Chapter 6

Well-posedness in the Gevrey classes

6.1 Gevrey well-posedness

We study the same operator

$$P(x,D) = -D_0^2 + \sum_{|\alpha|\le 2, \alpha_0<2} a_\alpha(x)D^\alpha = P_2 + P_1 + P_0$$
$$= -D_0^2 + A_1(x,D')D_0 + A_2(x,D')$$

as in the preceding chapter. As before we assume that p vanishes exactly of order 2 on a C^∞ manifold Σ on which σ has constant rank and p is noneffectively hyperbolic, that is we assume that $\Sigma = \{(x,\xi) \mid p(x,\xi) = 0, dp(x,\xi) = 0\}$ is a C^∞ manifold and (4.1.1) is satisfied.

We assume (5.1.1) but not (5.1.2). Thus the Hamilton flow H_p may touch Σ tangentially. If the Hamilton map really touches Σ tangentially then the Cauchy problem is no more C^∞ well posed even though under the Levi condition (which will be proved in Chapter 8). What is the best we can expect is the well-posedness in much smaller function space, that is in the Gevrey class s with $1 \le s \le 5$ under the Levi condition. We start with the definition of the Gevrey classes.

Definition 6.1.1 *We say $f(x) \in \gamma^{(s)}(\mathbb{R}^n)$, the Gevrey class of order s (≥ 1) if for any compact set $K \subset \mathbb{R}^n$ there exist $C > 0$, $h > 0$ such that*

$$|\partial_x^\alpha f(x)| \le C h^{-|\alpha|} |\alpha|!^s, \quad x \in K, \ \forall \alpha \in \mathbb{N}^n$$

holds. We also set

$$\gamma_0^{(s)}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n) \cap \gamma^{(s)}(\mathbb{R}^n).$$

Theorem 6.1.1 *Assume* (4.1.1), (5.1.1) *and that $P_{sub} = 0$ everywhere on Σ. Then the Cauchy problem for P is well posed in $\gamma^{(s)}$ with $1 \le s \le 5$, that is* *for any* $f(x) \in C^0(\mathbb{R}; \gamma_0^{(s)}(\mathbb{R}^n))$ *vanishing in* $x_0 \le 0$ *there is* $u(x)$ *which is* C^∞*, vanishing in* $x_0 \le 0$ *satisfying*

$$Pu = f$$

near the origin.

The optimality of the index 5 will be discussed in Chapter 8.

6.2 Preliminaries

Let fix $\rho \in \Sigma$ and we work near ρ. Thanks to Proposition 3.3.2, p admits a decomposition verifying the conditions in Proposition 3.3.2. We extend these ϕ_j (given in Proposition 3.3.2) outside a neighborhood of ρ so that they belong to $S(\langle\xi'\rangle, g_0)$ and zero outside another neighborhood of ρ. Using thus extended ϕ_j let us write

$$p = -(\xi_0 + \phi_1)(\xi_0 - \phi_1) + \sum_{j=2}^{r} \phi_j^2$$

which of course coincides with the original p near ρ. Take $0 \le \chi_1(x', \xi') \le 1$, homogeneous of degree 0 in ξ' ($|\xi'| \ge 1$), which is 1 in a conic neighborhood of ρ' where $\rho = (0, \rho')$ and supported in another small conic neighborhood of ρ' on which Proposition 3.3.2 holds. We now define $f(x, \xi')$ solving

$$\{\xi_0 - \phi_1, f\} = 0, \quad f(0, x', \xi') = (1 - \chi_1(x', \xi'))|\xi'|. \tag{6.2.1}$$

Note that $f(x, \xi') = |\xi'|$ outside some neighborhood of ρ' because $\phi_1 = 0$ and $\chi_1 = 0$ outside some neighborhood of ρ'.

Lemma 6.2.1 *Let* $f(x, \xi')$ *be as above. Taking* $M > 0$ *large and* $\tau > 0$ *small we have a decomposition*

$$p = -(\xi_0 + \phi_1)(\xi_0 - \phi_1) + \sum_{j=2}^{r+1} \phi_j^2$$

in $|x_0| < \tau$ *with* $\phi_{r+1} = Mf(x, \xi')$ *such that*

$$\{\xi_0 - \phi_1, \phi_j\} = \sum_{k=1}^{r+1} C_{jk}\phi_k$$

and

$$\{\phi_2, \phi_1\} + |\phi_{r+1}| \ge c|\xi'|$$

with some $c > 0$.

Proof: By a compactness argument there are $c > 0$ and $\tau > 0$ such that we have

$$f(x, \xi') \ge c|\xi'|$$

outside a neighborhood of ρ' if $|x_0| \le \tau$. We may assume that one can write $\{\xi_0 - \phi_1, \phi_j\} = \sum_{j=1}^{r} c_{jk}\phi_k$ in a neighborhood V and $|\phi_{r+1}| \ge c|\xi'|$ outside V. Thus one can write

$$\{\xi_0 - \phi_1, \phi_j\} - \sum_{j=1}^{r} c_{jk}\phi_k = c\phi_{r+1}$$

with a smooth c which proves the first assertion. The second assertion is obvious. $\square$

Lemma 6.2.2 *Assume that P satisfies the Levi condition on Σ. Then P_{sub} can be written*

$$P_{sub} = \sum_{j=0}^{r+1} C_j\phi_j$$

where $\phi_0 = \xi_0$.

Proof: Let $P_{sub} - C_0\xi_0$ be independent of ξ_0. Then in a neighborhood V of ρ', $|x_0| \le \tau$ we can write

$$P_{sub} - C_0\xi_0 = \sum_{j=1}^{r} C_j\phi_j$$

thanks to the Levi condition where $C_j \in S(1, g_0)$. The rest of the proof is just a repetition of the proof of Lemma 6.2.1. $\square$

We now make a dilation of the variable; $x_0 \to \mu x_0$ so that we have

$$\begin{aligned} &\mu^2 p(x, \xi, \mu) \\ &= -(\xi_0 + \phi_1(\mu x_0, x', \mu\xi'))(\xi_0 - \phi_1(\mu x_0, x', \mu\xi')) + \sum_{j=2}^{r+1} \phi_j(\mu x_0, x', \mu\xi')^2. \end{aligned}$$

We simply write $\phi_j(\mu x_0, x', \mu\xi')$ as $\phi_j(x, \xi', \mu)$ or sometimes $\phi_j(x, \xi')$. Let us put

$$g_1 = |dx'|^2 + \langle\xi'\rangle_\mu^{-2}|d\xi'|^2,$$
$$\langle\xi'\rangle_\mu^2 = \mu^{-2} + |\xi'|^2 = \mu^{-2}\langle\mu\xi'\rangle^2.$$

The symbol class $S(m, g_1)$ is defined in Definition 4.2.1 with $C_{\alpha\beta}$ which is independent of μ. Namely by $S(m, g_1)$ we denote the set of all smooth $a(x, \xi', \mu)$ satisfying

$$|\partial_x^\beta \partial_{\xi'}^\alpha a(x, \xi', \mu)| \le C_{\alpha,\beta} m(x, \xi', \mu)\langle\xi'\rangle_\mu^{-|\alpha|}$$

with some $C_{\alpha,\beta}$ independent of $0 < \mu < \mu_0$.

Lemma 6.2.3 *Let $a(x, \xi') \in S(\langle\xi'\rangle^k, g_0)$. Then we have*

$$a(\mu x_0, x', \mu\xi') \in S(\langle\mu\xi'\rangle^k, g_1).$$

Proof: Easy. □

Let us set

$$w=\sqrt{\langle\mu\xi'\rangle^{-2}\phi_1^2+\langle\mu\xi'\rangle^{-4/5}},\quad \Phi=\sqrt{1-aw}. \tag{6.2.2}$$

with some constant $a>0$ so that $1-aw\geq a_1>0$. Let us define

$$g=w^{-2}\big(|dx'|^2+\langle\xi'\rangle_\mu^{-2}|d\xi'|^2\big)=w^{-2}g_1$$

then we have

$$w\in S(w,g),\quad \phi_1\in S(\langle\mu\xi'\rangle w,g).$$

That is w verifies the estimates

$$|\partial_x^\beta\partial_{\xi'}^\alpha w(x,\xi',\mu)|\leq C_{\alpha,\beta}w(x,\xi',\mu)w^{-|\beta|}\big(w\langle\xi'\rangle_\mu\big)^{-|\alpha|}$$

with some $C_{\alpha,\beta}$ independent of $0<\mu<\mu_0$. We rewrite p as

$$p=-(\xi_0+\phi_1\Phi)(\xi_0-\phi_1\Phi)+\sum_{j=2}^{r+1}\phi_j^2+aw\phi_1^2$$

because $1-\Phi^2=aw$. Remark that

$$\xi_0-\phi_1\Phi=\xi_0-\phi_1-\phi_1(\Phi-1)=\xi_0-\phi_1+\phi_1\psi$$

where

$$\psi=1-\Phi=\frac{aw}{1+\sqrt{1-aw}}\in S(w,g).$$

Lemma 6.2.4 *We have*

$$\phi_1\psi\in S(w^2\langle\mu\xi'\rangle,g),\quad \phi_1(1+\psi)\in S(w\langle\mu\xi'\rangle,g),$$
$$\partial_x^\alpha(\phi_1\Phi)\in S(\langle\mu\xi'\rangle,g),\quad |\alpha|=2.$$

Proof: The first two assertions are clear. To check the third assertion it is enough to note that $\partial_x^\alpha\Phi\in S(1,g)$ for $|\alpha|=1$. □

Then Lemma 6.2.1 shows

$$\{\xi_0-\phi_1,\phi_j\}=\sum_{j=1}^{r+1}C_{jk}\phi_k,\ \ C_{jk}\in\mu S(1,g_1),$$

$$\{\phi_2,\phi_1\}+\mu\phi_{r+1}^2\langle\mu\xi'\rangle^{-1}\geq c\mu\langle\mu\xi'\rangle.$$

In what follows we set

$$\kappa=\frac{1}{5},\quad \delta=\frac{4}{5}$$

and assume that $a=1$ without restrictions. Since $w\geq\langle\mu\xi'\rangle^{-\delta/2}$ and hence $w^{-1/2}\leq\langle\mu\xi'\rangle^{\delta/4}=\langle\mu\xi'\rangle^{1/5}=\langle\mu\xi'\rangle^{\kappa}$ so that $w^{-1/2}\in S(\langle\mu\xi'\rangle^\kappa,g)$.

In the next lemma we summarize a few important properties of the calculus of μ dependent pseudodifferential operators introduced above.

Lemma 6.2.5 *Let $a \in S(m_1, g)$ and $b \in S(m_2, g_1)$. Then we have*

$$\begin{aligned} &a\#a - a^2 \in \mu^2 S(m_1^2 w^{-4}\langle\mu\xi'\rangle^{-2}, g),\\ &a\#b - b\#a - \frac{1}{i}\{a, b\} \in \mu^3 S(m_1 m_2 w^{-3}\langle\mu\xi'\rangle^{-3}, g),\\ &a\#b + b\#a - 2ab \in \mu^2 S(m_1 m_2 w^{-2}\langle\mu\xi'\rangle^{-2}, g).\end{aligned}$$

Corollary 6.2.1 *Let $a \in S(m_1, g)$ and $b \in S(m_2, g_1)$ be real. Then we have*

$$([ab]^w u, u) = \mathsf{Re}(b^w u, a^w u) + (T^w u, u)$$

with $T \in \mu^2 S(m_1 m_2 w^{-2}\langle\mu\xi'\rangle^{-2}, g)$.

We now prepare several lemmas which we use in Section 6.4 to derive energy estimates.

Lemma 6.2.6 *Let $a \in \mu S(1, g)$. Then we have*

$$\begin{aligned} &\mathsf{Re}([a\phi_1^2 w]^w u, u) \le C\mu \mathsf{Re}([\phi_1^2 w]^w u, u) + C\mu^3 \|\langle\mu D'\rangle^{\kappa} u\|^2,\\ &\mathsf{Re}([a\phi_j^2]^w u, u) \le C\mu \mathsf{Re}([\phi_j^2]^w u, u) + C\mu^3 \|\langle\mu D'\rangle^{2\kappa} u\|^2, \quad j \ge 2.\end{aligned}$$

Let $a \in \mu S(\langle\mu\xi'\rangle^{\kappa}, g)$. Then we have

$$\begin{aligned} &\mathsf{Re}([a\phi_1^2 w]^w u, u) \le C\mu \mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}\phi_1^2 w]^w u, u) + C\mu^3 \|\langle\mu D'\rangle^{3\kappa/2} u\|^2,\\ &\mathsf{Re}([a\phi_j^2]^w u, u) \le C\mu \mathsf{Re}([\phi_j^2\langle\mu\xi'\rangle^{\kappa}]^w u, u) + C\mu^3 \|\langle\mu D'\rangle^{1/2} u\|^2, \quad j \ge 2.\end{aligned}$$

Let $a \in \mu S(1, g)$ then we have

$$\begin{aligned} &\|[a\langle\mu\xi'\rangle^{\kappa/2}\phi_j]^w u\|^2 \le C\mu^2 \mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}\phi_j^2]^w u, u) + C\mu^4 \|\langle\mu D'\rangle^{1/2} u\|^2,\\ &\|[a\langle\mu\xi'\rangle^{\kappa/2}\sqrt{w}\phi_1]^w u\|^2 \le C\mu^2 \mathsf{Re}([\langle\mu\xi'\rangle^{\kappa} w\phi_1^2]^w u, u) + C\mu^4 \|\langle\mu D'\rangle^{3\kappa/2} u\|^2.\end{aligned}$$

Proof: It is enough to prove the case $a \in \mu S(\langle\mu\xi'\rangle^{\kappa}, g)$. Since $\mathsf{Re}(A^w u, u) = ([\mathsf{Re}\, A]^w u, u)$ we may assume that a is real. Let us consider

$$\begin{aligned} C\mu\langle\mu\xi'\rangle^{\kappa}\phi_1^2 w - a\phi_1^2 w &= C\mu\langle\mu\xi'\rangle^{\kappa}\phi_1^2 w\big(1 - C^{-1}\mu^{-1} a\langle\mu\xi'\rangle^{-\kappa}\big)\\ &= C\mu\langle\mu\xi'\rangle^{\kappa}\phi_1^2 w\psi^2 = C\mu\langle\mu\xi'\rangle^{\kappa/2}\phi_1\sqrt{w}\psi\#\langle\mu\xi'\rangle^{\kappa/2}\phi_1\sqrt{w}\psi\\ &\qquad + \mu^3 S(\langle\mu\xi'\rangle^{3\kappa}, g).\end{aligned}$$

Hence we have

$$C\mu \mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}\phi_1^2 w]^w u, u) - \mathsf{Re}([a\phi_1^2 w]^w u, u) \ge -C\mu^3 \|\langle\mu D'\rangle^{3\kappa/2} u\|^2$$

which shows the first assertion. From Corollary 6.2.1 we have

$$\begin{aligned} ([a\phi_j^2]^w u, u) = \mathsf{Re}([\mu^{1/2}\langle\mu\xi'\rangle^{\kappa/2}\phi_j]^w u, [\mu^{-1/2}\langle\mu\xi'\rangle^{-\kappa/2} a\phi_j]^w u)\\ + (T^w u, u)\end{aligned}$$

with $T \in \mu^3 S(\langle\mu\xi'\rangle^\kappa w^{-2}, g) \subset \mu^3 S(\langle\mu\xi'\rangle, g)$. Since one can write

$$\mu^{-1/2}\langle\mu\xi'\rangle^{-\kappa/2} a\phi_j = \mu^{-1/2} a\langle\mu\xi'\rangle^{-\kappa}\#\langle\mu\xi'\rangle^{\kappa/2}\phi_j + \mu^{3/2} S(\langle\mu\xi'\rangle^{\kappa/2} w^{-1}, g)$$

and $S(\langle\mu\xi'\rangle^{\kappa/2} w^{-1}, g) \subset S(\langle\mu\xi'\rangle^{1/2}, g)$ we have

$$\|[\mu^{-1/2}\langle\mu\xi'\rangle^{-\kappa/2} a\phi_j]^w u\|^2 \le C\mu\|[\langle\mu\xi'\rangle^{\kappa/2}\phi_j]^w u\|^2 + C\mu^3\|\langle\mu D'\rangle^{1/2} u\|^2.$$

This proves the assertion.

We turn to the next assertion. Note that

$$a\langle\mu\xi'\rangle^{\kappa/2}\phi_j = a\#\langle\mu\xi'\rangle^{\kappa/2}\phi_j + \mu^2 S(\langle\mu\xi'\rangle^{1/2}, g)$$

and hence

$$\|[a\langle\mu\xi'\rangle^{\kappa/2}\phi_j]^w u\|^2 \le C\mu^2\|[\langle\mu\xi'\rangle^{\kappa/2}\phi_j]^w u\|^2 + C\mu^4\|\langle\mu D'\rangle^{1/2} u\|^2.$$

Since $\langle\mu\xi'\rangle^{\kappa/2}\phi_j\#\langle\mu\xi'\rangle^{\kappa/2}\phi_j = \langle\mu\xi'\rangle^\kappa\phi_j^2 + \mu^2 S(\langle\mu\xi'\rangle^\kappa, g)$ we apply the assertion just proven to get the assertion. Finally we note

$$a\sqrt{w}\langle\mu\xi'\rangle^{\kappa/2}\phi_1\# a\langle\mu\xi'\rangle^{\kappa/2}\phi_1 = a^2 w\langle\mu\xi'\rangle^\kappa\phi_1^2 + \mu^4 S(\langle\mu\xi'\rangle^{3\kappa}, g)$$

which, together with the above assertion, proves the desired assertion. □

Lemma 6.2.7 *We have*

$$c([\mu\langle\mu\xi'\rangle^{1+\kappa}\sqrt{w}]^w u, u) \le \mathsf{Re}([\langle\mu\xi'\rangle^\kappa\phi_1^2 w]^w u, u)$$
$$+\mathsf{Re}([\langle\mu\xi'\rangle^\kappa\phi_2^2]^w u, u) + \mathsf{Re}([\phi_{r+1}^2]^w u, u) + C\mu\|\langle\mu D'\rangle^{3\kappa/2} u\|^2$$

with some $c > 0$.

Proof: Put $A = \langle\mu\xi'\rangle^{\kappa/2}\phi_1\sqrt{w}$, $B = \langle\mu\xi'\rangle^{\kappa/2}\phi_2$ and note that

$$A \in S(\langle\mu\xi'\rangle^{1+\kappa/2} w^{3/2}, g), \quad B \in S(\langle\mu\xi\rangle^{1+\kappa/2}, g_1).$$

Note that $|([A^w, B^w]u, u)| \le ([A\#A]^w u, u) + ([B\#B]^w u, u)$ and recall

$$[A^w, B^w] = \frac{1}{i}\{A, B\}^w + \mu^3 S(\langle\mu\xi'\rangle^{-1+\kappa} w^{-3/2}, g).$$

Thus we have

$$i[A^w, B^w] = \{A, B\}^w + \mu^3 S(\langle\mu\xi'\rangle^{-\kappa}, g).$$

Let us consider the first term in the right-hand side

$$\{A, B\} = \{\phi_1\sqrt{w}, \langle\mu\xi'\rangle^{\kappa/2}\}\phi_2\langle\mu\xi'\rangle^{\kappa/2} + \{\langle\mu\xi'\rangle^{\kappa/2}, \phi_2\}\sqrt{w}\phi_1\langle\mu\xi'\rangle^{\kappa/2}$$
$$+\{\sqrt{w}, \phi_2\}\phi_1\langle\mu\xi'\rangle^\kappa + \{\phi_1, \phi_2\}\sqrt{w}\langle\mu\xi'\rangle^\kappa = K_1 + K_2 + K_3 + K_4.$$

Since we have

$$\{\phi_1\sqrt{w}, \langle\mu\xi'\rangle^{\kappa/2}\} \in \mu S(\langle\mu\xi'\rangle^{\kappa/2}\sqrt{w}, g), \ \ \{\langle\mu\xi'\rangle^{\kappa/2}, \phi_2\} \in \mu S(\langle\mu\xi'\rangle^{\kappa/2}, g_1)$$

we see that the first and the second term can be written as

$$T_1\#B+R_1,\quad T_2\#A+R_2$$

where $T_i\in\mu S(\langle\mu\xi'\rangle^{\kappa/2},g)$ and $R_i\in\mu^2S(\langle\mu\xi'\rangle^{2\kappa},g)$. Then we have

$$\begin{aligned}\mathsf{Re}(K_1u,u)\geq -C\mu\big(\|\langle\mu D'\rangle^{\kappa/2}u\|^2+\|B^wu\|^2\big)-C\mu^2\|\langle\mu D'\rangle^{\kappa}u\|^2,\\ \mathsf{Re}(K_2u,u)\geq -C\mu\big(\|\langle\mu D'\rangle^{\kappa/2}u\|^2+\|A^wu\|^2\big)-C\mu^2\|\langle\mu D'\rangle^{\kappa}u\|^2.\end{aligned}$$

Consider K_3. Note that

$$\{\sqrt{w},\phi_2\}=\frac{1}{4}w^{-3/2}\big(2\{\phi_1,\phi_2\}\phi_1\langle\mu\xi'\rangle^{-2}+\{\langle\mu\xi'\rangle^{-2},\phi_2\}\phi_1^2+\{\langle\mu\xi'\rangle^{-\delta},\phi_2\}\big).$$

Since $w^{-5/2}\{\phi_1,\phi_2\}\langle\mu\xi'\rangle^{-2}\in\mu S(1,g)$ and $w^{-5/2}\{\langle\mu\xi'\rangle^{-2},\phi_2\}\phi_1\in\mu S(1,g)$ and $w^{-3/2}\{\langle\mu\xi'\rangle^{-\delta},\phi_2\}\in\mu S(w^{-3/2}\langle\mu\xi'\rangle^{-\delta},g)$ one can write

$$K_3=a\phi_1^2w\langle\mu\xi'\rangle^{\kappa}+b\phi_1$$

with $a\in\mu S(1,g)$, $b\in\mu S(1,g)$. We first consider $\mathsf{Re}\,([b\phi_1]^wu,u)$. Note that

$$2\mathsf{Re}([b\phi_1]^wu,u)\geq -\mu^{-1}\|[\langle\mu\xi'\rangle^{-\kappa/2}\#b\phi_1]^wu\|^2-\mu\|\langle\mu D'\rangle^{-\kappa/2}u\|^2$$

and $\langle\mu\xi'\rangle^{-\kappa/2}\#b\phi_1=b\langle\mu\xi'\rangle^{-\kappa/2}\phi_1+\mu^2S(\langle\mu\xi'\rangle^{-\kappa/2},g)$. Remarking that

$$b\langle\mu\xi'\rangle^{-\kappa/2}\#b\langle\mu\xi'\rangle^{-\kappa/2}=b^2\langle\mu\xi'\rangle^{-\kappa}\phi_1^2+\mu^4S(\langle\mu\xi'\rangle^{3\kappa},g)$$

and $b^2\langle\mu\xi'\rangle^{-\kappa}\phi_1^2=a\langle\mu\xi'\rangle^{\kappa}w\phi_1^2$ with $a\in\mu^2S(1,g)$ we conclude that, applying Lemma 6.2.6

$$2\mathsf{Re}([b\phi_1]^wu,u)\geq -C\mu\mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}w\phi_1^2]^wu,u)-C\mu\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.$$

From Lemma 6.2.6 again we see that

$$\mathsf{Re}(K_3^wu,u)\geq -C\mu\mathsf{Re}([\phi_1^2w\langle\mu\xi'\rangle^{\kappa}]^wu,u)-C\mu\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.$$

We turn to K_4. From the assumption we may assume that

$$C\mu\langle\mu\xi'\rangle\geq\{\phi_1,\phi_2\}+\mu\phi_{r+1}^2\langle\mu\xi'\rangle^{-1}=T\geq c\mu\langle\mu\xi'\rangle$$

with some $C>0$, $c>0$. Let us write

$$\begin{aligned}C(K_4+\mu\phi_{r+1}^2\sqrt{w}\langle\mu\xi'\rangle^{-1+\kappa})-\mu\sqrt{w}\langle\mu\xi'\rangle^{1+\kappa}\\ =C\sqrt{w}\langle\mu\xi'\rangle^{\kappa}T\big(1-C^{-1}\mu T^{-1}\langle\mu\xi'\rangle\big)\\ =\psi\#\psi+\mu^3S(\langle\mu\xi'\rangle^{3\kappa},g)\end{aligned}$$

from which we see that

$$\begin{aligned}C\mathsf{Re}(K_4^wu,u)\geq\mathsf{Re}([\mu\sqrt{w}\langle\mu\xi'\rangle^{1+\kappa}]^wu,u)\\ -C\mu\mathsf{Re}((\sqrt{w}\phi_{r+1}^2\langle\mu\xi'\rangle^{-1+\kappa})^wu,u)-C\mu^3\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.\end{aligned}$$

The above estimates show

$$\begin{aligned}\mathsf{Re}(\{A,B\}^w u,u) &\geq c\mathsf{Re}([\mu\sqrt{w}\langle\mu\xi'\rangle^{1+\kappa}]^w u,u)\\ &-C\mu\big(\|A^w u\|^2+\|B^w u\|^2+\|\langle\mu D'\rangle^{3\kappa/2}u\|^2\big)\\ &-C\mu\mathsf{Re}([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u,u)-C\mu\|[\phi_{r+1}^2]^w u\|^2\end{aligned}$$

because $\phi_{r+1}^2\sqrt{w}\langle\mu\xi'\rangle^{-1+\kappa} = \phi_{r+1}\#(\phi_{r+1}\sqrt{w}\langle\mu\xi'\rangle^{-1+\kappa}) + \mu^{4/5}S(\langle\mu\xi'\rangle^{2\kappa},g)$. Let us turn to $A\#A$, $B\#B$. Since $A\in S(w^{3/2}\langle\mu\xi'\rangle^{1+\kappa/2},g)$, $B\in S(\langle\mu\xi'\rangle^{1+\kappa/2},g_1)$ we see

$$\begin{aligned}A\#A &= w\phi_1^2\langle\mu\xi'\rangle^{\kappa}+\mu^2 S(\langle\mu\xi'\rangle^{3\kappa},g),\\ B\#B &= \phi_2^2\langle\mu\xi'\rangle^{\kappa}+\mu^2 S(\langle\mu\xi'\rangle^{\kappa},g_1)\end{aligned}$$

and hence

$$\begin{aligned}\|A^w u\|^2 &= ([A\#A]^w u,u)\leq \mathsf{Re}([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u,u)+C\mu^2\|\langle\mu D'\rangle^{3\kappa/2}u\|^2,\\ \|B^w u\|^2 &= ([B\#B]^w u,u,u)\leq \mathsf{Re}([\phi_2^2\langle\mu\xi'\rangle^{\kappa}]^w u,u)+C\mu^2\|\langle\mu D'\rangle^{\kappa/2}u\|^2.\end{aligned}$$

These prove the assertion. □

Corollary 6.2.2 *We have*

$$\begin{aligned}&\mu\|\langle\mu D'\rangle^{1/2}u\|^2\leq C\mathsf{Re}([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u,u)\\ &+C\mathsf{Re}([\phi_2^2\langle\mu\xi'\rangle^{\kappa}]^w u,u)+C\mathsf{Re}([\phi_{r+1}^2]^w u,u)+C\mu\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.\end{aligned}$$

Proof: Write

$$\begin{aligned}&C\mu\langle\mu\xi'\rangle^{1+\kappa}\sqrt{w}-\mu\langle\mu\xi'\rangle\\ &=C\mu\langle\mu\xi'\rangle^{1+\kappa}\sqrt{w}\big(1-C^{-1}w^{-1/2}\langle\mu\xi'\rangle^{-\kappa}\big)\\ &=C\mu\psi\#\psi+\mu^3 S(\langle\mu\xi'\rangle^{3\kappa},g)\end{aligned}$$

with $\psi=\langle\mu\xi'\rangle^{(1+\kappa)/2}w^{1/4}(1-C^{-1}w^{-1/2}\langle\mu\xi'\rangle^{-\kappa})^{1/2}$and hence

$$\begin{aligned}&C\mu\mathsf{Re}([\langle\mu\xi'\rangle^{1+\kappa}\sqrt{w}]^w u,u)-\mu\|\langle\mu D'\rangle^{1/2}u\|^2\\ &\geq -C\mu^3\|\langle\mu\xi'\rangle^{3\kappa/2}u\|^2\end{aligned}$$

which proves the assertion. □

Lemma 6.2.8 *Let $j\neq 1$ and $a\in\mu S(1,g)$. Then we have*

$$\begin{aligned}&\mathsf{Re}([a\phi_1\phi_j]^w u,u)\leq C\mu\mathsf{Re}([\phi_j^2\langle\mu\xi'\rangle^{\kappa}]^w u,u)\\ &+C\mu\mathsf{Re}([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u,u)+C\mu^3\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.\end{aligned}$$

Proof: We may assume that a is real. Consider

$$a\phi_1\phi_j=\mathsf{Re}(\mu^{1/2}\langle\mu\xi'\rangle^{\kappa/2}\phi_j\#\mu^{-1/2}a\langle\mu\xi'\rangle^{-\kappa/2}\phi_1)+\mu^3 S(\langle\mu\xi'\rangle^{2\kappa},g).$$

Since $\mu^{1/2}\langle\mu\xi'\rangle^{\kappa/2}\phi_j\#\mu^{1/2}\langle\mu\xi'\rangle^{\kappa/2}\phi_j=\mu\langle\mu\xi'\rangle^{\kappa}\phi_j^2+\mu^3S(\langle\mu\xi'\rangle^{\kappa},g_1)$ and

$$\mu^{-1/2}a\langle\mu\xi'\rangle^{-\kappa/2}\phi_1\#\mu^{-1/2}a\langle\mu\xi'\rangle^{-\kappa/2}\phi_1$$
$$=(\mu^{-1}a^2w^{-1}\langle\mu\xi'\rangle^{-2\kappa})\phi_1^2w\langle\mu\xi'\rangle^{\kappa}+\mu^3S(\langle\mu\xi'\rangle^{3\kappa},g)$$

we see that, noting $\mu^{-1}a^2w^{-1}\langle\mu\xi'\rangle^{-2\kappa}\in\mu S(1,g)$,

$$\mathsf{Re}([a\phi_1\phi_j]^wu,u)\le\frac{1}{2}\big(\mu\|[\phi_j\langle\mu\xi'\rangle^{\kappa/2}]^wu\|^2+\mu^{-1}\|[a\langle\mu\xi'\rangle^{-\kappa/2}\phi_1]^wu\|^2\big)$$
$$+C\mu^3\|\langle\mu D'\rangle^{\kappa}u\|^2\le C\mu\mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}\phi_j^2]^wu,u)$$
$$+C\mu\mathsf{Re}([\phi_1^2w\langle\mu\xi'\rangle^{\kappa}]^wu,u)+C\mu^3\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.$$

This is the assertion. □

Lemma 6.2.9 *Let $a\in\mu S(\langle\mu\xi'\rangle w,g)$. Then for $j\neq 1$ we have*

$$\mathsf{Re}([a\phi_j]^wu,u)\le C\mu^{1/2}\mathsf{Re}([\phi_j^2\langle\mu\xi'\rangle^{\kappa}]^wu,u)$$
$$+C\mu^{1/2}\mathsf{Re}([\phi_1^2w\langle\mu\xi'\rangle^{\kappa}]^wu,u)+C\mu^{1/2}\mathsf{Re}([\phi_2^2\langle\mu\xi'\rangle^{\kappa}]^wu,u)$$
$$+C\mu^{1/2}\mathsf{Re}([\phi_{r+1}^2]^wu,u)+C\mu^{3/2}\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.$$

Proof: Let us write

$$a\phi_j=\mathsf{Re}(\mu^{1/4}\langle\mu\xi'\rangle^{\kappa/2}\phi_j\#\mu^{-1/4}\langle\mu\xi'\rangle^{-\kappa/2}a)+\mu^3S(\langle\mu\xi'\rangle^{2\kappa},g)$$

and hence

$$\mathsf{Re}([a\phi_j]^wu,u)\le\mu^{1/2}\|[\langle\mu\xi'\rangle^{\kappa/2}\phi_j]^wu\|^2$$
$$+\mu^{-1/2}\|[\langle\mu\xi'\rangle^{-\kappa/2}a]^wu\|^2+C\mu^3\|\langle\mu D'\rangle^{\kappa}u\|^2.$$

Note that $\langle\mu\xi'\rangle^{-\kappa/2}a\#\langle\mu\xi'\rangle^{-\kappa/2}a=\langle\mu\xi'\rangle^{-\kappa}a^2+\mu^4S(\langle\mu\xi'\rangle^{3\kappa},g)$ and write

$$\langle\mu\xi'\rangle^{-\kappa}a^2=(w^{-2}a^2\langle\mu\xi'\rangle^{-2})w^2\langle\mu\xi'\rangle^{2-\kappa}$$
$$=b(\langle\mu\xi'\rangle^{-2}\phi_1^2+\langle\mu\xi'\rangle^{-\delta})\langle\mu\xi'\rangle^{2-\kappa}$$
$$=b\langle\mu\xi'\rangle^{-2\kappa}w^{-1}(\phi_1^2w\langle\mu\xi'\rangle^{\kappa})+b\langle\mu\xi'\rangle^{2-\delta-\kappa}$$

where $b=w^{-2}a^2\langle\mu\xi'\rangle^{-2}\in\mu^2S(1,g)$. Since $2-\delta-\kappa=1$ thanks to Lemma 6.2.6 we get

$$\mu^{-1/2}\|[\langle\mu\xi'\rangle^{-\kappa/2}a]^wu\|^2\le C\mu^{3/2}\mathsf{Re}([\phi_1^2w\langle\mu\xi'\rangle^{\kappa}]^wu,u)$$
$$+C\mu^{5/2}\|\langle\mu D'\rangle^{3\kappa/2}u\|^2+C\mu^{3/2}\|\langle\mu D'\rangle^{1/2}u\|^2.$$

Then applying Corollary 6.2.2 we get the assertion. □

We now estimate $\{\xi_0-\phi_1\Phi,\phi_j^2\}$, $\{\xi_0-\phi_1\Phi,w\phi_1^2\}$. Recall that

$$\{\xi_0-\phi_1\Phi,\phi_j^2\}=\{\xi_0-\phi_1,\phi_j^2\}+\{\phi_1\psi,\phi_j^2\}.$$

From Lemma 6.2.1 we have

$$\{\xi_0-\phi_1,\phi_j^2\}=2\{\xi_0-\phi_1,\phi_j\}\phi_j=\sum_{k=1}^{r+1}C_{jk}\phi_k\phi_j$$

where $C_{jk}\in\mu S(1,g_1)$. Note that for $j,k\geq 2$ one has

$$\begin{aligned}\mathsf{Re}([C_{jk}\phi_k\phi_j]^w u,u)&\leq C\mu\mathsf{Re}([\phi_k^2]^w u,u)\\&+C\mu\mathsf{Re}([\phi_j^2]^w u,u)+C\mu^3\|u\|^2.\end{aligned}$$

For $C_{j1}\phi_1\phi_j$ we apply Lemma 6.2.8 to get

$$\begin{aligned}\mathsf{Re}([C_{j1}\phi_1\phi_j]^w u,u)&\leq C\mu\mathsf{Re}([\phi_j^2\langle\mu\xi'\rangle^{\kappa}]^w u,u)\\+C\mu\mathsf{Re}([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u,u)&+C\mu^3\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.\end{aligned}$$

Let us consider for $j\geq 2$

$$\{\phi_1\psi,\phi_j^2\}=2\{\phi_1,\phi_j\}\phi_j\psi+2\{\psi,\phi_j\}\phi_j\phi_1.$$

Write

$$\{\phi_1,\phi_j\}\phi_j\psi=\mathsf{Re}(\langle\mu\xi'\rangle^{\kappa/2}\phi_j\#\{\phi_1,\phi_j\}\psi\langle\mu\xi'\rangle^{-\kappa/2})+\mu^2S(\langle\mu\xi'\rangle^{2\kappa},g)$$

and note that

$$\begin{aligned}\{\phi_1,\phi_j\}^2\psi^2\langle\mu\xi'\rangle^{-\kappa}&=(\{\phi_1,\phi_j\}^2\psi^2\langle\mu\xi'\rangle^{-2}w^{-2})w^2\langle\mu\xi'\rangle^{2-\kappa}\\&=Tw^2\langle\mu\xi'\rangle^{2-\kappa}=T(\langle\mu\xi'\rangle^{-2}\phi_1^2+\langle\mu\xi'\rangle^{-\delta})\langle\mu\xi'\rangle^{2-\kappa}\\&=(Tw^{-1}\langle\mu\xi'\rangle^{-2\kappa})w\langle\mu\xi'\rangle^{\kappa}\phi_1^2+T\langle\mu\xi'\rangle^{2-\delta-\kappa}\end{aligned}$$

with $T=\{\phi_1,\phi_j\}^2\psi^2\langle\mu\xi'\rangle^{-2}w^{-2}$ and hence we have $Tw^{-1}\langle\mu\xi'\rangle^{-2\kappa}\in\mu^2S(1,g)$, $T\langle\mu\xi'\rangle^{2-\delta-\kappa}\in\mu^2S(\langle\mu\xi'\rangle,g)$. We now apply Lemma 6.2.8 and Corollary 6.2.2 to get the desired estimate.

We turn to estimate $\{\xi_0-\phi_1\Phi,w\phi_1^2\}$. Consider

$$\{\xi_0-\phi_1,w\phi_1^2\}=2\{\xi_0-\phi_1,\phi_1\}\phi_1 w+\{\xi_0-\phi_1,w\}\phi_1^2.$$

For the first term of the right-hand side we remark that

$$\{\xi_0-\phi_1,\phi_1\}=\sum_{k=1}^{r+1}C_{1k}\phi_k$$

and apply Lemma 6.2.8 and Lemma 6.2.6 to get the estimates. Let us study

$$\begin{aligned}\{\xi_0-\phi_1,w\}&=\frac{1}{2}w^{-1}\{\xi_0-\phi_1,\langle\mu\xi'\rangle^{-2}\phi_1^2+\langle\mu\xi'\rangle^{-\delta}\}\\=-\frac{1}{2}w^{-1}\{\phi_1,\langle\mu\xi'\rangle^{-2}\}\phi_1^2&+w^{-1}\{\xi_0-\phi_1,\phi_1\}\phi_1\langle\mu\xi'\rangle^{-2}\\&-\frac{1}{2}w^{-1}\{\phi_1,\langle\mu\xi'\rangle^{-\delta}\}.\end{aligned}$$

Note that $w^{-1}\{\phi_1, \langle\mu\xi'\rangle^{-2}\}\phi_1^2$, $w^{-1}\{\phi_1, \langle\mu\xi'\rangle^{-\delta}\} \in \mu S(w,g)$ and apply Lemma 6.2.6 to $w^{-1}\{\phi_1, \langle\mu\xi'\rangle^{-2}\}\phi_1^4$ and $w^{-1}\{\phi_1, \langle\mu\xi'\rangle^{-\delta}\}\phi_1^2$ to obtain the estimates. As for the second term on the right-hand side it is enough to note that

$$w^{-1}\{\xi_0 - \phi_1, \phi_1\}\phi_1^3\langle\mu\xi'\rangle^{-2} = \sum_{k=1}^{r} T_k\phi_k\phi_1$$

with $T_k \in \mu S(w,g)$. We finally consider

$$\{\phi_1\psi, \phi_1^2 w\} = \{\phi_1, w\}\phi_1^2\psi + \{\psi, w\}\phi_1^3 + 2\{\psi, \phi_1\}\phi_1^2 w.$$

Note that $\{\phi_1, w\}$, $\{\psi, \phi_1\} \in \mu S(1,g)$ and apply Lemma 6.2.6. Recall that

$$\{w, \psi\} = \frac{1}{2}w^{-1}\big(2\{\phi_1, \psi\}\phi_1\langle\mu\xi'\rangle^{-2} + \{\langle\mu\xi'\rangle^{-2}, \psi\}\phi_1^2 + \{\langle\mu\xi'\rangle^{-\delta}, \psi\}\big).$$

Hence one can write $\{w, \psi\}\phi_1^3 = Tw\phi_1^2$ with $T \in \mu S(1,g)$ we apply Lemma 6.2.8 and Lemma 6.2.6 again to obtain the desired estimate.

Proposition 6.2.1 *We have*

$$\sum_{j=2}^{r+1}\big|\mathsf{Re}(\{\xi_0 - \phi_1\Phi, \phi_j^2\}^w u, u)\big|,\ \big|\mathsf{Re}(\{\xi_0 - \phi_1\Phi, w\phi_1^2\}^w u, u)\big|$$
$$\leq C\mu\Big(\sum_{j=2}^{r+1}\mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}\phi_j^2]^w u, u) + \mathsf{Re}([w\langle\mu\xi'\rangle^{\kappa}\phi_1^2]^w u, u)\Big)$$
$$+C\mu^2\|\langle\mu D'\rangle^{1/2}u\|^2.$$

6.3 A lemma on composition

In this section, to simplify notations we use ξ, x instead of ξ' and x'. Let us consider

$$e^{\phi(D,\mu)}b^w(x, D, \mu)e^{-\phi(D,\mu)}.$$

Since

$$e^{\phi(D,\mu)}b^w(x, D, \mu)v = \int e^{i(x\xi - z\xi + (z-y)\eta)}e^{\phi(\xi,\mu)}$$
$$\times b(\frac{z+y}{2}, \eta, \mu)v(y)dyd\eta dzd\xi$$

inserting

$$e^{-\phi(D,\mu)}u(y) = \int e^{iy\zeta - \phi(\zeta,\mu)}\hat{u}(\zeta)d\zeta$$

into v we have

$$e^{\phi(D,\mu)}b^w(x, D, \mu)e^{-\phi(D,\mu)}u = \int e^{ix\zeta}I(x, \zeta, \mu)\hat{u}(\zeta)d\zeta$$

where

$$I=\int e^{i(x\xi-z\xi+(z-y)\eta+y\zeta-x\zeta)}e^{\phi(\xi,\mu)}b(\frac{z+y}{2},\eta,\mu)e^{-\phi(\zeta,\mu)}dyd\eta dzd\xi.$$

Let us consider $I(x,\zeta,\mu)$. Make a change of variables such that

$$\tilde{z}=(y+z)/2,\quad \tilde{y}=(y-z)/2$$

and hence we have

$$\begin{aligned}I&=2^n\int e^{i(-\tilde{z}(\xi-\zeta)+\tilde{y}(\xi-2\eta+\zeta)+x(\xi-\zeta))}e^{\phi(\xi,\mu)}b(\tilde{z},\eta,\mu)e^{-\phi(\zeta,\mu)}d\tilde{y}d\eta d\tilde{z}d\xi\\&=2^n\int e^{i\tilde{y}(\xi-2\eta+\zeta)}d\tilde{y}\int e^{-i(\tilde{z}-x)(\xi-\zeta)}e^{\phi(\xi,\mu)}b(\tilde{z},\eta,\mu)e^{-\phi(\zeta,\mu)}d\eta d\tilde{z}d\xi\\&=2^n\int e^{-2i(\tilde{z}-x)(\eta-\zeta)}e^{\phi(2\eta-\zeta,\mu)}b(\tilde{z},\eta,\mu)e^{-\phi(\zeta,\mu)}d\eta d\tilde{z}\\&=\int e^{-i\tilde{z}\eta}e^{\phi(\sqrt{2}\eta+\zeta,\mu)-\phi(\zeta,\mu)}b(x+\frac{\tilde{z}}{\sqrt{2}},\zeta+\frac{\eta}{\sqrt{2}},\mu)d\eta d\tilde{z}.\end{aligned}$$

Thus we conclude that

$$\begin{aligned}e^{\phi(D,\mu)}b^w(x,D,\mu)e^{-\phi(D,\mu)}u&=\int e^{i(x-y)\xi}a(x,\xi,\mu)u(y)dyd\xi\\&=\int e^{i(x-y)\xi}a(x,\xi,\mu)u(y)dyd\xi=a(x,D,\mu)u\end{aligned}$$

with

$$a(x,\xi,\mu)=\int e^{-iy\eta}e^{\phi(\xi+\sqrt{2}\eta,\mu)-\phi(\xi,\mu)}b(x+\frac{y}{\sqrt{2}},\xi+\frac{\eta}{\sqrt{2}},\mu)dyd\eta.$$

Here we remark that with

$$q(x,\xi,\mu)=\int e^{iz\zeta}p(x+\frac{z}{\sqrt{2}},\xi+\frac{\zeta}{\sqrt{2}},\mu)dzd\zeta$$

we have

$$q^w(x,D,\mu)=p(x,D,\mu). \tag{6.3.1}$$

Indeed we see

$$\begin{aligned}q^w(x,D,\mu)u&=\int e^{i(x-y)\xi}q(\frac{x+y}{2},\xi,\mu)u(y)dyd\xi\\&=\int e^{i(x\xi-y\xi+z\zeta)}p(\frac{x+y}{2}+\frac{z}{\sqrt{2}},\xi+\frac{\zeta}{\sqrt{2}},\mu)u(y)dyd\xi dzd\zeta\\&=\int e^{i((x-y-z)\xi+z\zeta)}p(\frac{x+y+z}{2},\zeta,\mu)u(y)dyd\xi dzd\zeta\\&=\int e^{iz\zeta}p(x,\zeta,\mu)u(x-z)dzd\zeta\\&=\int e^{i(x-z)\zeta}p(x,\zeta,\mu)u(z)dzd\zeta=p(x,D,\mu)u.\end{aligned}$$

From (6.3.1) it follows that

$$c^w(x, D, \mu) = a(x, D, \mu)$$

with

$$c(x, \xi, \mu) = \int e^{iz\zeta} a(x + \frac{z}{\sqrt{2}}, \xi + \frac{\zeta}{\sqrt{2}}, \mu) dz d\zeta.$$

Insert the expression of $a(x, \xi, \mu)$ into the above formula to get

$$c(x, \xi, \mu) = \int e^{i(z\zeta - y\eta)} e^{\phi(\sqrt{2}\eta + \xi + \frac{\zeta}{\sqrt{2}}, \mu) - \phi(\xi + \frac{\zeta}{\sqrt{2}}, \mu)}$$
$$\times b(x + \frac{z+y}{\sqrt{2}}, \xi + \frac{\eta + \zeta}{\sqrt{2}}, \mu) dy d\eta dz d\zeta.$$

Make a change of variables

$$\tilde{z} = \frac{z+y}{\sqrt{2}}, \quad \tilde{y} = \frac{y-z}{\sqrt{2}}, \quad \tilde{\zeta} = \frac{\zeta+\eta}{\sqrt{2}}, \quad \tilde{\eta} = \frac{\eta-\zeta}{\sqrt{2}}$$

to get

$$\begin{aligned} c(x, \xi, \mu) &= \int e^{-i(\tilde{z}\tilde{\eta} + \tilde{y}\tilde{\zeta})} e^{\phi(\frac{3\tilde{\zeta}}{2} + \xi + \frac{\tilde{\eta}}{2}, \mu) - \phi(\xi + \frac{\tilde{\zeta}}{2} - \frac{\tilde{\eta}}{2}, \mu)} \\ &\qquad \times b(x + \tilde{z}, \xi + \tilde{\zeta}, \mu) d\tilde{y} d\tilde{\eta} d\tilde{z} d\tilde{\zeta} \\ &= \int e^{-i\tilde{z}\tilde{\eta}} e^{\phi(\xi + \frac{\tilde{\eta}}{2}, \mu) - \phi(\xi - \frac{\tilde{\eta}}{2}, \mu)} b(x + \tilde{z}, \xi, \mu) d\tilde{z} d\tilde{\eta}. \end{aligned}$$

Then we obtain

Lemma 6.3.1 *We have*

$$e^{\phi(D,\mu)} b^w(x, D, \mu) e^{-\phi(D,\mu)} = c^w(x, D, \mu)$$

where

$$c(x, \xi, \mu) = \int e^{-iy\eta} e^{\phi(\xi + \frac{\eta}{2}, \mu) - \phi(\xi - \frac{\eta}{2}, \mu)} b(x + y, \xi, \mu) dy d\eta.$$

Here we recall

$$\begin{aligned} \langle \xi' \rangle_\mu^2 &= \mu^{-2} + |\xi'|^2 = \mu^{-2} \langle \mu\xi' \rangle^2, \\ w &= \sqrt{\langle \mu\xi' \rangle^{-2} \phi_1^2 + \langle \mu\xi' \rangle^{-4/5}}. \end{aligned}$$

Let $0 < \sigma < 1$ and we define the metric $\bar{g}$

$$\bar{g} = \langle \mu\xi \rangle^\sigma \big(|dx|^2 + \langle \xi \rangle_\mu^{-2} |d\xi|^2 \big)$$

and recall

$$g = w(x, \xi, \mu)^{-2} \big(|dx|^2 + \langle \xi \rangle_\mu^{-2} |d\xi|^2 \big).$$

Definition 6.3.1 *We say $b(x,\xi,\mu) \in \gamma^{(s)}S(m(x,\xi,\mu),\bar{g})$ if $b(x,\xi,\mu)$ verifies the following estimates*

$$|\partial_x^\alpha \partial_\xi^\beta b(x,\xi,\mu)| \le C_\beta m(x,\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\beta|} \times A^{|\alpha|}|\alpha|!^{s/2}(|\alpha|^{s/2}+\langle\mu\xi\rangle^{\sigma/2})^{|\alpha|}$$

for every α, β.

We assume that $b(x,\xi,\mu)$ is independent of x for $|x| \ge M$ with a large M. Here we note that if

$$|\partial_x^\alpha \partial_\xi^\beta b(x,\xi,\mu)| \le C_\beta m(x,\xi,\mu)\langle\xi\rangle_\mu^{-|\beta|}A^{|\alpha|}|\alpha|!^s, \quad \forall \alpha,\beta$$

then it is clear that $b(x,\xi,\mu) \in \gamma^{(s)}S(m(x,\xi,\mu),\bar{g})$.

Lemma 6.3.2 *Let $a_i(x,\xi,\mu) \in \gamma^{(s)}S(m_i(x,\xi,\mu),\bar{g})$, $i = 1,2$. Then we have*

$$a_1(x,\xi,\mu)a_2(x,\xi,\mu) \in \gamma^{(s)}S(m_1(x,\xi,\mu)m_2(x,\xi,\mu),\bar{g}).$$

Proof: It is enough to note that

$$\sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}A_1^{|\alpha'|}|\alpha'|!^{s/2}A_2^{|\alpha-\alpha'|}|\alpha-\alpha'|!^{s/2} \le A_1^{1+|\alpha|}(A_1-A_2)^{-1}|\alpha|!^{s/2}$$

for $A_1 > A_2$ and

$$(|\alpha'|^{s/2}+\langle\mu\xi\rangle^{\sigma/2})^{|\alpha'|}(|\alpha-\alpha'|^{s/2}+\langle\mu\xi\rangle^{\sigma/2})^{|\alpha-\alpha'|} \le (|\alpha|^{s/2}+\langle\mu\xi\rangle^{\sigma})^{|\alpha|}$$

which is easily checked. □

We now show

Lemma 6.3.3 *Let $s \ge 4$. Assume that*

$$|\partial_x^\alpha \partial_\xi^\beta f(x,\xi,\mu)| \le C_\beta \langle\xi\rangle_\mu^{-|\beta|}A^{|\alpha|}|\alpha|!^s$$

for every α, β. Then we have

$$w(x,\xi,\mu) = \sqrt{f(x,\xi,\mu)^2+\langle\mu\xi\rangle^{-\sigma}} \in \gamma^{(s)}S(w(x,\xi,\mu),\bar{g}).$$

We first show the next lemma.

Lemma 6.3.4 *Let $s \ge 4$ and assume that*

$$|\partial_x^\alpha f(x)| \le C_1 C_2^{|\alpha|}|\alpha|!^s, \quad |\alpha| \ge 1, \quad x \in \mathbb{R}^n \tag{6.3.2}$$

with positive constants C_1, C_2. We define $w(x) = \sqrt{f(x)^2+B^{-2}}$ with a positive constant B. Then there exist positive constant A_i such that

$$\begin{cases} |\partial_x^\alpha w(x)| \le w(x)A_1^{|\alpha|}|\alpha|!^{s/2}(|\alpha|^{s/2}+B)^{|\alpha|}, & |\alpha| \ge 1, \\ |\partial_x^\alpha w^{-1}(x)| \le w^{-1}(x)A_2^{|\alpha|}|\alpha|!^{s/2}(|\alpha|^{s/2}+B)^{|\alpha|}, & |\alpha| \ge 1. \end{cases} \tag{6.3.3}$$

Proof: Note that

$$w(x)\partial_x^e w(x) = f(x)\partial_x^e f(x) = F(x), \quad |e| = 1 \tag{6.3.4}$$

where we may assume that

$$|\partial_x^\alpha F(x)| \le A^{|\alpha|+1}|\alpha|!^s, \quad |\alpha| \ge 1.$$

Assume that the inequalities (6.3.3) hold for $|\alpha| \le n$ and study $\partial_x^\alpha w(x)$ with $|\alpha| = n+1$. Using (6.3.4) we have

$$w\partial_x^{\alpha+e} w = -\sum_{\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!}\partial_x^\beta w \partial_x^{\alpha+e-\beta} w + \partial_x^\alpha F \tag{6.3.5}$$

which gives

$$\begin{aligned}|w\partial_x^{\alpha+e} w| \le \sum_{j=1}^{n}\binom{n}{j} w A_1^j j!^{s/2}(j^{s/2}+B)^j w A_1^{n+1-j}(n+1-j)!^{s/2}\\ \times((n+1-j)^{s/2}+B)^{n+1-j} + A^{n+1}n!^s\\ \le w^2 A_1^{n+1}\Big(\sum_{j=1}^{n}\binom{n}{j} j!^{s/2}(n+1-j)!^{s/2}(j^{s/2}+B)^j\\ \times((n+1-j)^{s/2}+B)^{n+1-j} + w^{-2}(A/A_1)^{n+1}n!^s\Big).\end{aligned}$$

Note that

$$(j^{s/2}+B)^j((n+1-j)^{s/2}+B)^{n+1-j} \le ((n+1)^{s/2}+B)^{n+1}$$

and $w^{-2} \le B^2$ and

$$\begin{aligned}&((n+1)^{s/2}+B)^{n+1}(n+1)!^{s/2}\\ &\ge \frac{(n+1)n}{2}(n+1)^{(n-1)s/2}B^2(n+1)!^{s/2}\\ &\ge \frac{(n+1)n}{2}B^2 n!^s.\end{aligned}$$

Thus one has

$$\begin{aligned}|w^{(n+1)}| \le w A_1^{n+1}((n+1)^{s/2}+B)^{n+1}(n+1)!^{s/2}\\ \times\Big(\sum_{j=1}^{n}\binom{n}{j}\binom{n+1}{j}^{-s/2} + \Big(\frac{A}{A_1}\Big)^{n+1}\frac{2}{(n+1)n}\Big).\end{aligned}$$

We now check that

$$\sum_{j=1}^{n}\binom{n}{j}\binom{n+1}{j}^{-s/2} + \Big(\frac{A}{A_1}\Big)^{n+1}\frac{2}{(n+1)n} \le 1$$

if $A/A_1 \le 1$. Indeed we have

$$\sum_{j=1}^{n} \binom{n}{j} \binom{n+1}{j}^{-s/2} = \sum_{j=1}^{n} \frac{n+1-j}{n} \binom{n+1}{j}^{1-s/2}$$
$$\le n \binom{n+1}{n}^{1-s/2} = \frac{n}{(n+1)^{s/2-1}} \le \frac{n}{n+1}$$

because $s \ge 4$ and hence the first assertion.

To check the second assertion we assume that

$$|\partial_x^\alpha w^{-1}| \le w^{-1} A_2^{|\alpha|} |\alpha|!^{s/2} (|\alpha|^{s/2} + B)^{|\alpha|}$$

holds for $|\alpha| \le n$. From $ww^{-1} = 1$ we have with $|\alpha| = n+1$

$$|\partial_x^\alpha w^{-1}| \le w^{-1} A_2^{n+1} (n+1)!^{s/2} ((n+1)^{s/2} + B)^{n+1}$$
$$\times \sum_{j=1}^{n+1} \binom{n+1}{j} \binom{n+1}{j}^{-s/2} (\frac{A_1}{A_2})^j.$$

Thus it suffices to take A_2 so that $A_1/A_2 \le 1/2$ to get the desired estimates for $|\partial_x^\alpha w^{-1}|$. □

Corollary 6.3.1 *Assume that $s \ge 4$. For any $m \in \mathbb{Z}$ we have*

$$|\partial_x^\alpha w^{-m}| \le w^{-m} A^{|\alpha|} |\alpha|!^{s/2} (|\alpha|^{s/2} + B)^{|\alpha|}$$

with some $A > 0$.

Proof: It is clear from Lemma 6.3.4 and the proof of Lemma 6.3.2. □

Proof of Lemma 6.3.3: Note that one can write

$$\partial_\xi^\beta w = \partial_\xi^\beta (w^2)^{1/2} = \sum C_{\beta_1,\ldots,\beta_k} w^{1-2k} (\partial_\xi^{\beta_1} w^2) \cdots (\partial_\xi^{\beta_k} w^2) \tag{6.3.6}$$

where $\beta_1 + \cdots + \beta_k = \beta$ and $|\beta_j| \ge 1$. Remarking

$$|\partial_\xi^\beta w^2| \le C_\beta w \langle \xi' \rangle_\mu^{-1} \le C_\beta w^2 \langle \mu\xi' \rangle^{\sigma/2} \langle \xi' \rangle_\mu^{-1}$$

for $|\beta| = 1$ and

$$|\partial_x^\alpha \partial_\xi^\beta w^2| \le C_\beta \langle \xi' \rangle_\mu^{-|\beta|} A^{|\alpha|} |\alpha|!^s \le C_\beta w^2 \langle \mu\xi' \rangle^{\sigma} \langle \xi' \rangle_\mu^{-|\beta|} A^{|\alpha|} |\alpha|!^s$$
$$\le C_\beta w^2 (\langle \mu\xi' \rangle^{-\sigma/2} \langle \xi' \rangle_\mu)^{-|\beta|} A_1^{|\alpha|} |\alpha|!^{s/2} (|\alpha|^{s/2} + \langle \mu\xi' \rangle^{\sigma/2})^{|\alpha|}$$

for $|\alpha + \beta| \ge 2$ ($|\beta| \ge 1$) we conclude that

$$|\partial_x^\alpha (\partial_\xi^{\beta_j} w^2)| \le C_{\beta_j} w^2 (\langle \mu\xi' \rangle^{-\sigma/2} \langle \xi' \rangle_\mu)^{-|\beta_j|} \tag{6.3.7}$$
$$\times A_1^{|\alpha|} |\alpha|!^{s/2} (|\alpha|^{s/2} + \langle \mu\xi' \rangle^{\sigma/2})^{|\alpha|}$$

for any α. Applying Corollary 6.3.1 to w^{1-2k} with $B = \langle \mu\xi' \rangle^{\sigma/2}$ we get

$$|\partial_x^\alpha w^{1-2k}| \le C w^{1-2k} A_1^{|\alpha|} |\alpha|!^{s/2} (|\alpha|^{s/2} + \langle \mu\xi' \rangle^{\sigma/2})^{|\alpha|}. \tag{6.3.8}$$

Then the assertion follows immediately from (6.3.7), (6.3.8) and (6.3.6). □

Let

$$\kappa = \frac{1}{s}$$

and assume that $\sigma + \kappa \le 1$. As for $\phi(\xi, \mu)$ we assume that

$$\begin{cases} \phi(\xi, \mu) \in S(\langle \mu\xi \rangle^\kappa, |dx|^2 + \langle \xi \rangle_\mu^{-2} |d\xi|^2), \\ \phi(\xi + \eta, \mu) - \phi(\xi - \eta, \mu) \le C \langle \mu\eta \rangle^\kappa. \end{cases} \tag{6.3.9}$$

Then we have

Proposition 6.3.1 *Let $\sigma + \kappa \le 1$ and*

$$b(x, \xi, \mu) \in \gamma^{(1/\kappa)} S(m(x, \xi, \mu), \bar{g}) \cap \gamma^{(1/\kappa)} S(\tilde{m}(\xi, \mu), \bar{g}).$$

Assume (6.3.9). *Let $e^{\phi(D,\mu)} b^w(x, D, \mu) e^{-\phi(D,\mu)} = c^w(x, D, \mu)$ then one can write*

$$c(x, \xi, \mu) = \sum_{j=0}^{N-1} c_j(x, \xi, \mu) + R_N(x, \xi, \mu)$$

where

$$c_j = \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\eta^\alpha e^{\phi(\xi + \frac{\eta}{2}, \mu) - \phi(\xi - \frac{\eta}{2}, \mu)} \Big|_{\eta=0} b_{(\alpha)}(x, \xi, \mu)$$

$$\in \mu^j S(m(x, \xi, \mu) \langle \mu\xi \rangle^{-j(1-\kappa-\sigma/2)}, \bar{g}),$$

$$R_N(x, \xi, \mu) \in \mu^N S(\tilde{m}(\xi, \mu) \langle \mu\xi \rangle^{-N(1-\kappa-\sigma/2)+n\sigma/2}, \bar{g}).$$

Proof: We divide the proof into two parts. In the first part we prove the assertions for c_j. Recall that

$$c(x, \xi, \mu) = \int e^{-iy\eta} e^{\phi(\xi + \frac{\eta}{2}, \mu) - \phi(\xi - \frac{\eta}{2}, \mu)} b(x + y, \xi, \mu) dy d\eta. \tag{6.3.10}$$

Let us write

$$b(x + y, \xi, \mu) = \sum_{|\alpha| < N} \frac{1}{\alpha!} b_{(\alpha)}(x, \xi, \mu)(iy)^\alpha$$

$$+ \sum_{|\alpha| = N} \frac{N}{\alpha!} (iy)^\alpha \int_0^1 (1 - s)^{N-1} b_{(\alpha)}(x + sy, \xi, \mu) ds$$

and insert this expression into (6.3.10) to get

$$
\begin{aligned}
c(x,\xi,\mu) = \sum_{|\alpha|<N} \frac{1}{\alpha!} \int e^{-iy\eta} e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)} b_{(\alpha)}(x,\xi,\mu)(iy)^{\alpha} dy d\eta \\
+ \sum_{|\alpha|=N} \frac{N}{\alpha!} \int e^{-iy\eta} e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)} (iy)^{\alpha} dy d\eta \\
\times \int_0^1 (1-s)^{N-1} b_{(\alpha)}(x+sy,\xi,\mu) ds.
\end{aligned} \tag{6.3.11}
$$

The first term in the right-hand side is

$$
\sum_{|\alpha|<N} \frac{1}{\alpha!} \partial_\eta^\alpha e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)}\Big|_{\eta=0} b_{(\alpha)}(x,\xi,\mu) \tag{6.3.12}
$$

because $e^{-iy\eta}(iy)^\alpha = (-\partial_\eta)^\alpha e^{-iy\eta}$. Let us put

$$
\partial_\eta^\alpha e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)} = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \partial_\eta^\beta e^{\phi(\xi+\frac{\eta}{2},\mu)} \partial_\eta^\gamma e^{-\phi(\xi-\frac{\eta}{2},\mu)}
$$

$$
= 2^{-|\alpha|} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \partial_\xi^\beta e^{\phi(\xi+\frac{\eta}{2},\mu)} (-\partial_\xi)^\gamma e^{-\phi(\xi-\frac{\eta}{2},\mu)} = 2^{-|\alpha|} \alpha! H_\alpha(\xi,\eta,\mu)
$$

that is

$$
H_\alpha(\xi,\eta,\mu) = \frac{2^{|\alpha|}}{\alpha!} \partial_\eta^\alpha e^{\phi(\xi+\eta/2,\mu)-\phi(\xi-\eta/2,\mu)}.
$$

Then the second term on the right-hand side of (6.3.11) yields

$$
2^{-N} \sum_{|\alpha|=N} N \int e^{-iy\eta} H_\alpha(\xi,\eta,\mu) dy d\eta \int_0^1 (1-s)^{N-1} b_{(\alpha)}(x+sy,\xi,\mu) ds
$$

$$
= 2^{-N} \sum_{|\alpha|=N} N \int \int_0^1 e^{ix\eta} (1-s)^{N-1} H_\alpha(\xi,s\eta,\mu) d\eta ds \int e^{-iy\eta} b_{(\alpha)}(y,\xi,\mu) dy.
$$

With

$$
B_\alpha(\eta,\xi,\mu) = \int e^{-iy\eta} b_{(\alpha)}(y,\xi,\mu) dy
$$

this can be written as

$$
2^{-N} \sum_{|\alpha|=N} N \int \int_0^1 e^{ix\eta} (1-s)^{N-1} H_\alpha(\xi,s\eta,\mu) B_\alpha(\eta,\xi,\mu) d\eta ds. \tag{6.3.13}
$$

It is easy to see that

$$
\partial_\eta^\alpha e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)}\Big|_{\eta=0}
$$

is a linear combination of terms such as

$$
\partial_\xi^{\alpha_1} \phi(\xi,\mu) \cdots \partial_\xi^{\alpha_s} \phi(\xi,\mu), \quad \sum \alpha_p = j, \ \ |\alpha_p| \ge 1
$$

which are in $S(\langle\mu\xi\rangle^{j\kappa}\langle\xi\rangle_\mu^{-j}, |dx|^2+\langle\xi\rangle_\mu^{-2}|d\xi|^2)$. Thus it is clear that

$$c_j = \sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_\eta^\alpha e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)}\Big|_{\eta=0} b_{(\alpha)}(x,\xi,\mu)$$
$$\in \mu^j S(m(x,\xi,\mu)\langle\mu\xi\rangle^{-j(1-\kappa-\sigma/2)}, \bar{g}).$$

In the second part we prove the assertion for R_N. We consider

$$R_N = 2^{-N}\sum_{|\alpha|=N} N\int\int_0^1 e^{ix\eta}(1-s)^{N-1}H_\alpha(\xi,s\eta,\mu)B_\alpha(\eta,\xi,\mu)d\eta ds.$$

Lemma 6.3.5 *One can choose $M>0$ so that we have*

$$|\partial_\xi^\delta B_\alpha(\eta,\xi,\mu)| \le C_{\alpha,\delta}\tilde{m}(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|}$$
$$\times e^{-c|\eta|^\kappa}, \quad |\eta| \ge M\langle\mu\xi\rangle^\sigma,$$
$$|\partial_\xi^\delta B_\alpha(\eta,\xi,\mu)| \le C_{\alpha,\delta}\tilde{m}(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|}$$
$$\times\langle\mu\xi\rangle^{|\alpha|\sigma/2}e^{-c(\langle\mu\xi\rangle^{-\sigma/2}|\eta|)^\kappa}, \quad |\eta| \le M\langle\mu\xi\rangle^\sigma$$

with some $c>0$.

Proof: Recall that we have

$$\eta^\nu\partial_\xi^\delta B_\alpha(\eta,\xi,\mu) = \int e^{-iy\eta}\partial_\xi^\delta b_{(\alpha+\nu)}(y,\xi,\mu)dy$$

and hence one gets

$$|\partial_\xi^\delta B_\alpha(\eta,\xi,\mu)| \le C_\delta\tilde{m}(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|}$$
$$\times A^{|\alpha+\nu|}|\alpha+\nu|!^{s/2}(|\alpha+\nu|^{s/2}+\langle\mu\xi\rangle^{\sigma/2})^{|\alpha+\nu|}|\eta|^{-|\nu|}$$
$$\le C_\delta\tilde{m}(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|}C_1C_2^{|\nu|}|\nu|!^{s/2}$$
$$\times(|\nu|^{s/2}+\langle\mu\xi\rangle^{\sigma/2})^{|\alpha+\nu|}|\eta|^{-|\nu|}$$

where $C_i = C_i(|\alpha|)$. We minimize $C_2^{|\nu|}|\nu|!^{s/2}(|\nu|^{s/2}+\langle\mu\xi\rangle^{\sigma/2})^{|\nu|}|\eta|^{-|\nu|}$. Note that

$$C_2^{|\nu|}|\nu|!^{s/2}(|\nu|^{s/2}+\langle\mu\xi\rangle^{\sigma/2})^{|\nu|} \le (2C_2)^{|\nu|}(|\nu|^s+\langle\mu\xi\rangle^\sigma)^{|\nu|}.$$

Choose ν so that $|\nu| = [e^{-s}(2C_2)^{-1}|\eta| - \langle\mu\xi\rangle^\sigma]^{1/s}$ assuming that

$$|\eta| \ge 4C_2e^s\langle\mu\xi\rangle^\sigma.$$

Then we see that we have

$$(2C_2)^{|\nu|}(|\nu|^s+\langle\mu\xi\rangle^\sigma)^{|\nu|}|\eta|^{-|\nu|} \le e^{-c|\eta|^{1/s}}$$

with some $c>0$. Then we conclude that

$$|\partial_\xi^\delta B_\alpha(\eta,\xi,\mu)| \le C_{\alpha,\delta}\tilde{m}(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|}$$
$$\times(C|\eta|^{1/2}+\langle\mu\xi\rangle^{\sigma/2})^{|\alpha|}e^{-c|\eta|^\kappa}$$
$$\le C_{\alpha,\delta}\tilde{m}(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|}e^{-c'|\eta|^\kappa}$$

with some $c' > 0$.

We turn to the case $|\eta| \le 4C_2 e^s \langle\mu\xi\rangle^\sigma$. Note that

$$C_2^{|\nu|}|\nu|!^{s/2}(|\nu|^{s/2} + \langle\mu\xi\rangle^{\sigma/2})^{|\nu|} \le (2C_2)^{|\nu|}(|\nu|^s\langle\mu\xi\rangle^{\sigma/2})^{|\nu|}.$$

Choose ν so that $|\nu| = e^{-1}(2C_2)^{-1/s}(|\eta|\langle\mu\xi\rangle^{-\sigma/2})^{1/s}$ then we have

$$(2C_2)^{|\nu|}(|\nu|^s\langle\mu\xi\rangle^{\sigma/2})^{|\nu|}|\eta|^{-|\nu|} \le e^{-c(|\eta|\langle\mu\xi\rangle^{-\sigma/2})^{1/s}}$$

with some $c > 0$. Thus we have

$$\begin{aligned}|\partial_\xi^\delta B_\alpha(\eta,\xi,\mu)| &\le C_{\alpha,\delta}\tilde{m}(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|}\\ &\times(C(|\eta|\langle\mu\xi\rangle^{-\sigma/2})^{1/2} + \langle\mu\xi\rangle^{\sigma/2})^{|\alpha|}e^{-c(|\eta|\langle\mu\xi\rangle^{-\sigma/2})^\kappa}\\ &\le C_{\alpha,\delta}\tilde{m}(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|}\langle\mu\xi\rangle^{\sigma|\alpha|/2}e^{-c'(|\eta|\langle\mu\xi\rangle^{-\sigma/2})^\kappa}\end{aligned}$$

with some $c' > 0$ which completes the proof. □

Note that $H_\alpha(\xi,\eta,\mu)$ is a linear combination of such terms

$$\begin{aligned}&\partial_\xi^{\beta_1}\phi(\xi+\frac{\eta}{2},\mu)\cdots\partial_\xi^{\beta_s}\phi(\xi+\frac{\eta}{2},\mu)\partial_\xi^{\gamma_1}\phi(\xi-\frac{\eta}{2},\mu)\cdots\partial_\xi^{\gamma_t}\phi(\xi-\frac{\eta}{2},\mu)\\ &\times e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)} = h_{\beta_1,\ldots,\beta_s,\gamma_1,\ldots,\gamma_t}(\xi,\eta,\mu)e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)}\end{aligned}$$

where $\sum\beta_j = \beta$, $\sum\gamma_j = \gamma$ and $|\beta_j| \ge 1$, $|\gamma_j| \ge 1$, $\beta+\gamma = \alpha$. It is easy to examine that

$$\begin{aligned}&|\partial_\xi^\delta h_{\beta_1,\ldots,\beta_s,\gamma_1,\ldots,\gamma_t}(\xi,\eta,\mu)|\\ &\le C_\delta\mu^{|\alpha|}\langle\mu\xi\rangle^{-|\alpha|(1-\kappa)}\langle\xi\rangle_\mu^{-|\delta|}\langle\mu\eta\rangle^{|\alpha|+|\delta|}.\end{aligned} \tag{6.3.14}$$

On the other hand noting that

$$\begin{aligned}&\partial_\xi^\alpha\phi(\xi+\frac{\eta}{2},\mu) - \partial_\xi^\alpha\phi(\xi-\frac{\eta}{2},\mu)\\ &= \sum_{k=1}^n \frac{1}{2}\eta_k\big(\partial_\xi^\alpha\partial_{\xi_k}\phi(\xi+\frac{\theta\eta}{2},\mu) + \partial_\xi^\alpha\partial_{\xi_k}\phi(\xi-\frac{\theta\eta}{2},\mu)\big)\end{aligned} \tag{6.3.15}$$

we see that

$$|\partial_\xi^\delta e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)}| \le C_\delta\langle\xi\rangle_\mu^{-|\delta|}\langle\mu\eta\rangle^{2|\delta|}\langle\eta\rangle^{|\delta|}e^{C\langle\mu\eta\rangle^\kappa}. \tag{6.3.16}$$

From Lemma 6.3.5, (6.3.14) and (6.3.16) it follows that for $|\eta| \ge M\langle\mu\xi\rangle^\sigma$, $|\alpha| = N$

$$\begin{aligned}|\partial_\xi^\delta(H_\alpha(\xi,\eta,\mu)B_\alpha(\eta,\xi,\mu))| &\le C_\delta\mu^N\tilde{m}(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|}\\ &\times\langle\mu\xi\rangle^{-N(1-\kappa)}e^{-c''\langle\mu\eta\rangle^\kappa}\end{aligned}$$

with some $c'' > 0$. We consider the case $|\eta| \le M\langle\mu\xi\rangle^\sigma$. Since $\sigma < 1$, taking μ small, we have $\langle\mu(\xi+\theta\eta)\rangle \sim \langle\mu\xi\rangle$ and $\langle\xi+\theta\eta\rangle_\mu \sim \langle\xi\rangle_\mu$ where $|\theta| \le 1$ and hence we have

$$|\partial_\xi^\delta h_{\beta_1,\ldots,\beta_s,\gamma_1,\ldots,\gamma_t}(\xi,\eta,\mu)| \le C_\delta\mu^{|\alpha|}\langle\mu\xi\rangle^{-|\alpha|(1-\kappa)}\langle\xi\rangle_\mu^{-|\delta|}.$$

On the other hand, since $\kappa+\sigma\le 1$ one sees from (6.3.15) and $\langle\mu\eta\rangle\le C\langle\mu\xi\rangle^{\sigma}$ that

$$\big|\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)\big|\le C\langle\mu\eta\rangle\langle\mu\xi\rangle^{\kappa-1}\le C'.$$

Then we see

$$|\partial_\xi^\delta e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)}|\le C_\delta\langle\xi\rangle_\mu^{-|\delta|}$$

using (6.3.15) again. Thus we conclude

$$|\partial_\xi^\delta(H_\alpha(\xi,\eta,\mu)B_\alpha(\eta,\xi,\mu))|\le C_\delta\mu^N\tilde m(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|}$$
$$\times\langle\mu\xi\rangle^{N\sigma/2-N(1-\kappa)}e^{-c''(|\eta|\langle\mu\xi\rangle^{-\sigma/2})^\kappa}$$

with some $c''>0$. Since

$$|\eta^\beta e^{-c''(|\eta|\langle\mu\xi\rangle^{-\sigma/2})^\kappa}|\le C_\beta\langle\mu\xi\rangle^{\sigma|\beta|/2}e^{-c'''(|\eta|\langle\mu\xi\rangle^{-\sigma/2})^\kappa}$$

we have

$$|\partial_x^\alpha\partial_\xi^\delta R_N(x,\xi,\mu)|\le C_{\alpha,\delta}\mu^N\tilde m(\xi,\mu)\langle\mu\xi\rangle^{-N(1-\kappa-\sigma/2)}\langle\mu\xi\rangle^{\sigma n/2}$$
$$\times\langle\mu\xi\rangle^{\sigma|\alpha|/2}(\langle\mu\xi\rangle^{\sigma/2}\langle\xi\rangle_\mu^{-1})^{|\delta|}$$

because $\int e^{-c'''(|\eta|\langle\mu\xi\rangle^{-\sigma/2})^\kappa}d\eta\le C\langle\mu\xi\rangle^{n\sigma/2}$. That is

$$R_N\in\mu^N S(\tilde m(\xi,\mu)\langle\mu\xi\rangle^{-N(1-\kappa-\sigma/2)+n\sigma/2},\bar g).$$

This completes the proof of Proposition 6.3.1. □

6.4 Energy inequality

Let us denote $\Lambda=D_0+(\phi_1\Phi)^w=D_0-\lambda^w$, $M=D_0-(\phi_1\Phi)^w=D_0-m^w$ then thanks to Lemma 6.2.4 we see that

$$(\xi_0+\phi_1\Phi)\#(\xi_0-\phi_1\Phi)=\xi_0^2-\phi_1^2\Phi^2+\frac{1}{i}\{\xi_0+\phi_1\Phi,\xi_0-\phi_1\Phi\}+T$$

with $T\in\mu^2S(1,g)$. From Lemma 6.2.1 one can write $\{\xi_0,\phi_1\}=\mu\sum_{j=1}^{r+1}C_j\phi_j$ with some $C_j\in S(1,g_1)$ and hence

$$\{\xi_0+\phi_1\Phi,\xi_0-\phi_1\Phi\}=\mu\sum_{j=1}^{r+1}C_j\phi_j$$

with some $C_j\in S(1,g)$. On the other hand Lemma 6.2.2 shows

$$P_{sub}=C_0(\xi_0-\phi_1\Phi)+\sum_{j=1}^{r+1}C_j\phi_j$$

with some $C_j \in S(1, \bar{g})$. Noting that $P = (p + P_{sub})^w + R$, $R \in \mu^2 S(1, g_1)$ one can write

$$P = -M\Lambda + B_0\Lambda + Q,$$
$$Q = \Big[\sum_{j=2}^{r+1} \phi_j^2 + w\phi_1^2 + R\Big]^w + R_0, \quad R_0 \in \mu^2 S(\langle \mu\xi' \rangle^{2\kappa}, \bar{g})$$

where $B_0 \in \mu S(1, g)$ and

$$R = \sum_{j=1}^{r+1} c_j \phi_j \tag{6.4.1}$$

with some $c_j \in \mu S(1, g)$. We now conjugate $e^{-x_0 \langle \mu D' \rangle^\kappa} = e^\phi$ to P

$$e^\phi P e^{-\phi} = -e^\phi M e^{-\phi} e^\phi \Lambda e^{-\phi} + e^\phi B_0 e^{-\phi} e^{-\phi} e^\phi \Lambda e^{-\phi} + e^\phi Q e^{-\phi}.$$

Let us denote $e^\phi M e^{-\phi}$, $e^\phi \Lambda e^{-\phi}$, $e^\phi B_0 e^{-\phi}$, $e^\phi Q e^{-\phi}$ by M, Λ, B_0, Q again. Let us consider

$$M = e^\phi (D_0 - m^w) e^{-\phi} = D_0 - i\langle \mu D' \rangle^\kappa - e^\phi m^w e^{-\phi}.$$

Since $m \in S(w\langle \mu\xi' \rangle, g)$ we apply Proposition 6.3.1 with $\sigma = 4/5 = 4\kappa$. Then we have

$$e^\phi m^w e^{-\phi} = [m_0 + m_1 + m_2]^w, \quad m_0 = -\phi_1 \Phi \tag{6.4.2}$$

where $m_1 \in \mu S(\langle \mu\xi' \rangle^\kappa, g)$ is pure imaginary and $m_2 \in \mu^2 S(\langle \mu\xi' \rangle^{-\kappa}, \bar{g})$ by Lemma 6.3.1. Let us consider

$$\Lambda = e^\phi (D_0 - \lambda^w) e^{-\phi} = D_0 - i\langle \mu D' \rangle^\kappa - e^\phi \lambda e^{-\phi}.$$

Since $\lambda \in S(w\langle \mu\xi' \rangle, g)$ repeating the same arguments we have

$$e^\phi \lambda^w e^{-\phi} = [\lambda_0 + \lambda_1 + \lambda_2]^w, \quad \lambda_0 = \phi_1 \Phi \tag{6.4.3}$$

where $\lambda_1 \in \mu S(\langle \mu\xi' \rangle^\kappa, g)$ is pure imaginary and $\lambda_2 \in \mu^2 S(\langle \mu\xi' \rangle^{-\kappa}, \bar{g})$. Consider $e^\phi Q e^{-\phi}$. Note that

$$e^\phi [\phi_j^2]^w e^{-\phi} = [\phi_j^2 + a_j \phi_j + r_j]^w$$

where $a_j \in \mu S(\langle \mu\xi' \rangle^\kappa, g)$ is pure imaginary and $r_j \in \mu^2 S(\langle \mu\xi' \rangle^{2\kappa}, \bar{g})$. We next consider

$$e^\phi [w\phi_1^2]^w e^{-\phi} = [w\phi_1^2 + a_1 w\phi_1 + r_1]^w$$

where $a_1 \in \mu S(\langle \mu\xi' \rangle^\kappa, g)$ is pure imaginary and $r_1 \in \mu^2 S(w\langle \mu\xi' \rangle^{2\kappa}, \bar{g})$. Remark that

$$e^\phi R e^{-\phi} = [\sum_{j=1}^{r+1} c_j \phi_j + r]^w$$

where $c_j \in \mu S(1, \bar{g})$ and $r \in \mu^2 S(\langle \mu\xi' \rangle^{2\kappa}, \bar{g})$. Hence one can write

$$e^{\phi} Q e^{-\phi} = [\sum_{j=2}^{r+1} \phi_j^2 + w\phi_1^2 + \sum_{j=2}^{r+1} a_j \phi_j + a_1 w \phi_1 + \sum_{j=1}^{r+1} c_j \phi_j + r]^w \tag{6.4.4}$$

where $a_j \in \mu S(\langle \mu\xi' \rangle^{\kappa}, g)$ and $r \in \mu^2 S(\langle \mu\xi' \rangle^{2\kappa}, \bar{g})$. We summarize

Proposition 6.4.1 *We can write*

$$e^{\phi} P e^{-\phi} = -M\Lambda + B_0 \Lambda + Q$$

with $B_0 \in \mu S(1, \bar{g})$ *and*

$$M = D_0 - i\langle \mu D' \rangle^{\kappa} - [m_0 + m_1 + m_2]^w = D_0 - i\langle \mu D' \rangle^{\kappa} - m^w,$$
$$\Lambda = D_0 - i\langle \mu D' \rangle^{\kappa} - [\lambda_0 + \lambda_1 + \lambda_2]^w = D_0 - i\langle \mu D' \rangle^{\kappa} - \lambda^w$$

where m_1, $\lambda_1 \in \mu S(\langle \mu\xi' \rangle^{\kappa}, g)$ *are pure imaginary and* m_2, $\lambda_2 \in \mu^2 S(\langle \mu\xi' \rangle^{-\kappa}, \bar{g})$. *As for* Q *we have*

$$Q = [q + q_1 + q_2 + r]^w, \quad q = \sum_{j=2}^{r+1} \phi_j^2 + w\phi_1^2,$$
$$q_1 = \sum_{j=2}^{r+1} a_j \phi_j + a_1 w \phi_1, \quad q_2 = \sum_{j=1}^{r+1} c_j \phi_j$$

where $a_j \in \mu S(\langle \mu\xi' \rangle^{\kappa}, g)$ *are pure imaginary,* $c_j \in \mu S(1, \bar{g})$ *and* $r \in \mu^2 S(\langle \mu\xi' \rangle^{2\kappa}, \bar{g})$.

Here we recall the energy identity. Let us put

$$P = -M\Lambda + B_0 \Lambda + Q$$

where

$$M = D_0 - i\langle \mu D' \rangle^{\kappa} - m^w, \quad \Lambda = D_0 - i\langle \mu D' \rangle^{\kappa} - \lambda^w.$$

Repeating the proof of Proposition 4.3.1 we get

Proposition 6.4.2 *We have*

$$\begin{aligned}
2\mathsf{Im}(Pu, \Lambda u) = \frac{d}{dx_0}(\|\Lambda u\|^2 + ((\mathsf{Re}\, Q)u, u)) + 2||\langle \mu D' \rangle^{\kappa/2} \Lambda u|| \\
+2((\mathsf{Im} B_0)\Lambda u, \Lambda u) + 2\mathsf{Re}(\langle \mu D' \rangle^{\kappa} (\mathsf{Re}\, Q)u, u) \\
+2((\mathsf{Im}\, m)\Lambda u, \Lambda u) + 2\mathsf{Re}(\Lambda u, (\mathsf{Im}\, Q)u) \\
+\mathsf{Im}([D_0 - \mathsf{Re}\, \lambda, \mathsf{Re}\, Q]u, u) + 2\mathsf{Re}((\mathsf{Re}\, Q)u, (\mathsf{Im}\, \lambda)u).
\end{aligned}$$

On the other hand from (4.3.1) we have

$$-2\mathsf{Im}\, (\Lambda v, v) = \frac{d}{dx_0}\|v\|^2 + 2\|\langle \mu D' \rangle^{\kappa/2} v\|^2 + 2((\mathsf{Im}\, \lambda)v, v).$$

From this it follows that

$$\delta^{-1}\|\langle\mu D'\rangle^{-\kappa/2}\Lambda\langle\mu D'\rangle^{\kappa}u\|^2 \geq \frac{d}{dx_0}\|\langle\mu D'\rangle^{\kappa}u\|^2 + (2-\delta)\|\langle\mu D'\rangle^{3\kappa/2}u\|^2 + 2((\mathsf{Im}\,\lambda)\langle\mu D'\rangle^{\kappa}u, \langle\mu D'\rangle^{\kappa}u). \tag{6.4.5}$$

Since $\mathsf{Im}\lambda \in \mu S(\langle\mu\xi'\rangle^{\kappa}, g)$ one sees

$$\delta^{-1}\|\langle\mu D'\rangle^{-\kappa/2}\Lambda\langle\mu D'\rangle^{\kappa}u\|^2 \geq \frac{d}{dx_0}\|\langle\mu D'\rangle^{\kappa}u\|^2 + (2-\delta-C\mu)\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.$$

Since $\langle\mu D'\rangle^{-\kappa/2}\Lambda\langle\mu D'\rangle^{\kappa} = \langle\mu D'\rangle^{\kappa/2}\Lambda + \langle\mu D'\rangle^{-\kappa/2}[\Lambda, \langle\mu D'\rangle^{\kappa}]$ and noting $\lambda_0 \in S(w\langle\mu\xi'\rangle, g)$ and hence $[\Lambda, \langle\mu D'\rangle^{\kappa}] \in \mu S(\langle\mu\xi'\rangle^{\kappa}, \bar{g})$ we have

$$\|\langle\mu D'\rangle^{-\kappa/2}[\Lambda, \langle\mu D'\rangle^{\kappa}]u\|^2 \leq C\mu\|\langle\mu D'\rangle^{\kappa/2}u\|^2.$$

Then we have

Lemma 6.4.1 *We have*

$$\|\langle\mu D'\rangle^{\kappa/2}\Lambda u\|^2 \geq \frac{d}{dx_0}\|\langle\mu D'\rangle^{\kappa}u\|^2 + (1-C\mu)\|\langle\mu D'\rangle^{3\kappa/2}u\|^2. \tag{6.4.6}$$

Since $\mathsf{Im}\, m \in \mu S(\langle\mu\xi'\rangle^{\kappa}, \bar{g})$ it follows that

$$|2((\mathsf{Im}\, m)\Lambda u, \Lambda u)| \leq C\mu\|\langle\mu D'\rangle^{\kappa/2}\Lambda u\|^2. \tag{6.4.7}$$

Let us study $2\mathsf{Re}(\Lambda u, (\mathsf{Im}\, Q)u)$. Recall that $\mathsf{Im}\, Q = [q_1 + \mathsf{Im}\, q_2 + r_1]^w$ with $r_1 \in \mu^2 S(\langle\mu\xi'\rangle^{2\kappa}, \bar{g})$. Then one can estimate

$$|2\mathsf{Re}(\Lambda u, (\mathsf{Im}\, Q)u)| \leq \mu\|\langle\mu D'\rangle^{\kappa/2}\Lambda u\|^2 + \mu^{-1}\|\langle\mu D'\rangle^{-\kappa/2}(\mathsf{Im}\, Q)u\|^2.$$

Note that with $q_2' = \mathsf{Im}\, q_2$

$$\langle\mu\xi'\rangle^{-\kappa/2}\#(q_1 + q_2' + r_1) = \langle\mu\xi'\rangle^{-\kappa/2}(q_1 + q_2') + T$$

with $T \in \mu^2 S(\langle\mu\xi'\rangle^{1/2}, \bar{g})$ because $q_1 \in \mu S(\langle\mu\xi'\rangle^{1+\kappa}, g)$. Here we remark that

$$\langle\mu\xi'\rangle^{-\kappa/2}c_1\phi_1 = (c_1\langle\mu\xi'\rangle^{-\kappa}w^{-1/2})(\langle\mu\xi'\rangle^{\kappa/2}w^{1/2}\phi_1)$$

where $c_1\langle\mu\xi'\rangle^{-\kappa}w^{-1/2} \in \mu S(1, g)$. Applying Lemma 6.2.6 to $\|[\langle\mu\xi'\rangle^{-\kappa/2}(q_1 + q_2')]^w u\|^2$ we get

Lemma 6.4.2 *We have*

$$|2\mathsf{Re}(\Lambda u, (\mathsf{Im}\, Q)u)| \leq \mu\|\langle\mu D'\rangle^{\kappa/2}\Lambda u\|^2 + C\mu\Big\{\sum_{j=2}^{r+1}\mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}\phi_j^2]^w u, u) + \mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}w\phi_1^2]^w u, u)\Big\} + C\mu^3\|\langle\mu D'\rangle^{1/2}u\|^2.$$

Let us consider $\mathsf{Re}((\mathsf{Re}\, Q)u, (\mathsf{Im}\,\lambda)u)$. From Proposition 6.4.1 we have

$$\mathsf{Re}\, Q = q + q_2'' + r, \quad q_2'' = \sum_{j=1}^{r+1} c_j \phi_j$$

with $c_j \in \mu S(1, \bar{g})$, $r \in \mu^2 S(\langle \mu\xi'\rangle^{2\kappa}, \bar{g})$ and $\mathsf{Im}\,\lambda \in \mu S(\langle \mu\xi'\rangle^{\kappa}, \bar{g})$. From Lemma 6.2.6 it is clear that $|([q_2'']^w u, (\mathsf{Im}\,\lambda)u)|$ is bounded by constant times

$$\mu \sum_{j=2}^{r+1} \mathsf{Re}([\langle \mu\xi'\rangle^{\kappa} \phi_j^2]^w u, u) + \mathsf{Re}\,([\langle \mu\xi'\rangle^{\kappa} w\phi_1^2]^w u, u) + \mu^3 \|\langle \mu D'\rangle^{3\kappa/2} u\|^2 \tag{6.4.8}$$

because $\langle \mu\xi'\rangle^{-\kappa/2} w^{-1/2} \mathsf{Im}\,\lambda \in \mu S(\langle \mu\xi'\rangle^{3\kappa/2}, \bar{g})$. Thus it suffices to study the term $\mathsf{Re}(q^w u, (\mathsf{Im}\,\lambda)u)$ modulo (6.4.8). Since one can write

$$\mathsf{Im}\,\lambda = \lambda_1 + R_N, \quad \lambda_1 \in \mu S(\langle \mu\xi'\rangle^{\kappa}, g), \quad R_N \in \mu^N S(\langle \mu\xi'\rangle^{1-2N\kappa+2n\kappa}, \bar{g})$$

for any N we may assume $\mathsf{Im}\,\lambda = \lambda_1 \in \mu S(\langle \mu\xi'\rangle^{\kappa}, g)$ modulo $\mu^3 \|\langle \mu D'\rangle^{3\kappa/2} u\|^2$. Note that

$$\mathsf{Re}(\lambda_1 \# q) - \lambda_1 q \in \mu^3 S(\langle \mu\xi'\rangle, g)$$

because $5\kappa = 1$. Applying Lemma 6.2.6 we get

Lemma 6.4.3 *We have*

$$|2\mathsf{Re}((\mathsf{Re}\, Q)u, (\mathsf{Im}\,\lambda)u)| \le C\mu\Big\{\sum_{j=2}^{r+1} \mathsf{Re}([\langle \mu\xi'\rangle^{\kappa} \phi_j^2]^w u, u)$$
$$+\mathsf{Re}([\langle \mu\xi'\rangle^{\kappa} w\phi_1^2]^w u, u)\Big\} + C\mu^3 \|\langle \mu D'\rangle^{1/2} u\|^2.$$

We now estimate $\mathsf{Im}([D_0 - \mathsf{Re}\,\lambda, \mathsf{Re}\, Q]u, u)$. Note that one can write

$$\mathsf{Re}\, Q = q + q_2'' + r + R_N, \quad r \in \mu^2 S(\langle \mu\xi'\rangle^{2\kappa}, g),$$
$$\mathsf{Re}\,\lambda = -\lambda_0 - \lambda_2 + R_N', \quad \lambda_2 \in \mu^2 S(\langle \mu\xi'\rangle^{-\kappa}, g)$$

where $R_N \in \mu^N S(\langle \mu\xi'\rangle^{2-2N\kappa+2n\kappa}, \bar{g})$ and $R_N' \in \mu^N S(\langle \mu\xi'\rangle^{1-2N\kappa+2n\kappa}, \bar{g})$. This proves that $|\mathsf{Im}([D_0 - \mathsf{Re}\,\lambda, \mathsf{Re}\, Q]u, u)| = |\mathsf{Re}(\{\xi_0 - \mathsf{Re}\,\lambda, \mathsf{Re}\, Q\}^w u, u)|$ modulo $O(\mu^3 \|\langle \mu D'\rangle^{2\kappa} u\|^2)$. Note that

$$\{\xi_0 - \mathsf{Re}\,\lambda, \mathsf{Re}\, Q\} = \{\xi_0 - \lambda_0, q + q_2''\} - \{\lambda_2, q\} + T, \quad T \in \mu^3 S(\langle \mu\xi'\rangle^{4\kappa}, \bar{g}).$$

Since one can write $\{\lambda_2, q\} = \sum_{j=2}^{r+1} a_j \langle \mu\xi'\rangle^{\kappa} \phi_j + a_1 \langle \mu\xi'\rangle^{\kappa} w\phi_1$ with $a_j \in \mu^3 S(1, g)$ and $\{\xi_0 - \lambda_0, q_2''\} - \sum_{j=1}^{r+1} c_j \phi_j \in \mu^2 S(\langle \mu\xi'\rangle, \bar{g})$ with $c_j \in \mu^2 S(\langle \mu\xi'\rangle^{2\kappa}, \bar{g})$ we have

$$|\mathsf{Im}([D_0 - \mathsf{Re}\,\lambda, \mathsf{Re}\, Q]u, u)| \le |\mathsf{Re}(\{\xi_0 - \lambda_0, q\}^w u, u)|$$
$$+C\mu^3 \{\sum_{j=2}^{r+1} \|[\phi_j]^w u\|^2 + \|[\sqrt{w}\phi_1]^w u\|^2 + \|\langle \mu D'\rangle^{1/2} u\|^2\}$$
$$\le |\mathsf{Re}(\{\xi_0 - \mathsf{Re}\,\lambda_0, q\}^w u, u)|$$
$$+C\mu^3 \{\sum_{j=2}^{r+1} \mathsf{Re}([\phi_j^2]^w u, u) + \mathsf{Re}([w\phi_1^2]^w u, u) + \|\langle \mu D'\rangle^{1/2} u\|^2\}.$$

Thanks to Proposition 6.2.1 we conclude that

Lemma 6.4.4 *We have*

$$|\mathsf{Im}([D_0 - \mathsf{Re}\,\lambda, \mathsf{Re}\,Q]u, u)| \le C\mu\Big\{\sum_{j=2}^{r+1}\mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}\phi_j^2]^w u, u)$$
$$+\mathsf{Re}([w\langle\mu\xi'\rangle^{\kappa}\phi_1^2]^w u, u)\Big\} + C\mu^2\|\langle\mu D'\rangle^{1/2}u\|^2.$$

It remains to estimate $\mathsf{Re}(\langle\mu D'\rangle^{\kappa}(\mathsf{Re}\,Q)u, u)$. We first note that

$$|\mathsf{Re}(\langle\mu D'\rangle^{\kappa}[q_2'']^w u, u)| \le C\mu\{\sum_{j=2}^{r+1}\|[\phi_j]^w u\|^2$$
$$+\|[\langle\mu\xi'\rangle^{\kappa/2}\sqrt{w}\phi_1]^w u\|^2 + \mu\|\langle\mu D'\rangle^{3\kappa/2}u\|^2\}$$

and hence it suffices to estimate $\mathsf{Re}(\langle\mu D'\rangle^{\kappa}q^w u, u)$ modulo $O(\mu^2\|\langle\mu D'\rangle^{3\kappa/2}u\|^2)$. Note that

$$\mathsf{Re}(\langle\mu\xi'\rangle^{\kappa}\#\phi_j^2) = \langle\mu\xi'\rangle^{\kappa}\phi_j^2 + T_1,\;\; T_1 \in \mu^2 S(\langle\mu\xi'\rangle^{\kappa}, g),$$
$$\mathsf{Re}\,(\langle\mu\xi'\rangle^{\kappa}\#w\phi_1^2) = \langle\mu\xi'\rangle^{\kappa}w\phi_1^2 + T_2,\;\; T_2 \in \mu^2 S(w\langle\mu\xi'\rangle^{\kappa}, g)$$

and hence

$$\mathsf{Re}(\langle\mu D'\rangle^{\kappa}q^w u, u) \ge \sum_{j=2}^{r+1}\mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}\phi_j^2]^w u, u)$$
$$+\mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}w\phi_1^2]^w u, u) - C\mu^2\|\langle\mu D'\rangle^{\kappa/2}u\|^2.$$

This proves that

$$\mathsf{Re}(\langle\mu D'\rangle^{\kappa}(\mathsf{Re}\,Q)u, u) \ge \sum_{j=2}^{r+1}\mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}\phi_j^2]^w u, u)$$
$$+\mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}w\phi_1^2]^w u, u) - C\mu^2\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.$$

From Lemmas 6.4.1, 6.4.2, 6.4.3, 6.4.4 and (6.4.7) we have

Proposition 6.4.3 *There exist $\mu_0 > 0$, $C > 0$, $c > 0$ such that we have*

$$C\|\langle\mu D'\rangle^{-\kappa/2}Pu\|^2 \ge \frac{d}{dx_0}\{\|\Lambda u\|^2 + ((\mathsf{Re}\,Q)u, u) + \|\langle\mu D'\rangle^{\kappa}u\|^2\}$$
$$+c\|\langle\mu D'\rangle^{\kappa/2}\Lambda u\|^2 + c\{\sum_{j=2}^{r+1}\mathsf{Re}\,([\phi_j^2\langle\mu\xi'\rangle^{\kappa}]^w u, u) + \mathsf{Re}([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u, u)\}$$
$$+c\|\langle\mu D'\rangle^{3\kappa/2}u\|^2 + c\mu\|\langle\mu D'\rangle^{1/2}u\|^2$$

for $0 < \mu < \mu_0$.

From Lemma 6.2.6 it follows that

$$\sum_{j=2}^{r+1} \mathsf{Re}([\phi_j^2 \langle \mu\xi' \rangle^{\kappa}]^w u, u) + \mathsf{Re}([\phi_1^2 w \langle \mu\xi' \rangle^{\kappa}]^w u, u)$$
$$\geq -C\mu^2 \|\langle \mu D' \rangle^{1/2} u\|^2 - C\mu^2 \|\langle \mu D' \rangle^{3\kappa/2} u\|^2.$$

Taking into account that $\phi_j^2 - \phi_j \# \phi_j \in \mu^2 S(1, g_1)$, $w\phi_1^2 = \sqrt{w}\phi_1 \# \sqrt{w}\phi_1 + T$, $T \in \mu^2 S(w^{-1}, g) \subset \mu^2 S(\langle \mu\xi' \rangle^{2\kappa}, g)$ it follows that

$$((\mathsf{Re}\, Q)u, u) \geq -C\mu^2 \|\langle \mu D' \rangle^{\kappa} u\|^2.$$

Integrating the inequality in Proposition 6.4.3 from T_2 to t with respect to x_0 we get

Proposition 6.4.4 *We have*

$$C \int_{T_2}^{t} \|\langle \mu D' \rangle^{-\kappa/2} P u\|^2 dx_0 \geq \big(\|\Lambda u(t, \cdot)\|^2 + c\|\langle \mu D' \rangle^{\kappa} u(t, \cdot)\|^2\big)$$
$$+ c \int_{T_2}^{t} \big(\|\langle \mu D' \rangle^{\kappa/2} \Lambda u\|^2 + \|\langle \mu D' \rangle^{3\kappa/2} u\|^2 + \mu\|\langle \mu D' \rangle^{1/2} u\|^2\big) dx_0$$

for $0 < \mu < \mu_0$.

Recalling $P^* = (p + \bar{P}_{sub})^w + R$, $R \in S(1, g_1)$ we have the same energy estimates for P^*. Repeating the same arguments on functional analysis in the end of Section 4.4 we conclude that for any given $F \in C^0([-T, T]; H^{\infty}(\mathbb{R}^n))$ vanishing in $x_0 \leq 0$ there is a unique $U \in C^2([-T, T]; H^{\infty}(\mathbb{R}^n))$ vanishing for $x_0 \leq 0$ such that

$$PU = F, \quad \|D_0^j U(t)\|_{(p)}^2 \leq C_p \int_{-T}^{t} \|F(x_0)\|_{(p)}^2 dx_0 \tag{6.4.9}$$

for any $p \in \mathbb{R}$. Let us denote $U = GF$. Then repeating similar (in fact easier because we do not need to take care of the positive trace here) arguments as in Section 6.5 one can prove that the parametrix G has finite propagation speed of WF. Let Γ_i $(i = 0, 1, 2)$ be open conic sets in $\mathbb{R}^{2n} \setminus \{0\}$ with relatively compact basis such that $\Gamma_0 \subset\subset \Gamma_1 \subset\subset \Gamma_2$. Let us take $h_i(x', \xi') \in S(1, g_1)$ with $\operatorname{supp} h_1 \subset \Gamma_0$ and $\operatorname{supp} h_2 \subset \Gamma_2 \setminus \Gamma_1$. We consider the solution $U \in C^1(I; H^{\infty})$ to $PU = h_1 F$ with $F \in C^0(I; H^{\infty})$ where $U = F = 0$ in $x_0 < 0$. Then we have

Proposition 6.4.5 *Let h_1 and h_2 be as above. Then there is $\delta = \delta(\Gamma_i) > 0$ such that*

$$\|D_0^j h_2 G h_1 F(t)\|_{(p)} \leq C_{pq} \int^{t} \|F(x_0)\|_{(q)} dx_0$$

for $j = 0, 1$ and $0 \leq t \leq \delta$ and any p, $q \in \mathbb{R}$.

To avoid notational confusions, in what follows we denote by $\hat{P}$ the original operator and by P the transformed one so that

$$e^{(\tau-x_0)\langle\mu D'\rangle^\kappa}\hat{P}e^{-(\tau-x_0)\langle\mu D'\rangle^\kappa}=P.$$

Let $1\le s\le 5$ and $f\in C^0([-T,T];\gamma_0^{(s)}(\mathbb{R}^n))$ be such that $f=0$ for $x_0\le 0$. It is clear that $F=e^{(\tau-x_0)\langle\mu D'\rangle^\kappa}f\in C^0([-\tau,\tau];H^\infty(\mathbb{R}^n))$ for small $\tau>0$. Then as observed above, there exists a unique $U\in C^2([-\tau,\tau];H^\infty(\mathbb{R}^n))$ vanishing in $x_0\le 0$ such that

$$e^{(\tau-x_0)\langle\mu D'\rangle^\kappa}\hat{P}e^{-(\tau-x_0)\langle\mu D'\rangle^\kappa}U=PU=F=e^{(\tau-x_0)\langle\mu D'\rangle^\kappa}f.$$

This implies that

$$\hat{P}(e^{-(\tau-x_0)\langle\mu D'\rangle^\kappa}U)=f.$$

It is easy to check that $u=e^{-(\tau-x_0)\langle\mu D'\rangle^\kappa}U\in C^2([-\tau,\tau];H^\infty(\mathbb{R}^n))$. Let us denote

$$u=\hat{G}f=e^{-(\tau-x_0)\langle\mu D'\rangle^\kappa}Ge^{(\tau-x_0)\langle\mu D'\rangle^\kappa}f.$$

Lemma 6.4.5 *Let h_1 and h_2 be as above. Then we have*

$$\|e^{(\tau-t)\langle\mu D'\rangle^\kappa}D_0^jh_2\hat{G}h_1f(t)\|_{(p)}\le C_{pq}\int^t\|e^{(\tau-x_0)\langle\mu D'\rangle^\kappa}f(x_0)\|_{(q)}dx_0$$

for $j=0,1$ and $0\le t\le\delta$ with small $\delta>0$ and for any p, $q\in\mathbb{R}$, for any $f\in C^0([-\tau,\tau];\gamma_0^{(1/\kappa)}(\mathbb{R}^n))$ vanishing in $x_0\le 0$.

Proof: Applying Proposition 6.3.1 with $\sigma=0$ one can write

$$h_2\hat{G}h_1=e^{-(\tau-x_0)\langle\mu D'\rangle^\kappa}\tilde{h}_2G\tilde{h}_1e^{(\tau-x_0)\langle\mu D'\rangle^\kappa}$$

where $\tilde{h}_i\in S(1,\bar{g})$. Moreover for any $N\in\mathbb{N}$ we can write

$$\tilde{h}_i=\tilde{h}_{i0}+\tilde{h}_{iN},\quad \tilde{h}_{iN}\in\mu^NS(\langle\mu\xi'\rangle^{-N(1-\kappa)},\bar{g})$$

where $\operatorname{supp}\tilde{h}_{i0}\subset\operatorname{supp}h_i$. To prove the assertion for $e^{(\tau-x_0)\langle\mu D'\rangle^\kappa}D_0^jh_2\hat{G}h_1f$ it suffices to consider

$$D_0^j\tilde{h}_{20}G\tilde{h}_{10}e^{(\tau-x_0)\langle\mu D'\rangle^\kappa}f.$$

Taking $N=N(p,q)$ large then from Proposition 6.4.5 and (6.4.9) it follows that

$$\|D_0^j\tilde{h}_{20}G\tilde{h}_{10}e^{(\tau-t)\langle\mu D'\rangle^\kappa}f(t)\|_{(p)}\le C_{pq}\int^t\|e^{(\tau-x_0)\langle\mu D'\rangle^\kappa}f(x_0)\|_{(q)}dx_0$$

which shows the desired assertion. □

By a compactness argument we can assume that there are finite number of $h_\alpha(x',\xi')\in S(1,g_1)$ such that $\sum_\alpha h_\alpha=1$ in a neighborhood of the origin and $\hat{P}_\alpha$ which coincides with $\hat{P}$ in a conic neighborhood of the support of h_α and to $P_\alpha=e^{(\tau-x_0)\langle\mu D'\rangle^\kappa}\hat{P}_\alpha e^{-(\tau-x_0)\langle\mu D'\rangle^\kappa}$ one can apply Propositions 6.4.4 and 6.4.5

so that there exists a parametrix G_α with finite propagation speed of WF. Let us define

$$\hat{G} = \sum_\alpha \hat{G}_\alpha h_\alpha = \sum_\alpha e^{-(\tau - x_0)\langle \mu D' \rangle^\kappa} G_\alpha e^{(\tau - x_0)\langle \mu D' \rangle^\kappa} h_\alpha.$$

Let f have small support with respect to x' and consider

$$\hat{P}\hat{G}f = \sum_\alpha \hat{P}\hat{G}_\alpha h_\alpha f = \sum_\alpha \hat{P}_\alpha \hat{G}_\alpha h_\alpha f$$
$$+ \sum_\alpha (\hat{P} - \hat{P}_\alpha)\hat{G}_\alpha h_\alpha f = \sum_\alpha h_\alpha f - Rf = f - Rf$$

where

$$Rf = \sum_\alpha (\hat{P}_\alpha - \hat{P})\hat{G}_\alpha h_\alpha f.$$

From Lemma 6.4.5 we see that

$$\|e^{(\tau - t)\langle \mu D' \rangle^\kappa} Rf(t)\| \le C \int^t \|e^{(\tau - x_0)\langle \mu D' \rangle^\kappa} f(x_0)\| dx_0.$$

Take $\tau > 0$ small so that $C\tau < 1$ and put

$$[f] = \sup_{0 \le x_0 \le \tau} \|e^{(\tau - x_0)\langle \mu D' \rangle^\kappa} f(x_0)\|$$

and hence $[Rf] \le C\tau[f]$. Since $\sum_{j=0}^\infty [R^j f] \le \sum_{j=0}^\infty (C\tau)^j [f]$ then $(1 - R)^{-1} f = \sum_{j=0}^\infty R^j f$ exists and $[(1 - R)^{-1} f] < \infty$ for any f with $[f] < +\infty$ vanishing in $x_0 \le 0$. We now conclude that

$$\hat{P}\hat{G}(1 - R)^{-1} f = f.$$

Since $\hat{G}(1 - R)^{-1} f = 0$ for $x_0 \le 0$ we get a desired solution. It is clear that

$$e^{(\tau - x_0)\langle \mu D' \rangle^\kappa} (1 - R)^{-1} f \in C^0([-\tau_1, \tau_1]; H^\infty(\mathbb{R}^n))$$

for small $\tau_1 > 0$ and hence we have $\hat{G}(1 - R)^{-1} f \in C^2([-\tau_1, \tau_1]; H^\infty(\mathbb{R}^n))$. This proves Theorem 6.1.1.