In formulas (1) and (3), in which b is any term at all, we might introduce the sign $\prod$ with respect to b. In the following formula, it becomes necessary to make use of this sign.

4. $$\prod_x \left\{[a < (b < x)] < x\right\} = ab.$$

Demonstration:

$$\left\{[a < (b < x)] < x\right\} = \left\{[a' + (b < x)] < x\right\}$$
$$= [(a' + b' + x) < x] = abx' + x = ab + x.$$

We must now form the product $\prod_x (ab + x)$, where x can assume every value, including 0 and 1. Now, it is clear that the part common to all the terms of the form $(ab + x)$ can only be ab. For, (1) ab is contained in each of the sums $(ab + x)$ and therefore in the part common to all; (2) the part common to all the sums $(ab + x)$ must be contained in $(ab + 0)$, that is, in ab. Hence this common part is equal to ab[1], which proves the theorem.

59. Reduction of Inequalities to Equalities.—As we have said, the principle of assertion enables us to reduce inequalities to equalities by means of the following formulas:

$$(a \neq 0) = (a = 1), \qquad (a \neq 1) = (a = 0),$$
$$(a \neq b) = (a = b').$$

For,

$$(a \neq b) = (ab' + a'b \neq 0) = (ab' + ab' = 1) = (a = b').$$

Consequently, we have the paradoxical formula

$$(a \neq b) = (a = b').$$

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1 This argument is general and from it we can deduce the formula

$$\prod_x (a + x) = a,$$

whence may be derived the correlative formula

$$\sum_x ax = a.$$

This is easily understood, for, whatever the proposition b, either it is true and its negative is false, or it is false and its negative is true. Now, whatever the proposition a may be, it is true or false; hence it is necessarily equal either to b or to b'. Thus to deny an equality (between propositions) is to affirm the *opposite* equality.

Thence it results that, in the calculus of propositions, we do not need to take inequalities into consideration—a fact which greatly simplifies both theory and practice. Moreover, just as we can combine alternative equalities, we can also combine simultaneous inequalities, since they are reducible to equalities.

For, from the formulas previously established (§ 57)

$$(ab = 0) = (a = 0) + (b = 0),$$
$$(a + b = 1) = (a = 1) + (b = 1),$$

we deduce by contraposition

$$(a \neq 0)\ (b \neq 0) = (ab \neq 0),$$
$$(a \neq 1)\ (b \neq 1) = (a + b \neq 1).$$

These two formulas, moreover, according to what we have just said, are equivalent to the known formulas

$$(a = 1)\ (b = 1) = (ab = 1),$$
$$(a = 0)\ (b = 0) = (a + b = 0).$$

Therefore, in the calculus of propositions, we can solve all simultaneous systems of equalities or inequalities and all alternative systems of equalities or inequalities, which is not possible in the calculus of classes. To this end, it is necessary only to apply the following rule:

First reduce the inclusions to equalities and the non-inclusions to inequalities; then reduce the equalities so that their second members will be 1, and the inequalities so that their second members will be 0, and transform the latter into equalities having 1 for a second member; finally, suppress the second members 1 and the signs of equality, *i. e.*, form the product of the first members of the simultaneous equalities and the sum of the first members of the alternative equalities, retaining the parentheses.