Chapter 4

Riemann zeta function

In this chapter, following Kanemitsu [17], let us view the Riemann zeta function. For those readers who are not familiar with this, we here give detailed proofs for almost all. The matter in this chapter is necessary for the Bohr-Jessen limit theorem stated in the next chapter. Since this limit theorem is concerned with the Riemann zeta function (precisely the log zeta function), we may well view this function here.

4.1 Euler-Maclaurin summation formula

Definition 4.1 We define the *Bernoulli number* B_n $(n \geq 0)$ by

$$\begin{cases} B_0 := 1, \\ B_n := \dfrac{-1}{n+1} \displaystyle\sum_{0 \leq k < n} \binom{n+1}{k} B_k \quad (n \geq 1). \end{cases}$$

We call B_n the nth Bernoulli number.

Claim 4.1 (i) $\displaystyle\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \ \ (n \geq 2)$.

(ii) $\dfrac{z}{e^z - 1} = \displaystyle\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$ ($z \in \mathbb{C}$ *with* $|z| < 2\pi$).

(iii) $B_1 = -\dfrac{1}{2}$, $B_2 = \dfrac{1}{6}$, $B_n = 0$ $(^\forall n \in 2\mathbb{N} + 1)$.

Proof. (i) For $n \geq 2$,

$$\begin{aligned} \sum_{k=0}^{n-1} \binom{n}{k} B_k &= \sum_{0 \leq k < n-1} \binom{n}{k} B_k + \binom{n}{n-1} B_{n-1} \\ &= \sum_{0 \leq k < n-1} \binom{n}{k} B_k + n \cdot \frac{-1}{n} \sum_{0 \leq k < n-1} \binom{n}{k} B_k \\ &= 0. \end{aligned}$$

(ii) $\frac{z}{e^z-1}$ is meromorphic on $\mathbb{C}$ and $z=0$ is a removable singularity. Thus it is holomorphic on $\{z \in \mathbb{C}; |z| < 2\pi\}$. Let us denote its Taylor expansion about $z=0$ by

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n.$$

Then

$$\begin{aligned} z = (e^z-1)\sum_{n=0}^{\infty}\frac{c_n}{n!}z^n &= \sum_{m=1}^{\infty}\frac{z^m}{m!}\sum_{n=0}^{\infty}\frac{c_n}{n!}z^n \\ &= \sum_{m\geq 1, n\geq 0}\frac{c_n}{m!n!}z^{m+n} \\ &= \sum_{k=1}^{\infty}\Big(\sum_{\substack{m\geq 1,n\geq 0;\\ m+n=k}}\frac{c_n}{m!n!}\Big)z^k \\ &= \sum_{k=1}^{\infty}\Big(\sum_{\substack{m\geq 1,n\geq 0;\\ m+n=k}}\frac{k!}{m!n!}c_n\Big)\frac{z^k}{k!} \\ &= \sum_{k=1}^{\infty}\Big(\sum_{0\leq n<k}\binom{k}{n}c_n\Big)\frac{z^k}{k!}. \end{aligned}$$

This shows that

$$\begin{cases} c_0 = 1, \\ \displaystyle\sum_{0\leq n<k}\binom{k}{n}c_n = 0 \quad (k\geq 2), \end{cases}$$

from which it follows that $c_k = B_k$ $(k \geq 0)$.
(iii) By definition,

$$B_1 = -\frac{1}{2}\sum_{0\leq k<1}\binom{2}{k}B_k = -\frac{1}{2}B_0 = -\frac{1}{2},$$

$$B_2 = -\frac{1}{3}\sum_{0\leq k<2}\binom{3}{k}B_k = -\frac{1}{3}(B_0+3B_1) = -\frac{1}{3}\Big(1-\frac{3}{2}\Big) = -\frac{1}{3}\cdot\frac{-1}{2} = \frac{1}{6}.$$

Let, for $|z| < 2\pi$,

$$f(z) := \frac{z}{e^z-1} - 1 + \frac{z}{2} = \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n - 1 + \frac{z}{2} = \sum_{n=2}^{\infty}\frac{B_n}{n!}z^n.$$

Since

$$f(-z) = \frac{-z}{e^{-z}-1} - 1 - \frac{z}{2} = -\frac{ze^z}{1-e^z} - 1 - \frac{z}{2}$$

$$
\begin{aligned}
&= z\frac{e^z - 1 + 1}{e^z - 1} - 1 - \frac{z}{2} \\
&= z\Big(1 + \frac{1}{e^z - 1}\Big) - 1 - \frac{z}{2} \\
&= \frac{z}{e^z - 1} - 1 + \frac{z}{2} = f(z),
\end{aligned}
$$

we have

$$
\begin{aligned}
0 = f(-z) - f(z) &= \sum_{n=2}^{\infty} \frac{B_n}{n!}\big((-z)^n - z^n\big) \\
&= \sum_{n=2}^{\infty} \frac{B_n}{n!}\big((-1)^n - 1\big)z^n \\
&= -2 \sum_{n\in 2\mathbb{N}+1} \frac{B_n}{n!} z^n,
\end{aligned}
$$

which implies the assertion (iii). ■

Definition 4.2 We define the *Bernoulli polynomial* $B_n(x)$ $(n \geq 0)$ by

$$B_n(x) := \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k, \quad n \geq 0.$$

We call $B_n(x)$ the nth Bernoulli polynomial.

Claim 4.2 (i) $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$.
(ii) $B'_k(x) = kB_{k-1}(x)$ $(k \geq 1)$.
(iii) $B_n(x+1) - B_n(x) = nx^{n-1}$ $(n \geq 1)$, $B_n(1-x) = (-1)^n B_n(x)$ $(n \geq 0)$.

Proof. (i) By definition,

$$
\begin{aligned}
B_0(x) &= \sum_{0\leq k\leq 0} \binom{0}{k} B_{0-k} x^k = \binom{0}{0} B_0 x^0 = 1, \\
B_1(x) &= \sum_{0\leq k\leq 1} \binom{1}{k} B_{1-k} x^k = \binom{1}{0} B_1 x^0 + \binom{1}{1} B_0 x = -\frac{1}{2} + x = x - \frac{1}{2}, \\
B_2(x) &= \sum_{0\leq k\leq 2} \binom{2}{k} B_{2-k} x^k = \binom{2}{0} B_2 x^0 + \binom{2}{1} B_1 x + \binom{2}{2} B_0 x^2 \\
&= \frac{1}{6} + 2\cdot\frac{-1}{2}x + x^2 = x^2 - x + \frac{1}{6}.
\end{aligned}
$$

(ii) For $n \geq 1$,

$$B'_n(x) = \sum_{k=1}^{n} \binom{n}{k} B_{n-k} k x^{k-1}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} k B_{n-k} x^{k-1} \\
&= \sum_{k=1}^{n} \frac{n(n-1)!}{(k-1)!(n-1-(k-1))!} B_{n-1-(k-1)} x^{k-1} \\
&= n \sum_{k=1}^{n} \binom{n-1}{k-1} B_{n-1-(k-1)} x^{k-1} \\
&= n \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-1-k} x^{k} \\
&= n B_{n-1}(x).
\end{aligned}
$$

(iii) We first note that for $x \in \mathbb{R}$ and $z \in \mathbb{C}$, $|z| < 2\pi$,

$$
\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n = \frac{z e^{zx}}{e^z - 1}.
$$

Because, by the definition of $B_n(\cdot)$,

$$
\begin{aligned}
\sum_{n=0}^{N} \frac{B_n(x)}{n!} z^n &= \sum_{n=0}^{N} \frac{z^n}{n!} \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k \\
&= \sum_{0 \le k \le n \le N} \frac{z^n}{n!} \frac{n!}{k!(n-k)!} B_{n-k} x^k \\
&= \sum_{k=0}^{N} \frac{(zx)^k}{k!} \sum_{n=k}^{N} \frac{B_{n-k}}{(n-k)!} z^{n-k} \\
&= \sum_{k=0}^{N} \frac{(zx)^k}{k!} \sum_{l=0}^{N-k} \frac{B_l}{l!} z^l \\
&= \sum_{k=0}^{\infty} \frac{(zx)^k}{k!} \mathbf{1}_{k \le N} \sum_{l=0}^{N-k} \frac{B_l}{l!} z^l.
\end{aligned}
$$

Since, by Claim 4.1(ii),

$$
\begin{aligned}
&\lim_{N \to \infty} \mathbf{1}_{k \le N} \sum_{l=0}^{N-k} \frac{B_l}{l!} z^l = \frac{z}{e^z - 1} \quad ({}^{\forall} k \ge 0), \\
&\sup_{N,k} \left| \mathbf{1}_{k \le N} \sum_{l=0}^{N-k} \frac{B_l}{l!} z^l \right| = \sup_{M} \left| \sum_{l=0}^{M} \frac{B_l}{l!} z^l \right| < \infty,
\end{aligned}
$$

it follows from Lebesgue's convergence theorem that

$$
\lim_{N \to \infty} \sum_{n=0}^{N} \frac{B_n(x)}{n!} z^n = \sum_{k=0}^{\infty} \frac{(zx)^k}{k!} \frac{z}{e^z - 1} = \frac{z e^{zx}}{e^z - 1}.
$$

Now

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{B_n(x+1)-B_n(x)}{n!}z^n &= \frac{ze^{z(x+1)}}{e^z-1}-\frac{ze^{zx}}{e^z-1}\\ &= \frac{ze^{zx}(e^z-1)}{e^z-1}\\ &= \sum_{n=0}^{\infty}\frac{z(zx)^n}{n!}\\ &= \sum_{n=0}^{\infty}\frac{x^n}{n!}z^{n+1}\\ &= \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}z^n,\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{B_n(1-x)}{n!}z^n &= \frac{ze^{z(1-x)}}{e^z-1}\\ &= \frac{ze^ze^{(-z)x}}{e^z-1}\\ &= \frac{ze^{(-z)x}}{1-e^{-z}}\\ &= \frac{(-z)e^{(-z)x}}{e^{-z}-1}\\ &= \sum_{n=0}^{\infty}\frac{B_n(x)}{n!}(-z)^n\\ &= \sum_{n=0}^{\infty}\frac{(-1)^nB_n(x)}{n!}z^n.\end{aligned}$$

Comparing the coefficients in the above series, we have the assertion (iii). ■

Definition 4.3 Put $\overline{B_n}(x) := B_n(\{x\})$, $n \geq 0$. Here $\{x\} =$ the fractional part of $x = x - \lfloor x \rfloor$. Note that $\overline{B_n}(\cdot)$ is periodic, with period 1.

Theorem 4.1 (Euler-Maclaurin summation formula) *Let* $-\infty < a < b < \infty$ *and* $n \in \mathbb{N}$. *For* $^\forall f \in C^n([a,b])$,

$$\sum_{a<k\leq b} f(k) = \int_a^b f(x)dx + \sum_{k=1}^{n}\frac{(-1)^k}{k!}\left[\overline{B_k}f^{(k-1)}\right]_a^b + \frac{(-1)^{n+1}}{n!}\int_a^b \overline{B_n}(x)f^{(n)}(x)dx.$$

Proof. We divide the proof into four steps:

1° For $k \geq 2$, $B_k(1) = B_k(0) = B_k$.

$\because$

$$B_k(0) = B_k,$$
$$B_k(1) = \sum_{l=0}^{k} \binom{k}{l} B_{k-l} = \sum_{l=0}^{k} \binom{k}{k-l} B_{k-l} = \sum_{l=0}^{k} \binom{k}{l} B_l = B_k \quad [\because \text{ Claim 4.1(i)}].$$

<u>2°</u> For $g \in C^1([a,b])$ and $k \geq 1$,

$$\int_a^b \overline{B_k}(x)g(x)dx = \frac{1}{k+1}\Big[\overline{B_{k+1}}(x)g(x)\Big]_a^b - \frac{1}{k+1}\int_a^b \overline{B_{k+1}}(x)g'(x)dx.$$

$\because$ The case where $a+1 \leq b$. Since $\lfloor a \rfloor + 1 \leq \lfloor b \rfloor$,

$$\begin{aligned}
&\text{L.H.S.}\\
&= \Big(\int_a^{\lfloor a \rfloor+1} + \int_{\lfloor a \rfloor+1}^{\lfloor b \rfloor} + \int_{\lfloor b \rfloor}^{b}\Big)\overline{B_k}(x)g(x)dx\\
&= \int_a^{\lfloor a \rfloor+1} B_k(x-\lfloor a \rfloor)g(x)dx\\
&\quad + \sum_{l=\lfloor a \rfloor+1}^{\lfloor b \rfloor-1} \int_l^{l+1} B_k(x-l)g(x)dx\\
&\quad + \int_{\lfloor b \rfloor}^{b} B_k(x-\lfloor b \rfloor)g(x)dx\\
&= \int_a^{\lfloor a \rfloor+1} \Big(\frac{1}{k+1}B_{k+1}(x-\lfloor a \rfloor)\Big)' g(x)dx\\
&\quad + \sum_{l=\lfloor a \rfloor+1}^{\lfloor b \rfloor-1} \int_l^{l+1} \Big(\frac{1}{k+1}B_{k+1}(x-l)\Big)' g(x)dx\\
&\quad + \int_{\lfloor b \rfloor}^{b} \Big(\frac{1}{k+1}B_{k+1}(x-\lfloor b \rfloor)\Big)' g(x)dx \quad [\because \text{ Claim 4.2(ii)}]\\
&= \Big[\frac{1}{k+1}B_{k+1}(x-\lfloor a \rfloor)g(x)\Big]_a^{\lfloor a \rfloor+1} - \int_a^{\lfloor a \rfloor+1} \frac{1}{k+1}B_{k+1}(x-\lfloor a \rfloor)g'(x)dx\\
&\quad + \sum_{l=\lfloor a \rfloor+1}^{\lfloor b \rfloor-1} \Big(\Big[\frac{1}{k+1}B_{k+1}(x-l)g(x)\Big]_l^{l+1} - \int_l^{l+1} \frac{1}{k+1}B_{k+1}(x-l)g'(x)dx\Big)\\
&\quad + \Big[\frac{1}{k+1}B_{k+1}(x-\lfloor b \rfloor)g(x)\Big]_{\lfloor b \rfloor}^{b} - \int_{\lfloor b \rfloor}^{b} \frac{1}{k+1}B_{k+1}(x-\lfloor b \rfloor)g'(x)dx\\
&\qquad [\because \text{ integration by parts}]\\
&= \frac{1}{k+1}\Big(B_{k+1}(1)g(\lfloor a \rfloor+1) - B_{k+1}(\{a\})g(a)\Big)\\
&\quad + \sum_{l=\lfloor a \rfloor+1}^{\lfloor b \rfloor-1} \frac{1}{k+1}\Big(B_{k+1}(1)g(l+1) - B_{k+1}(0)g(l)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{k+1}\Big(B_{k+1}(\{b\})g(b) - B_{k+1}(0)g(\lfloor b \rfloor)\Big) \\
&- \frac{1}{k+1}\Big(\int_a^{\lfloor a \rfloor+1} B_{k+1}(\{x\})g'(x)dx + \sum_{l=\lfloor a \rfloor+1}^{\lfloor b \rfloor-1}\int_l^{l+1} B_{k+1}(\{x\})g'(x)dx \\
&\qquad + \int_{\lfloor b \rfloor}^{b} B_{k+1}(\{x\})g'(x)dx\Big) \\
=& \frac{1}{k+1}B_{k+1}\Big(g(\lfloor a \rfloor+1) + g(\lfloor b \rfloor) - g(\lfloor a \rfloor+1) - g(\lfloor b \rfloor)\Big) \\
&+ \frac{1}{k+1}\Big(\overline{B_{k+1}}(b)g(b) - \overline{B_{k+1}}(a)g(a)\Big) \\
&- \frac{1}{k+1}\int_a^b \overline{B_{k+1}}(x)g'(x)dx \quad \Big[\because \text{ By } 1^{\circ},\ B_{k+1}(1) = B_{k+1}(0) = B_{k+1}\Big] \\
=& \text{ R.H.S.}
\end{aligned}
$$

The case where $a+1 > b$. Since $\lfloor a \rfloor + 1 \geq \lfloor b \rfloor \geq \lfloor a \rfloor$ by $a+1 > b > a$, either $\lfloor b \rfloor = \lfloor a \rfloor$ or $\lfloor b \rfloor = \lfloor a \rfloor + 1$. When $\lfloor b \rfloor = \lfloor a \rfloor$,

$$
\begin{aligned}
\text{L.H.S.} &= \int_a^b B_k(x - \lfloor x \rfloor)g(x)dx \\
&= \int_a^b B_k(x - \lfloor a \rfloor)g(x)dx \\
&\qquad \Big[\because \text{ Since } \lfloor a \rfloor \leq a \leq x \leq b < \lfloor b \rfloor + 1 = \lfloor a \rfloor + 1,\ \lfloor x \rfloor = \lfloor a \rfloor\Big] \\
&= \int_a^b \Big(\frac{1}{k+1}B_{k+1}(x - \lfloor a \rfloor)\Big)' g(x)dx \\
&= \Big[\frac{1}{k+1}B_{k+1}(x - \lfloor a \rfloor)g(x)\Big]_a^b - \int_a^b \frac{1}{k+1}B_{k+1}(x - \lfloor a \rfloor)g'(x)dx \\
&= \frac{1}{k+1}\Big[\overline{B_{k+1}}(x)g(x)\Big]_a^b - \frac{1}{k+1}\int_a^b \overline{B_{k+1}}(x)g'(x)dx \\
&= \text{R.H.S.}
\end{aligned}
$$

When $\lfloor b \rfloor = \lfloor a \rfloor + 1$,

$$
\begin{aligned}
\text{L.H.S.} &= \Big(\int_a^{\lfloor a \rfloor+1} + \int_{\lfloor b \rfloor}^{b}\Big)\overline{B_k}(x)g(x)dx \\
&= \int_a^{\lfloor a \rfloor+1} B_k(x - \lfloor a \rfloor)g(x)dx + \int_{\lfloor b \rfloor}^{b} B_k(x - \lfloor b \rfloor)g(x)dx \\
&= \int_a^{\lfloor a \rfloor+1} \Big(\frac{1}{k+1}B_{k+1}(x - \lfloor a \rfloor)\Big)' g(x)dx \\
&\quad + \int_{\lfloor b \rfloor}^{b} \Big(\frac{1}{k+1}B_{k+1}(x - \lfloor b \rfloor)\Big)' g(x)dx \\
&= \Big[\frac{1}{k+1}B_{k+1}(x - \lfloor a \rfloor)g(x)\Big]_a^{\lfloor a \rfloor+1} - \int_a^{\lfloor a \rfloor+1} \frac{1}{k+1}B_{k+1}(x - \lfloor a \rfloor)g'(x)dx
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{1}{k+1}B_{k+1}(x-\lfloor b\rfloor)g(x)\right]_{\lfloor b\rfloor}^{b}-\int_{\lfloor b\rfloor}^{b}\frac{1}{k+1}B_{k+1}(x-\lfloor b\rfloor)g'(x)dx\\
&=\frac{1}{k+1}\Big(B_{k+1}(1)g(\lfloor a\rfloor+1)-\overline{B_{k+1}}(a)g(a)\Big)\\
&\quad-\frac{1}{k+1}\int_{a}^{\lfloor a\rfloor+1}\overline{B_{k+1}}(x)g'(x)dx\\
&\quad+\frac{1}{k+1}\Big(\overline{B_{k+1}}(b)g(b)-B_{k+1}(0)g(\lfloor b\rfloor)\Big)\\
&\quad-\frac{1}{k+1}\int_{\lfloor b\rfloor}^{b}\overline{B_{k+1}}(x)g'(x)dx\\
&=\frac{1}{k+1}\Big[\overline{B_{k+1}}(x)g(x)\Big]_{a}^{b}-\frac{1}{k+1}\int_{a}^{b}\overline{B_{k+1}}(x)g'(x)dx\\
&=\text{R.H.S.}
\end{aligned}
$$

3° Let $n \geq 2$. By 2° with $g = f^{(k)}$ $(k = 1, \ldots, n-1)$,

$$
\begin{aligned}
&\frac{(-1)^k}{k!}\int_a^b \overline{B_k}(x)f^{(k)}(x)dx-\frac{(-1)^{k+1}}{(k+1)!}\int_a^b \overline{B_{k+1}}(x)f^{(k+1)}(x)dx\\
&=\frac{(-1)^k}{k!}\left(\int_a^b \overline{B_k}(x)f^{(k)}(x)dx+\frac{1}{k+1}\int_a^b \overline{B_{k+1}}(x)\big(f^{(k)}(x)\big)'dx\right)\\
&=\frac{(-1)^k}{(k+1)!}\Big[\overline{B_{k+1}}(x)f^{(k)}(x)\Big]_a^b.
\end{aligned}
$$

Adding this over $k \in \{1, \ldots, n-1\}$, we have

$$
\begin{aligned}
&-\int_a^b \overline{B_1}(x)f'(x)dx-\frac{(-1)^n}{n!}\int_a^b \overline{B_n}(x)f^{(n)}(x)dx\\
&=\sum_{k=1}^{n-1}\frac{(-1)^k}{(k+1)!}\Big[\overline{B_{k+1}}(x)f^{(k)}(x)\Big]_a^b\\
&=\sum_{k=2}^{n}\frac{(-1)^{k-1}}{k!}\Big[\overline{B_k}(x)f^{(k-1)}(x)\Big]_a^b,
\end{aligned}
$$

so that

$$
\begin{aligned}
&\int_a^b \overline{B_1}(x)f'(x)dx\\
&=\sum_{k=2}^{n}\frac{(-1)^k}{k!}\Big[\overline{B_k}(x)f^{(k-1)}(x)\Big]_a^b+\frac{(-1)^{n+1}}{n!}\int_a^b \overline{B_n}(x)f^{(n)}(x)dx.
\end{aligned}
$$

4° By integration by parts,

$$
\Big[\overline{B_1}(x)f(x)\Big]_a^b=\int_{(a,b]} d\big(\overline{B_1}(x)f(x)\big)
$$

$$= \int_{(a,b]} \left(\overline{B_1}(x) f'(x)dx + f(x)\overline{B_1}(dx)\right)$$
$$= \int_a^b \overline{B_1}(x) f'(x)dx + \int_a^b f(x)dx - \sum_{a<k\le b} f(k)$$
$$\left[\because\ \overline{B_1}(dx) = dx - \textstyle\sum_{k\in\mathbb{Z}} \delta_k(dx)\right].$$

Thus

$$\sum_{a<k\le b} f(k) = \int_a^b f(x)dx - \left[\overline{B_1}(x) f(x)\right]_a^b + \int_a^b \overline{B_1}(x) f'(x)dx,$$

from which and 3°, it follows that

$$\sum_{a<k\le b} f(k) = \int_a^b f(x)dx$$
$$+ \sum_{k=1}^{n} \frac{(-1)^k}{k!} \left[\overline{B_k}(x) f^{(k-1)}(x)\right]_a^b$$
$$+ \frac{(-1)^{n+1}}{n!} \int_a^b \overline{B_n}(x) f^{(n)}(x)dx. \qquad \blacksquare$$

4.2 Analytic continuation to the entire complex plane

Definition 4.4 For $s = \sigma + \sqrt{-1}t$ $(\sigma > 1, t \in \mathbb{R})$, we define

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\text{where } n^s := e^{s\log n}).$$

We call this the *Riemann zeta function*.

The Dirichlet series in R.H.S. is absolutely convergent on $\{s \in \mathbb{C}; \operatorname{Re} s > 1\}$. Because

$$\sum_{n=1}^{\infty}\left|\frac{1}{n^s}\right| = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} = 1 + \sum_{n=2}^{\infty} \int_{n-1}^{n} \left(\frac{1}{\lceil x\rceil}\right)^\sigma dx$$
$$= 1 + \int_1^\infty \left(\frac{1}{\lceil x\rceil}\right)^\sigma dx$$
$$\le 1 + \int_1^\infty x^{-\sigma} dx$$
$$\left[\because\ x \le \lceil x\rceil \Rightarrow \left(\tfrac{1}{\lceil x\rceil}\right)^\sigma \le \left(\tfrac{1}{x}\right)^\sigma = x^{-\sigma}\right]$$
$$= 1 + \left[\frac{-1}{\sigma-1}\left(\frac{1}{x}\right)^{\sigma-1}\right]_1^\infty$$
$$= 1 + \frac{1}{\sigma-1} = \frac{\sigma}{\sigma-1} < \infty.$$

Thus $\zeta(\cdot)$ is holomorphic there.

Claim 4.3 $\zeta(s) \neq 0$. *Moreover* $\zeta(s) = \prod_{p:prime} \dfrac{1}{1-\frac{1}{p^s}}$. *This is the Euler product expression of* $\zeta(\cdot)$.

Proof. Let $\{p_i\}_{i=1}^{\infty}$ be an arrangement of prime numbers in ascending order. Fix $s \in \mathbb{C}$, $\operatorname{Re} s > 1$. We divide the proof into three steps:

$\underline{1^{\circ}}$ $\zeta(s)\prod_{i=1}^{k}\Big(1-\dfrac{1}{p_i^s}\Big) = 1 + \sum_{\substack{n\geq 2;\\ p_1\nmid n,\dots,p_k\nmid n}} \dfrac{1}{n^s}.$

$\because$ Note that[†1] for sets $A_1, \dots, A_k$,

$$\begin{aligned}
\mathbf{1}_{A_1\cup\cdots\cup A_k} &= 1 - \prod_{i=1}^{k}\big(1-\mathbf{1}_{A_i}\big)\\
&= 1 - \Big(1 + \sum_{r=1}^{k}(-1)^r \sum_{1\leq i_1<\cdots<i_r\leq k} \mathbf{1}_{A_{i_1}\cap\cdots\cap A_{i_r}}\Big)\\
&= -\sum_{r=1}^{k}(-1)^r \sum_{1\leq i_1<\cdots<i_r\leq k} \mathbf{1}_{A_{i_1}\cap\cdots\cap A_{i_r}}.
\end{aligned}$$

Let $A_i = p_i\mathbb{N} = \big\{p_i m; m \in \mathbb{N}\big\}$. Then

$$\begin{aligned}
\sum_{\substack{n\geq 2;\\ 1\leq{}^{\exists}i\leq k \text{ s.t. } n\in A_i}} \frac{1}{n^s} &= \sum_{n\geq 2} \mathbf{1}_{A_1\cup\cdots\cup A_k}(n)\frac{1}{n^s}\\
&= \sum_{n\geq 2}\frac{1}{n^s}\Big(-\sum_{r=1}^{k}(-1)^r \sum_{1\leq i_1<\cdots<i_r\leq k} \mathbf{1}_{A_{i_1}\cap\cdots\cap A_{i_r}}(n)\Big)\\
&= -\sum_{r=1}^{k}(-1)^r \sum_{1\leq i_1<\cdots<i_r\leq k} \sum_{n\geq 2} \mathbf{1}_{A_{i_1}\cap\cdots\cap A_{i_r}}(n)\frac{1}{n^s}\\
&= -\sum_{r=1}^{k}(-1)^r \sum_{1\leq i_1<\cdots<i_r\leq k} \sum_{\substack{n\geq 2;\\ n\in p_{i_1}\cdots p_{i_r}\mathbb{N}}} \frac{1}{n^s}\\
&\left[\begin{aligned}\because\ & n \in A_{i_1}\cap\cdots\cap A_{i_r}\\ &\Leftrightarrow p_{i_1} \mid n, \dots, p_{i_r} \mid n\\ &\Leftrightarrow^{\dagger 2} p_{i_1}\cdots p_{i_r} \mid n\\ &\Leftrightarrow n \in p_{i_1}\cdots p_{i_r}\mathbb{N}\end{aligned}\right]\\
&= -\sum_{r=1}^{k}(-1)^r \sum_{1\leq i_1<\cdots<i_r\leq k} \sum_{m=1}^{\infty} \frac{1}{(p_{i_1}\cdots p_{i_r} m)^s}
\end{aligned}$$

[†1] We call this identity (relation) the *inclusion-exclusion formula*.

$$= -\sum_{r=1}^{k}(-1)^r \sum_{1\le i_1<\cdots<i_r\le k} \frac{1}{p_{i_1}^s\cdots p_{i_r}^s}\sum_{m=1}^{\infty}\frac{1}{m^s}$$
$$= \Bigl(-\sum_{r=1}^{k}(-1)^r \sum_{1\le i_1<\cdots<i_r\le k} \frac{1}{p_{i_1}^s\cdots p_{i_r}^s}\Bigr)\zeta(s).$$

From this it follows that

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = 1+\sum_{n\ge 2}\frac{1}{n^s}$$
$$= 1+\sum_{\substack{n\ge 2;\\ n\notin A_i\ (1\le{}^{\forall} i\le k)}}\frac{1}{n^s}+\sum_{\substack{n\ge 2;\\ 1\le{}^{\exists} i\le k \text{ s.t. } n\in A_i}}\frac{1}{n^s}$$
$$= 1+\sum_{\substack{n\ge 2;\\ p_1\nmid n,\dots,p_k\nmid n}}\frac{1}{n^s}$$
$$+\Bigl(-\sum_{r=1}^{k}(-1)^r \sum_{1\le i_1<\cdots<i_r\le k} \frac{1}{p_{i_1}^s\cdots p_{i_r}^s}\Bigr)\zeta(s),$$

which implies that

$$1+\sum_{\substack{n\ge 2;\\ p_1\nmid n,\dots,p_k\nmid n}}\frac{1}{n^s} = \zeta(s)\Biggl(1+\sum_{r=1}^{k}(-1)^r \sum_{1\le i_1<\cdots<i_r\le k} \frac{1}{p_{i_1}^s\cdots p_{i_r}^s}\Biggr)$$
$$= \zeta(s)\prod_{i=1}^{k}\Bigl(1-\frac{1}{p_i^s}\Bigr).$$

$\underline{2^{\circ}}$ $\displaystyle\lim_{k\to\infty}\sum_{\substack{n\ge 2;\\ p_1\nmid n,\dots,p_k\nmid n}}\frac{1}{n^s}=0.$

∵ For $2 \le n \le p_k$, the following implications hold:

$$p \text{ is a prime factor of } n \Rightarrow p\le n\le p_k$$
$$\Rightarrow p = p_i \text{ for some } i\in\{1,\dots,k\}.$$

Taking its contraposition, we have that for $n\ge 2$,

$$p_1\nmid n,\dots,p_k\nmid n \Rightarrow n > p_k.$$

This implies that

$$\Biggl|\sum_{\substack{n\ge 2;\\ p_1\nmid n,\dots,p_k\nmid n}}\frac{1}{n^s}\Biggr| \le \sum_{\substack{n\ge 2;\\ p_1\nmid n,\dots,p_k\nmid n}}\Bigl|\frac{1}{n^s}\Bigr| \le \sum_{n>p_k}\frac{1}{n^{\operatorname{Re} s}}\to 0 \quad \text{as } k\to\infty.$$

[†2]"$\Leftarrow$" is clear. "$\Rightarrow$" follows from Euclid's lemma (Euclid's first theorem): *For prime p and integers* $a,b,\ p\mid ab \underset{\textit{iff}}{\Leftrightarrow} p\mid a$ *or* $p\mid b$.

3° By 1° and 2°, we have

$$\lim_{k\to\infty} \zeta(s) \prod_{i=1}^{k} \Big(1 - \frac{1}{p_i^s}\Big) = 1,$$

from which, the assertion of the claim is obvious. ■

Corollary 4.1 $\zeta(\sigma)^{-1} \le |\zeta(s)| \le \zeta(\sigma)$. *Here* $\sigma = \operatorname{Re} s > 1$.

Proof. Clearly

$$|\zeta(s)| = \Big|\sum_{n=1}^{\infty} \frac{1}{n^s}\Big| \le \sum_{n=1}^{\infty} \Big|\frac{1}{n^s}\Big| = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \zeta(\sigma).$$

Since

$$\begin{aligned}\Big|\prod_{i=1}^{k} \Big(1 - \frac{1}{p_i^s}\Big)\Big| = \prod_{i=1}^{k} \Big|1 - \frac{1}{p_i^s}\Big| &\le \prod_{i=1}^{k} \Big(1 + \Big|\frac{1}{p_i^s}\Big|\Big) \\ &= \prod_{i=1}^{k} \Big(1 + \frac{1}{p_i^\sigma}\Big) \\ &\le \prod_{i=1}^{k} \frac{1}{1 - \frac{1}{p_i^\sigma}} \\ &\quad \Big[\because \Big(1 + \tfrac{1}{p^\sigma}\Big)\Big(1 - \tfrac{1}{p^\sigma}\Big) = 1 - \tfrac{1}{p^{2\sigma}} \le 1\Big],\end{aligned}$$

we have by Claim 4.3 that

$$\Big|\frac{1}{\zeta(s)}\Big| = \lim_{k\to\infty} \Big|\prod_{i=1}^{k} \Big(1 - \frac{1}{p_i^s}\Big)\Big| \le \lim_{k\to\infty} \prod_{i=1}^{k} \frac{1}{1 - \frac{1}{p_i^\sigma}} = \zeta(\sigma),$$

which shows $\zeta(\sigma)^{-1} \le |\zeta(s)|$. ■

Theorem 4.2 $\zeta(\cdot)$ *is analytically continuable to a meromorphic function on the entire complex plane which is holomorphic on* $\mathbb{C} \setminus \{1\}$ *and has a simple pole at* $s = 1$ *with residue* 1. (*This meromorphic function is denoted by the same* $\zeta(\cdot)$.)

Proof. We divide the proof into two steps:

1° Let $n \in \mathbb{N}$.

(a) For $\forall s \in \mathbb{C}$ with $\operatorname{Re} s > -n + 1$, $\displaystyle\int_1^\infty \big|\overline{B_n}(x) x^{-s-n}\big| dx < \infty$.

(b) $s \mapsto \displaystyle\int_1^\infty \overline{B_n}(x) x^{-s-n} dx$ is holomorphic on $\big\{s \in \mathbb{C}; \operatorname{Re} s > -n + 1\big\}$.

$\because$ (a) Let $s \in \mathbb{C}$ with $\operatorname{Re} s > -n + 1$. Since

$$\big|\overline{B_n}(x) x^{-s-n}\big| = \big|B_n(\{x\})\big| \, \big|x^{-s-n}\big| \le \Big(\max_{0\le y\le 1} |B_n(y)|\Big) x^{-\operatorname{Re} s - n},$$

we have

$$\int_1^\infty \left|\overline{B_n}(x)x^{-s-n}\right|dx \le \left(\max_{0\le y\le 1}|B_n(y)|\right)\int_1^\infty \left(\frac{x^{-\operatorname{Re}s-n+1}}{-\operatorname{Re}s-n+1}\right)'dx$$
$$= \left(\max_{0\le y\le 1}|B_n(y)|\right)\left[\frac{-1}{\operatorname{Re}s+n-1}\left(\frac{1}{x}\right)^{\operatorname{Re}s+n-1}\right]_1^\infty$$
$$= \left(\max_{0\le y\le 1}|B_n(y)|\right)\frac{1}{\operatorname{Re}s+n-1} < \infty.$$

(b) Fix $s \in \mathbb{C}$ with $\operatorname{Re}s > -n+1$, and let $0 < \delta < \operatorname{Re}s + n - 1$. By noting that for $h \in \mathbb{C}$ with $0 < |h| < \delta$ and $0 \le t \le 1$,

$$\begin{aligned}\operatorname{Re}(s+th+n) &= \operatorname{Re}s + t\operatorname{Re}h + n\\ &\ge \operatorname{Re}s - t|\operatorname{Re}h| + n\\ &\ge \operatorname{Re}s - |h| + n\\ &\ge \operatorname{Re}s - \delta + n,\end{aligned}$$

the following estimate is obtained:

$$\begin{aligned}\left|\frac{1}{h}\left(x^{-s-h-n} - x^{-s-n}\right)\right| &= \left|\frac{1}{h}\left(e^{-(s+h+n)\log x} - e^{-(s+n)\log x}\right)\right|\\ &= \left|\frac{1}{h}\int_0^1 \left(e^{-(s+th+n)\log x}\right)'dt\right|\\ &= \left|\frac{1}{h}\int_0^1 e^{-(s+th+n)\log x}\cdot(-h\log x)dt\right|\\ &= \left|\int_0^1 e^{-(s+th+n)\log x}dt\log x\right|\\ &\le \int_0^1 \left|e^{-(s+th+n)\log x}\right|dt\log x\\ &= \int_0^1 e^{-\left(\operatorname{Re}(s+th+n)\log x\right)}dt\log x\\ &\le e^{-(\operatorname{Re}s-\delta+n)\log x}\log x, \quad 0<|h|<\delta.\end{aligned}$$

Since

$$\begin{aligned}\int_1^\infty e^{-(\operatorname{Re}s-\delta+n)\log x}\log x\,dx &= \int_0^\infty e^{-(\operatorname{Re}s-\delta+n)y}ye^y dy\\ &\qquad [\because \text{ change of variable: } y = \log x]\\ &= \int_0^\infty e^{-(\operatorname{Re}s-\delta+n-1)y}y\,dy\\ &= \frac{1}{(\operatorname{Re}s-\delta+n-1)^2} < \infty,\end{aligned}$$

it follows from Lebesgue's convergence theorem that

$$\lim_{h\to 0}\frac{1}{h}\left(\int_1^\infty \overline{B_n}(x)x^{-s-h-n}dx - \int_1^\infty \overline{B_n}(x)x^{-s-n}dx\right)$$

$$= \lim_{h\to 0}\int_1^\infty \overline{B_n}(x)\frac{1}{h}\left(x^{-s-h-n} - x^{-s-n}\right)dx$$
$$= \int_1^\infty \overline{B_n}(x)x^{-s-n}\left(-\log x\right)dx.$$

This shows that the function in question is holomorphic on $\{s \in \mathbb{C}; \operatorname{Re} s > -n+1\}$.

$\underline{2^\circ}$ Fix $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$. Consider $f : (0,\infty) \to \mathbb{C}$ so that $f(x) = x^{-s} = e^{-s\log x}$. $f \in C^\infty(0,\infty)$ and

$$f^{(n)}(x) = (-s)(-s-1)\cdots(-s-n+1)x^{-s-n},\ \ n \geq 1.$$

Theorem 4.1 with this f gives that for $0 < {}^\forall\varepsilon < 1$, ${}^\forall X \geq 1$, ${}^\forall n \in \mathbb{N}$,

$$\begin{aligned}\sum_{1\leq k\leq X}\frac{1}{k^s} &= \int_\varepsilon^X x^{-s}dx \\ &\quad + \sum_{k=1}^n \frac{(-1)^k}{k!}\left[\overline{B_k}(x)(-s)(-s-1)\cdots(-s-k+2)x^{-s-k+1}\right]_\varepsilon^X \\ &\quad + \frac{(-1)^{n+1}}{n!}\int_\varepsilon^X \overline{B_n}(x)(-s)(-s-1)\cdots(-s-n+1)x^{-s-n}dx \\ &= -\frac{1}{s-1}\left(\frac{1}{X}\right)^{s-1} + \frac{\varepsilon^{1-s}}{s-1} \\ &\quad + \sum_{k=1}^n\frac{(-1)^k}{k!}\Bigg(B_k(\{X\})(-s)(-s-1)\cdots(-s-k+2)\left(\frac{1}{X}\right)^{s+k-1} \\ &\qquad\qquad - B_k(\varepsilon)(-s)(-s-1)\cdots(-s-k+2)\varepsilon^{-s-k+1}\Bigg) \\ &\quad + \frac{(-1)^{n+1}}{n!}(-s)(-s-1)\cdots(-s-n+1)\int_\varepsilon^X \overline{B_n}(x)x^{-s-n}dx.\end{aligned}$$

Letting $X \nearrow \infty$ and $\varepsilon \nearrow 1$, we have

$$\begin{aligned}\zeta(s) &= \frac{1}{s-1} - \sum_{k=1}^n \frac{(-1)^k}{k!}B_k(1)(-s)(-s-1)\cdots(-s-k+2) \\ &\quad + \frac{(-1)^{n+1}}{n!}(-s)(-s-1)\cdots(-s-n+1)\int_1^\infty \overline{B_n}(x)x^{-s-n}dx.\end{aligned}\tag{4.1}$$

By 1°, the function of R.H.S. is meromorphic on $\{s \in \mathbb{C}; \operatorname{Re} s > -n+1\}$, is holomorphic except $s = 1$ and has a simple pole at $s = 1$ with residue 1. Since $\{s \in \mathbb{C}; \operatorname{Re} s > -n+1\} \nearrow \mathbb{C}$ as $n \to \infty$, the assertion of the theorem is obvious. ■

Remark 4.1 $\overline{\zeta(s)} = \zeta(\overline{s}),\ s \in \mathbb{C}\setminus\{1\}$. Here $\overline{z}$ is the conjugate of $z \in \mathbb{C}$.

Proof. By (4.1),

$$\overline{\zeta(s)} = \frac{1}{\overline{s}-1} - \sum_{k=1}^n \frac{(-1)^k}{k!}B_k(1)(-\overline{s})(-\overline{s}-1)\cdots(-\overline{s}-k+2)$$

$$+\frac{(-1)^{n+1}}{n!}(-\overline{s})(-\overline{s}-1)\cdots(-\overline{s}-n+1)\int_1^\infty \overline{B_n}(x)x^{-\overline{s}-n}dx$$
$$=\zeta(\overline{s}).$$

Here $\overline{B_n}(x) = B_n(\{x\})$ [cf. Definition 4.3]. Note that $\overline{B_n}(x)$ is not the conjugate of $B_n(x)$. ■

4.3 Functional equation

Theorem 4.3 (Functional equation for $\zeta(\cdot)$) (i) $\zeta(s) = 2\Gamma(1-s)\sin\left(\frac{\pi}{2}s\right)(2\pi)^{s-1}\zeta(1-s)$.

(ii) $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$.

In (i) *and* (ii), $\Gamma(\cdot)$ *is the gamma function* [*cf. Definition* A.6 *and Claim* A.9(ii)].

Proof. (i) We divide the proof into five steps:

$\underline{1^{\circ}}$ (4.1) with $n=3$ yields that on $\{s; \operatorname{Re} s > -2\}$,

$$\begin{aligned}\zeta(s) &= \frac{1}{s-1} - \left(\frac{(-1)^1}{1!}B_1(1) + \frac{(-1)^2}{2!}B_2(1)(-s) + \frac{(-1)^3}{3!}B_3(1)(-s)(-s-1)\right)\\ &\quad + \frac{(-1)^4}{3!}(-s)(-s-1)(-s-2)\int_1^\infty \overline{B_3}(x)x^{-s-3}dx\\ &= \frac{1}{s-1} - \left(-\frac{1}{2} - \frac{s}{12}\right) - \frac{1}{6}s(s+1)(s+2)\int_1^\infty \overline{B_3}(x)x^{-s-3}dx\\ &\qquad \left[\because B_1(1) = 1 - \tfrac{1}{2} = \tfrac{1}{2}, B_2(1) = B_2 = \tfrac{1}{6}, B_3(1) = B_3 = 0\right]\\ &= \frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - \frac{1}{6}s(s+1)(s+2)\int_1^\infty \overline{B_3}(x)x^{-s-3}dx. \end{aligned} \tag{4.2}$$

Here, for $X \geq 1$,

$$\begin{aligned}&\int_1^X \overline{B_2}(x)x^{-s-2}dx\\ &= \int_1^{\lfloor X\rfloor+1} \overline{B_2}(x)x^{-s-2}dx - \int_X^{\lfloor X\rfloor+1} \overline{B_2}(x)x^{-s-2}dx\\ &= \sum_{n=1}^{\lfloor X\rfloor}\int_n^{n+1} B_2(x-n)x^{-s-2}dx - \int_X^{\lfloor X\rfloor+1} B_2\big(x-\lfloor x\rfloor\big)x^{-s-2}dx\\ &= \sum_{n=1}^{\lfloor X\rfloor}\int_n^{n+1}\left(\frac{B_3(x-n)}{3}\right)' x^{-s-2}dx - \int_X^{\lfloor X\rfloor+1}\left(\frac{B_3\big(x-\lfloor x\rfloor\big)}{3}\right)' x^{-s-2}dx\\ &\qquad [\because \text{Claim 4.2(ii)}]\\ &= \sum_{n=1}^{\lfloor X\rfloor}\left(\left[\frac{B_3(x-n)}{3}x^{-s-2}\right]_n^{n+1} - \int_n^{n+1}\frac{B_3(x-n)}{3}(-s-2)x^{-s-3}dx\right)\end{aligned}$$

$$- \left(\left[\frac{B_3(x - \lfloor x \rfloor)}{3} x^{-s-2} \right]_X^{\lfloor X \rfloor + 1} - \int_X^{\lfloor X \rfloor + 1} \frac{B_3(x - \lfloor x \rfloor)}{3} (-s-2) x^{-s-3} dx \right)$$
$[\because$ integration by parts$]$

$$= \sum_{n=1}^{\lfloor X \rfloor} \left(\frac{1}{3} \big(B_3(1)(n+1)^{-s-2} - B_3(0) n^{-s-2} \big) + \frac{s+2}{3} \int_n^{n+1} \overline{B_3}(x) x^{-s-3} dx \right)$$
$$- \left(\frac{1}{3} \big(B_3(1) (\lfloor X \rfloor + 1)^{-s-2} - \overline{B_3}(X) X^{-s-2} \big) + \frac{s+2}{3} \int_X^{\lfloor X \rfloor + 1} \overline{B_3}(x) x^{-s-3} dx \right)$$
$$= \frac{s+2}{3} \int_1^X \overline{B_3}(x) x^{-s-3} dx + \frac{1}{3} \overline{B_3}(X) \left(\frac{1}{X} \right)^{s+2}$$
$[\because B_3(1) = B_3(0) = B_3 = 0]$.

Since $\left(\frac{1}{X}\right)^{s+2} \to 0$ as $X \nearrow \infty$ by $\operatorname{Re} s > -2$,

$$\lim_{X \nearrow \infty} \int_1^X \overline{B_2}(x) x^{-s-2} dx = \frac{s+2}{3} \int_1^\infty \overline{B_3}(x) x^{-s-3} dx.$$

Putting this into (4.2), we have

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - \frac{s(s+1)}{2} \int_1^\infty \overline{B_2}(x) x^{-s-2} dx, \quad \operatorname{Re} s > -2,$$

where the integral of the 4th term in R.H.S. is improper. As for the other terms in R.H.S., note that for $-2 < \operatorname{Re} s < -1$,

$$\begin{aligned}
\frac{s(s+1)}{2} \int_0^1 \overline{B_2}(x) x^{-s-2} dx &= \frac{s(s+1)}{2} \int_0^1 \big(x^2 - x + \frac{1}{6} \big) x^{-s-2} dx \\
&= \frac{s(s+1)}{2} \int_0^1 \big(x^{-s} - x^{-s-1} + \frac{1}{6} x^{-s-2} \big) dx \\
&= \frac{s(s+1)}{2} \left[\frac{x^{1-s}}{1-s} - \frac{x^{-s}}{-s} + \frac{1}{6} \frac{x^{-s-1}}{-s-1} \right]_0^1 \\
&= \frac{s(s+1)}{2} \Big(\frac{1}{1-s} + \frac{1}{s} - \frac{1}{6} \frac{1}{s+1} \Big) \\
&= \frac{s}{2} \frac{s+1}{1-s} + \frac{s+1}{2} - \frac{s}{12} \\
&= \frac{s}{2} \frac{s-1+2}{1-s} + \frac{s}{2} + \frac{1}{2} - \frac{s}{12} \\
&= -\frac{s}{2} + \frac{s-1+1}{1-s} + \frac{s}{2} + \frac{1}{2} - \frac{s}{12} \\
&= -\frac{1}{s-1} - \frac{1}{2} - \frac{s}{12}.
\end{aligned}$$

Substituting this into R.H.S. in the above, we obtain

$$\zeta(s) = -\frac{s(s+1)}{2} \int_0^1 \overline{B_2}(x) x^{-s-2} dx - \frac{s(s+1)}{2} \int_1^\infty \overline{B_2}(x) x^{-s-2} dx$$

$$= -\frac{s(s+1)}{2}\int_0^\infty \overline{B_2}(x)x^{-s-2}dx, \quad -2 < \operatorname{Re} s < -1. \tag{4.3}$$

$\underline{2^{\circ}}$ $\overline{B_2}(x) = \dfrac{1}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{\cos 2\pi nx}{n^2}$, $x \in \mathbb{R}$.

∵ Since $B_2(\cdot) \in C^\infty[0,1]$ and $B_2(0) = B_2(1) = \frac{1}{6}$, it follows that

$$B_2(x) = \sum_{n\in\mathbb{Z}} \widehat{B_2}(n)e^{\sqrt{-1}2\pi nx},$$

whose convergence is uniform on $[0,1]$. Here $\widehat{B_2}(n)$ are the Fourier coefficients of $B_2(\cdot)$. In this case, they are computed as follows:

$$\begin{aligned}
\widehat{B_2}(n) &= \int_0^1 B_2(x)e^{-\sqrt{-1}2\pi nx}dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} B_2\big(y+\tfrac{1}{2}\big)e^{-\sqrt{-1}2\pi n(y+\frac{1}{2})}dy \quad \big[\because \text{ change of variable: } y = x - \tfrac{1}{2}\big] \\
&= e^{-\sqrt{-1}\pi n}\int_{-\frac{1}{2}}^{\frac{1}{2}} \big(y^2 - \tfrac{1}{12}\big)e^{-\sqrt{-1}2\pi ny}dy \\
&\qquad \big[\because B_2\big(y+\tfrac{1}{2}\big) = \big(y+\tfrac{1}{2}\big)\big(y-\tfrac{1}{2}\big) + \tfrac{1}{6} = y^2 - \tfrac{1}{4} + \tfrac{1}{6} = y^2 - \tfrac{1}{12}\big] \\
&= (-1)^n \int_{-\frac{1}{2}}^{\frac{1}{2}} \big(y^2 - \tfrac{1}{12}\big)\cos 2\pi ny\,dy \quad \big[\because \big(y^2 - \tfrac{1}{12}\big)\sin 2\pi ny \text{ is odd}\big] \\
&= 2(-1)^n \int_0^{\frac{1}{2}} \big(y^2 - \tfrac{1}{12}\big)\cos 2\pi|n|y\,dy.
\end{aligned}$$

Thus, when $n = 0$,

$$\widehat{B_2}(0) = 2\int_0^{\frac{1}{2}}\big(y^2 - \tfrac{1}{12}\big)dy = 2\Big[\frac{y^3}{3} - \frac{y}{12}\Big]_0^{\frac{1}{2}} = 2\Big(\frac{1}{3}\cdot\frac{1}{8} - \frac{1}{12}\cdot\frac{1}{2}\Big) = 0;$$

when $n \neq 0$,

$$\begin{aligned}
\widehat{B_2}(n) &= 2(-1)^n \int_0^{\frac{1}{2}} \big(y^2 - \tfrac{1}{12}\big)\Big(\frac{\sin 2\pi|n|y}{2\pi|n|}\Big)'dy \\
&= 2(-1)^n\Big(\Big[\big(y^2 - \tfrac{1}{12}\big)\frac{\sin 2\pi|n|y}{2\pi|n|}\Big]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} 2y\frac{\sin 2\pi|n|y}{2\pi|n|}dy\Big) \\
&= \frac{-2(-1)^n}{\pi|n|}\int_0^{\frac{1}{2}} y\Big(\frac{-\cos 2\pi|n|y}{2\pi|n|}\Big)'dy \\
&= \frac{-2(-1)^n}{\pi|n|}\Big(\Big[y\frac{-\cos 2\pi|n|y}{2\pi|n|}\Big]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}}\frac{-\cos 2\pi|n|y}{2\pi|n|}dy\Big) \\
&= \frac{-2(-1)^n}{\pi|n|}\Big(\frac{1}{2}\cdot\frac{-\cos\pi|n|}{2\pi|n|} + \frac{1}{2\pi|n|}\int_0^{\frac{1}{2}}\cos 2\pi|n|y\,dy\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-2(-1)^n}{\pi|n|}\left(\frac{1}{2}\cdot\frac{-(-1)^n}{2\pi|n|} + \frac{1}{2\pi|n|}\left[\frac{\sin 2\pi|n|y}{2\pi|n|}\right]_0^{\frac{1}{2}}\right)\\
&= \frac{-2(-1)^n}{\pi|n|}\cdot\frac{1}{2}\cdot\frac{-(-1)^n}{2\pi|n|}\\
&= \frac{1}{2}\frac{1}{\pi^2 n^2}.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
B_2(x) = \sum_{n\neq 0}\frac{1}{2\pi^2n^2}e^{\sqrt{-1}2\pi nx} &= \sum_{n\neq 0}\frac{1}{2\pi^2n^2}\left(\cos 2\pi nx + \sqrt{-1}\sin 2\pi nx\right)\\
&= \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos 2\pi nx}{n^2}.
\end{aligned}
$$

<u>3°</u> For $\tau \in \mathbb{C}$ with $0 < \operatorname{Re}\tau < 1$, $R > 0$,

$$
\left|\int_0^R (\cos x)x^{\tau-1}dx - \int_0^R e^{-y}y^{\tau-1}dy\,\cos\!\left(\frac{\pi}{2}\tau\right)\right| \leq e^{|\operatorname{Im}\tau|\frac{\pi}{2}}\frac{\pi}{2}\left(\frac{1}{R}\right)^{1-\operatorname{Re}\tau}.
$$

∵ For $0 < \varepsilon < R$, we consider contours $C^{\pm}_{\varepsilon,R}$ as in Figure 4.1. Then, by Cauchy's

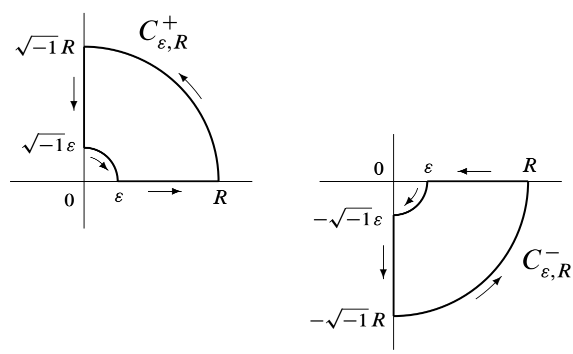

Figure 4.1: $C^+_{\varepsilon,R}$ and $C^-_{\varepsilon,R}$

integral theorem,

$$
\begin{aligned}
0 &= \int_{C^+_{\varepsilon,R}} e^{\sqrt{-1}z}\frac{e^{\tau\log z}}{z}dz\\
&= \int_{\varepsilon}^{R} e^{\sqrt{-1}x}\frac{e^{\tau\log x}}{x}dx\\
&\quad + \int_0^{\frac{\pi}{2}} e^{\sqrt{-1}Re^{\sqrt{-1}\theta}}\frac{e^{\tau\log Re^{\sqrt{-1}\theta}}}{Re^{\sqrt{-1}\theta}}Re^{\sqrt{-1}\theta}\sqrt{-1}d\theta\\
&\quad - \int_{\varepsilon}^{R} e^{\sqrt{-1}\sqrt{-1}y}\frac{e^{\tau\log\sqrt{-1}y}}{\sqrt{-1}y}\sqrt{-1}dy
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^{\frac{\pi}{2}} e^{\sqrt{-1}\varepsilon e^{\sqrt{-1}\theta}} \frac{e^{\tau \log \varepsilon e^{\sqrt{-1}\theta}}}{\varepsilon e^{\sqrt{-1}\theta}} \varepsilon e^{\sqrt{-1}\theta}\sqrt{-1}d\theta \\
&= \int_\varepsilon^R e^{\sqrt{-1}x} \frac{e^{\tau \log x}}{x} dx \\
&\quad + \int_0^{\frac{\pi}{2}} e^{\sqrt{-1}R(\cos\theta+\sqrt{-1}\sin\theta)} e^{\tau(\log R+\sqrt{-1}\theta)}\sqrt{-1}d\theta \\
&\quad - \int_\varepsilon^R e^{-y} \frac{e^{\tau(\log y+\sqrt{-1}\frac{\pi}{2})}}{y} dy \\
&\quad - \int_0^{\frac{\pi}{2}} e^{\sqrt{-1}\varepsilon(\cos\theta+\sqrt{-1}\sin\theta)} e^{\tau(\log \varepsilon+\sqrt{-1}\theta)}\sqrt{-1}d\theta \\
&= \int_\varepsilon^R e^{\sqrt{-1}x} \frac{e^{\tau \log x}}{x} dx \\
&\quad + \int_0^{\frac{\pi}{2}} e^{\sqrt{-1}R\cos\theta} e^{\sqrt{-1}\tau\theta}\sqrt{-1}e^{-R\sin\theta} R^\tau d\theta \\
&\quad - \int_\varepsilon^R e^{-y} y^{\tau-1} dy e^{\sqrt{-1}\frac{\pi}{2}\tau} \\
&\quad - \int_0^{\frac{\pi}{2}} e^{\sqrt{-1}\varepsilon\cos\theta} e^{\sqrt{-1}\tau\theta}\sqrt{-1}e^{-\varepsilon\sin\theta} d\theta \varepsilon^\tau,
\end{aligned}
$$

$$
\begin{aligned}
0 &= \int_{C^-_{\varepsilon,R}} e^{-\sqrt{-1}z} \frac{e^{\tau \log z}}{z} dz \\
&= -\int_\varepsilon^R e^{-\sqrt{-1}x} \frac{e^{\tau \log x}}{x} dx \\
&\quad + \int_{-\frac{\pi}{2}}^0 e^{-\sqrt{-1}Re^{\sqrt{-1}\theta}} \frac{e^{\tau \log Re^{\sqrt{-1}\theta}}}{Re^{\sqrt{-1}\theta}} Re^{\sqrt{-1}\theta}\sqrt{-1}d\theta \\
&\quad - \int_{-\frac{\pi}{2}}^0 e^{-\sqrt{-1}\varepsilon e^{\sqrt{-1}\theta}} \frac{e^{\tau \log \varepsilon e^{\sqrt{-1}\theta}}}{\varepsilon e^{\sqrt{-1}\theta}} \varepsilon e^{\sqrt{-1}\theta}\sqrt{-1}d\theta \\
&\quad + \int_\varepsilon^R e^{\sqrt{-1}\sqrt{-1}y} \frac{e^{\tau \log(-\sqrt{-1}y)}}{-\sqrt{-1}y} (-\sqrt{-1}) dy \\
&= -\int_\varepsilon^R e^{-\sqrt{-1}x} \frac{e^{\tau \log x}}{x} dx \\
&\quad + \int_{-\frac{\pi}{2}}^0 e^{-\sqrt{-1}R(\cos\theta+\sqrt{-1}\sin\theta)} e^{\tau(\log R+\sqrt{-1}\theta)}\sqrt{-1}d\theta \\
&\quad - \int_{-\frac{\pi}{2}}^0 e^{-\sqrt{-1}\varepsilon(\cos\theta+\sqrt{-1}\sin\theta)} e^{\tau(\log \varepsilon+\sqrt{-1}\theta)}\sqrt{-1}d\theta \\
&\quad + \int_\varepsilon^R e^{-y} \frac{e^{\tau(\log y-\sqrt{-1}\frac{\pi}{2})}}{y} dy
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_\varepsilon^R e^{-\sqrt{-1}x}\frac{e^{\tau\log x}}{x}dx \\
&\quad + \int_0^{\frac{\pi}{2}} e^{-\sqrt{-1}R\cos\theta}e^{-\sqrt{-1}\tau\theta}\sqrt{-1}e^{-R\sin\theta}R^\tau d\theta \\
&\quad - \int_0^{\frac{\pi}{2}} e^{-\sqrt{-1}\varepsilon\cos\theta}e^{-\sqrt{-1}\tau\theta}\sqrt{-1}e^{-\varepsilon\sin\theta}d\theta\varepsilon^\tau \\
&\quad + \int_\varepsilon^R e^{-y}y^{\tau-1}dye^{-\sqrt{-1}\frac{\pi}{2}\tau}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left|\int_\varepsilon^R e^{\pm\sqrt{-1}x}\frac{e^{\tau\log x}}{x}dx - \int_\varepsilon^R e^{-y}y^{\tau-1}dye^{\pm\sqrt{-1}\frac{\pi}{2}\tau}\right| \\
&= \left|\int_0^{\frac{\pi}{2}} e^{\pm\sqrt{-1}\varepsilon\cos\theta}e^{\pm\sqrt{-1}\tau\theta}\sqrt{-1}e^{-\varepsilon\sin\theta}d\theta\varepsilon^\tau\right. \\
&\quad \left.-\int_0^{\frac{\pi}{2}} e^{\pm\sqrt{-1}R\cos\theta}e^{\pm\sqrt{-1}\tau\theta}\sqrt{-1}e^{-R\sin\theta}R^\tau d\theta\right| \\
&\le \int_0^{\frac{\pi}{2}} |e^{\pm\sqrt{-1}\tau\theta}|\, e^{-\varepsilon\sin\theta}d\theta|\varepsilon^\tau| + \int_0^{\frac{\pi}{2}} |e^{\pm\sqrt{-1}\tau\theta}|\, e^{-R\sin\theta}|R^\tau|d\theta \\
&\le \int_0^{\frac{\pi}{2}} e^{\mp(\operatorname{Im}\tau)\theta}d\theta\varepsilon^{\operatorname{Re}\tau} + \int_0^{\frac{\pi}{2}} e^{\mp(\operatorname{Im}\tau)\theta}e^{-\frac{2}{\pi}R\theta}R^{\operatorname{Re}\tau}d\theta \\
&\qquad \left[\because\ \tfrac{2}{\pi}\theta \le \sin\theta \le \theta\ \ (0 \le \theta \le \tfrac{\pi}{2})\right] \\
&\le e^{|\operatorname{Im}\tau|\frac{\pi}{2}}\left(\frac{\pi}{2}\varepsilon^{\operatorname{Re}\tau} + \int_0^{\frac{\pi}{2}} e^{-\frac{2}{\pi}R\theta}d\theta R^{\operatorname{Re}\tau}\right) \\
&\le e^{|\operatorname{Im}\tau|\frac{\pi}{2}}\left(\frac{\pi}{2}\varepsilon^{\operatorname{Re}\tau} + \int_0^{\infty} e^{-\frac{2}{\pi}R\theta}d\theta R^{\operatorname{Re}\tau}\right) \\
&= e^{|\operatorname{Im}\tau|\frac{\pi}{2}}\frac{\pi}{2}\left(\varepsilon^{\operatorname{Re}\tau} + \frac{1}{R^{1-\operatorname{Re}\tau}}\right).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\left|\int_\varepsilon^R (\cos x)x^{\tau-1}dx - \int_\varepsilon^R e^{-y}y^{\tau-1}dy\cos\left(\frac{\pi}{2}\tau\right)\right| \\
&= \left|\int_\varepsilon^R \frac{e^{\sqrt{-1}x}+e^{-\sqrt{-1}x}}{2}x^{\tau-1}dx - \int_\varepsilon^R e^{-y}y^{\tau-1}dy\frac{e^{\sqrt{-1}\frac{\pi}{2}\tau}+e^{-\sqrt{-1}\frac{\pi}{2}\tau}}{2}\right| \\
&= \frac{1}{2}\left|\int_\varepsilon^R e^{\sqrt{-1}x}x^{\tau-1}dx - \int_\varepsilon^R e^{-y}y^{\tau-1}dye^{\sqrt{-1}\frac{\pi}{2}\tau}\right. \\
&\quad \left.+\int_\varepsilon^R e^{-\sqrt{-1}x}x^{\tau-1}dx - \int_\varepsilon^R e^{-y}y^{\tau-1}dye^{-\sqrt{-1}\frac{\pi}{2}\tau}\right| \\
&\le e^{|\operatorname{Im}\tau|\frac{\pi}{2}}\frac{\pi}{2}\left(\varepsilon^{\operatorname{Re}\tau} + \frac{1}{R^{1-\operatorname{Re}\tau}}\right).
\end{aligned}
$$

Letting $\varepsilon \searrow 0$, we have the assertion of 3°.

4° Fix $s \in \mathbb{C}$ with $-2 < \operatorname{Re} s < -1$. By (4.3),

$$\zeta(s) = -\frac{s(s+1)}{2} \lim_{X\to\infty} \int_0^X \overline{B_2}(x) x^{-s-2} dx.$$

On the other hand, by 2°,

$$\begin{aligned}
\int_0^X \overline{B_2}(x) x^{-s-2} dx &= \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^X (\cos 2\pi n x) x^{-s-2} dx \\
&= \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{2n\pi X} (\cos y) \Big(\frac{y}{2\pi n}\Big)^{-s-2} \frac{dy}{2\pi n} \\
&\quad \big[\because \text{ change of variable: } y = 2\pi n x\big] \\
&= \frac{4}{(2\pi)^2} \Big(\frac{1}{2\pi}\Big)^{-s-1} \sum_{n=1}^{\infty} \Big(\frac{1}{n}\Big)^{2-s-1} \int_0^{2\pi n X} (\cos y) y^{-s-2} dy \\
&= 4(2\pi)^{s-1} \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \int_0^{2\pi n X} (\cos x) x^{-s-1-1} dx.
\end{aligned}$$

Since $\operatorname{Re}(-s-1) = -(\operatorname{Re} s) - 1 \in (0,1)$, it follows from 3° that for each $n \in \mathbb{N}$,

$$\begin{aligned}
\lim_{X\to\infty} \int_0^{2\pi n X} (\cos x) x^{-s-1-1} dx &= \int_0^{\infty} e^{-y} y^{-s-1-1} dy \cos \frac{\pi}{2}(-s-1) \\
&= \Gamma(-s-1) \cos \frac{\pi}{2}(s+1),
\end{aligned}$$

$$\begin{aligned}
&\Big| \int_0^{2\pi n X} (\cos x) x^{-s-1-1} dx \Big| \\
&= \Big| \int_0^{2\pi n X} (\cos x) x^{-s-1-1} dx - \int_0^{2\pi n X} e^{-y} y^{-s-1-1} dy \cos \frac{\pi}{2}(-s-1) \\
&\quad + \int_0^{2\pi n X} e^{-y} y^{-s-1-1} dy \cos \frac{\pi}{2}(-s-1) \Big| \\
&\le \Big| \int_0^{2\pi n X} (\cos x) x^{-s-1-1} dx - \int_0^{2\pi n X} e^{-y} y^{-s-1-1} dy \cos \frac{\pi}{2}(-s-1) \Big| \\
&\quad + \Big| \int_0^{2\pi n X} e^{-y} y^{-s-1-1} dy \cos \frac{\pi}{2}(-s-1) \Big| \\
&\le e^{|\operatorname{Im}(-s-1)|\frac{\pi}{2}} \frac{\pi}{2} \Big(\frac{1}{2\pi n X}\Big)^{1-\operatorname{Re}(-s-1)} + \int_0^{2\pi n X} e^{-y} y^{\operatorname{Re}(-s-1)-1} dy \Big|\cos \frac{\pi}{2}(-s-1)\Big| \\
&\le e^{|\operatorname{Im}(s+1)|\frac{\pi}{2}} \frac{\pi}{2} \Big(\frac{1}{2\pi X}\Big)^{2+\operatorname{Re} s} + \Gamma\big(\operatorname{Re}(-s-1)\big) \Big|\cos \frac{\pi}{2}(s+1)\Big|.
\end{aligned}$$

Thus Lebesgue's convergence theorem gives that

$$\lim_{X\to\infty} \int_0^X \overline{B_2}(x) x^{-s-2} dx = 4(2\pi)^{s-1} \Big(\sum_{n=1}^{\infty} \frac{1}{n^{1-s}}\Big) \Gamma(-s-1) \cos \frac{\pi}{2}(s+1)$$

$$= -4(2\pi)^{s-1}\zeta(1-s)\Gamma(-s-1)\sin\frac{\pi}{2}s$$
$$\left[\because\ \cos\tfrac{\pi}{2}(s+1) = \cos\left(\tfrac{\pi}{2}s+\tfrac{\pi}{2}\right) = -\sin\tfrac{\pi}{2}s\right],$$

so that we have

$$\begin{aligned}\zeta(s) &= -\frac{s(s+1)}{2}\cdot(-4)(2\pi)^{s-1}\zeta(1-s)\Gamma(-s-1)\sin\frac{\pi}{2}s\\ &= 2(2\pi)^{s-1}\left(\sin\frac{\pi}{2}s\right)(-s)(-s-1)\Gamma(-s-1)\zeta(1-s)\\ &= 2\Gamma(1-s)(2\pi)^{s-1}\left(\sin\frac{\pi}{2}s\right)\zeta(1-s)\end{aligned}$$

$$\left[\begin{aligned}&\because \text{ By } \Gamma(x+1) = x\Gamma(x),\\ &(-s)(-s-1)\Gamma(-s-1) = (-s)\Gamma(-s-1+1)\\ &\qquad\qquad\qquad\qquad\quad = (-s)\Gamma(-s)\\ &\qquad\qquad\qquad\qquad\quad = \Gamma(-s+1) = \Gamma(1-s)\end{aligned}\right].$$

5° By 4°,

$$\zeta(s) = 2\Gamma(1-s)(2\pi)^{s-1}\left(\sin\frac{\pi}{2}s\right)\zeta(1-s) \quad \text{on } \left\{s \in \mathbb{C}; -2 < \operatorname{Re} s < -1\right\}.$$

The function of L.H.S. is holomorphic on $\mathbb{C}\setminus\{1\}$, and so is the function of R.H.S. on $\mathbb{C}\setminus\{0,1,2,\ldots\}$. By the uniqueness theorem, the identity above holds on $\mathbb{C}\setminus\{0,1,2,\ldots\}$.

When $\operatorname{Re} s < 0$,

$$\begin{aligned}\zeta(s) = 0 &\Leftrightarrow \Gamma(1-s)(2\pi)^{s-1}\left(\sin\frac{\pi}{2}s\right)\zeta(1-s) = 0\\ &\Leftrightarrow \sin\frac{\pi}{2}s = 0\end{aligned}$$

$$\left[\begin{array}{l}\because \text{ Clearly } (2\pi)^{s-1} \neq 0. \text{ Also } \Gamma(1-s) \neq 0,\\ \zeta(1-s) \neq 0, \text{ since } \operatorname{Re}(1-s) = 1-\operatorname{Re} s > 1\\ [\text{cf. Claim 4.3}]\end{array}\right]$$

$$\Leftrightarrow s \in \left\{-2,-4,-6,\ldots\right\}.$$

Moreover $s = -2n$ ($n \in \mathbb{N}$) is a zero of $\zeta(\cdot)$ of order 1.

$s = 0$ is a simple pole of $\zeta(1-\cdot)$ and a zero of $\sin\frac{\pi}{2}\cdot$ of order 1, thus it is a removable singularity of the function of R.H.S. $s = 1$ is a simple pole of $\Gamma(1-\cdot)$, thus it is a simple pole of the function of R.H.S. $s = 2n$ ($n \in \mathbb{N}$) is a simple pole of $\Gamma(1-\cdot)$ and a zero of $\sin\frac{\pi}{2}\cdot$ of order 1, thus it is a removable singularity of the function of R.H.S. $s = 2n+1$ ($n \in \mathbb{N}$) is a simple pole of $\Gamma(1-\cdot)$ and a zero of $\zeta(1-\cdot)$ of order 1, thus it is a removable singularity of the function of R.H.S. Therefore, putting all together, we see that the functional equation

$$\zeta(s) = 2\Gamma(1-s)(2\pi)^{s-1}\left(\sin\frac{\pi}{2}s\right)\zeta(1-s)$$

is valid for $^\forall s \in \mathbb{C}$.

(ii) We divide the proof into three steps:

1° For $s \in \mathbb{C}\setminus\mathbb{Z}$, $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\sin\pi s}$.

∵ By the uniqueness theorem, it suffices to verify that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \quad (0 < s < 1).$$

In the following, fix $0 < s < 1$.

First note that

$$\Gamma(s)\Gamma(1-s) = \int_0^\infty \frac{x^{s-1}}{1+x} dx.$$

Because

$$\begin{aligned}
\text{R.H.S.} &= \int_0^\infty x^{s-1} dx \int_0^\infty e^{-(1+x)t} dt \\
&= \int_0^\infty e^{-t} dt \int_0^\infty x^{s-1} e^{-xt} dx \\
&= \int_0^\infty e^{-t} dt \int_0^\infty \left(\frac{y}{t}\right)^{s-1} e^{-y} \frac{dy}{t} \\
&\qquad \left[\because \text{ change of variable: } y = xt\right] \\
&= \int_0^\infty t^{-s} e^{-t} dt \int_0^\infty y^{s-1} e^{-y} dy \\
&= \Gamma(1-s)\Gamma(s) \\
&= \text{L.H.S.}
\end{aligned}$$

Next we show that

$$\int_0^\infty \frac{x^{s-1}}{1+x} dx = \frac{\pi}{\sin \pi s}, \tag{4.4}$$

from this and the above expression, the assertion of 1° is obvious. To this end, we introduce the logarithm function defined on $\mathbb{C} \setminus [0, \infty)$ by

$$\log z := \int_{-1}^{z} \frac{dw}{w} \quad \left[\text{cf. (3.1)}\right].$$

When $z = re^{\sqrt{-1}\theta}$ $(r > 0, 0 < \theta < 2\pi)$,

$$\log re^{\sqrt{-1}\theta} = \log r + \sqrt{-1}(\theta - \pi).$$

$\frac{e^{(s-1)\log z}}{1+z}$ is holomorphic on $\mathbb{C} \setminus \big([0, \infty) \cup \{-1\}\big)$ and has a simple pole at $z = -1$ with residue $e^{(s-1)\log(-1)} = 1$.

Now, for $0 < \varepsilon < 1 < R$ and $0 < \varphi < \frac{\pi}{2}$, take a contour $C_{\varepsilon,R,\varphi}$ as in Figure 4.2. Then

$$\begin{aligned}
2\pi\sqrt{-1} &= \int_{C_{\varepsilon,R,\varphi}} \frac{e^{(s-1)\log z}}{1+z} dz \\
&= \int_\varepsilon^R \frac{e^{(s-1)\log(xe^{\sqrt{-1}\varphi})}}{1+xe^{\sqrt{-1}\varphi}} e^{\sqrt{-1}\varphi} dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\varphi}^{2\pi-\varphi}\frac{e^{(s-1)\log(Re^{\sqrt{-1}\theta})}}{1+Re^{\sqrt{-1}\theta}}Re^{\sqrt{-1}\theta}\sqrt{-1}d\theta\\
&-\int_{\varepsilon}^{R}\frac{e^{(s-1)\log(xe^{\sqrt{-1}(2\pi-\varphi)})}}{1+xe^{\sqrt{-1}(2\pi-\varphi)}}e^{\sqrt{-1}(2\pi-\varphi)}dx\\
&-\int_{\varphi}^{2\pi-\varphi}\frac{e^{(s-1)\log(\varepsilon e^{\sqrt{-1}\theta})}}{1+\varepsilon e^{\sqrt{-1}\theta}}\varepsilon e^{\sqrt{-1}\theta}\sqrt{-1}d\theta\\
=&\int_{\varepsilon}^{R}\frac{e^{(s-1)(\log x+\sqrt{-1}(\varphi-\pi))}}{1+xe^{\sqrt{-1}\varphi}}dxe^{\sqrt{-1}\varphi}\\
&+\int_{\varphi}^{2\pi-\varphi}\frac{e^{(s-1)(\log R+\sqrt{-1}(\theta-\pi))}}{1+Re^{\sqrt{-1}\theta}}Re^{\sqrt{-1}\theta}\sqrt{-1}d\theta\\
&-\int_{\varepsilon}^{R}\frac{e^{(s-1)(\log x+\sqrt{-1}(\pi-\varphi))}}{1+xe^{\sqrt{-1}(2\pi-\varphi)}}dxe^{\sqrt{-1}(2\pi-\varphi)}\\
&-\int_{\varphi}^{2\pi-\varphi}\frac{e^{(s-1)(\log\varepsilon+\sqrt{-1}(\theta-\pi))}}{1+\varepsilon e^{\sqrt{-1}\theta}}\varepsilon e^{\sqrt{-1}\theta}\sqrt{-1}d\theta\\
=&\int_{\varepsilon}^{R}\frac{x^{s-1}}{1+xe^{\sqrt{-1}\varphi}}dxe^{\sqrt{-1}(s-1)(\varphi-\pi)}e^{\sqrt{-1}\varphi}\\
&+\int_{\varphi}^{2\pi-\varphi}\frac{R^{s}}{1+Re^{\sqrt{-1}\theta}}e^{\sqrt{-1}(s-1)(\theta-\pi)}e^{\sqrt{-1}\theta}\sqrt{-1}d\theta\\
&-\int_{\varepsilon}^{R}\frac{x^{s-1}}{1+xe^{\sqrt{-1}(2\pi-\varphi)}}dxe^{\sqrt{-1}(s-1)(\pi-\varphi)}e^{\sqrt{-1}(2\pi-\varphi)}\\
&-\int_{\varphi}^{2\pi-\varphi}\frac{\varepsilon^{s}}{1+\varepsilon e^{\sqrt{-1}\theta}}e^{\sqrt{-1}(s-1)(\theta-\pi)}e^{\sqrt{-1}\theta}\sqrt{-1}d\theta.
\end{aligned}
$$

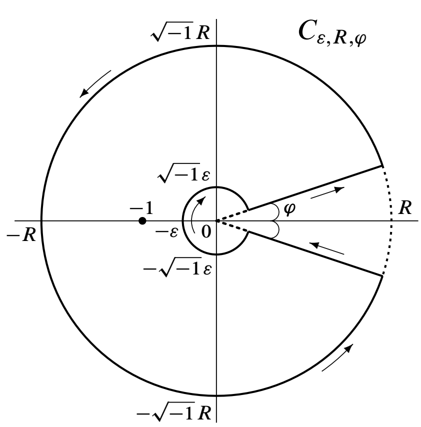

Figure 4.2: $C_{\varepsilon,R,\varphi}$

Letting $\varphi \searrow 0$, we have

$$\begin{aligned}
2\pi\sqrt{-1} &= \int_\varepsilon^R \frac{x^{s-1}}{1+x}dx e^{-\sqrt{-1}(s-1)\pi} \\
&\quad + \int_0^{2\pi} \frac{R^s}{1+Re^{\sqrt{-1}\theta}} e^{\sqrt{-1}(s-1)(\theta-\pi)} e^{\sqrt{-1}\theta}\sqrt{-1}d\theta \\
&\quad - \int_\varepsilon^R \frac{x^{s-1}}{1+x}dx e^{\sqrt{-1}(s-1)\pi} \\
&\quad - \int_0^{2\pi} \frac{\varepsilon^s}{1+\varepsilon e^{\sqrt{-1}\theta}} e^{\sqrt{-1}(s-1)(\theta-\pi)} e^{\sqrt{-1}\theta}\sqrt{-1}d\theta \\
&= \int_\varepsilon^R \frac{x^{s-1}}{1+x}dx\big(-2\sqrt{-1}\sin(s-1)\pi\big) \\
&\quad + \int_0^{2\pi} \frac{R^s}{1+Re^{\sqrt{-1}\theta}} e^{\sqrt{-1}(s-1)(\theta-\pi)} e^{\sqrt{-1}\theta}\sqrt{-1}d\theta \\
&\quad - \int_0^{2\pi} \frac{\varepsilon^s}{1+\varepsilon e^{\sqrt{-1}\theta}} e^{\sqrt{-1}(s-1)(\theta-\pi)} e^{\sqrt{-1}\theta}\sqrt{-1}d\theta \\
&= 2\sqrt{-1}\Big(\int_\varepsilon^R \frac{x^{s-1}}{1+x}dx\Big)\sin\pi s \\
&\quad + \int_0^{2\pi} \frac{R^s}{1+Re^{\sqrt{-1}\theta}} e^{\sqrt{-1}(s-1)(\theta-\pi)} e^{\sqrt{-1}\theta}\sqrt{-1}d\theta \\
&\quad - \int_0^{2\pi} \frac{\varepsilon^s}{1+\varepsilon e^{\sqrt{-1}\theta}} e^{\sqrt{-1}(s-1)(\theta-\pi)} e^{\sqrt{-1}\theta}\sqrt{-1}d\theta.
\end{aligned}$$

Thus

$$\begin{aligned}
&2\Big|\pi - \Big(\int_\varepsilon^R \frac{x^{s-1}}{1+x}dx\Big)\sin\pi s\Big| \\
&= \Big|\int_0^{2\pi} \frac{R^s}{1+Re^{\sqrt{-1}\theta}} e^{\sqrt{-1}(s-1)(\theta-\pi)} e^{\sqrt{-1}\theta}d\theta \\
&\quad - \int_0^{2\pi} \frac{\varepsilon^s}{1+\varepsilon e^{\sqrt{-1}\theta}} e^{\sqrt{-1}(s-1)(\theta-\pi)} e^{\sqrt{-1}\theta}d\theta\Big| \\
&\le \int_0^{2\pi} \frac{R^s}{|1+Re^{\sqrt{-1}\theta}|}d\theta + \int_0^{2\pi} \frac{\varepsilon^s}{|1+\varepsilon e^{\sqrt{-1}\theta}|}d\theta \\
&\le 2\pi\Big(\frac{R^s}{R-1} + \frac{\varepsilon^s}{1-\varepsilon}\Big) \\
&= 2\pi\Big(\frac{1}{R^{1-s}-R^{-s}} + \frac{\varepsilon^s}{1-\varepsilon}\Big) \underset{R\nearrow\infty,\,\varepsilon\searrow 0}{\to} 0,
\end{aligned}$$

which shows (4.4).

2° For $s \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, $\Gamma(s) = 2^{s-1}\pi^{-\frac{1}{2}}\Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{s+1}{2}\Big)$.

$\because$ Fix $s \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. By Gauss's product formula $\big[$cf. Claim A.9(ii)$\big]$:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\},$$

we have

$$\begin{aligned}
&\Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{s+1}{2}\Big)\\
&= \lim_{n\to\infty} \frac{n!n^{\frac{s}{2}}}{\frac{s}{2}(\frac{s}{2}+1)\cdots(\frac{s}{2}+n)} \frac{n!n^{\frac{s+1}{2}}}{\frac{s+1}{2}(\frac{s+1}{2}+1)\cdots(\frac{s+1}{2}+n)}\\
&= \lim_{n\to\infty} \frac{2^{n+1}n!n^{\frac{s}{2}}}{s(s+2)\cdots(s+2n)} \frac{2^{n+1}n!n^{\frac{s+1}{2}}}{(s+1)(s+3)\cdots(s+1+2n)}\\
&= \lim_{n\to\infty} \frac{(2n+1)!(2n+1)^s}{s(s+1)(s+2)\cdots(s+2n)(s+2n+1)} \frac{2^{n+1}n!2^{n+1}n!n^{s+\frac{1}{2}}}{(2n+1)!(2n+1)^s}\\
&= \lim_{n\to\infty} \frac{(2n+1)!(2n+1)^s}{s(s+1)\cdots(s+2n+1)} \frac{2^{n+1}n!2^n n!n^{\frac{1}{2}}}{1\cdot 3\cdot 5\cdots(2n+1)\cdot 2\cdot 4\cdots 2n}\Big(\frac{n}{2n+1}\Big)^s 2\\
&= \lim_{n\to\infty} \frac{(2n+1)!(2n+1)^s}{s(s+1)\cdots(s+2n+1)} \frac{n!n^{\frac{1}{2}}}{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)\cdots(\frac{1}{2}+n)}\Big(\frac{1}{2+\frac{1}{n}}\Big)^s 2\\
&= \Gamma(s)\Gamma\Big(\frac{1}{2}\Big)2^{1-s}\\
&= 2^{1-s}\pi^{\frac{1}{2}}\Gamma(s) \quad \big[\because\ \Gamma\big(\tfrac{1}{2}\big) = \sqrt{\pi}^{\dagger 3}\big],
\end{aligned}$$

which is the assertion of 2°.

<u>3°</u> By 1° and 2°, it follows that for $0 < \operatorname{Re} s < 1$,

$$\begin{aligned}
&2\Gamma(1-s)(2\pi)^{s-1}\Big(\sin\frac{\pi}{2}s\Big)\\
&= 2\cdot 2^{1-s-1}\pi^{-\frac{1}{2}}\Gamma\Big(\frac{1-s}{2}\Big)\Gamma\Big(\frac{1-s+1}{2}\Big)(2\pi)^{s-1}\frac{\pi}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}\\
&= 2^{1-s}\cdot 2^{s-1}\pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\\
&= \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}\pi^{-\frac{1-s}{2}}\Gamma\Big(\frac{1-s}{2}\Big),
\end{aligned}$$

and thus

$$\begin{aligned}
\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s) &= \pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)2\Gamma(1-s)(2\pi)^{s-1}\Big(\sin\frac{\pi}{2}s\Big)\zeta(1-s)\\
&= \pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}\pi^{-\frac{1-s}{2}}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s)\\
&= \pi^{-\frac{1-s}{2}}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s).
\end{aligned}$$

†3 By 1°, $\Gamma(\frac{1}{2})^2 = \pi$. Since $\Gamma(\frac{1}{2}) > 0$, we have $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

By the uniqueness theorem, this identity holds on $\mathbb{C} \setminus \{\ldots, -6, -4, -2, 0, 1, 3, 5, \ldots\}$.

$s = -2n$ ($n \in \mathbb{N}$) is a removable singularity of the function of L.H.S., and so is $s = 2n + 1$ ($n \in \mathbb{N}$) of the function of R.H.S. $s = 0$ is a simple pole of the functions of L.H.S. and R.H.S., and so is $s = 1$. Therefore the identity above is valid for $^{\forall}s \in \mathbb{C}$. ■

4.4 No zeros on the line Re $s = 1$

Theorem 4.4 $\zeta(s) \neq 0$ *on the line* Re $s = 1$.

Proof. We divide the proof into three steps:

$\underline{1^{\circ}}$ For $0 < \eta < 1$ and $z \in \mathbb{C}$ with $|z| = 1$, $(1-\eta)^3|1-\eta z|^4|1-\eta z^2|^2 < 1$.

∵ Let $0 < \eta < 1$ and $z = e^{\sqrt{-1}\theta}$ ($\theta \in \mathbb{R}$). It is observed that

$$\begin{aligned}
&|1-\eta z|^4|1-\eta z^2|^2 \\
&= |1-\eta e^{\sqrt{-1}\theta}|^4|1-\eta e^{\sqrt{-1}2\theta}|^2 \\
&= \left(|1-\eta\cos\theta - \sqrt{-1}\eta\sin\theta|^2\right)^2|1-\eta\cos 2\theta - \sqrt{-1}\eta\sin 2\theta|^2 \\
&= \left(1-2\eta\cos\theta+\eta^2\right)^2(1-2\eta\cos 2\theta+\eta^2) \\
&\leq \left(\frac{1}{3}(2(1-2\eta\cos\theta+\eta^2)+1-2\eta\cos 2\theta+\eta^2)\right)^3 \\
&\quad \left[\because \text{ the inequality of the arithmetic and geometric means: } \tfrac{\alpha+\beta+\gamma}{3} \geq (\alpha\beta\gamma)^{\frac{1}{3}} \ (\alpha,\beta,\gamma \geq 0)\right] \\
&= \left(1-\frac{2}{3}\eta(2\cos\theta+\cos 2\theta)+\eta^2\right)^3 \\
&= \left(1+\eta+\eta^2-\frac{\eta}{3}(3+4\cos\theta+2\cos 2\theta)\right)^3 \\
&= \left(1+\eta+\eta^2-\frac{\eta}{3}(4\cos^2\theta+4\cos\theta+1)\right)^3 \\
&= \left(1+\eta+\eta^2-\frac{\eta}{3}(2\cos\theta+1)^2\right)^3 \\
&\leq \left(1+\eta+\eta^2\right)^3 \\
&= \left(\frac{1-\eta^3}{1-\eta}\right)^3 \\
&= \frac{(1-\eta^3)^3}{(1-\eta)^3} \\
&< \frac{1}{(1-\eta)^3},
\end{aligned}$$

which shows the assertion of 1°.

$\underline{2^{\circ}}$ From Claim 4.3, it follows that for $\varepsilon > 0$ and $t \in \mathbb{R}$,

$$\zeta(1+\varepsilon) = \prod_p \frac{1}{1-\frac{1}{p^{1+\varepsilon}}},$$

$$\zeta(1+\varepsilon+\sqrt{-1}t) = \prod_p \frac{1}{1-\frac{1}{p^{1+\varepsilon+\sqrt{-1}t}}} = \prod_p \frac{1}{1-\frac{e^{\sqrt{-1}t\log\frac{1}{p}}}{p^{1+\varepsilon}}},$$

$$\zeta(1+\varepsilon+\sqrt{-1}2t) = \prod_p \frac{1}{1-\frac{e^{\sqrt{-1}2t\log\frac{1}{p}}}{p^{1+\varepsilon}}},$$

and thus

$$\zeta(1+\varepsilon)^3\big|\zeta(1+\varepsilon+\sqrt{-1}t)\big|^4\big|\zeta(1+\varepsilon+\sqrt{-1}2t)\big|^2 = \prod_p \frac{1}{\big(1-\frac{1}{p^{1+\varepsilon}}\big)^3\big|1-\frac{e^{\sqrt{-1}t\log\frac{1}{p}}}{p^{1+\varepsilon}}\big|^4\big|1-\frac{e^{\sqrt{-1}2t\log\frac{1}{p}}}{p^{1+\varepsilon}}\big|^2}.$$

Since letting $\eta = \frac{1}{p^{1+\varepsilon}}$ and $z = e^{\sqrt{-1}t\log\frac{1}{p}}$ in 1° yields that

$$\Big(1-\frac{1}{p^{1+\varepsilon}}\Big)^3\bigg|1-\frac{e^{\sqrt{-1}t\log\frac{1}{p}}}{p^{1+\varepsilon}}\bigg|^4\bigg|1-\frac{(e^{\sqrt{-1}t\log\frac{1}{p}})^2}{p^{1+\varepsilon}}\bigg|^2 < 1,$$

it is seen that

$$\zeta(1+\varepsilon)^3\big|\zeta(1+\varepsilon+\sqrt{-1}t)\big|^4\big|\zeta(1+\varepsilon+\sqrt{-1}2t)\big|^2 > 1.$$

<u>3°</u> Fix $^\forall t \in \mathbb{R}\setminus\{0\}$. Let $a, b \in \mathbb{N}\cup\{0\}$ be

$$a = \min\{n \geq 0; \zeta^{(n)}(1+\sqrt{-1}t) \neq 0\},$$
$$b = \min\{n \geq 0; \zeta^{(n)}(1+\sqrt{-1}2t) \neq 0\}.$$

Then the Taylor expansions of $\zeta(\cdot)$ about $z = 1+\sqrt{-1}t$ and $z = 1+\sqrt{-1}2t$ are

$$\zeta(z) = \sum_{n=a}^{\infty} \frac{\zeta^{(n)}(1+\sqrt{-1}t)}{n!}\big(z-(1+\sqrt{-1}t)\big)^n,$$

$$\zeta(z) = \sum_{n=b}^{\infty} \frac{\zeta^{(n)}(1+\sqrt{-1}2t)}{n!}\big(z-(1+\sqrt{-1}2t)\big)^n,$$

respectively, which give

$$\lim_{\varepsilon\searrow 0}\frac{\zeta(1+\varepsilon+\sqrt{-1}t)}{\varepsilon^a} = \frac{\zeta^{(a)}(1+\sqrt{-1}t)}{a!} \neq 0,$$

$$\lim_{\varepsilon\searrow 0}\frac{\zeta(1+\varepsilon+\sqrt{-1}2t)}{\varepsilon^b} = \frac{\zeta^{(b)}(1+\sqrt{-1}2t)}{b!} \neq 0.$$

Now, by 2°,

$$\Big(\frac{1}{\varepsilon^2}\Big)^{2a+b-\frac{3}{2}} = \varepsilon^3\cdot\Big(\frac{1}{\varepsilon^a}\Big)^4\cdot\Big(\frac{1}{\varepsilon^b}\Big)^2$$
$$< \big(\varepsilon\zeta(1+\varepsilon)\big)^3\bigg|\frac{\zeta(1+\varepsilon+\sqrt{-1}t)}{\varepsilon^a}\bigg|^4\bigg|\frac{\zeta(1+\varepsilon+\sqrt{-1}2t)}{\varepsilon^b}\bigg|^2.$$

Here note that $\zeta(1+\varepsilon) \sim \frac{1}{\varepsilon}$ as $\varepsilon \searrow 0$ $\big[$cf. (4.1)$\big]$. From this and the convergences above, it follows that

$$\infty > \left|\frac{\zeta^{(a)}(1+\sqrt{-1}t)}{a!}\right|^4 \left|\frac{\zeta^{(b)}(1+\sqrt{-1}2t)}{b!}\right|^2 \geq \lim_{\varepsilon \searrow 0}\left(\frac{1}{\varepsilon^2}\right)^{2a+b-\frac{3}{2}}$$
$$= \begin{cases} \infty, & 2a+b > \frac{3}{2}, \\ 0, & 2a+b < \frac{3}{2}, \end{cases}$$

where the case when $2a + b = \frac{3}{2}$ is excluded since $2a + b \in \mathbb{N} \cup \{0\}$, which shows that $2a + b < \frac{3}{2}$. This implies that $a < \frac{3}{4}$ because $b \geq 0$, so that $a = 0$ because $a \in \mathbb{N} \cup \{0\}$. Therefore $\zeta(1 + \sqrt{-1}t) \neq 0$ by definition of a. ■

By Theorem 4.4, $\zeta(s) \neq 0$ on $\{s \in \mathbb{C}; \operatorname{Re} s \geq 1\}$[†4]. Also, by Theorem 4.3(i), together with this, $\zeta(s) \neq 0$ on $\{s \in \mathbb{C}; \operatorname{Re} s = 0\}$. Therefore it turns out that

$$\big\{\text{zeros of } \zeta(\cdot)\big\} \cap \Big(\big\{s \in \mathbb{C}; \operatorname{Re} s \leq 0\big\} \cup \big\{s \in \mathbb{C}; \operatorname{Re} s \geq 1\big\}\Big) = \{-2, -4, -6, \ldots\}.$$

These zeros are called the *trivial zeros*. The zeros in $\{s \in \mathbb{C}; 0 < \operatorname{Re} s < 1\}$ are called the *non-trivial zeros*. The Riemann hypothesis states that

$$\big\{\text{non-trivial zeros}\big\} \subset \big\{s \in \mathbb{C}; \operatorname{Re} s = \tfrac{1}{2}\big\},$$

which remains open.

[†4]Since $s = 1$ is a simple pole of $\zeta(\cdot)$, we understand that $\zeta(1) \neq 0$.