Chapter 2

Hyperbolic double characteristics

2.1 Hamilton map

Let us denote by $T^*\Omega$ the cotangent bundle over Ω with a system of local coordinates $x = (x_0, x_1, ..., x_n)$. Let (x, ξ) be a system of canonical coordinates on $T^*\Omega$, then the canonical 2-form σ on $T^*\Omega$ is given by

$$\sigma = \sum_{j=0}^{n} d\xi_j \wedge dx_j$$

in these coordinates. This 2-form gives a symplectic structure on $T^*\Omega$.

Let $f \in C^\infty(T^*\Omega)$. Then the Hamilton vector field H_f of f is defined by

$$df(\cdot) = \sigma(\cdot, H_f). \tag{2.1.1}$$

In the canonical coordinates (x, ξ), denoting $X = \alpha\partial/\partial x + \beta\partial/\partial\xi$, $H_f = a\partial/\partial x + b\partial/\partial\xi$ we have

$$df(X) = \alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial\xi} = d\xi \wedge dx(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial\xi}, H_f) = \langle\beta, a\rangle - \langle\alpha, b\rangle.$$

That is $b = -\partial f/\partial x$, $a = \partial f/\partial\xi$ and hence

$$H_f = \frac{\partial f}{\partial\xi}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial\xi}.$$

It is clear that

$$\sigma((x, \xi), (y, \eta)) = \langle\xi, y\rangle - \langle x, \eta\rangle$$

in a system of canonical coordinates.

Let $P(x, D)$ be a differential operator of order m on Ω and let

$$P(x, D) = P_m(x, D) + P_{m-1}(x, D) + \cdots .$$

Then the principal symbol $P_m(x,\xi) \in C^\infty(T^*\Omega)$, that is if $y = (y_0, y_1, ..., y_n)$ is a second system of local coordinates and (y,η) is the system of canonical coordinates on $T^*\Omega$ then

$$P_m(x,\xi) = P_m(y,\eta) \quad \text{if} \quad \eta = {}^t\left(\frac{\partial x}{\partial y}\right)\xi.$$

Definition 2.1.1 *Let $p(x,\xi) \in C^\infty(T^*\Omega)$. Then a null bicharacteristic of p is an integral curve of H_p lying on $\{p = 0\}$.*

In what follows we assume that $P_m(x,\xi)$ is hyperbolic and monic with respect to ξ_0 and we write $p(x,\xi) = P_m(x,\xi)$. By definition, a null bicharacteristic of p is an integral curve of the following Hamilton system

$$\left\{\begin{array}{l} \dot{x} = \dfrac{\partial p}{\partial \xi}(x,\xi), \\ \dot{\xi} = -\dfrac{\partial p}{\partial x}(x,\xi) \end{array}\right. \tag{2.1.2}$$

on which $p = 0$.

Let $\rho = (\bar{x},\bar{\xi})$ be a multiple characteristic of $p(x,\xi)$. Then it is clear that ρ is a singular (stationary) point of the Hamilton system (2.1.2). To make a closer look on behaviors of null bicharacteristics near double characteristics we linearize the Hamilton system at the reference double characteristic $\rho = (\bar{x},\bar{\xi})$. Let

$$x(t) = \bar{x} + \epsilon y(t), \ \xi(t) = \bar{\xi} + \epsilon\eta(t)$$

and plug this into (2.1.2). Then the linear term in ϵ gives

$$\frac{d}{dt}\begin{pmatrix} y \\ \eta \end{pmatrix} = \begin{pmatrix} \dfrac{\partial^2 p}{\partial\xi\partial x} & \dfrac{\partial^2 p}{\partial\xi\partial\xi} \\ -\dfrac{\partial^2 p}{\partial x\partial x} & -\dfrac{\partial^2 p}{\partial x\partial\xi} \end{pmatrix}\begin{pmatrix} y \\ \eta \end{pmatrix}.$$

Definition 2.1.2 *We call*

$$F_p(\rho) = \frac{1}{2}\begin{pmatrix} \dfrac{\partial^2 p}{\partial\xi\partial x}(\rho) & \dfrac{\partial^2 p}{\partial\xi\partial\xi}(\rho) \\ -\dfrac{\partial^2 p}{\partial x\partial x}(\rho) & -\dfrac{\partial^2 p}{\partial x\partial\xi}(\rho) \end{pmatrix}$$

the Hamilton map of p at ρ.

Hamilton map is first studied in [22] by the word *fundamental matrix*[1]

[1] one of the authors of [22] told me the history of the word "fundamental matrix" as follows: At this time I was a grad student and among mathematical students we had the following definitions: *Derivative* of the drunken party is the party financed through deposit bottles. [i.e. if I remember correctly the cheap booze was 1.52 per bottle, while returning the bottle intact to the store one could recover 0.12, so multiplier was 12/152 and in order to be able to get one bottle in the second round one should consume 13 in the first.]

Fundamental drunken party is a party with non-zero second derivative.

Let $Q(X,Y)$ be the quadratic form (polar form) corresponding to the localization p_ρ of p at ρ (see Definition 1.2.1)

$$2Q(X,Y) = p_\rho(X+Y) - p_\rho(X) - p_\rho(Y)$$

then $F_p(\rho)$ is given by

$$\frac{1}{2}Q(X,Y) = \sigma(X, F_pY), \quad \forall X, Y \in T_\rho(T^*\Omega). \tag{2.1.3}$$

Note that this definition is coordinates free.

2.2 Ivrii-Petkov-Hörmander condition

Thanks to Corollary 1.2.1 it follows that p_ρ is a hyperbolic polynomial and hence Q is a hyperbolic quadratic form, that is a quadratic form with the signature $(-1, 1, ..., 1, 0, ..., 0)$. This hyperbolicity implies a special spectral structure of the Hamilton map F_p.

Lemma 2.2.1 ([22], [18]) *Let $\rho \in T^*\Omega \setminus \{0\}$ be a double characteristic of p. Then all eigenvalues of $F_p(\rho)$ are on the pure imaginary axis possibly with one exception of a pair of $\pm\lambda$, $\lambda \in \mathbb{R}^+$. More precisely the eigenvalues of $F_p(\rho)$ consists of*

$$\pm\lambda, \ \pm i\mu_1, ..., \pm i\mu_k, \quad \lambda \in \mathbb{R}^+, \ \mu_j \in \mathbb{R}^+ \cup \{0\}.$$

Definition 2.2.1 *We say that p is effectively hyperbolic at ρ if F_p has non zero real eigenvalues otherwise we say that p is noneffectively hyperbolic at ρ.*

We now characterize effective hyperbolicity in a more geometrical way.

Definition 2.2.2 *We define the hyperbolic cone Γ_ρ of p_ρ as the connected component of $\theta = (0, ..., 0, 1, 0, ..., 0) = -H_{x_0}$ of the set*

$$\{X \in T_\rho(T^*\Omega) \mid p_\rho(X) \neq 0\}.$$

Definition 2.2.3 *The propagation cone C_ρ of the localization $p_\rho(x,\xi)$ is given by*

$$C_\rho = \{X \in T_\rho(T^*\Omega) \mid \sigma(X,Y) \leq 0, \forall Y \in \Gamma_\rho\}.$$

When ρ is a simple characteristic, that is $p(\rho) = 0$, $dp(\rho) \neq 0$ then p_ρ is a linear function in $X = (x,\xi)$ and $p_\rho(X) = dp(\rho; X)$. Then it is clear that $\Gamma_\rho = \{X \mid dp(\rho;\theta)dp(\rho;X) > 0\}$ for $dp(\rho;\theta) \neq 0$ by Lemma 1.2.3 and hence $C_\rho = \mathbb{R}^+ \cdot dp(\rho;\theta)H_p(\rho)$. We note that C_ρ is the *minimal* cone including every null bicharacteristic which has ρ as a limit point in the following sense

Lemma 2.2.2 *Let $\rho \in T^*\Omega \setminus \{0\}$ be a multiple characteristic of p. Assume that there are simple characteristics ρ_j and positive numbers γ_j such that*

$$\rho_j \to \rho \ \text{ and } \ \gamma_j H_p(\rho_j) \to X(\neq 0), \quad j \to \infty, \ \pm dp(\rho_j;\theta) > 0, \ \forall j.$$

Then $\pm X \in C_\rho$.

Proof: Let K be any compact set in Γ_ρ. Then from Theorem 3 in [54] (see also [53]) it follows that for sufficiently large j we have $K \subset \Gamma_{\rho_j}$. Since

$$dp(\rho_j;\theta)dp(\rho_j;Y) = dp(\rho_j;\theta)\sigma(Y, H_p(\rho_j)) > 0, \quad \forall Y \in K$$

it follows that $\pm X \in C_\rho$. □

Definition 2.2.4 *Let $t(x,\xi)$ be homogeneous of degree 0 in ξ, C^∞ near ρ. We say that t is a time function for p near ρ if $t(\rho) = 0$ and*

$$-H_t(\rho) \in \Gamma_\rho. \tag{2.2.1}$$

Note that $t(x,\xi)$ is a time function near ρ for p if and only if

$$C_\rho \cap T_\rho(\{t(x,\xi) = 0\}) = \{0\}.$$

Definition 2.2.5 ([12]) *We define the linearity of p_ρ by*

$$\Lambda_\rho = \{X \in T_\rho(T^*\Omega) \mid p_\rho(tX + Y) = p_\rho(Y),\ \forall t \in \mathbb{R}, \forall Y \in T_\rho(T^*\Omega)\}.$$

When ρ is a simple characteristic then

$$p_\rho(tX + Y) = dp(\rho; tX + Y) = tdp(\rho; X) + dp(\rho; Y)$$

hence it is clear that $\Lambda_\rho = \{X \in T_\rho(T^*\Omega) \mid dp(\rho; X) = 0\}$. When ρ is a double characteristic of p then we have

$$\Lambda_\rho = \mathrm{Ker}\, F_p(\rho).$$

Indeed $p_\rho(tX+Y) = p_\rho(Y)$ for any $t \in \mathbb{R}$ and $Y \in \mathbb{R}^{2(n+1)}$ implies $Q(Y,X) = 0$ for any Y. Since $Q(Y,X) = 2\sigma(Y, F_pX)$ and σ is non degenerate we have $F_pX = 0$.

Lemma 2.2.3 *The following conditions are equivalent.*

(a) $F_p(\rho)$ *has non zero real eigenvalues,*

(b) $(\mathrm{Ker}\, F_p(\rho))^\sigma \cap \Gamma_\rho \neq \emptyset$,

(c) $C_\rho \cap \Lambda_\rho = \{0\}$

where $(\mathrm{Ker}\, F_p(\rho))^\sigma = \{X \in T_\rho(T^*\Omega) \mid \sigma(X,Y) = 0,\ \forall Y \in \mathrm{Ker}\, F_p(\rho)\}$.

Proof: We give a proof in the next section. □

Let P be a second order differential operator. Then we give another useful characterization of effective hyperbolicity. Write p as

$$p(x,\xi) = -(\xi_0 - a(x,\xi'))^2 + q(x,\xi') \tag{2.2.2}$$

where $\xi' = (\xi_1, ..., \xi_n)$ and $q(x,\xi') \geq 0$ by Theorem 1.1.1. Then we have

Lemma 2.2.4 ([44]) *Assume that p is effectively hyperbolic at ρ. Then there is a time function $t(x,\xi')$ near ρ for p verifying*

$$q(x,\xi') \geq ct(x,\xi')^2|\xi'|^2 \quad \text{near} \quad \rho \tag{2.2.3}$$

with some $c > 0$. Conversely if (2.2.3) holds with some time function $t(x,\xi')$ then p is effectively hyperbolic at ρ.

Proposition 2.2.1 *Let p be as in (2.2.2). Assume that p is effectively hyperbolic at ρ and there is a null bicharacteristic γ having ρ as a limit point. Then γ is transversal to some hypersurface containing the double characteristic set near ρ.*

Proof: Let $t(x,\xi')$ be a time function given in Lemma 2.2.4. Then it is clear that the double characteristic set is contained in

$$\{\xi_0 = a, q = 0\} \subset \{q = 0\} \subset \{t = 0\}.$$

Hence from Lemma 2.2.2 and (2.2.1) the assertion is clear. □

Thanks to Corollary 1.2.2 if p is strongly hyperbolic then every multiple characteristic is double. The importance of effective hyperbolicity is clear from the following result.

Theorem 2.2.1 *In order that $p(x,D)$ is strongly hyperbolic it is necessary and sufficient that $p(x,\xi)$ is effectively hyperbolic at every double characteristic.*

The necessary part follows from Theorem 2.2.2 below. The proof of sufficiency part is due to [25], [29], [30], [42]. For a detailed exposition, see [31], [48].

Theorem 2.2.1 implies, in particular, that if p is noneffectively hyperbolic at ρ then in order that the Cauchy problem for $P(x,D)$ is C^∞ well posed the lower order terms must verify some conditions. We discuss about one of such necessary conditions.

Definition 2.2.6 *Let $\rho \in T^*\Omega \setminus \{0\}$ be a double characteristic. We define $\mathrm{Tr}^+F_p(\rho)$ by*

$$\mathrm{Tr}^+F_p(\rho) = \sum \mu_j$$

where $i\mu_j$ are the eigenvalues of $F_p(\rho)$ on the positive imaginary axis repeated according to their multiplicities.

Definition 2.2.7 *The subprincipal symbol of $P(x,D)$ is defined by*

$$P_{sub}(x,\xi) = P_{m-1}(x,\xi) + \frac{i}{2}\sum_{j=0}^{n} \frac{\partial^2 p}{\partial x_j \partial \xi_j}(x,\xi)$$

which is well defined on the double characteristics.

The proof of the next result was given in part in [22] and completed in [18].

Theorem 2.2.2 (Ivrii-Petkov-Hörmander) *Assume that the Cauchy problem for P is C^∞ well posed near the origin. Let $\rho \in T_0^*\Omega \setminus \{0\}$ be a multiple characteristic and p is not effectively hyperbolic at ρ. Then we have*

$$\mathsf{Im}\, P_{sub}(\rho) = 0, \quad -\mathrm{Tr}^+ F_p(\rho) \le P_{sub}(\rho) \le \mathrm{Tr}^+ F_p(\rho).$$

We end this subsection with a conjecture proposed by Melrose [37]. Let Σ_e be the subset in $T^*\Omega \setminus \{0\}$ on which p is effectively hyperbolic. Denote by $e(\rho)$ the positive eigenvalue of $F_p(\rho)$ when $\rho \in \Sigma_e$. Set

$$s(\rho) = |\mathsf{Im}\, P_{sub}(\rho)| + \inf\{|\mathsf{Re}\, P_{sub}(\rho) - s| \mid |s| \le \mathrm{Tr}^+ F_p(\rho)\}.$$

Then

CONJECTURE:([37]) Assume that the Cauchy problem for P is C^∞ well posed near the origin. Then there is a neighborhood U of the origin such that $s(\rho)/e(\rho)$ is uniformly bounded in $\Sigma \cap (T^*U \setminus \{0\})$.

2.3 Hyperbolic quadratic form

Since p_ρ is a hyperbolic quadratic form it is natural to make more detailed studies about general hyperbolic quadratic forms.

Here we restate Theorem 21.5.3 in [19] about symplectic equivalence of hyperbolic quadratic forms.

Theorem 2.3.1 *Let Q be a hyperbolic quadratic form on $\mathbb{R}^{2(n+1)}$, that is a quadratic form with the signature $(-1, 1, ..., 1, 0, ..., 0)$ and let F be the Hamilton map of the quadratic form Q*

$$\frac{1}{2} Q(X, Y) = \sigma(X, FY), \quad \forall X, Y \in \mathbb{R}^{2(n+1)}.$$

Then one can choose symplectic coordinates $(x, \xi) = (x_0, x_1, ..., x_n, \xi_0, \xi_1, ..., \xi_n)$ so that

(1) $Q = -\xi_0^2 + \sum_{j=1}^k \mu_j (x_j^2 + \xi_j^2) + \sum_{j=k+1}^l \xi_j^2$,

(2) $Q = (-\xi_0^2 + 2\xi_0\xi_1 + x_1^2)/\sqrt{2} + \sum_{j=2}^k \mu_j (x_j^2 + \xi_j^2) + \sum_{j=k+1}^l \xi_j^2$,

(3) $Q = \lambda(x_0^2 - \xi_0^2)/\sqrt{2} + \sum_{j=1}^k \mu_j (x_j^2 + \xi_j^2) + \sum_{j=k+1}^l \xi_j^2$

where F has only pure imaginary eigenvalues in the case (1) and (2) while F has a pair of non zero real eigenvalues in the case (3) and

$$\sum \mu_j = \mathrm{Tr}^+ F.$$

In (1) and (3) we have $\mathrm{Ker}\, F^2 \cap \mathrm{Im}\, F^2 = \{0\}$ while one has $\mathrm{Ker}\, F^2 \cap \mathrm{Im}\, F^2 \ne \{0\}$ in the case (2).

For symplectic coordinates, see Chapter 10 where we gathered basic facts about them.

Corollary 2.3.1 *Assume that Q is a hyperbolic quadratic form. Then the following conditions are equivalent.*

(i) *F has non zero real eigenvalues,*

(ii) *there is $v \in V_0^\sigma$ such that $Q(v) < 0$,*

(iii) *there is $v \in (\operatorname{Ker} F)^\sigma$ such that $Q(v) < 0$,*

(iv) *for any $v \in \operatorname{Ker} F$ there is $w \in \mathbb{R}^{2(n+1)}$ such that $\sigma(v, w) = 0$, $Q(w) < 0$,*

where V_0 denotes the space of generalized eigenvectors belonging to the zero eigenvalue.

Proof: The implication (i)$\Longrightarrow$(ii) follows from Theorem 2.3.1. Indeed assume the case (3) occurs. Then we have $V_0 = \{x_0 = \cdots = x_k = 0, \xi_0 = \cdots = \xi_\ell = 0\}$ and hence $V_0^\sigma = \{\xi_0, ..., \xi_k, x_0, ..., x_\ell\}$ so that $v = H_{x_0}$ is a desired vector. The implications (ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv) are trivial. We now prove (iv)$\Longrightarrow$ (i). By Theorem 2.3.1, Q has one of the forms (1), (2) and (3) in suitable symplectic coordinates. Suppose now (2) occurs. Working in $\{x_0, x_1, \xi_0, \xi_1\}$ space $\operatorname{Ker} F$ is given by $\{x_1 = \xi_0 = \xi_1 = 0\}$. If $\sigma(v, w) = 0$, $\forall v \in \operatorname{Ker} F$ it follows that the ξ_0 coordinate of w is zero. Hence we get $Q(w) \geq 0$ and this shows that if (iv) holds then (2) never occurs.

Suppose that the case (1) occurs. Working in $\{x_0, \xi_0\}$ space we have $\operatorname{Ker} F = \{\xi_0 = 0\}$. If $\sigma(v, w) = 0$, $\forall v \in \operatorname{Ker} F$ then we see that the ξ_0 coordinate of w is zero and hence $Q(w) \geq 0$. This shows that the case (1) also never happens if (iv) holds.

Thus we proved that (iv) implies that only the case (3) happens. This proves the assertion. $\square$

Proof of Lemma 2.2.3: Apply Corollary 2.3.1 to $Q = -F_p(\rho)$ which is a hyperbolic quadratic form. Then (iii) of Corollary 2.3.1 shows that (a) and (b) are equivalent. It is sufficient to prove the equivalence of (b) and (c). Recall that $\operatorname{Ker} F_p(\rho) = \Lambda_\rho$ in this case. Assume $\Gamma_\rho \cap \Lambda_\rho^\sigma = \emptyset$. Then by the Hahn-Banach theorem there is $0 \neq Y \in T_\rho(T^*\Omega)$ such that

$$\sigma(Y, X) \leq 0, \ \forall X \in \Gamma_\rho, \quad \sigma(Y, X) \geq 0, \ \forall X \in \Lambda_\rho^\sigma.$$

This implies that $Y \in C_\rho \cap \Lambda_\rho$ which would give a contradiction to (c). Thus (c)$\Longrightarrow$(b). Suppose $0 \neq Y \in \Gamma_\rho \cap \Lambda_\rho^\sigma$. Then it is clear that $\langle Y \rangle^\sigma \supset \Lambda_\rho$ and $\langle Y \rangle^\sigma \cap C_\rho = \{0\}$ because Γ_ρ is open where $\langle Y \rangle = \mathbb{R} \cdot Y$. This implies obviously $C_\rho \cap \Lambda_\rho = \{0\}$ and hence (b)$\Longrightarrow$(c). $\square$