Appendix

A.1 Several facts from probability theory

In this section, we gather several facts from probability theory that are necessary in this monograph.

A.1.1 Convergence of probability measures

Let (S, d) be a metric space and $\mathcal{B}(S)$ the Borel σ-algebra of S, i.e., the smallest σ-algebra on S containing all open sets of S. (In this monograph, $S = \mathbb{R}^d$ or $\mathbb{C}$ in most cases.) By a *probability measure* on S we mean a measure on $\big(S, \mathcal{B}(S)\big)$ with total measure 1. For simplicity, put

$$\begin{aligned}\mathcal{P}(S) &:= \text{the set of all probability measures on } S,\\ C_{\mathrm{b}}(S) &:= \text{the set of all bounded continuous functions of } S \text{ to } \mathbb{R}.\end{aligned}$$

Definition A.1 Let $\nu_n \in \mathcal{P}(S)$ $(n \geq 1)$ and $\nu \in \mathcal{P}(S)$. Then

$$\begin{aligned}&\nu_n \to \nu \text{ weakly as } n \to \infty\\ &\underset{\text{def}}{\Longleftrightarrow} \int_S f(x)\nu_n(dx) \to \int_S f(x)\nu(dx) \quad \text{as } n \to \infty \quad \text{for } {}^\forall f \in C_{\mathrm{b}}(S).\end{aligned}$$

In this case, we say that ν_n *converges weakly* to ν as $n \to \infty$.

Claim A.1 *Let* $\nu_n \in \mathcal{P}(S)$ $(n \geq 1)$ *and* $\nu \in \mathcal{P}(S)$. *The following conditions* (i) $\sim$ (iv) *are equivalent to each other:*

(i) $\nu_n \to \nu$ *weakly as* $n \to \infty$,

(ii) *For every closed set* F *of* S, $\overline{\lim}_{n\to\infty} \nu_n(F) \leq \nu(F)$,

(iii) *For every open set* O *of* S, $\underline{\lim}_{n\to\infty} \nu_n(O) \geq \nu(O)$,

(iv) *For every continuity set* B *of* ν, *i.e.,* $B \in \mathcal{B}(S)$ *satisfying* $\nu(\partial B) = 0$, $\lim_{n\to\infty} \nu_n(B) = \nu(B)$.

For the proof, cf. Kotani [20, Proposition 9.2], H. Sato [29, Theorem 11.2], Stroock [31, Theorem 3.1.5].

A.1.2 Characteristic functions

In this subsection, let $S = \mathbb{R}^d$ or $\mathbb{C}^d$.

Definition A.2 (i) For $\nu \in \mathcal{P}(\mathbb{R}^d)$, put

$$\widehat{\nu}(\xi) := \int_{\mathbb{R}^d} e^{\sqrt{-1}\langle \xi, x\rangle} \nu(dx), \quad \xi \in \mathbb{R}^d.$$

$\widehat{\nu} : \mathbb{R}^d \to \mathbb{C}$ is called the *characteristic function* of ν. Here $\langle \xi, x\rangle$ is the inner product of ξ and x, i.e., $\langle \xi, x\rangle = \sum_{i=1}^d \xi_i x_i$ (ξ_i and x_i are the ith component of ξ and x, respectively).
(ii) For $\nu \in \mathcal{P}(\mathbb{C}^d)$, put

$$\widehat{\nu}(w) := \int_{\mathbb{C}^d} e^{\sqrt{-1}\langle w, z\rangle} \nu(dz), \quad w \in \mathbb{C}^d.$$

$\widehat{\nu} : \mathbb{C}^d \to \mathbb{C}$ is called the characteristic function of ν. Here, for $w = (w_1, \dots, w_d)$, $z = (z_1, \dots, z_d) \in \mathbb{C}^d$,

$$\langle w, z\rangle := \sum_{i=1}^d \big((\mathrm{Re}\, w_i) \cdot (\mathrm{Re}\, z_i) + (\mathrm{Im}\, w_i) \cdot (\mathrm{Im}\, z_i)\big).$$

Claim A.2 *Let $S = \mathbb{R}^d$ or $\mathbb{C}^d$. For $\nu_n \in \mathcal{P}(S)$ $(n \geq 1)$ and $\nu \in \mathcal{P}(S)$,*

$$\nu_n \to \nu \text{ weakly as } n \to \infty \underset{\text{iff}}{\Longleftrightarrow} \widehat{\nu_n} \to \widehat{\nu} \text{ pointwise as } n \to \infty.$$

For the proof, cf. Kotani [20, Theorem 9.16], H. Sato [29, Theorem 13.2], Stroock [31, Lemma 2.2.8].

Claim A.3 (Lévy's continuity theorem) *Let $S = \mathbb{R}^d$ or $\mathbb{C}^d$. Let $\nu_n \in \mathcal{P}(S)$ $(n \geq 1)$ and $\varphi : S \to \mathbb{C}$. Suppose*

- *$\widehat{\nu_n} \to \varphi$ pointwise as $n \to \infty$,* • *φ is continuous at origin.*

Then there exists a unique $\nu \in \mathcal{P}(S)$ such that $\nu_n \to \nu$ weakly as $n \to \infty$. (Thus $\widehat{\nu} = \varphi$ by Claim A.2.)

For the proof, cf. Durrett [8, Theorem 3.3.6], Kotani [20, Theorem 9.16], H. Sato [29, Theorem 13.3].

Definition A.3 For a function $\varphi : \mathbb{R}^d \to \mathbb{C}$,

φ is *positive definite*

$$\underset{\text{def}}{\Longleftrightarrow} \sum_{i,j=1}^n z_i \overline{z_j} \varphi(\xi_i - \xi_j) \geq 0 \text{ for } {}^\forall \xi_1, \dots, {}^\forall \xi_n \in \mathbb{R}^d \text{ and } {}^\forall z_1, \dots, {}^\forall z_n \in \mathbb{C}.$$

Claim A.4 (Bochner's theorem) *Suppose $\varphi : \mathbb{R}^d \to \mathbb{C}$ satisfies that*

- *φ is positive definite,* • *φ is continuous at $\xi = 0$,* • *$\varphi(0) = 1$.*

Then there exists a unique $\nu \in \mathcal{P}(\mathbb{R}^d)$ such that $\widehat{\nu} = \varphi$.

For the proof, cf. Itô [15, Theorem 2.6.6], Kotani [20, Corollary 9.17], H. Sato [29, Theorem 13.4].

A.1.3 Kolmogorov's extension theorem

In this subsection, let (S, d) be a complete separable metric space. Let T be a non-empty set.

Definition A.4 For $\emptyset \subsetneqq \Lambda_1 \subset \Lambda_2 \subset T$, we define $\pi_{\Lambda_1,\Lambda_2} : S^{\Lambda_2} \to S^{\Lambda_1}$ by

$$\pi_{\Lambda_1,\Lambda_2}\big((x_f)_{f\in\Lambda_2}\big) := (x_f)_{f\in\Lambda_1}, \quad (x_f)_{f\in\Lambda_2} \in S^{\Lambda_2}.$$

Definition A.5 Given a probability measure μ_Λ on $(S^\Lambda, \mathcal{B}(S^\Lambda))$ for each $\Lambda \subset T$ with $1 \le \operatorname{card}\Lambda < \infty$, $\{\mu_\Lambda\}$ is said to satisfy the *consistency condition* if, for any $\Lambda_1 \subset \Lambda_2$ with $1 \le \operatorname{card}\Lambda_1 \le \operatorname{card}\Lambda_2 < \infty$,

$$\mu_{\Lambda_2} \circ \pi_{\Lambda_1,\Lambda_2}^{-1} = \mu_{\Lambda_1}.$$

Claim A.5 (Kolmogorov's extension theorem) *Suppose $\{\mu_\Lambda; \Lambda \subset T$ is non-empty and finite$\}$ satisfies the consistency condition. Then*

$$\begin{aligned} &{}^{\exists 1}\mathbf{P}\text{: a probability measure on } \big(S^T, \sigma(\pi_f; f \in T)\big) \\ &\text{s.t. } \mathbf{P} \circ \pi_{\Lambda,T}^{-1} = \mu_\Lambda,\ \emptyset \subsetneqq {}^\forall\Lambda \subset T \text{ finite.} \end{aligned}$$

Here $\pi_f = \pi_{\{f\},T}$, i.e.,

$$\begin{array}{rccc} \pi_f : & S^T & \to & S \\ & \Psi & & \Psi \\ & (x_g)_{g\in T} & \mapsto & x_f \end{array}$$

and $\sigma(\pi_f; f \in T)$ is the smallest σ-algebra on S^T such that all π_f's are measurable.

For the proof, cf. Kotani [20, Theorem 4.22] (or Durrett [8, Theorem A.3.1] or Itô [15, Theorem 2.9.1]).

A.1.4 Almost sure convergence theorem for independent random variables

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Claim A.6 (Almost sure convergence theorem) *Let $\{X_n\}_{n=1}^\infty$ be a sequence of real random variables defined on $(\Omega, \mathcal{F}, P)$. Suppose that*

- *$\{X_n; n = 1, 2, \dots\}$ are independent,*
- *X_n is square-integrable, i.e., $E[X_n^2] < \infty$ $({}^\forall n)$,*
- *$\sum_{n=1}^\infty Var(X_n) < \infty$, where $Var(X_n)$ is the variance of X_n, i.e., $Var(X_n) = E\big[(X_n - E[X_n])^2\big]$.*

Then $\sum_{n=1}^\infty \big(X_n - E[X_n]\big)$ is convergent P-a.e., i.e., $\sum_{n=1}^N \big(X_n - E[X_n]\big)$ is convergent as $N \to \infty$ P-a.e.

For the proof, cf. Itô [15, Theorem 4.2.1], H. Sato [29, Theorem 10.1], Stroock [31, Theorem 1.4.2].

A.1.5 Lindeberg's central limit theorem

Similarly as above, let $(\Omega, \mathcal{F}, P)$ be a probability space.

Claim A.7 (Lindeberg's central limit theorem) *Let* $\{X_{nj}; j = 1, \ldots, k_n, n = 1, 2, \ldots\}$ *be a triangular array of real random variables defined on* $(\Omega, \mathcal{F}, P)$*. Suppose that*

- $\{X_{nj}; j = 1, \ldots, k_n\}$ *are independent* ($^\forall n \geq 1$),
- X_{nj} *is square-integrable and of mean zero* ($1 \leq {}^\forall j \leq k_n$, $^\forall n \geq 1$),
- $\lim_{n\to\infty} \sum_{j=1}^{k_n} E\big[X_{nj}^2\big] = v \in [0, \infty)$ *and* $\lim_{n\to\infty} \sum_{j=1}^{k_n} E\big[X_{nj}^2; |X_{nj}| \geq \varepsilon\big] = 0$ ($^\forall \varepsilon > 0$).

Then

$$\text{the distribution of } \sum_{j=1}^{k_n} X_{nj} \\ \to \text{the normal distribution } N(0, v) \text{ weakly as } n \to \infty.$$

Namely

$$E\left[e^{\sqrt{-1}\xi \sum_{j=1}^{k_n} X_{nj}}\right] \to e^{-\frac{v\xi^2}{2}} \quad \text{as } n \to \infty, \ ^\forall \xi \in \mathbb{R}.$$

For the proof, cf. Durrett [8, Theorem 3.4.5].

A.2 Gauss's product formula of the gamma function

Definition A.6 We define the *gamma function* $\Gamma(\cdot)$ by

$$\Gamma(s) := \int_0^\infty e^{-x} x^{s-1} dx, \quad s \in \mathbb{C} \text{ with } \operatorname{Re} s > 0.$$

Since $|e^{-x}x^{s-1}| = e^{-x}x^{\operatorname{Re} s - 1}$ and $\operatorname{Re} s > 0$, this integral is absolutely convergent on $\big\{s \in \mathbb{C}; \operatorname{Re} s > 0\big\}$.

Claim A.8 (i) $\Gamma(\cdot)$ *is holomorphic on* $\{s \in \mathbb{C}; \operatorname{Re} s > 0\}$.
(ii) $\Gamma(s+1) = s\Gamma(s)$*. In particular,* $\Gamma(n) = (n-1)!$ *for* $n \in \mathbb{N}$.

Proof. (i) For $s, h \in \mathbb{C}$ with $0 < |h| < \frac{1}{2} \operatorname{Re} s$,

$$\begin{aligned} e^{-x} \frac{x^{s+h-1} - x^{s-1}}{h} &= e^{-x} \frac{1}{h} \int_0^1 (x^{s+th-1})' dt \\ &= e^{-x} \frac{1}{h} \int_0^1 x^{s+th-1} (\log x) h \, dt \\ &= e^{-x} \log x \int_0^1 x^{s+th-1} dt. \end{aligned}$$

Taking the absolute value, we have

$$\begin{aligned}
\left|e^{-x}\frac{x^{s+h-1}-x^{s-1}}{h}\right| &\le e^{-x}|\log x|\int_0^1 x^{\operatorname{Re}(s+th)-1}dt \\
&= e^{-x}|\log x|\int_0^1 x^{\operatorname{Re}s+t\operatorname{Re}h-1}dt \\
&\le \mathbf{1}_{x>1}e^{-x}(\log x)x^{\frac{3}{2}\operatorname{Re}s-1} + \mathbf{1}_{0<x<1}e^{-x}|\log x|x^{\frac{1}{2}\operatorname{Re}s-1} \\
&\quad \left[\because\ \operatorname{Re}s+t\operatorname{Re}h-1 \begin{cases}\le \operatorname{Re}s+t|h|-1<\frac{3}{2}\operatorname{Re}s-1,\\ \ge \operatorname{Re}s-t|h|-1>\frac{1}{2}\operatorname{Re}s-1\end{cases}\right] \\
&\le \mathbf{1}_{x>1}e^{-x}x^{\frac{3}{2}\operatorname{Re}s+1-1} + \mathbf{1}_{0<x<1}e^{-x}\frac{4}{\operatorname{Re}s}x^{\frac{1}{4}\operatorname{Re}s-1} \\
&\quad \left[\begin{array}{l}\because\ \text{Since } 0\le \log y<y\ (y\ge 1),\\ 0\le \log\frac{1}{x}=\frac{4}{\operatorname{Re}s}\log(\frac{1}{x})^{\frac{\operatorname{Re}s}{4}}<\frac{4}{\operatorname{Re}s}x^{-\frac{\operatorname{Re}s}{4}} \text{ for } 0<x\le 1\end{array}\right].
\end{aligned}$$

Thus, by Lebesgue's convergence theorem

$$\begin{aligned}
\lim_{h\to 0}\frac{\Gamma(s+h)-\Gamma(s)}{h} &= \lim_{h\to 0}\int_0^\infty e^{-x}\frac{x^{s+h-1}-x^{s-1}}{h}dx \\
&= \int_0^\infty e^{-x}x^{s-1}\log x\,dx.
\end{aligned}$$

(ii) By integration by parts,

$$\begin{aligned}
\text{L.H.S.} = \int_0^\infty e^{-t}t^s dt &= \int_0^\infty (-e^{-t})' t^s dt \\
&= \left[-e^{-t}t^s\right]_0^\infty + \int_0^\infty e^{-t}st^{s-1}dt \\
&= \text{R.H.S.} \quad \left[\begin{array}{l}\because\ \lim_{t\searrow 0}e^{-t}t^s=0,\\ \lim_{t\to\infty}e^{-t}t^s=\lim_{t\to\infty}\frac{t^s}{e^t}=0\end{array}\right].
\end{aligned}$$

■

Lemma A.1 (i) *As* $n\to\infty$,

$$\prod_{k=1}^{n}\Big(1+\frac{s}{k}\Big)e^{-\frac{s}{k}}$$

is uniformly convergent on every compact set of $\mathbb{C}$. *Thus the limit function is holomorphic on* $\mathbb{C}$.

(ii) $\displaystyle\prod_{k=1}^{\infty}\Big(1+\frac{s}{k}\Big)e^{-\frac{s}{k}}\begin{cases}\ne 0, & s\in\mathbb{C}\setminus(-\mathbb{N}),\\ = 0, & s\in -\mathbb{N}.\end{cases}$

Proof. For simplicity, put

$$\begin{aligned}
a_k(s) := \Big(1+\frac{s}{k}\Big)e^{-\frac{s}{k}}-1 &= \int_0^1\left(\Big(1+\frac{ts}{k}\Big)e^{-\frac{ts}{k}}\right)' dt \\
&= \frac{s^2}{k^2}\int_0^1(-t)e^{-\frac{ts}{k}}dt.
\end{aligned}$$

Taking the absolute value, we have

$$|a_k(s)| \le \frac{|s|^2}{k^2}\int_0^1 t\left|e^{-\frac{ts}{k}}\right|dt \le \frac{|s|^2}{k^2}\int_0^1 te^{\frac{t|s|}{k}}dt \le \frac{|s|^2}{k^2}e^{\frac{|s|}{k}}. \tag{A.1}$$

(i) For $m > n \ge 1$,

$$\begin{aligned}
&\left|\prod_{k=1}^{m}\Big(1+\frac{s}{k}\Big)e^{-\frac{s}{k}} - \prod_{k=1}^{n}\Big(1+\frac{s}{k}\Big)e^{-\frac{s}{k}}\right| \\
&= \left|\prod_{k=1}^{m}\big(1+a_k(s)\big) - \prod_{k=1}^{n}\big(1+a_k(s)\big)\right| \\
&= \left|\prod_{k=1}^{n}\big(1+a_k(s)\big)\Big(\prod_{k=n+1}^{m}\big(1+a_k(s)\big)-1\Big)\right| \\
&= \left|\prod_{k=1}^{n}\big(1+a_k(s)\big)\right|\left|\prod_{k=n+1}^{m}\big(1+a_k(s)\big)-1\right| \\
&= \Big(\prod_{k=1}^{n}|1+a_k(s)|\Big)\left|1+\sum_{r=1}^{m-n}\sum_{n+1\le k_1<\cdots<k_r\le m} a_{k_1}(s)\cdots a_{k_r}(s)-1\right| \\
&\le \Big(\prod_{k=1}^{n}(1+|a_k(s)|)\Big)\Big(\sum_{r=1}^{m-n}\sum_{n+1\le k_1<\cdots<k_r\le m} |a_{k_1}(s)|\cdots|a_{k_r}(s)|\Big) \\
&= \Big(\prod_{k=1}^{n}(1+|a_k(s)|)\Big)\Big(1+\sum_{r=1}^{m-n}\sum_{n+1\le k_1<\cdots<k_r\le m} |a_{k_1}(s)|\cdots|a_{k_r}(s)|-1\Big) \\
&= \Big(\prod_{k=1}^{n}(1+|a_k(s)|)\Big)\Big(\prod_{k=n+1}^{m}(1+|a_k(s)|)-1\Big) \\
&\le \Big(\prod_{k=1}^{n}e^{|a_k(s)|}\Big)\Big(\prod_{k=n+1}^{m}e^{|a_k(s)|}-1\Big) \quad \big[\because\ 1+x\le e^x\ ({}^\forall x\in\mathbb{R})\big] \\
&= e^{\sum_{k=1}^{n}|a_k(s)|}\Big(e^{\sum_{k=n+1}^{m}|a_k(s)|}-1\Big) \\
&\le e^{\sum_{k=1}^{n}|a_k(s)|}\Big(\sum_{k=n+1}^{m}|a_k(s)|\Big)e^{\sum_{k=n+1}^{m}|a_k(s)|} \\
&\qquad \Big[\because\ \text{For } x\ge 0,\ 0\le e^x-1=\textstyle\int_0^x (e^y)'dy=\int_0^x e^y dy\le xe^x\Big] \\
&= \Big(\sum_{k=n+1}^{m}|a_k(s)|\Big)e^{\sum_{k=1}^{m}|a_k(s)|} \\
&\le \Big(\sum_{k=n+1}^{m}\frac{|s|^2}{k^2}e^{\frac{|s|}{k}}\Big)e^{\sum_{k=1}^{m}\frac{|s|^2}{k^2}e^{\frac{|s|}{k}}} \quad [\because\ (A.1)] \\
&\le \Big(|s|^2e^{\frac{|s|}{n+1}}\sum_{k=n+1}^{m}\frac{1}{k^2}\Big)e^{|s|^2e^{|s|}\sum_{k=1}^{m}\frac{1}{k^2}}
\end{aligned}$$

$$\leq |s|^2 e^{\frac{|s|}{n+1}} e^{|s|^2 e^{|s|}\zeta(2)} \Big(\sum_{k=n+1}^{m} \frac{1}{k^2} \Big),$$

from which the assertion (i) follows.
(ii) Let $C \subset \mathbb{C}$ be a compact set such that $C \subset \mathbb{C} \setminus (-\mathbb{N})$. Take $\varepsilon > 0$ and $R > 0$ so that

$$\Big|1 + \frac{s}{k}\Big| \geq \varepsilon, \quad |s| \leq R \quad (^\forall s \in C, {}^\forall k \in \mathbb{N}).$$

Then, since, for $s \in C$, $k \in \mathbb{N}$,

$$\begin{aligned}
\Big|\frac{1}{1 + a_k(s)}\Big| = \Big|\frac{1 + a_k(s) - a_k(s)}{1 + a_k(s)}\Big| &= \Big|1 - \frac{a_k(s)}{1 + a_k(s)}\Big| \\
&\leq 1 + \Big|\frac{a_k(s)}{1 + a_k(s)}\Big| \\
&= 1 + \left|\frac{\frac{s^2}{k^2}\int_0^1 (-t) e^{-\frac{ts}{k}}\, dt}{(1 + \frac{s}{k}) e^{-\frac{s}{k}}}\right| \\
&= 1 + \left|\frac{1}{k^2} \frac{s^2}{1 + \frac{s}{k}} \int_0^1 (-t) e^{\frac{s}{k}(1-t)} dt\right| \\
&\leq 1 + \frac{1}{k^2} \frac{|s|^2}{|1 + \frac{s}{k}|} \int_0^1 t e^{\frac{|s|}{k}(1-t)} dt \\
&\leq 1 + \frac{1}{k^2} \frac{R^2}{\varepsilon} e^{\frac{R}{k}} \\
&\leq e^{\frac{1}{k^2}\frac{R^2}{\varepsilon} e^R},
\end{aligned}$$

we have

$$\begin{aligned}
\Big|\prod_{k=1}^{n}\Big(1 + \frac{s}{k}\Big) e^{-\frac{s}{k}}\Big| = \Big|\prod_{k=1}^{n}\big(1 + a_k(s)\big)\Big| &= \prod_{k=1}^{n}|1 + a_k(s)| \\
&\geq \prod_{k=1}^{n} e^{-\frac{1}{k^2}\frac{R^2}{\varepsilon} e^R} \\
&= e^{-\frac{R^2}{\varepsilon} e^R \sum_{k=1}^{n} \frac{1}{k^2}} \\
&\geq e^{-\frac{R^2}{\varepsilon} e^R \zeta(2)},
\end{aligned}$$

which implies that

$$\inf_{s \in C}\Big|\prod_{k=1}^{\infty}\Big(1 + \frac{s}{k}\Big) e^{-\frac{s}{k}}\Big| \geq e^{-\frac{R^2}{\varepsilon} e^R \zeta(2)} > 0. \qquad \blacksquare$$

Claim A.9 (i) *On* $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$,

$$\lim_{n \to \infty} \frac{n! n^s}{s(s+1)\cdots(s+n)} = \frac{e^{-\gamma s}}{s} \frac{1}{\prod_{k=1}^{\infty}(1 + \frac{s}{k}) e^{-\frac{s}{k}}}.$$

Here γ is Euler's constant. Thus the limit function is holomorphic on $\mathbb{C}\setminus\{0,-1,-2,\dots\}$.
(ii) *For* $\operatorname{Re} s > 0$,

$$\Gamma(s) = \lim_{n\to\infty} \frac{n!n^s}{s(s+1)\cdots(s+n)}.$$

Thus $\Gamma(\cdot)$ is analytically continuable to a holomorphic function on $\mathbb{C}\setminus\{0,-1,-2,\dots\}$ which is denoted by the same $\Gamma(\cdot)$.
(iii) *For each $m \in \{0,1,2,\dots\}$, $s=-m$ is a simple pole of $\Gamma(\cdot)$, and the residue at this point is $\frac{(-1)^m}{m!}$.*

Proof. (i) First, for $s \in \mathbb{C}\setminus\{0,-1,-2,\dots\}$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\frac{n!n^s}{s(s+1)\cdots(s+n)}\\
&= \frac{1}{s}\frac{1\cdot 2\cdot\cdots\cdot n}{(s+1)(s+2)\cdots(s+n)}e^{s\log n}\\
&= \frac{1}{s}\frac{e^{s(1+\frac{1}{2}+\cdots+\frac{1}{n})}}{\frac{1+s}{1}\cdot\frac{2+s}{2}\cdot\cdots\cdot\frac{n+s}{n}}e^{-s(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n)}\\
&= \frac{1}{s}e^{-s(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n)}\frac{1}{(1+\frac{s}{1})e^{-s}\cdot(1+\frac{s}{2})e^{-\frac{s}{2}}\cdot\cdots\cdot(1+\frac{s}{n})e^{-\frac{s}{n}}}\\
&= \frac{1}{s}e^{-s\left(\sum_{k=1}^n\frac{1}{k}-\log n\right)}\frac{1}{\prod_{k=1}^n(1+\frac{s}{k})e^{-\frac{s}{k}}}.
\end{aligned}$$

Since, as $n\to\infty$,

$$\sum_{k=1}^n \frac{1}{k} - \log n \to \gamma,$$

$$\prod_{k=1}^n\Bigl(1+\frac{s}{k}\Bigr)e^{-\frac{s}{k}} \to \prod_{k=1}^\infty\Bigl(1+\frac{s}{k}\Bigr)e^{-\frac{s}{k}},$$

we have

$$\lim_{n\to\infty}\frac{n!n^s}{s(s+1)\cdots(s+n)} = \frac{e^{-\gamma s}}{s}\frac{1}{\prod_{k=1}^\infty(1+\frac{s}{k})e^{-\frac{s}{k}}}.$$

(ii) It suffices to show the identity for $s>0$, whence the assertion (ii) follows by the uniqueness theorem.

Fix $s>0$. For $n\in\mathbb{N}$, put

$$\Gamma_n(s) := \int_0^n \Bigl(1-\frac{x}{n}\Bigr)^n x^{s-1}dx.$$

Since

$$0 \le \mathbf{1}_{(0,n)}(x)\Bigl(1-\frac{x}{n}\Bigr)^n x^{s-1} \le \mathbf{1}_{(0,n)}(x)\bigl(e^{-\frac{x}{n}}\bigr)^n x^{s-1}$$

$$\bigl[\because\ 0\le 1-y\le e^{-y}\ \ (0\le{}^\forall y\le 1)\bigr]$$

$$
\begin{aligned}
&= \mathbf{1}_{(0,n)}(x)e^{-x}x^{s-1} \\
&\le e^{-x}x^{s-1} \quad ({}^{\forall}n \in \mathbb{N},\ {}^{\forall}x > 0),
\end{aligned}
$$

$$
\lim_{n\to\infty} \mathbf{1}_{(0,n)}(x)\Big(1-\frac{x}{n}\Big)^n x^{s-1} = e^{-x}x^{s-1} \quad ({}^{\forall}x > 0),
$$

$$
\int_0^\infty e^{-x}x^{s-1}dx = \Gamma(s) < \infty,
$$

it follows from Lebesgue's convergence theorem that $\lim_{n\to\infty} \Gamma_n(s) = \Gamma(s)$. On the other hand, integration by parts yields that

$$
\begin{aligned}
\Gamma_n(s) &= \int_0^n \Big(1-\frac{x}{n}\Big)^n x^{s-1}dx \\
&= \int_0^1 (1-y)^n (ny)^{s-1} n\,dy \quad \Big[\because \text{ change of variable: } y = \tfrac{x}{n}\Big] \\
&= n^s \int_0^1 (1-y)^n y^{s-1} dy \\
&= n^s \int_0^1 (1-y)^n \Big(\frac{y^s}{s}\Big)' dy \\
&= n^s \Big\{ \Big[(1-y)^n \frac{y^s}{s}\Big]_0^1 + \frac{n}{s}\int_0^1 (1-y)^{n-1} y^s dy \Big\} \\
&= n^s \frac{n}{s} \int_0^1 (1-y)^{n-1} \Big(\frac{y^{s+1}}{s+1}\Big)' dy \\
&= n^s \frac{n}{s} \Big\{ \Big[(1-y)^{n-1} \frac{y^{s+1}}{s+1}\Big]_0^1 + \frac{n-1}{s+1}\int_0^1 (1-y)^{n-2} y^{s+1} dy \Big\} \\
&= n^s \frac{n(n-1)}{s(s+1)} \int_0^1 (1-y)^{n-2} \Big(\frac{y^{s+2}}{s+2}\Big)' dy \\
&\ \ \vdots \\
&= n^s \frac{n(n-1)\cdots 2}{s(s+1)\cdots(s+n-2)} \int_0^1 (1-y) \Big(\frac{y^{s+n-1}}{s+n-1}\Big)' dy \\
&= n^s \frac{n(n-1)\cdots 2}{s(s+1)\cdots(s+n-2)} \Big\{ \Big[(1-y)\frac{y^{s+n-1}}{s+n-1}\Big]_0^1 \\
&\qquad\qquad\qquad\qquad + \frac{1}{s+n-1}\int_0^1 y^{s+n-1} dy \Big\} \\
&= n^s \frac{n!}{s(s+1)\cdots(s+n)}.
\end{aligned}
$$

Therefore we have

$$
\lim_{n\to\infty} \frac{n!\,n^s}{s(s+1)\cdots(s+n)} = \Gamma(s).
$$

(iii) First note that for $s \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$,

$$\begin{aligned} s\Gamma(s) &= \lim_{n\to\infty} s\frac{(n+1)!(n+1)^s}{s(s+1)\cdots(s+n+1)} \\ &= \lim_{n\to\infty} \frac{n!(n+1)^{s+1}}{(s+1)(s+1+1)\cdots(s+1+n)} \\ &= \lim_{n\to\infty} \frac{n!n^{s+1}}{(s+1)(s+1+1)\cdots(s+1+n)}\left(1+\frac{1}{n}\right)^{s+1} \\ &= \Gamma(s+1). \end{aligned}$$

From this identity it follows that for $m \in \mathbb{N}$, $s \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$,

$$\Gamma(s+m) = \left(\prod_{i=0}^{m-1}(s+i)\right)\Gamma(s).$$

For each $m \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} \lim_{s\to -m}\big(s-(-m)\big)\Gamma(s) &= \lim_{s\to -m}(s+m)\frac{\Gamma(s+m)}{\prod_{i=0}^{m-1}(s+i)} \\ &\qquad \left[\text{when } m=0, \text{ we let } \textstyle\prod_{i=0}^{-1}(s+i)=1\right] \\ &= \lim_{s\to -m}(s+m)\Gamma(s+m)\frac{1}{\prod_{i=0}^{m-1}\big(s+m+(i-m)\big)} \\ &= \lim_{s\to 0} s\Gamma(s)\frac{1}{\prod_{i=0}^{m-1}\big(s-(m-i)\big)} \\ &= \lim_{s\to 0}\Gamma(s+1)\frac{1}{\prod_{j=1}^{m}(s-j)} \\ &= \frac{1}{(-1)^m m!} \\ &= \frac{(-1)^m}{m!}, \end{aligned}$$

which shows the assertion (iii). ■

R.H.S. in Claim A.9(i) is called *Weierstrass's formula* of the gamma function and the identity in Claim A.9(ii) is called *Gauss's product formula* of the gamma function.

A.3 A proof of $\zeta(2) = \frac{\pi^2}{6}$

To find the value of $\zeta(2)$ is historically known as the *Basel problem*. In 1735, L. Euler solved this problem by showing that $\zeta(2) = \frac{\pi^2}{6}$. After Euler, there are many proofs of this. In fact, from 2° in the proof of Theorem 4.3, we can immediately see it. In this section, we introduce another proof of it due to Fujita ([7, 10]). This proof is simple, but not elementary. In other words, it is a senior or junior level in college.

Claim A.10 $\zeta(2) = \frac{\pi^2}{6}$, *i.e.*, $\sum_{n=1}^{\infty} \frac{1}{n^2}$[†1] $= \frac{\pi^2}{6}$.

Proof. We divide the proof into three steps:

1° Let $f, g : (0, \infty) \to [0, \infty)$ be Borel measurable, and

$$\int_0^\infty f(x)dx = \int_{(0,\infty)} f(x)dx < \infty, \quad \int_0^\infty g(x)dx = \int_{(0,\infty)} g(x)dx < \infty.$$

Define $h : (0, \infty) \to [0, \infty]$ by

$$h(x) := \int_{(0,\infty)} f(u) g\Big(\frac{x}{u}\Big)\frac{du}{u}.$$

By Fubini's theorem, $h(\cdot)$ is Borel measurable and

$$\begin{aligned}\int_{(0,\infty)} h(x)dx &= \int_{(0,\infty)} f(u)du \int_{(0,\infty)} g\Big(\frac{x}{u}\Big)\frac{dx}{u} \\ &= \int_{(0,\infty)} f(u)du \int_{(0,\infty)} g(v)dv \quad \Big[\because \text{ change of variable: } v = \tfrac{x}{u}\Big] \\ &< \infty. \end{aligned} \tag{A.2}$$

Thus $h(x) < \infty$ a.e. x.

2° Take $f(x) = g(x) = \frac{1}{x^2+1}$. Then

$$\int_0^\infty f(x)dx = \int_0^\infty g(x)dx = \big[\tan^{-1} x\big]_0^\infty = \frac{\pi}{2}.$$

Let us find the h above for these f, g: For $x \neq 1$,

$$\begin{aligned} h(x) &= \int_{(0,\infty)} \frac{1}{u^2+1}\frac{1}{\big(\frac{x}{u}\big)^2+1}\frac{du}{u} \\ &= \int_0^\infty \frac{u}{(u^2+1)(u^2+x^2)}du \\ &= \frac{1}{2}\int_0^\infty \frac{1}{(v+1)(v+x^2)}dv \quad \big[\because \text{ change of variable: } v = u^2\big] \\ &= \frac{1}{2}\int_0^\infty \Big(\frac{1}{v+1} - \frac{1}{v+x^2}\Big)\frac{dv}{x^2-1} \\ &= \frac{1}{2}\frac{1}{x^2-1}\int_0^\infty \Big(\log\frac{v+1}{v+x^2}\Big)' dv \\ &= \frac{1}{2}\frac{1}{x^2-1}\Big[\log\frac{v+1}{v+x^2}\Big]_0^\infty \\ &= \frac{1}{2}\frac{1}{x^2-1}\Big(\log 1 - \log\frac{1}{x^2}\Big) \\ &= \frac{\log x}{x^2-1}. \end{aligned}$$

[†1] The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is sometimes called the *Euler series*.

By (A.2), we have

$$\int_0^\infty \frac{\log x}{x^2-1}dx = \frac{\pi^2}{4}.$$

3° We compute the integral above in a different way:

$$\begin{aligned}
&\int_0^\infty \frac{\log x}{x^2-1}dx \\
&= \int_0^1 \frac{\log x}{x^2-1}dx + \int_1^\infty \frac{\log x}{x^2-1}dx \\
&= \int_0^1 \frac{-\log x}{1-x^2}dx + \int_0^1 \frac{\log \frac{1}{y}}{\frac{1}{y^2}-1}\frac{dy}{y^2} \quad \left[\because \text{ change of variable: } y = \tfrac{1}{x}\right] \\
&= 2\int_0^1 \frac{-\log x}{1-x^2}dx \\
&= 2\int_0^1 (-\log x)\sum_{k=0}^\infty x^{2k}dx \quad \left[\because \textstyle\sum_{k=0}^\infty r^k = \frac{1}{1-r}\ (|r|<1)\right] \\
&= 2\sum_{k=0}^\infty \int_0^1 (-\log x)x^{2k}dx \quad \left[\because \text{ termwise integration theorem}\right] \\
&= 2\sum_{k=0}^\infty \frac{1}{(2k+1)^2}\int_0^\infty e^{-v}v\,dv \quad \left[\because \text{ change of variable: } v = -(2k+1)\log x\right] \\
&= 2\Gamma(2)\sum_{k=0}^\infty \frac{1}{(2k+1)^2} \\
&= 2\sum_{k=0}^\infty \frac{1}{(2k+1)^2} \quad \left[\because \Gamma(2) = \Gamma(1+1) = 1\cdot\Gamma(1) = 1\right].
\end{aligned}$$

Thus

$$\sum_{k=0}^\infty \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Finally, noting that

$$\begin{aligned}
\zeta(2) = \sum_{n=1}^\infty \frac{1}{n^2} = \sum_{k=1}^\infty \frac{1}{(2k)^2} + \sum_{k=0}^\infty \frac{1}{(2k+1)^2} &= \frac{1}{4}\sum_{k=1}^\infty \frac{1}{k^2} + \sum_{k=0}^\infty \frac{1}{(2k+1)^2} \\
&= \frac{\zeta(2)}{4} + \sum_{k=0}^\infty \frac{1}{(2k+1)^2},
\end{aligned}$$

we have $\zeta(2) = \frac{\pi^2}{6}$. ■

A.4 The second mean value theorem for integrals

This theorem is used several times in the proof of Claim 6.2. Although this can be found in textbooks of fundamental calculus (= differential and integral calculus), we here prove it.

Claim A.11 (Second mean value theorem) *Let* $-\infty < a < b < \infty$*,* $f : [a,b] \to \mathbb{R}$ *Riemann integrable and* $\varphi : [a,b] \to \mathbb{R}$ *monotonic, i.e., nondecreasing or nonincreasing. Then*

$$a \leq {}^{\exists}\xi \leq b \ \ s.t. \ \ \int_a^b f(x)\varphi(x)dx = \varphi(a)\int_a^{\xi} f(x)dx + \varphi(b)\int_{\xi}^{b} f(x)dx.$$

Proof. We give a proof due to Takagi [32]. Suppose $\varphi \in \searrow$, i.e., φ is nonincreasing. Let $\psi(x) := \varphi(x) - \varphi(b)$. Clearly $\psi \in \searrow, \ \geq 0$ on $[a,b]$. For a partition $\Delta : a = t_0 < t_1 < \cdots < t_n = b$, put

$$s_j = \sum_{i=1}^{j} f(t_{i-1})(t_i - t_{i-1}) \quad (0 \leq j \leq n).$$

Then

$$\begin{aligned}
\sum_{j=1}^{n} f(t_{j-1})\psi(t_{j-1})(t_j - t_{j-1}) &= \sum_{j=1}^{n} f(t_{j-1})(t_j - t_{j-1})\psi(t_{j-1}) \\
&= \sum_{j=1}^{n} (s_j - s_{j-1})\psi(t_{j-1}) \\
&= \sum_{j=1}^{n} s_j\psi(t_{j-1}) - \sum_{j=1}^{n} s_{j-1}\psi(t_{j-1}) \\
&= \sum_{j=1}^{n} s_j\psi(t_{j-1}) - \sum_{j=1}^{n-1} s_j\psi(t_j) \\
&= s_n\psi(t_{n-1}) + \sum_{j=1}^{n-1} s_j\big(\psi(t_{j-1}) - \psi(t_j)\big).
\end{aligned}$$

Since, by $\psi(t_{n-1}) \geq 0$, $\psi(t_{j-1}) - \psi(t_j) \geq 0 \ \ (1 \leq j \leq n-1)$,

$$s_n\psi(t_{n-1}) + \sum_{j=1}^{n-1} s_j\big(\psi(t_{j-1}) - \psi(t_j)\big)$$

$$\left\{\begin{aligned}
&\leq \big(\max_{1\leq j\leq n} s_j\big)\Big(\psi(t_{n-1}) + \sum_{j=1}^{n-1}\big(\psi(t_{j-1}) - \psi(t_j)\big)\Big) = \big(\max_{1\leq j\leq n} s_j\big)\psi(t_0), \\
&\geq \big(\min_{1\leq j\leq n} s_j\big)\Big(\psi(t_{n-1}) + \sum_{j=1}^{n-1}\big(\psi(t_{j-1}) - \psi(t_j)\big)\Big) = \big(\min_{1\leq j\leq n} s_j\big)\psi(t_0),
\end{aligned}\right.$$

we see that

$$\min_{1\le j\le n}\Big(\sum_{i=1}^{j} f(t_{i-1})(t_i-t_{i-1})\Big)\psi(a) \le \sum_{j=1}^{n} f(t_{j-1})\psi(t_{j-1})(t_j-t_{j-1})$$
$$\le \max_{1\le j\le n}\Big(\sum_{i=1}^{j} f(t_{i-1})(t_i-t_{i-1})\Big)\psi(a).$$

Now, since f is Riemann integrable on $[a,b]$,

$$\forall\varepsilon>0,\ \exists\delta>0\ \text{s.t.}\ \begin{cases}\text{for any partition } \Delta: a=t_0<t_1<\cdots<t_n=b\\ \text{with } |\Delta|=\max_{1\le i\le n}(t_i-t_{i-1})<\delta,\\ \displaystyle\sum_{i=1}^{n}\Big(\sup_{[t_{i-1},t_i]} f-\inf_{[t_{i-1},t_i]} f\Big)(t_i-t_{i-1})<\varepsilon.\end{cases}$$

This implies that for $j=1,\dots,n$,

$$\Big|\sum_{i=1}^{j} f(t_{i-1})(t_i-t_{i-1})-\int_a^{t_j} f(t)dt\Big| = \Big|\sum_{i=1}^{j}\int_{t_{i-1}}^{t_i}\big(f(t_{i-1})-f(t)\big)dt\Big|$$
$$\le \sum_{i=1}^{j}\int_{t_{i-1}}^{t_i}\big|f(t_{i-1})-f(t)\big|dt$$
$$\le \sum_{i=1}^{j}\Big(\sup_{[t_{i-1},t_i]} f-\inf_{[t_{i-1},t_i]} f\Big)(t_i-t_{i-1})$$
$$\le \sum_{i=1}^{n}\Big(\sup_{[t_{i-1},t_i]} f-\inf_{[t_{i-1},t_i]} f\Big)(t_i-t_{i-1})$$
$$<\varepsilon,$$

and thus

$$\Big(\Big(\min_{a\le x\le b}\int_a^x f(t)dt\Big)-\varepsilon\Big)\psi(a)$$
$$\le \sum_{j=1}^{n} f(t_{j-1})\psi(t_{j-1})(t_j-t_{j-1})$$
$$\le \Big(\Big(\max_{a\le x\le b}\int_a^x f(t)dt\Big)+\varepsilon\Big)\psi(a),\quad \forall\Delta \text{ with } |\Delta|<\delta.$$

Letting $|\Delta|\to 0$, we have

$$\Big(\min_{a\le x\le b}\int_a^x f(t)dt\Big)\psi(a) \le \int_a^b f(t)\psi(t)dt$$
$$\le \Big(\max_{a\le x\le b}\int_a^x f(t)dt\Big)\psi(a).$$

By the intermediate value theorem,

$$a \leq {}^{\exists}\xi \leq b \ \text{ s.t. } \int_a^b f(t)\psi(t)dt = \Big(\int_a^{\xi} f(t)dt\Big)\psi(a).$$

Putting $\psi(\cdot) = \varphi(\cdot) - \varphi(b)$, we obtain

$$\int_a^b f(t)\varphi(t)dt = \varphi(a)\int_a^{\xi} f(t)dt + \varphi(b)\int_{\xi}^b f(t)dt.$$

In case $\varphi \in \nearrow$, i.e., φ is nondecreasing, since $-\varphi \in \searrow$, it follows from the above that

$$a \leq {}^{\exists}\eta \leq b$$
$$\text{s.t. } \int_a^b f(t)\big(-\varphi(t)\big)dt = \big(-\varphi(a)\big)\int_a^{\eta} f(t)dt + \big(-\varphi(b)\big)\int_{\eta}^b f(t)dt.$$

Multiplying it by -1, we have

$$\int_a^b f(t)\varphi(t)dt = \varphi(a)\int_a^{\eta} f(t)dt + \varphi(b)\int_{\eta}^b f(t)dt. \qquad \blacksquare$$