Chapter 1

Necessary conditions for well-posedness

1.1 Lax-Mizohata theorem

Let $P(x, D)$ be a differential operator of order m defined in a neighborhood Ω of the origin of $\mathbb{R}^{n+1}$ with coordinates $x = (x_0, x_1, ..., x_n) = (x_0, x')$

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \tag{1.1.1}$$

where $D^\alpha = D_0^{\alpha_0} \cdots D_n^{\alpha_n}$, $D_j = -i\partial/\partial x_j$ and $a_\alpha(x) \in C^\infty(\Omega)$. We assume that hyperplanes $x_0 =$ const. are non characteristic for P. Thus we may assume $a_{(m,0,...,0)}(x) = 1$.

Definition 1.1.1 *We say that the Cauchy problem for P is C^∞ well posed near the origin if there are $\epsilon > 0$ and a neighborhood ω of the origin such that for any $|\tau| \leq \epsilon$ and for any $f(x) \in C_0^\infty(\omega)$ vanishing in $x_0 < \tau$ there is a unique solution $u(x) \in H^\infty(\omega)$ to $Pu = f$ in ω vanishing in $x_0 < \tau$, where $H^\infty(\omega) = \cap_{p=0}^\infty H^p(\omega)$ and $H^p(\omega)$ denotes the usual Sobolev space of order p.*

Assume that $u \in H^\infty(\omega)$ vanishes in $x_0 < \tau$ with $|\tau| < \epsilon$. If $Pu = 0$ in $x_0 < t$ ($|t| < \epsilon$) then we conclude that $u = 0$ in $x_0 < t$. To see this, take $\chi \in C_0^\infty(\omega)$ and note that the equation $Pw = P(\chi u)$ has a solution $w \in H^\infty(\omega)$ vanishing in $x_0 < t$. Since $w - \chi u = 0$ in $x_0 < \min\{\tau, t\}$, and $P(w - \chi u) = 0$, by the uniqueness we get $w = \chi u$ and hence $u = 0$ in $x_0 < t$. Since $\chi \in C_0^\infty(\omega)$ is arbitrary we conclude $u = 0$ in $x_0 < t$.

Lemma 1.1.1 *Assume that the Cauchy problem for P is C^∞ well posed near the origin. Then the following classical Cauchy problem has a unique solution $u \in H^\infty(\omega)$*

$$\begin{cases} Pu = f \quad in \quad \omega \cap \{x_0 > \tau\} \\ D_0^j u(\tau, x') = u_j(x'), \quad j = 0, 1, ..., m-1 \end{cases} \tag{1.1.2}$$

for any given $f(x) \in C_0^\infty(\omega)$ *and* $u_j(x') \in C_0^\infty(\omega \cap \{x_0 = \tau\})$.

Proof: Since $x_0 = \tau$ is non characteristic, we can compute $u_j(x') = D_0^j u(\tau, x')$ for $j = m, m+1, \ldots$ from $u_j(x')$, $j = 0, \ldots, m-1$ and the equation $Pu = f$. By a Borel's lemma, we can take $\hat{u} \in C_0^\infty(\omega)$ so that $D_0^j \hat{u}(\tau, x') = u_j(x')$ for all $j \in \mathbb{N}$. Clearly we have $D_0^j(P\hat{u} - f) = 0$ on $\{x_0 = \tau\}$ for all $j \in \mathbb{N}$. The function g, defined by $g = P\hat{u} - f$ in $x_0 > \tau$ and zero in $x_0 < \tau$ is in $C_0^\infty(\omega)$. By the assumption there is $v \in H^\infty(\omega)$ such that $Pv = g$ in ω and $v = 0$ in $x_0 < \tau$. This shows that

$$\begin{cases} P(\hat{u} - v) = f \quad \text{in} \quad \omega \cap \{x_0 > \tau\}, \\ D_0^j(\hat{u} - v) = u_j(x') \quad \text{on} \quad \omega \cap \{x_0 = \tau\} \end{cases}$$

so that $\hat{u} - v \in H^\infty(\omega)$ is a desired solution to (1.1.2). □

For the operator (1.1.1) we write

$$P(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha = P_m(x, \xi) + P_{m-1}(x, \xi) + \cdots$$

where $P_j(x, \xi)$ denotes the homogeneous part of degree j with respect to ξ. Recall that $P_m(x, \xi)$ is called the principal symbol and we often write $p(x, \xi) = P_m(x, \xi)$ if the order m is clear from the context.

Definition 1.1.2 *Let P be as above. We say that P or p is strongly hyperbolic near the origin if for any differential operator Q of order less than m defined in Ω, the Cauchy problem for $P + Q$ is C^∞ well posed near the origin.*

REMARK: In this definition, the open set ω may depend on Q. In conclusion, at least in the scalar case, ω is independent of Q.

Definition 1.1.3 *Let $P(\zeta)$ be a (monic) polynomial in ζ of degree m. Then we say that $P(\zeta)$ is a hyperbolic polynomial if and only if the all zeros of $P(\zeta)$ are real.*

We note that the coefficients of $P(\zeta)$ are real.

Lemma 1.1.2 *Let $P(\zeta)$ be a hyperbolic polynomial. Then*

$$\left(\frac{\partial}{\partial \zeta}\right)^j P(\zeta), \quad 0 \le j \le m-1$$

are also hyperbolic polynomials.

Proof: For a proof, see [1] for example. □

The following result is due to [32] for the case of simple characteristic roots and to [38] in general and called the Lax-Mizohata theorem.

Theorem 1.1.1 (Lax-Mizohata) *Assume that the Cauchy problem for P is C^∞ well posed near the origin. Then there is a neighborhood U of the origin such that for any $x \in U$, the polynomial $p(x,\xi)$ is hyperbolic polynomial with respect to ξ_0.*

Proof: See, for example, Theorem 23.3.1 in [19]. □

Thus if the Cauchy problem is C^∞ well posed near the origin then the characteristic equation

$$p(x,\xi) = 0$$

has only real roots with respect to ξ_0 for any $x \in U$ and any $\xi' = (\xi_1, ..., \xi_n) \in \mathbb{R}^n \setminus \{0\}$.

Definition 1.1.4 *We say that p (or P) is strictly hyperbolic with respect to the direction x_0 if $p(x,\xi) = 0$ has mutually different real roots for any $x \in \Omega$ and for any $\xi' \in \mathbb{R}^n \setminus \{0\}$.*

There is a beautiful theory for strictly hyperbolic operators, see for example, [33], [13]. Here we only refer to the well-posedness of the Cauchy problem.

Theorem 1.1.2 *Assume that p is strictly hyperbolic in Ω in the direction x_0. Then the Cauchy problem is C^∞ well posed for any lower order term. In particular strictly hyperbolic operator is strongly hyperbolic.*

Hence we are mainly interested in what happens if we have a (real) multiple characteristic root.

1.2 Ivrii-Petkov condition

Before stating the Ivrii-Petkov condition, we study multiple roots of a hyperbolic polynomial. Let us study

$$f(t,s) = t^r + f_1(s)t^{r-1} + \cdots + f_r(s) \tag{1.2.1}$$

where $f_i(s) \in C^\infty(J)$ and J is an open interval containing the origin. We assume that $f(t,s) = 0$ has only real roots with respect to t for any $s \in J$. We also assume that

$$f_i(0) = 0, \quad i = 1, 2, ..., r \tag{1.2.2}$$

that is $t = 0$ is a r folded zero of $f(t,0)$.

Lemma 1.2.1 *Assume* (1.2.2)*. Then we have*

$$f_i(s) = O(s^i) \text{ as } s \to 0, \quad i = 1, 2, ..., r$$

and one can write

$$f(t,s) = f_{(0,0)}(t,s) + O(|t| + |s|)^{r+1}$$

where $f_{(0,0)}(t,s)$ is of homogeneous of degree r and hyperbolic with respect to t for all $s \in \mathbb{R}$.

REMARK: Note that $f_{(0,0)}(t,s)$ is given by

$$f(\mu t, \mu s) = \mu^r\{f_{(0,0)}(t,s) + O(\mu)\}, \quad \mu \to 0.$$

Proof: Take $\sigma_j \in \mathbb{N}$ such that $f_j(s) = O(s^{\sigma_j})$ (if $f_j(s) = O(s^k)$ for any k then we take σ_j sufficiently large). Put

$$\min_{1\le j\le r} \frac{\sigma_j}{j} = \lambda = \frac{q}{p} > 0$$

where p, q are relatively prime. We first prove $f_i(s) = O(s^i)$. It is enough to prove $\lambda \ge 1$. We suppose $0 < \lambda < 1$ and derive a contradiction. Plug $t = w|s|^\lambda$ into $f(t,s) = 0$ which yields

$$0 = \sum_{j=0}^{r} w^j |s|^{\lambda j} f_{r-j}(s), \;\; f_0(s) = 1.$$

Multiplying $|s|^{-\lambda r}$ we get

$$0 = \sum_{j=0}^{r} w^j f_{r-j}(s)|s|^{-\lambda(r-j)}.$$

Let $s \to \pm 0$ then we have

$$0 = \sum_{j=0}^{r} w^j f^{\pm}_{r-j}, \quad f^{\pm}_{r-j} = \lim_{s\to\pm 0} |s|^{-\lambda(r-j)} f_{r-j}(s). \tag{1.2.3}$$

By the assumption there is at least one $0 \le j \le r-1$ such that $f^{\pm}_{r-j} \neq 0$. We first note that the equation (1.2.3) has r real roots. Otherwise since $f^{\pm}_0 = f_0(s) = 1$, by Rouché's theorem, $f(t,s) = 0$ would have a non real root for small s which contradicts the assumption. We first treat the case $q > 2$. If $f^{\pm}_j \neq 0$ then $\sigma_j q = pj$ and hence $j = nq$ with some $n \in \mathbb{N}$. Then (1.2.3) with $+$ sign is reduced to

$$w^r + a_1 w^{r-q} + \cdots + a_l w^{r-lq} = 0.$$

The equation (1.2.3) with $-$ sign is reduced to a similar equation. One can express

$$w^r\left(1 + a_1(\frac{1}{w})^q + \cdots + a_l(\frac{1}{w})^{lq}\right) = 0, \;\; (a_l \neq 0).$$

With $W = (1/w)^q$ this turns out to be

$$a_l W^l + \cdots + a_1 W + 1 = 0. \tag{1.2.4}$$

Noting that (1.2.4) has a non zero root, W, we get a non real root w from $w^q = 1/W$ because $q > 2$ and hence a contradiction. We turn to the case $q = 2$ and hence $p = 1$. From the same arguments (1.2.3) is reduced to

$$w^r + a^{\pm}_1 w^{r-2} + \cdots + a^{\pm}_l w^{r-2l} = 0.$$

Since $f_{2k}(s) = s^k(a_{2k} + O(s))$, $s \to 0$ we see that $a_k^+ = a_k^-$ if k is even and $a_k^+ = -a_k^-$ if k is odd. As before we are led to

$$w^r \left(1 + a_1^\pm (\frac{1}{w})^2 + \cdots + a_l^\pm (\frac{1}{w})^{2l}\right) = 0.$$

With $W = (1/w)^2$ we have

$$a_l^\pm W^l + \cdots + a_1^\pm W + 1 = 0. \tag{1.2.5}$$

As observed above, W and $-W$ are the root of (1.2.5) at the same time and hence from $w^2 = 1/W$ we get a non real root and a contradiction. Thus we have proved that $\lambda \geq 1$ and hence the result.

We turn to the second assertion. Set $t = ws$ and plug this into $f(t, s) = 0$. Then we have

$$\begin{aligned} s^{-r} f(t, s) &= w^r + a_1 w^{r-1} + \cdots + a_r + sg(w, s) \\ &= f_{(0,0)}(w, 1) + sg(w, s). \end{aligned}$$

From this we see that $f_{(0,0)}(w, 1) = 0$ has only real roots. Since

$$f_{(0,0)}(t, s) = s^r f_{(0,0)}(\frac{t}{s}, 1)$$

we get the desired assertion. □

Definition 1.2.1 *Let $P(\zeta)$ be a polynomial and assume that $P(\eta) = 0$. We define $P_\eta(\zeta)$ by*

$$P(\eta + \mu\zeta) = \mu^r\{P_\eta(\zeta) + O(\mu)\},\ \mu \to 0,\ P_\eta(\zeta) \not\equiv 0.$$

We call $P_\eta(\zeta)$ the localization of $P(\zeta)$ at η. This is nothing but

$$P_\eta(\zeta) = \sum_{|\alpha|=r} \frac{1}{\alpha!}(\frac{\partial}{\partial\zeta})^\alpha P(\eta)\zeta^\alpha$$

and hence of homogeneous of degree r.

We study now hyperbolic polynomial with parameter $x \in \mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be a neighborhood of the origin of $\mathbb{R}^n$ and

$$P(t, s) = t^r + f_1(x)t^{r-1} + \cdots + f_r(x), \quad f_i(0) = 0$$

which is hyperbolic with respect to t for all $x \in U$.

Lemma 1.2.2 *Under the above assumption we have*

$$(\frac{\partial}{\partial x})^\alpha f_j(0) = 0,\ |\alpha| \leq j - 1$$

and the localization $P_{(0,0)}(t, x)$ at $(0, 0)$ is hyperbolic with respect to t for all $x \in U$.

Proof: Fix $\eta \in \mathbb{R}^n$, $\eta \neq 0$ and consider

$$P(t, s\eta) = f(t, s; \eta) = t^r + f_1(s\eta)t^{r-1} + \cdots + f_r(s\eta).$$

For $|s| < \delta$, $f(t, s; \eta)$ is hyperbolic with respect to t. From Lemma 1.2.1 it follows that

$$(\frac{d}{ds})^k f_j(s\eta)|_{s=0} = \langle \eta, \frac{\partial}{\partial x}\rangle^k f_j(0) = 0, \quad k \leq j-1.$$

Since η is arbitrary this shows the assertion. On the other hand

$$P(\mu t, \mu x) = f(\mu t, \mu; x) = \mu^r\{f_{(0,0)}(t, 1; x) + O(\mu)\}$$

where $f_{(0,0)}(t, 1; x)$ is hyperbolic with respect to t by Lemma 1.2.1. This shows that $P_{(0,0)}(t, x) = f_{(0,0)}(t, 1; x)$ and hence the result. □

We finally study the general case

$$P(t, x) = t^m + a_1(x)t^{m-1} + \cdots + a_m(x)$$

which is hyperbolic with respect to t for any $x \in U$. Assume that

(1.2.6) $$(\frac{\partial}{\partial t})^k P(\hat{t}, \hat{x}) = 0, \ k = 0, 1, ..., r-1, \ \ (\frac{\partial}{\partial t})^r P(\hat{t}, \hat{x}) \neq 0.$$

Then we have

Corollary 1.2.1 *Assume* (1.2.6). *Then*

$$(\frac{\partial}{\partial t})^j (\frac{\partial}{\partial x})^\alpha P(\hat{t}, \hat{x}) = 0, \quad j + |\alpha| \leq r-1$$

and the localization $P_{(\hat{t},\hat{x})}(t, x)$ is hyperbolic with respect to t for any $x \in \mathbb{R}^n$.

Proof: We first note that there is a neighborhood V of $\hat{x}$ such that one can write

$$P(t, x) = Q(t, x)R(t, x), \quad x \in V$$

where Q and R are hyperbolic polynomials in t of degree r and $m-r$ respectively and

$$(\frac{\partial}{\partial t})^k Q(\hat{t}, \hat{x}) = 0, \ k = 0, ..., r-1, \ \ R(\hat{t}, \hat{x}) \neq 0.$$

Applying Lemma 1.2.1 to $Q(\hat{t} + t, \hat{x} + x)$ to get

$$Q(\hat{t} + t, \hat{x} + x) = Q_{(\hat{t},\hat{x})}(t, x) + O(|t| + |x|)^{r+1}.$$

Since $R(\hat{t}, \hat{x}) \neq 0$ we get

$$P(\hat{t} + t, \hat{x} + x) = R(\hat{t}, \hat{x})Q_{(\hat{t},\hat{x})}(t, x) + O(|t| + |x|)^{r+1}.$$

This proves the assertion. □

Definition 1.2.2 *A point $(\bar{x}, \bar{\xi})$ is called a characteristic (point) of order r if*

$$\partial_\xi^\alpha \partial_x^\beta p(\bar{x}, \bar{\xi}) = 0,\ |\alpha + \beta| < r,\ \partial_\xi^\alpha \partial_x^\beta p(\bar{x}, \bar{\xi}) \neq 0 \text{ some } |\alpha + \beta| = r.$$

Lemma 1.2.3 *Assume that $p(x, \xi)$ is a hyperbolic polynomial with respect to ξ_0 and*

$$\left(\frac{\partial}{\partial \xi_0}\right)^j p(\bar{x}, \bar{\xi}) = 0,\ 0 \le j \le r-1, \quad \left(\frac{\partial}{\partial \xi_0}\right)^r p(\bar{x}, \bar{\xi}) \neq 0.$$

Then $(\bar{x}, \bar{\xi})$ is a characteristic of order r.

Proof: By Corollary 1.2.1. □

The next result is due to [22] and called the Ivrii-Petkov (necessary) condition.

Theorem 1.2.1 (Ivrii-Petkov) *Assume that the Cauchy problem for P is C^∞ well posed near the origin and $(0, \bar{\xi})$ is a characteristic of order r. Then we have*

$$\partial_x^\beta \partial_\xi^\alpha P_{m-j}(0, \bar{\xi}) = 0, \quad |\alpha + \beta| < r - 2j.$$

We give a proof of this theorem in Section 2.4.

Corollary 1.2.2 *Assume that p is strongly hyperbolic near the origin. Then every multiple characteristic is at most double.*

1.3 Implications of well posedness

We start with the next lemma.

Lemma 1.3.1 *Assume that the Cauchy problem for P is C^∞ well posed near the origin. Then there are open neighborhood ω and $\epsilon > 0$ such that for any compact set $K \subset\subset \omega$ and $p \in \mathbb{N}$ there are $C > 0$, $q \in \mathbb{N}$ such that*

$$\|u\|_{H^p(K^t)} \le C\|Pu\|_{H^q(K^t)}$$

for any $u \in C_0^\infty(K_{-\epsilon})$ and $|t| < \epsilon$ where $K^t = \{x \in K \mid x_0 \le t\}$ and similarly $K_t = \{x \in K \mid x_0 \ge t\}$.

Proof: Take an open set V so that $K \subset\subset V \subset\subset \omega$. Let us define F_M, $M = 1, 2, ...$ by

$$F_M = \{f \in C_0^\infty(\overline{V_{-\epsilon}}) \mid \exists u \in H^p(\omega) \text{ such that}$$
$$Pu = f \text{ in } \omega, \|u\|_{H^p(\omega)} \le M, u = 0 \text{ in } x_0 \le -\epsilon\}.$$

By the well-posedness assumption, it is clear that

$$\bigcup_{M=1}^\infty F_M = C_0^\infty(\overline{V_{-\epsilon}}).$$

It is also clear that F_M is symmetric and convex. Let $F_M \ni f_j \to f$ in $C_0^\infty(\overline{V_{-\epsilon}})$. Then there exist u_j such that $Pu_j = f_j$ and $\|u_j\|_{H^p(\omega)} \le M$, taking a subsequence, we may suppose that

$$u_j \to u \text{ in } H^{p-1}_{loc}(\omega) \text{ and } u_j \to u \text{ weak in } H^p(\omega),\ u \in H^p(\omega).$$

It is clear that $Pu = f$ in ω and $u = 0$ in $x_0 \le -\epsilon$. This shows that F_M is closed. Since $C_0^\infty(\overline{V_{-\epsilon}})$ is a complete metric space, then from the Baire's category theorem some F_M contains a neighborhood of 0 in $C_0^\infty(\overline{V_{-\epsilon}})$. That is, there exist $q \ge 0$ and $\delta > 0$ such that

$$f \in C_0^\infty(\overline{V_{-\epsilon}}),\ \ \|f\|_{H^q(V)} \le \delta \Longrightarrow f \in F_M.$$

For any $f \in C_0^\infty(\overline{V_{-\epsilon}})$, taking $\delta f/\|f\|_{H^q(V)}$ which is now in F_M, we get

$$\|u\|_{H^p(\omega)} \le M\delta^{-1}\|f\|_{H^q(V)}. \tag{1.3.1}$$

We summarize that for any $f \in C_0^\infty(\overline{V_{-\epsilon}})$, the solution u to $Pu = f$ in ω vanishing in $x_0 \le -\epsilon$ satisfies (1.3.1).

Let $u \in C_0^\infty(K_{-\epsilon})$. Take $\chi \in C_0^\infty(V)$ so that $\chi = 1$ on K. Take $g \in \mathcal{S}(\mathbb{R}^{n+1})$ so that $Pu = g$ in $x_0 < t$. Then the solution v, vanishing in $x_0 \le -\epsilon$, to $Pv = \chi g$ coincides with u in $x_0 < t$ as we remarked after Definition 1.1.1. Hence

$$\begin{aligned}\|v\|_{H^p(V^t)} = \|u\|_{H^p(V^t)} &\le C_0\|\chi g\|_{H^q(V)}\\ &\le C_0'\|g\|_{H^q(V)} \le C_0''\|g\|_{H^q(\mathbb{R}^{n+1})}.\end{aligned}$$

Since this holds for any $g \in \mathcal{S}(\mathbb{R}^{n+1})$ provided $g = Pu$ in $x_0 < t$, this shows that

$$\|u\|_{H^p(V^t)} \le C_0''\|Pu\|_{H^q(\{x_0<t\})} = C_0''\|Pu\|_{H^q(K^t)}$$

and hence the result. □

Corollary 1.3.1 *Assume that the Cauchy problem for P is C^∞ well posed near the origin. Then there are a neighborhood ω of the origin and $\epsilon > 0$ such that for any compact set $K \subset\subset \omega$ one can find $C > 0$ and $p \in \mathbb{N}$ such that*

$$|u|_{C^0(K^t)} \le C|Pu|_{C^p(K^t)}$$

for any $u \in C_0^\infty(K_{-\epsilon})$, $|t| < \epsilon$ where $|u|_{C^p(K)} = \sup_{x\in K,|\alpha|\le p}|\partial_x^\alpha u(x)|$.

Proof: By Sobolev embedding theorem. □

Let $P(x,\xi)$ be the full symbol of $P(x,D)$. Let us set

$$P_\lambda(x,\xi) = P(y(\lambda) + \lambda^{-\sigma}x, \lambda^\kappa\eta(\lambda) + \lambda^\sigma\xi)$$

where $\sigma = (\sigma_0, ..., \sigma_n)$, $\lambda^{-\sigma}x = (\lambda^{-\sigma_0}x_0, ..., \lambda^{-\sigma_n}x_n)$ and $y(\lambda)$, $\eta(\lambda)$ are $\mathbb{R}^{n+1}$ valued continuous functions defined near $\lambda = \infty$.

Lemma 1.3.2 *Assume that $0 \in \Omega$ and $y(\infty) = 0$ and the Cauchy problem for P is C^∞ well posed near the origin. Then for every compact set $W \subset \mathbb{R}^{n+1}$ and every positive $T > 0$ there are $C > 0$, $\bar\lambda > 0$ and $p \in \mathbb{N}$ such that*

$$|u|_{C^0(W^t)} \le C\lambda^{(\kappa+\bar\sigma)p}|P_\lambda u|_{C^p(W^t)}$$

for any $u \in C_0^\infty(W)$, $\lambda \ge \bar\lambda$, $|t| < T$ where $\bar\sigma = \max_j \sigma_j$.

Proof: Let $u \in C_0^\infty(K)$ and put $v(x) = e^{i\lambda^\kappa\langle\eta(\lambda),x\rangle}u(x) \in C_0^\infty(K)$. Then from Corollary 1.3.1 we get

$$\begin{aligned} |v|_{C^0(K^t)} &= |u|_{C^0(K^t)} \le C|Pv|_{C^p(K^t)} \\ &= C|e^{i\lambda^\kappa\langle\eta(\lambda),x\rangle}\tilde P u|_{C^p(K^t)} \le C'\lambda^{\kappa p}|\tilde P u|_{C^p(K^t)} \end{aligned} \tag{1.3.2}$$

where $\tilde P(x,D) = e^{-i\lambda^\kappa\langle\eta(\lambda),x\rangle}P(x,D)e^{i\lambda^\kappa\langle\eta(\lambda),x\rangle} = P(x,\lambda^\kappa\eta(\lambda)+D)$. Take a compact set K so that K contains the origin and $\inf\{x_0 \mid x \in K\} > -\epsilon$. Let $W \subset \mathbb{R}^{n+1}$ be a given compact set. Then there is λ_1 such that for any $u \in C_0^\infty(W)$ we have

$$u(\lambda^\sigma(x-y(\lambda))) \in C_0^\infty(K) \quad \text{if} \quad \lambda \ge \lambda_1$$

remarking $y(\infty) = 0$. Set $v(x) = u(\lambda^\sigma(x-y(\lambda)))$ and $t(\lambda) = \lambda^{-\sigma_0}s + y_0(\lambda)$. Take λ_2 so that $\lambda \ge \lambda_2$ and $|s| < T$ implies $|t(\lambda)| < \epsilon$. Then from (1.3.2) it follows that

$$|v|_{C^0(K^{t(\lambda)})} \le C\lambda^{\kappa p}|\tilde P v|_{C^p(K^{t(\lambda)})}.$$

This shows that, by the change of coordinates $z = \lambda^\sigma(x-y(\lambda))$,

$$|u|_{C^0(W^s)} \le C'\lambda^{\kappa p+\bar\sigma p}|P_\lambda u|_{C^p(W^s)}$$

and hence the result. □

1.4 Proof of Ivrii-Petkov condition

We give a proof of Theorem 1.2.1 following [22] but some techniques of the proof are little bit refined which are taken from [8]. Recall that we are interested in a differential operator of order m

$$P(x,D) = P_m(x,D) + P_{m-1}(x,D) + \cdots.$$

Assume that $(\bar x,\bar\xi)$ is a characteristic of order r. Taking a new system of local coordinates, we may assume, without restrictions, that $(\bar x,\bar\xi) = (0,e_n)$ where $e_n = (0,...,0,1)$.

Let us denote $z = (0,e_n)$ and put

$$P_{m-j}(z+\mu(x,\xi)) = \mu^{s_j}\{P_{m-j,z}(x,\xi) + O(\mu)\}$$

where $P_{m-j,z}(x,\xi)$ is the localization of P_{m-j} at z (Definition 1.1.1). Assume that Theorem 1.2.1 does not hold, that is there is $j \geq 1$ such that $s_j < r - 2j$. Let us define

$$\theta_0 = \min_{j, s_j < r-2j} \left\{ \frac{j}{r - s_j} \right\}$$

then $\theta_0 < 1/2$ by assumption. Put

$$\hat{P}(x,\xi) = \sum_{m-r\theta_0 = m-j-s_j\theta_0, j\geq 0} P_{m-j,z}(x,\xi).$$

Lemma 1.4.1 *There is $(\bar{x}, \bar{\xi}')$ near $(0,0)$ such that $\hat{P}(\bar{x}, \xi_0, \bar{\xi}') = 0$ has a non real root.*

Proof: Let us write

$$q(x,\xi) = \sum_{m-r\theta_0 = m-j-s_j\theta_0, j\geq 1} P_{m-j,z}(x,\xi).$$

We first assume $q(0, \xi_0, 0) \not\equiv 0$. Then one can write

$$\hat{P}(0, \xi_0, 0) = \xi_0^r + \sum_{r\theta_0 = j + s_j\theta_0} a_j \xi_0^{s_j}.$$

Put $\theta_0 = p/q$ where p and q are relatively prime. Since $\theta_0 < 1/2$ we get $q \geq 3$. Then $(r - s_j)\theta_0 = j$ implies $r - s_j = nq$ with some $n \in \mathbb{N}$. Hence we can express

$$\hat{P}(0, \xi_0, 0) = \xi_0^r \left(1 + \sum \tilde{a}_l \left(\frac{1}{\xi_0}\right)^{lq} \right).$$

Now to prove that $\hat{P}(0, \xi_0, 0) = 0$ has a non real root it is enough to repeat the same arguments as in the proof of Lemma 1.2.1.

We now assume that $q(0, \xi_0, 0) \equiv 0$. It is clear that there is $(\bar{x}, \bar{\xi}')$ such that $q(s\bar{x}, \xi_0, s\bar{\xi}')$ is not identically zero in (ξ_0, s) by the assumption. Consider

$$f(\xi_0, s) = \hat{P}(s\bar{x}, \xi_0, s\bar{\xi}') = \xi_0^r + \sum_{j\geq 1} f_j(s) \xi_0^{r-j}$$

which verifies $f(\xi_0, 0) = \xi_0^r$. From Lemma 1.2.1 it follows that $f_j(s) = O(s^j)$ if $f(\xi_0, s) = 0$ has only real roots for small s. But this is not the case by the assumption again. This ends the proof. □

Put $\sigma_0 = 1 - 2\theta_0$ and study

$$P(\lambda^{-\theta_0}x, \lambda e_n + \lambda^{\theta_0}D) = \sum_{j=0}^{m} P_{m-j}(\lambda^{-\theta_0}x, \lambda e_n + \lambda^{\theta_0}D)$$

$$= \sum_{j=0}^{m} \lambda^{m-j} P_{m-j}(\lambda^{-\theta_0}x, e_n + \lambda^{-\theta_0-\sigma_0}D)$$

$$= \sum_{j=0}^{m} \sum_{k\geq 0, m-j-s_j\theta_0-k\theta_0>-M} \lambda^{m-j-s_j\theta_0-k\theta_0}$$

$$\times \sum_{|\alpha+\beta|=s_j+k} \frac{1}{\alpha!\beta!} P^{(\alpha)}_{(\beta)}(0, e_n) x^{\beta} (\lambda^{-\sigma_0}D)^{\alpha} + O(\lambda^{-M})$$

where M is a sufficiently large integer and by $O(\lambda^{-M})$ we denote a differential operator whose coefficients are bounded by λ^{-M} on any preassigned open set U in $\mathbb{R}^{n+1}$ and $P^{(\alpha)}_{(\beta)}(x,\xi) = \partial_x^{\beta}\partial_{\xi}^{\alpha}P(x,\xi)$. Let us set

$$G^{(0)}(x,\xi;\lambda) = \sum_{j=0}^{m} \sum_{m-j-s_j\theta_0-k\theta_0>-M} \lambda^{-j+(r-s_j)\theta_0-k\theta_0}$$

$$\times \sum_{|\alpha+\beta|=s_j+k} \frac{1}{\alpha!\beta!} P^{(\alpha)}_{(\beta)}(0, e_n) x^{\beta}\xi^{\alpha}$$

so that

$$P_{\lambda}(x,D) = \lambda^{m-r\theta_0} G^{(0)}(x, \lambda^{-\sigma_0}D; \lambda) + O(\lambda^{-M}).$$

It is useful to rewrite $G^{(0)}(x,\xi;\lambda)$ in the following way

$$G^{(0)}(x,\xi;\lambda) = \sum_{j\geq 0} \lambda^{\delta_j(G^{(0)})} G^{(0)}_j(x,\xi)$$

where $0 = \delta_0(G^{(0)}) < \delta_1(G^{(0)}) \cdots$. It is clear that

$$G^{(0)}_0(x,\xi) = \sum_{j-(r-s_j)\theta_0=0} P_{m-j,z}(x,\xi). \tag{1.4.1}$$

Definition 1.4.1 *We say that a differential operator $P(x,D;\lambda)$ with a parameter λ is in $\mathcal{R}(U)$ if there are $\kappa \in \mathbb{Q}_+$ and differential operators $P_j(x,D)$ with coefficients in $C^{\infty}(U)$ such that*

$$P(x,D;\lambda) = \sum_{j=0} \lambda^{-\kappa j} P_j(x,D)$$

where it is understood that the sum is finite.

Lemma 1.4.2 *Let $G(x,D)$ be a differential operator with coefficients in $C^{\infty}(U)$ and let σ, $\theta \in \mathbb{Q}_+$ be such that $\sigma \geq \theta > 0$ and let $\phi \in C^{\infty}(U)$. Then*

(i) *we have*

$$e^{-i\lambda^\theta \phi} G(x, \lambda^{-\sigma} D) e^{i\lambda^\theta \phi} = G(x, \lambda^{-(\sigma-\theta)}(\phi_x + \lambda^{-\theta} D)) + \lambda^{-\theta} r(x, \lambda^{-\theta} D; \lambda)$$

with $r(x, \xi; \lambda) \in \mathcal{R}(U)$.

(ii) *If* $G(x, \xi) = O(|\xi|^q)$ *as* $\xi \to 0$ *uniformly with respect to* $x \in U$, *then*

$$\begin{aligned} e^{-i\lambda^\theta \phi} G(x, \lambda^{-\sigma} D) e^{i\lambda^\theta \phi} &= G(x, \lambda^{-(\sigma-\theta)}(\phi_x + \lambda^{-\theta} D)) \\ &\quad + \lambda^{-(\sigma-\theta)q-\theta} r(x, \lambda^{-\theta} D; \lambda) \end{aligned}$$

with $r(x, \xi; \lambda) \in \mathcal{R}(U)$.

REMARK: It is important to remark that in the notation above the quantity $G(x, \lambda^{-(\sigma-\theta)}(\phi_x + \lambda^{-\theta} D))$ does not contain any term in which the derivatives land on $\phi_x(x)$, as will be clear from the proof, these terms are pushed into the "error" terms r and thus $G(x, \lambda^{-(\sigma-\theta)}(\phi_x + \lambda^{-\theta} D))$ is to be thought as a commutative expression.

Proof: Denote by $\psi(x, y) = \phi(x) - \phi(y) - \langle y - x, \phi_x(x) \rangle$. Then if $u(x)$ is smooth we have

$$\begin{aligned} & e^{-i\lambda^\theta \phi(x)} G(x, \lambda^{-\sigma} D) e^{i\lambda^\theta \phi(x)} u(x) \\ = & \sum_{\alpha \geq 0} \frac{1}{\alpha!} G^{(\alpha)}(x, \lambda^{-(\sigma+\theta)} \phi_x(x)) (\lambda^{-\sigma} D_y)^\alpha \left[e^{\lambda^\theta \psi(x,y)} u(y) \right]_{y=x} \\ & = \sum_{\alpha \geq 0} \frac{1}{\alpha!} G^{(\alpha)}(x, \lambda^{-(\sigma+\theta)} \phi_x(x)) (\lambda^{-\sigma} D_x)^\alpha u(x) \\ & + \sum_{\alpha \geq 0} \frac{1}{\alpha!} G^{(\alpha)}(x, \lambda^{-(\sigma+\theta)} \phi_x(x)) \\ \times & \sum_{2 \leq |\beta| \leq |\alpha|} \binom{\alpha}{\beta} \left[(\lambda^{-\sigma} D_y)^\beta e^{i\lambda^\theta \psi(x,y)} (\lambda^{-\sigma} D_y)^{\alpha-\beta} u(y) \right]_{y=x}. \end{aligned}$$

The first term is what has been called $G(x, \lambda^{-(\sigma-\theta)}(\phi_x(x) + \lambda^{-\sigma} D))$. Let us take a closer look at the second term. Due to the vanishing of $\psi(x, y)|_{y=x}$ and $\nabla \psi(x, y)|_{y=x}$ the quantity $D^\beta e^{i\lambda^\theta \psi(x,y)}|_{y=x}$ is a polynomial in the variable λ^θ of degree less than or equal to $[|\beta|/2]$, the integral part of $|\beta|/2$. Factoring out $\lambda^{\theta|\beta|/2}$ we obtain a polynomial of the same degree in the variable λ^θ. Thus the second sum above can be written as

$$\sum_{\alpha\geq 0}\frac{1}{\alpha!}G^{(\alpha)}(x,\lambda^{-(\sigma-\theta)}\phi_x(x))\lambda^{-(\sigma-\theta)|\alpha|}$$
$$\times\sum_{2\leq|\beta|\leq|\alpha|}\lambda^{-\theta|\alpha|+\theta|\beta|/2}P_{\alpha,\beta,\phi}(x;\lambda^{-\theta})D^{\alpha-\beta}u(x)$$
$$=\sum_{\alpha\geq 0}\frac{1}{\alpha!}G^{(\alpha)}(x,\lambda^{-(\sigma-\theta)}\phi_x(x))\lambda^{-(\sigma-\theta)|\alpha|}$$
$$\times\sum_{\nu=1}^{[|\alpha|/2]}\lambda^{-\theta\nu}\sum_{|\beta|=2\nu}P_{\alpha,\beta,\phi}(x;\lambda^{-\theta})(\lambda^{-\theta}D)^{\alpha-\beta}u(x).$$

With

$$b_{k\nu}(x,D;\lambda)=\lambda^{-(\sigma-\theta)k}\sum_{|\alpha|=k}\frac{1}{\alpha!}G^{(\alpha)}(x,\lambda^{-(\sigma-\theta)}\phi_x(x))$$
$$\times\sum_{|\beta|=2\nu}P_{\alpha,\beta,\phi}(x;\lambda^{-\theta})(\lambda^{-\theta}D)^{\alpha-\beta}$$

this can be rewritten as

$$\lambda^{-\theta}r(x,\lambda^{-\theta}D;\lambda),\quad r(x,D;\lambda)=\sum_{k=0}\sum_{\nu=1}^{[k/2]}b_{k\nu}(x,D;\lambda).$$

It is clear that $r(x,\xi;\lambda)\in\mathcal{R}(U)$.

We turn to the second assertion. It is obvious that nothing is changed in the first term, so that all we have to do is just look at the second term. If $k\geq q$ then

$$\lambda^{-(\sigma-\theta)k}\leq\lambda^{-(\sigma-\theta)q}.$$

If $k<q$ then our assumption implies that

$$G^{(\alpha)}(x,\xi)=O(|\xi|^{q-k})$$

and hence $G^{(\alpha)}(x,\lambda^{-(\sigma-\theta)}\phi_x)=O(\lambda^{-(\sigma-\theta)(q-k)})$. This proves the assertion. □

From the assumption we may start off assuming that there is an analytic function $\tau_0(x,\xi')$ with $\mathsf{Im}\,\tau_0\neq 0$ such that

$$\det G_0^{(0)}(x,\xi)=(\xi_0-\tau_0(x,\xi'))^{q_0}\Delta_0(x,\xi),\ \Delta_0(x,\tau_0(x,\xi'),\xi')\neq 0$$

in some open set $U\times V$ in $\mathbb{R}^{n+1}\times\mathbb{R}^n$. In the sequel, U and V stands for an open set in $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$ respectively which may differ from line to line but "*the subsequent one is contained in the preceding one*". Denote by $\phi^{(0)}(x)$ a real analytic function in U such that

$$\partial_{x_0}\phi^{(0)}(x)=\tau_0(x,\partial_{x'}\phi^{(0)}(x)).$$

Then we have

$$
\begin{aligned}
&e^{-i\lambda^{\sigma_0}\phi^{(0)}(x)}G^{(0)}(x,\lambda^{-\sigma_0}D;\lambda)e^{i\lambda^{\sigma_0}\phi^{(0)}(x)}\\
&= G^{(0)}(x,\phi_x^{(0)}(x)+\lambda^{-\sigma_0}D;\lambda)+\lambda^{-\sigma_0}R^{(0)}(x,\lambda^{-\sigma_0}D;\lambda)
\end{aligned}
$$

with $R^{(0)}(x,\xi;\lambda)\in\mathcal{R}(U)$ by Lemma 1.4.2.

We prepare the following lemma for our induction.

Lemma 1.4.3 *Consider a differential operator*

$$
G^{(p)}(x,\phi_x^{(p)}(x)+\lambda^{-\sigma_p}D;\lambda)+\lambda^{-\sigma_p}R^{(p)}(x,\lambda^{-\sigma_p}D;\lambda) \tag{1.4.2}
$$

where $\sigma_p=\sigma_{p-1}-\theta_p$, $\theta_p\in\mathbb{Q}_+$ with $\sigma_{-1}=1-\theta_0$, $R^{(p)}(x,\xi;\lambda)\in\mathcal{R}(U)$, which is verifying

$(i)_p$ $$G^{(p)}(x,\xi;\lambda)=\sum_{j=0}\lambda^{-\delta_j(G^{(p)})}G_j^{(p)}(x,\xi),\ 0=\delta_0(G^{(p)})<\delta_1(G^{(p)})<\cdots$$

the sum being finite and $G_j^{(p)}(x,D)$ denoting a differential operator with real analytic coefficients and $\phi^{(p)}(x)$ is a real analytic function in U such that $\phi_x^{(p)}(x)$ is a root of $G^{(p)}(x,\xi)=0$ with uniform multiplicity q_p, that is

$(ii)_p$ $$\begin{cases}\partial_{x_0}\phi^{(p)}(x)=\tau_p(x,\partial_{x'}\phi^{(p)}(x)) \quad in \quad U,\\ G_0^{(p)}(x,\xi)=(\xi_0-\tau_p(x,\xi'))^{q_p}\Delta_p(x,\xi),\\ \Delta_p(x,\tau_p(x,\xi'),\xi')\neq 0 \ in\ U\times V\end{cases}$$

where $\tau_p(x,\xi')$ is a real analytic in $U\times V$ and there is $k(p)\in\mathbb{N}$ such that

$(iii)_p$ $$\sigma_p,\ \theta_p,\ \delta_j(G^{(p)})\ (j\geq 1)\in\mathbb{N}/k(p).$$

Then we can find $\theta_{p+1}\in\mathbb{Q}_+$ and a real analytic $\phi^{(p+1)}(x)$ in U such that with $\sigma_{p+1}=\sigma_p-\theta_{p+1}$ that

$$
\begin{aligned}
&e^{-i\lambda^{\sigma_{p+1}}\phi^{(p+1)}(x)}\big[G^{(p)}(x,\phi_x^{(p)}(x)+\lambda^{-\sigma_p}D;\lambda)\\
&\qquad+\lambda^{-\sigma_p}R^{(p)}(x,\lambda^{-\sigma_p}D;\lambda)\big]e^{i\lambda^{\sigma_{p+1}}\phi^{(p+1)}(x)}\\
&=\lambda^{-\theta_{p+1}q_p}\big[G^{(p+1)}(x,\phi_x^{(p+1)}(x)+\lambda^{-\sigma_{p+1}}D;\lambda)\\
&\qquad+\lambda^{-\sigma_{p+1}}R^{(p+1)}(x,\lambda^{-\sigma_{p+1}}D;\lambda)\big]
\end{aligned} \tag{1.4.3}
$$

where

$(i)_{p+1}$ $$\begin{cases}G^{(p+1)}(x,\xi;\lambda)=\sum_{j=0}\lambda^{-\delta_j(G^{(p+1)})}G_j^{(p+1)}(x,\xi),\\ 0=\delta_0(G^{(p+1)})<\delta_1(G^{(p+1)})<\cdots\end{cases}$$

the sum being finite and $G_j^{(p+1)}(x,D)$ denoting a differential operator with analytic coefficients and $\phi^{(p+1)}(x)$ is a real analytic function such that $\phi_x^{(p+1)}$ is a *root of* $G_0^{(p+1)}(x,\xi)=0$ *with uniform multiplicity* q_{p+1} *that is*

$$(ii)_{p+1} \qquad \begin{cases} \partial_{x_0}\phi^{(p+1)}(x)=\tau_{p+1}(x,\partial_{x'}\phi^{(p+1)}(x)) \quad \textit{in} \quad U, \\ G_0^{(p+1)}(x,\xi)=(\xi_0-\tau_{p+1}(x,\xi'))^{q_{p+1}}\Delta_{p+1}(x,\xi), \\ \Delta_{p+1}(x,\tau_{p+1}(x,\xi'),\xi')\neq 0 \ \textit{in} \ U\times V \end{cases}$$

where $\tau_{p+1}(x,\xi')$ *is real analytic in* $U\times V$ *and there is* $k(p+1)\in\mathbb{N}$ *such that*

$$(iii)_{p+1} \qquad \sigma_{p+1},\ \theta_{p+1},\ \delta_j(G^{(p+1)})\ (j\geq 1)\in\mathbb{N}/k(p+1).$$

Proof: Set

$$\tilde{G}^{(p)}(x,\xi;\lambda)=G^{(p)}(x,\phi_x^{(p)}(x)+\xi;\lambda)+\lambda^{-\sigma_p}R^{(p)}(x,\xi;\lambda)$$

then $\tilde{G}^{(p)}(x,\xi;\lambda)$ can be written as

$$\tilde{G}^{(p)}(x,\xi;\lambda)=\sum_{j=0}\lambda^{-\delta_j(\tilde{G}^{(p)})}\tilde{G}_j^{(p)}(x,\xi)$$

where $\delta_0(\tilde{G}^{(p)})=\delta_0(G^{(p)})=0$ and

$$\tilde{G}_0^{(p)}(x,\xi)=G_0^{(p)}(x,\phi_x^{(p)}(x)+\xi)$$

and hence $\tilde{G}_0^{(p)}(x,0)=0$. Thus one can write

$$(1.4.4) \qquad \tilde{G}_j^{(p)}(x,\lambda^{-\theta}\xi)=\lambda^{-\theta s_j^p}\big[\hat{G}_j^{(p)}(x,\xi)+O(\lambda^{-\theta})\big]$$

for any $\theta>0$. From $(ii)_p$ it is clear that $s_0^p=q_p$ and

$$\tilde{G}_0^{(p)}(x,\lambda^{-\theta}\xi)=\hat{G}_0^{(p)}(x,\phi_x^{(p)}(x)+\lambda^{-\theta}\xi)$$
$$=\lambda^{-\theta q_p}\Big[\sum_{|\alpha|\leq q_p}\frac{1}{\alpha!}(G_0^{(p)})^{(\alpha)}(x,\phi_x^{(p)}(x))\xi^\alpha+O(\lambda^{-\theta})\Big]$$

hence

$$(1.4.5) \qquad \hat{G}_0^{(p)}(x,\xi)=\sum_{|\alpha|=q_p}\frac{1}{\alpha!}(G_0^{(p)})^{(\alpha)}(x,\phi_x^{(p)}(x))\xi^\alpha.$$

We now define

$$\theta_{p+1}=\min_{j\geq 1,s_j^p\leq s_0^p}\left\{\frac{\delta_j(\tilde{G}^{(p)})}{s_0^p-s_j^p},\theta_p\right\}$$

so that, in particular, $\theta_{p+1}\leq\theta_p$. Let

$$\sigma_{p+1}=\sigma_p-\theta_{p+1}.$$

For our present purpose we shall assume that $\sigma_{p+1}>0$. If $\sigma_{p+1}\leq 0$ we make a different argument in the following.

Let now $\phi^{(p+1)}(x)$ be a real analytic function in U, which we shall precise in the following. Applying Lemma 1.4.2 we compute

$$
\begin{aligned}
&e^{-i\lambda^{\sigma_{p+1}}\phi^{(p+1)}(x)}\tilde{G}^{(p)}(x,\lambda^{-\sigma_p}D;\lambda)e^{i\lambda^{\sigma_{p+1}}\phi^{(p+1)}}\\
&=\sum_{j\geq 0}\lambda^{-\delta_j(\tilde{G}^{(p)})}\tilde{G}_j^{(p)}(x,\lambda^{-\theta_{p+1}}(\phi_x^{(p+1)}(x)+\lambda^{-\sigma_{p+1}}D))\\
&\quad+\sum_{j\geq 0}\lambda^{-\delta_j(\tilde{G}^{(p)})-\theta_{p+1}s_j^p-\sigma_{p+1}}\tilde{R}_j^{(p+1)}(x,\lambda^{-\sigma_{p+1}}D;\lambda)
\end{aligned}
\tag{1.4.6}
$$

where $\tilde{R}_j^{(p+1)}(x,\xi;\lambda)\in\mathcal{R}(U)$. Define $G^{(p+1)}(x,\xi;\lambda)$ and $R^{(p+1)}(x,\xi;\lambda)$ by

$$
\begin{aligned}
\tilde{G}^{(p)}(x,\lambda^{-\theta_{p+1}}\xi;\lambda)&=\lambda^{-\theta_{p+1}s_0^p}G^{(p+1)}(x,\xi;\lambda)\\
&=\lambda^{-\theta_{p+1}s_0^p}\sum_{j\geq 0}\lambda^{-\delta_j(G^{(p+1)})}G_j^{(p+1)}(x,\xi)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{j\geq 0}\lambda^{-\delta_j(\tilde{G}^{(p)})-\theta_{p+1}s_j^p}\tilde{R}_j^{(p+1)}(x,\xi;\lambda)\\
&\qquad=\lambda^{-\theta_{p+1}s_0^p}R^{(p+1)}(x,\xi;\lambda).
\end{aligned}
$$

(Note that $\delta_j(\tilde{G}^{(p)})+\theta_{p+1}s_j^p\geq\theta_{p+1}s_0^p$). Then this proves (1.4.4). From (1.4.4) we obtain

$$
G_0^{(p+1)}(x,\xi)=\sum_{\theta_{p+1}s_0^p=\theta_{p+1}s_j^p+\delta_j(\tilde{G}^{(p)})}\hat{G}_j^{(p)}(x,\xi)
\tag{1.4.7}
$$

where $\hat{G}_j^{(p)}(x,\xi)$ being homogeneous of degree s_j^p with respect to ξ. Then it is clear that $\hat{G}_0^{(p+1)}(x,\xi)$ is a polynomial in ξ of degree $q_0=s_0^p$ and the coefficient of $\xi_0^{q_0}$ is different from zero. Then one can find some open sets U and V and real analytic $\tau_{p+1}(x,\xi')$ defined in $U\times V$, and real analytic $\phi^{(p+1)}(x)$ in U such that $(ii)_{p+1}$ holds. This proves the lemma. □

Lemma 1.4.4 *Assume that there exists a* $\bar{p}\in\mathbb{N}$ *such that*

$$q_{\bar{p}}=q_{\bar{p}+1}=\cdots=q.$$

Then there exists a $k=k(\bar{p})\in\mathbb{N}$ *such that for all* $p\geq\bar{p}$

$$\sigma_p,\ \theta_p,\ \delta_j(G^{(p)}),\ j\geq 1 \text{ belong to } \mathbb{N}/k.$$

Proof: The fact that $q_p=q_{p+1}$ implies that there is no roots of $G_0^{(p+1)}(x,\xi)=0$ with respect to ξ_0 with uniform multiplicity less than q_p. Two cases may occur; either the sum in (1.4.7) has $\hat{G}_0^{(p)}$ as the only summand or there are also other summands. In the former case we have

$$\theta_{p+1}s_0^p<\theta_{p+1}s_j^p+\delta_j(\tilde{G}^{(p)})\text{ for every }j\geq 0$$

which implies that

$$\theta_{p+1} < \frac{\delta_j(\tilde{G}^{(p)})}{s_0^p - s_j^p}$$

that is, $\theta_{p+1} = \theta_p$. Assume now that there are terms other than $\hat{G}_0^{(p)}$, corresponding to $j > 0$. Then the condition defining the sum implies that there is $j \geq 1$ such that

$$\delta_j(\tilde{G}^{(p)}) = \theta_{p+1}$$

because of the following lemma.

Lemma 1.4.5 *Let*

$$f(\tau) = \sum_{j=0}^{s} a_j \tau^{q_j}$$

where $0 = q_0 < q_1 < \cdots < q_s$ *and* $a_j \neq 0$. *Then the roots of* $f(\tau) = 0$ *have multiplicity at most* s.

In both cases we conclude that either $\theta_{p+1} = \theta_p$ or $\theta_{p+1} = \delta_j(\tilde{G}^{(p)})$ holds. In particular this implies $\theta_{p+1} \in \mathbb{N}/k(p)$ and hence $k(p+1) = k(p)$ since $\delta_j(G^{(p+1)})$ are obtained summing and multiplying rational numbers whose denominator is $k(p)$. □

From Lemma 1.4.4 the above iteration procedure occurs only a finite number of times before reaching a point where

$$\sigma_{p+1} = \sigma_0 - \sum_{i=1}^{\bar{p}} \theta_i \leq 0$$

for a suitable integer $\bar{p}$. We may assume for a certain $t > 0$ that

$$\sigma_t > 0, \quad \sigma_{t+1} = \sigma_t - \theta_{t+1} \leq 0$$

that is

$$\theta_{t+1} = \min\left\{\frac{\delta_j(\tilde{G}^{(t)})}{s_0^t - s_j^t}, \theta_t\right\} \geq \sigma_t.$$

Our purpose is to construct an asymptotic null solution to the operator

$$\tilde{G}^{(t)}(x, \lambda^{-\sigma_t} D; \lambda) = G^{(t)}(x, \phi_x^{(t)}(x) + \lambda^{-\sigma_t} D; \lambda) + \lambda^{-\sigma_t} R^{(t)}(x, \lambda^{-\sigma_t} D; \lambda)$$

where $R^{(t)}(x, \xi; \lambda) \in \mathcal{R}(U)$. With

$$\tilde{G}^{(t)}(x, \xi; \lambda) = \sum_{j \geq 0} \lambda^{-\delta_j(\tilde{G}^{(t)})} \tilde{G}_j^{(t)}(x, \xi)$$

repeating the same arguments as in the proof of Lemma 1.4.3 one can write

$$\begin{aligned} \tilde{G}^{(t)}(x, \lambda^{-\sigma_t} D; \lambda) &= \sum_{j \geq 0} \lambda^{-\delta_j(\tilde{G}^{(t)}) - s_j^t \sigma_t} \left[\hat{G}_j^{(t)}(x, D) + O(\lambda^{-\sigma_t})\right] \\ &= \lambda^{-\sigma_t s_0^t} \Big[\sum_{\sigma_t s_0^t = \sigma_t s_j^t + \delta_j(\tilde{G}^{(t)})} \hat{G}_j^{(t)}(x, D) + \sum_{j \geq 1} \lambda^{-\bar{\delta}_j} K_j(x, D) \Big] \end{aligned}$$

because $\delta_j(\tilde{G}^{(p)}) + \sigma_t s_j^t \geq \sigma_t s_0^t$ where $0 < \delta_1 < \delta_2 < \cdots$. Since $\hat{G}_0^{(t)}(x,D)$ is a differential operator of order s_0^t which is non characteristic with respect to $x_0 =$const, disposing of the power λ in front of the operator in square brackets, we are left with the operator

$$P(x,D) + \sum_{j\geq 1} \lambda^{-j/k} P_j(x,D) \tag{1.4.8}$$

where $P(x,D)$ has the principal part $\hat{G}^{(t)}(x,D)$ and $P_j(x,D)$ are differential operators. One can then seek an asymptotic null solution to (1.4.8) in the form

$$\sum_{j\geq 0} \lambda^{-j/k} u_j(x).$$

By the Cauchy-Kowalevski theorem we solve $u_j(x)$ successively with $u_0(x) \neq 0$. Note that we may assume that

$$\mathsf{Im}\, \tau_0(x,\xi') \leq -c \quad \text{in} \quad U \times V$$

with some $c > 0$ where $(\hat{x}, \hat{\xi}') \in U \times V$. We solve $\phi^{(0)}(x)$ under the condition

$$\phi^{(0)}(\hat{x}_0, x') = i|x' - \hat{x}'|^2 + \langle x', \hat{\xi}' \rangle.$$

Then it is easy to see that $\phi^{(0)}(x)$ verifies

$$\mathsf{Im}\, \phi^{(0)}(x) \geq c\{\hat{x}_0 - x_0 + |x' - \hat{x}'|^2\}, \;\; x_0 \leq \hat{x}_0$$

near $\bar{x}$ with some $c > 0$. The rest of the proof is a repetition of standard arguments (e.g. Theorem 23.3.1 in [19]). □