δ so small that for all methods of division for which $\delta_i \leqq \Delta$, the sums S and s shall differ from their frontier values L and l by less than any preassigned ϵ.

To prove the first theorem note that although (A) is written for the whole interval from a to b and for the sums constructed on it, yet it applies equally to any part of the interval and to the sums constructed on that part. Hence if $S_i = M_i\delta_i$ be the part of S due to the interval δ_i and if S'_i be the part of S' due to this interval after the introduction of some of the additional points into it, $m_i\delta_i \leqq S'_i \leqq S_i = M_i\delta_i$. Hence S'_i is not greater than S_i (and as this is true for each interval δ_i, S' is not greater than S) and, moreover, $S_i - S'_i$ is not greater than $O_i\delta_i$ and a fortiori not greater than $O\Delta$. As there are only n' new points, not more than n' of the intervals δ_i can be affected, and hence the total decrease $S - S'$ in S cannot be more than $n'O\Delta$. The treatment of s is analogous.

Inasmuch as (A) shows that the sums S and s are limited, it follows from Theorem 4 that they possess the frontiers required in Theorem 20. To prove Theorem 21 note first that as L is a frontier for all the sums S, there is some particular sum S which differs from L by as little as desired, say $\frac{1}{2}\epsilon$. For this S let n be the number of divisions. Now consider S' as any sum for which each δ_i is less than $\Delta = \frac{1}{2}\epsilon/nO$. If the sum S'' be constructed by adding the n points of division for S to the points of division for S', S'' cannot be greater than S and hence cannot differ from L by so much as $\frac{1}{2}\epsilon$. Also S'' cannot be greater than S' and cannot be less than S' by more than $nO\Delta$, which is $\frac{1}{2}\epsilon$. As S'' differs from L by less than $\frac{1}{2}\epsilon$ and S' differs from S'' by less than $\frac{1}{2}\epsilon$, S' cannot differ from L by more than ϵ, which was to be proved. The treatment of s and l is analogous.

29. If indices are introduced to indicate the interval for which the frontiers L and l are calculated and if β lies in the interval from a to b, then L_a^β and l_a^β will be functions of β.

THEOREM 22. The equations $L_a^b = L_a^c + L_c^b$, $a < c < b$; $L_a^b = -L_b^a$; $L_a^b = \mu(b-a)$, $m \leqq \mu \leqq M$, hold for L, and similar equations for l. As functions of β, L_a^β and l_a^β are continuous, and if $f(x)$ is continuous, they are differentiable and have the common derivative $f(\beta)$.

To prove that $L_a^b = L_a^c + L_c^b$, consider c as one of the points of division of the interval from a to b. Then the sums S will satisfy $S_a^b = S_a^c + S_c^b$, and as the limit of a sum is the sum of the limits, the corresponding relation must hold for the frontier L. To show that $L_a^b = -L_b^a$ it is merely necessary to note that $S_a^b = -S_b^a$ because in passing from b to a the intervals δ_i must be taken with the sign opposite to that which they have when the direction is from a to b. From (A) it appears that $m(b-a) \leqq S_a^b \leqq M(b-a)$ and hence in the limit $m(b-a) \leqq L_a^b \leqq M(b-a)$.

Hence there is a value μ, $m \leqq \mu \leqq M$, such that $L_a^b = \mu(b-a)$. To show that L_a^β is a continuous function of β, take $K > |M|$ and $|m|$, and consider the relations

$$L_a^{\beta+h} - L_a^\beta = L_a^\beta + L_\beta^{\beta+h} - L_a^\beta = L_\beta^{\beta+h} = \mu h; \qquad |\mu| < K,$$
$$L_a^{\beta-h} - L_a^\beta = L_a^{\beta-h} - L_a^{\beta-h} - L_{\beta-h}^\beta = - L_{\beta-h}^\beta = - \mu' h, \qquad |\mu'| < K.$$

Hence if ϵ is assigned, a δ may be found, namely $\delta < \epsilon/K$, so that $|L_a^{\beta\pm h} - L_a^\beta| < \epsilon$ when $h < \delta$ and L_a^β is therefore continuous. Finally consider the quotients

$$\frac{L_a^{\beta+h} - L_a^\beta}{h} = \mu \quad \text{and} \quad \frac{L_a^{\beta-h} - L_a^\beta}{-h} = \mu',$$

where μ is some number between the maximum and minimum of $f(x)$ in the interval $\beta \leqq x \leqq \beta + h$ and, if f is continuous, is some value $f(\xi)$ of f in that interval and where $\mu' = f(\xi')$ is some value of f in the interval $\beta - h \leqq x \leqq \beta$. Now let $h \doteq 0$. As the function f is continuous, $\lim f(\xi) = f(\beta)$ and $\lim f(\xi') = f(\beta)$. Hence the right-hand and left-hand derivatives exist and are equal and the function L_a^β has the derivative $f(\beta)$. The treatment of l is analogous.

Theorem 23. For a given interval and function f, the quantities l and L satisfy the relation $l \leqq L$; and the necessary and sufficient condition that $L = l$ is that there shall be some division of the interval which shall make $\Sigma(M_i - m_i)\delta_i = \Sigma O_i \delta_i < \epsilon$.

If $L_a^b = l_a^b$, the function f is said to be integrable over the interval from a to b and the integral $\int_a^b f(x)\,dx$ is defined as the common value $L_a^b = l_a^b$. Thus the definite integral is defined.

Theorem 24. If a function is integrable over an interval, it is integrable over any part of the interval and the equations

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx,$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx, \qquad \int_a^b f(x)\,dx = \mu(b-a)$$

hold; moreover, $\int_a^\beta f(x)\,dx = F(\beta)$ is a continuous function of β; and if $f(x)$ is continuous, the derivative $F'(\beta)$ will exist and be $f(\beta)$.

By (A) the sums S and s constructed for the same division of the interval satisfy the relation $S - s \geqq 0$. By Darboux's Theorem the sums S and s will approach the values L and l when the divisions are indefinitely decreased. Hence $L - l \geqq 0$. Now if $L = l$ and a Δ be found so that when $\delta_i < \Delta$ the inequalities $S - L < \frac{1}{2}\epsilon$ and $l - s < \frac{1}{2}\epsilon$ hold, then $S - s = \Sigma(M_i - m_i)\delta_i = \Sigma O_i\delta_i < \epsilon$; and hence the condition $\Sigma O_i \delta_i < \epsilon$ is seen to be necessary. Conversely if there is any method of division such that $\Sigma O_i\delta_i < \epsilon$, then $S - s < \epsilon$ and the lesser quantity $L - l$ must also be less than ϵ. But if the difference between two constant quantities can be made less than ϵ, where ϵ is arbitrarily assigned, the constant quantities are equal; and hence the condition is seen to be also sufficient. To show that if a function is integrable over an interval, it is integrable over any part of the interval, it is merely necessary to show that if $L_a^b = l_a^b$, then $L_\alpha^\beta = l_\alpha^\beta$ where α and β are two points of the interval. Here the condition $\Sigma O_i \delta_i < \epsilon$ applies; for if $\Sigma O_i \delta_i$ can be made less than ϵ for the whole interval, its value for any part of the interval, being less than for the whole, must be less than ϵ. The rest of Theorem 24 is a corollary of Theorem 22.

30. Theorem 25. A function is integrable over the interval $a \leqq x \leqq b$ if it is continuous in that interval.

Theorem 26. If the interval $a \leqq x \leqq b$ over which $f(x)$ is defined and limited contains only a finite number of points at which f is discontinuous or if it contains an infinite number of points at which f is discontinuous but these points have only a finite number of points of condensation, the function is integrable.

Theorem 27. If $f(x)$ is integrable over the interval $a \leqq x \leqq b$, the sum $\sigma = \Sigma f(\xi_i)\,\delta_i$ will approach the limit $\int_a^b f(x)\,dx$ when the individual intervals δ_i approach the limit zero, it being immaterial how they approach that limit or how the points ξ_i are selected in their respective intervals δ_i.

Theorem 28. If $f(x)$ is continuous in an interval $a \leqq x \leqq b$, then $f(x)$ has an indefinite integral, namely $\int_a^x f(x)\,dx$, in the interval.

Theorem 25 may be reduced to Theorem 23. For as the function is continuous, it is possible to find a Δ so small that the oscillation of the function in any interval of length Δ shall be as small as desired (Theorem 9). Suppose Δ be chosen so that the oscillation is less than $\epsilon/(b-a)$. Then $\Sigma O_i \delta_i < \epsilon$ when $\delta_i < \Delta$; and the function is integrable. To prove Theorem 26, take first the case of a finite number of discontinuities. Cut out the discontinuities surrounding each value of x at which f is discontinuous by an interval of length δ. As the oscillation in each of these intervals is not greater than O, the contribution of these intervals to the sum $\Sigma O_i \delta_i$ is not greater than $On\delta$, where n is the number of the discontinuities. By taking δ small enough this may be made as small as desired, say less than $\frac{1}{2}\epsilon$. Now in each of the remaining parts of the interval $a \leqq x \leqq b$, the function f is continuous and hence integrable, and consequently the value of $\Sigma O_i \delta_i$ for these portions may be made as small as desired, say $\frac{1}{2}\epsilon$. Thus the sum $\Sigma O_i \delta_i$ for the whole interval can be made as small as desired and $f(x)$ is integrable. When there are points of condensation they may be treated just as the isolated points of discontinuity were treated. After they have been surrounded by intervals, there will remain over only a finite number of discontinuities. Further details will be left to the reader.

For the proof of Theorem 27, appeal may be taken to the fundamental relation (A) which shows that $s \leqq \sigma \leqq S$. Now let the number of divisions increase indefinitely and each division become indefinitely small. As the function is integrable, S and s approach the same limit $\int_a^b f(x)\,dx$, and consequently σ which is included between them must approach that limit. Theorem 28 is a corollary of Theorem 24 which states that as $f(x)$ is continuous, the derivative of $\int_a^x f(x)\,dx$ is $f(x)$. By definition, the indefinite integral is any function whose derivative is the integrand. Hence $\int_a^x f(x)\,dx$ is an indefinite integral of $f(x)$, and any other may be obtained by adding to this an arbitrary constant (Theorem 16). Thus it is seen that the proof of the existence of the indefinite integral for any given continuous function is made to depend on the theory of definite integrals.

EXERCISES

1. Rework some of the proofs in the text with l replacing L.

2. Show that the L obtained from $Cf(x)$, where C is a constant, is C times the L obtained from f. Also if u, v, w are all limited in the interval $a \leqq x \leqq b$, the L for the combination $u + v - w$ will be $L(u) + L(v) - L(w)$, where $L(u)$ denotes the L for u, etc. State and prove the corresponding theorems for definite integrals and hence the corresponding theorems for indefinite integrals.

3. Show that $\Sigma O_i\delta_i$ can be made less than an assigned ϵ in the case of the function of Ex. 6, p. 44. Note that $l = 0$, and hence infer that the function is integrable and the integral is zero. The proof may be made to depend on the fact that there are only a finite number of values of the function greater than any assigned value.

4. State with care and prove the results of Exs. 3 and 5, p. 29. What restriction is to be placed on $f(x)$ if $f(\xi)$ may replace μ?

5. State with care and prove the results of Ex. 4, p. 29, and Ex. 13, p. 30.

6. If a function is limited in the interval $a \leqq x \leqq b$ and never decreases, show that the function is integrable. This follows from the fact that $\Sigma O_i \leqq O$ is finite.

7. More generally, let $f(x)$ be such a function that ΣO_i remains less than some number K, no matter how the interval be divided. Show that f is integrable. Such a function is called a *function of limited variation* (§ 127).

8. *Change of variable.* Let $f(x)$ be continuous over $a \leqq x \leqq b$. Change the variable to $x = \phi(t)$, where it is supposed that $a = \phi(t_1)$ and $b = \phi(t_2)$, and that $\phi(t)$, $\phi'(t)$, and $f[\phi(t)]$ are continuous in t over $t_1 \leqq t \leqq t_2$. Show that

$$\int_a^b f(x)\,dx = \int_{t_1}^{t_2} f[\phi(t)]\,\phi'(t)\,dt \quad \text{or} \quad \int_{\phi(t_1)}^{\phi(t)} f(x)\,dx = \int_{t_1}^{t} f[\phi(t)]\,\phi'(t)\,dt.$$

Do this by showing that the derivatives of the two sides of the last equation with respect to t exist and are equal over $t_1 \leqq t \leqq t_2$, that the two sides vanish when $t = t_1$ and are equal, and hence that they must be equal throughout the interval.

9. *Osgood's Theorem.* Let α_i be a set of quantities which differ uniformly from $f(\xi_i)\,\delta_i$ by an amount $\zeta_i\delta_i$, that is, suppose

$$\alpha_i = f(\xi_i)\,\delta_i + \zeta_i\delta_i, \quad \text{where} \quad |\zeta_i| < \epsilon \quad \text{and} \quad a \leqq \xi \leqq b.$$

Prove that if f is integrable, the sum $\Sigma\alpha_i$ approaches a limit when $\delta_i \doteq 0$ and that the limit of the sum is $\int_a^b f(x)\,dx$.

10. Apply Ex. 9 to the case $\Delta f = f'\Delta x + \zeta\Delta x$ where f' is continuous to show directly that $f(b) - f(a) = \int_a^b f'(x)\,dx$. Also by regarding $\Delta x = \phi'(t)\,\Delta t + \zeta\Delta t$, apply to Ex. 8 to prove the rule for change of variable.