CHAPTER 5

Reduction of Fuchsian differential equations

Additions and middle convolutions introduced in Chapter 1 are transformations within Fuchsian differential operators and in this chapter we examine how their Riemann schemes change under the transformations.

PROPOSITION 5.1. i) *Let $Pu = 0$ be a Fuchsian differential equation. Suppose there exists $c \in \mathbb{C}$ such that $P \in (\partial - c)W[x]$. Then $c = 0$.*

ii) *For $\phi(x) \in \mathbb{C}(x)$, $\lambda \in \mathbb{C}$, $\mu \in \mathbb{C}$ and $P \in W[x]$, we have*

$$P \in \mathbb{C}[x]\,\mathrm{RAdei}\big(-\phi(x)\big) \circ \mathrm{RAdei}\big(\phi(x)\big)P, \tag{5.1}$$

$$P \in \mathbb{C}[\partial]\,\mathrm{RAd}\big(\partial^{-\mu}\big) \circ \mathrm{RAd}\big(\partial^{\mu}\big)P. \tag{5.2}$$

In particular, if the equation $Pu = 0$ is irreducible and $\operatorname{ord} P > 1$, $\mathrm{RAd}\big(\partial^{-\mu}\big) \circ \mathrm{RAd}\big(\partial^{\mu}\big)P = cP$ with $c \in \mathbb{C}^\times$.

PROOF. i) Put $P = (\partial - c)Q$. Then there is a function $u(x)$ satisfying $Qu(x) = e^{cx}$. Since $Pu = 0$ has at most a regular singularity at $x = \infty$, there exist $C > 0$ and $N > 0$ such that $|u(x)| < C|x|^N$ for $|x| \gg 1$ and $0 \le \arg x \le 2\pi$, which implies $c = 0$.

ii) This follows from the fact

$$\begin{aligned}&\mathrm{Adei}\big(-\phi(x)\big) \circ \mathrm{Adei}\big(\phi(x)\big) = \mathrm{id},\\ &\mathrm{Adei}\big(\phi(x)\big)f(x)P = f(x)\,\mathrm{Adei}\big(\phi(x)\big)P \quad (f(x) \in \mathbb{C}(x))\end{aligned}$$

and the definition of $\mathrm{RAdei}\big(\phi(x)\big)$ and $\mathrm{RAd}(\partial^\mu)$. □

The addition and the middle convolution transform the Riemann scheme of the Fuchsian differential equation as follows.

THEOREM 5.2. *Let $Pu = 0$ be a Fuchsian differential equation with the Riemann scheme* (4.15). *We assume that P has the normal form* (4.43).

i) (addition) *The operator $\mathrm{Ad}\big((x - c_j)^\tau\big)P$ has the Riemann scheme*

$$\left\{\begin{matrix} x = c_0 = \infty & c_1 & \cdots & c_j & \cdots & c_p \\ [\lambda_{0,1} - \tau]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{j,1} + \tau]_{(m_{j,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0} - \tau]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{j,n_j} + \tau]_{(m_{j,1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{matrix}\right\}.$$

ii) (middle convolution) *Fix $\mu \in \mathbb{C}$. By allowing the condition $m_{j,1} = 0$, we may assume*

$$\mu = \lambda_{0,1} - 1 \quad \text{and} \quad \lambda_{j,1} = 0 \text{ for } j = 1, \dots, p \tag{5.3}$$

and $\#\{j\,;\, m_{j,1} < n\} \ge 2$ and P is of the normal form (4.43). *Putting*

$$d := \sum_{j=0}^{p} m_{j,1} - (p-1)n, \tag{5.4}$$

we suppose

$$m_{j,1} \geq d \quad \textit{for} \quad j = 0, \dots, p, \tag{5.5}$$

$$\begin{cases} \lambda_{0,\nu} \notin \{0, -1, -2, \dots, m_{0,1} - m_{0,\nu} - d + 2\} \\ \textit{if} \quad m_{0,\nu} + \cdots + m_{p,1} - (p-1)n \geq 2, \ m_{1,1} \cdots m_{p,1} \neq 0 \quad \textit{and} \quad \nu \geq 1, \end{cases} \tag{5.6}$$

$$\begin{cases} \lambda_{0,1} + \lambda_{j,\nu} \notin \{0, -1, -2, \dots, m_{j,1} - m_{j,\nu} - d + 2\} \\ \textit{if} \quad m_{0,1} + \cdots + m_{j-1,1} + m_{j,\nu} + m_{j+1,1} + \cdots + m_{p,1} - (p-1)n \geq 2, \\ m_{j,1} \neq 0, \ 1 \leq j \leq p \quad \textit{and} \quad \nu \geq 2. \end{cases} \tag{5.7}$$

Then $S := \partial^{-d} \operatorname{Ad}(\partial^{-\mu}) \prod_{j=1}^{p} (x - c_j)^{-m_{j,1}} P \in W[x]$ *and the Riemann scheme of* S *equals*

$$\begin{Bmatrix} x = c_0 = \infty & c_1 & \cdots & c_p \\ [1-\mu]_{(m_{0,1}-d)} & [0]_{(m_{1,1}-d)} & \cdots & [0]_{(m_{p,1}-d)} \\ [\lambda_{0,2}-\mu]_{(m_{0,2})} & [\lambda_{1,2}+\mu]_{(m_{1,2})} & \cdots & [\lambda_{p,2}+\mu]_{(m_{p,2})} \\ \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0}-\mu]_{(m_{0,n_0})} & [\lambda_{1,n_1}+\mu]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}+\mu]_{(m_{p,n_p})} \end{Bmatrix}. \tag{5.8}$$

More precisely, the condition (5.5) *and the condition* (5.6) *for* $\nu = 1$ *assure* $S \in W[x]$*. In this case the condition* (5.6) (resp. (5.7) for a fixed j) *assures that the sets of characteristic exponents of* P *at* $x = \infty$ (resp. c_j) *are equal to the sets given in* (5.8)*, respectively.*

Here we have $\operatorname{RAd}(\partial^{-\mu}) \operatorname{R} P = S$*, if*

$$\begin{cases} \lambda_{j,1} + m_{j,1} \quad \textit{are not characteristic exponents of } P \\ \qquad \textit{at } x = c_j \textit{ for } j = 0, \dots, p, \ \textit{respectively,} \end{cases} \tag{5.9}$$

and moreover

$$m_{0,1} = d \quad \textit{or} \quad \lambda_{0,1} \notin \{-d, -d-1, \dots, 1 - m_{0,1}\}. \tag{5.10}$$

Using the notation in Definition 1.3*, we have*

$$\begin{aligned} S = {} & \operatorname{Ad}\bigl((x-c_1)^{\lambda_{0,1}-2}\bigr)(x-c_1)^d T^*_{\frac{1}{x-c_1}} (-\partial)^{-d} \operatorname{Ad}(\partial^{-\mu}) T^*_{\frac{1}{x}+c_1} \\ & \cdot (x-c_1)^d \prod_{j=1}^{p} (x-c_j)^{-m_{j,1}} \operatorname{Ad}\bigl((x-c_1)^{\lambda_{0,1}}\bigr) P \end{aligned} \tag{5.11}$$

under the conditions (5.5) *and*

$$\begin{cases} \lambda_{0,\nu} \notin \{0, -1, -2, \dots, m_{0,1} - m_{0,\nu} - d + 2\} \\ \textit{if} \quad m_{0,\nu} + m_{1,1} + \cdots + m_{p,1} - (p-1)n \geq 2, \ m_{1,1} \neq 0 \quad \textit{and} \quad \nu \geq 1. \end{cases} \tag{5.12}$$

iii) *Suppose* $\operatorname{ord} P > 1$ *and* P *is irreducible in* ii)*. Then the conditions* (5.5)*,* (5.6)*,* (5.7) *are valid. The condition* (5.10) *is also valid if* $d \geq 1$*.*

All these conditions in ii) *are valid if* $\#\{j\,;\, m_{j,1} < n\} \geq 2$ *and* $\mathbf{m}$ *is realizable and moreover* $\lambda_{j,\nu}$ *are generic under the Fuchs relation with* $\lambda_{j,1} = 0$ *for* $j = 1, \dots, p$*.*

iv) *Let* $\mathbf{m} = \bigl(m_{j,\nu}\bigr)_{\substack{j=0,\dots,p \\ \nu=1,\dots,n_j}} \in \mathcal{P}^{(n)}_{p+1}$*. Define* d *by* (5.4)*. Suppose* $\lambda_{j,\nu}$ *are complex numbers satisfying* (5.3)*. Suppose moreover* $m_{j,1} \geq d$ *for* $j = 1, \dots, p$*.*

Defining $\mathbf{m}' \in \mathcal{P}_{p+1}^{(n)}$ *and* $\lambda'_{j,\nu}$ *by*

$$m'_{j,\nu} = m_{j,\nu} - \delta_{\nu,1} d \quad (j = 0,\dots,p,\ \nu = 1,\dots,n_j), \tag{5.13}$$

$$\lambda'_{j,\nu} = \begin{cases} 2 - \lambda_{0,1} & (j = 0,\ \nu = 1), \\ \lambda_{j,\nu} - \lambda_{0,1} + 1 & (j = 0,\ \nu > 1), \\ 0 & (j > 0,\ \nu = 1), \\ \lambda_{j,\nu} + \lambda_{0,1} - 1 & (j > 0,\ \nu > 1), \end{cases} \tag{5.14}$$

we have

$$\operatorname{idx}\mathbf{m} = \operatorname{idx}\mathbf{m}', \quad |\{\lambda_{\mathbf{m}}\}| = |\{\lambda'_{\mathbf{m}'}\}|. \tag{5.15}$$

Proof. The claim i) is clear from the definition of the Riemann scheme.

ii) Suppose (5.5), (5.6) and (5.7). Then

$$P' := \Bigl(\prod_{j=1}^{p} (x - c_j)^{-m_{j,1}}\Bigr) P \in W[x]. \tag{5.16}$$

Note that $\mathrm{R}\,P = P'$ under the condition (5.9). Put $Q := \partial^{(p-1)n - \sum_{j=1}^p m_{j,1}} P'$. Here we note that (5.5) assures $(p-1)n - \sum_{j=1}^p m_{j,1} \ge 0$.

Fix a positive integer j with $j \le p$. For simplicity suppose $j = 1$ and $c_j = 0$. Since $P' = \sum_{j=0}^n a_j(x)\partial^j$ with $\deg a_j(x) \le (p-1)n + j - \sum_{j=1}^p m_{j,1}$, we have

$$x^{m_{1,1}} P' = \sum_{\ell=0}^{N} x^{N-\ell} r_\ell(\vartheta) \prod_{\substack{1\le\nu\le n_0 \\ 0\le i<m_{0,\nu}-\ell}} (\vartheta + \lambda_{0,\nu} + i)$$

and

$$N := (p-1)n - \sum_{j=2}^{p} m_{j,1} = m_{0,1} + m_{1,1} - d$$

with suitable polynomials r_ℓ such that $r_0 \in \mathbb{C}^\times$. Suppose

$$\prod_{\substack{1\le\nu\le n_0 \\ 0\le i<m_{0,\nu}-\ell}} (\vartheta + \lambda_{0,\nu} + i) \notin xW[x] \ \text{ if } \ N - m_{1,1} + 1 \le \ell \le N. \tag{5.17}$$

Since $P' \in W[x]$, we have

$$x^{N-\ell} r_\ell(\vartheta) = x^{N-\ell} x^{\ell-N+m_{1,1}} \partial^{\ell-N+m_{1,1}} s_\ell(\vartheta) \ \text{ if } \ N - m_{1,1} + 1 \le \ell \le N$$

for suitable polynomials s_ℓ. Putting $s_\ell = r_\ell$ for $0 \le \ell \le N - m_{1,1}$, we have

$$\begin{aligned} P' = &\sum_{\ell=0}^{N-m_{1,1}} x^{N-m_{1,1}-\ell} s_\ell(\vartheta) \prod_{\substack{1\le\nu\le n_0 \\ 0\le i<m_{0,\nu}-\ell}} (\vartheta + \lambda_{0,\nu} + i) \\ &+ \sum_{\ell=N-m_{1,1}+1}^{N} \partial^{\ell-N+m_{1,1}} s_\ell(\vartheta) \prod_{\substack{1\le\nu\le n_0 \\ 0\le i<m_{0,\nu}-\ell}} (\vartheta + \lambda_{0,\nu} + i). \end{aligned} \tag{5.18}$$

Note that $s_0 \in \mathbb{C}^\times$ and the condition (5.17) is equivalent to the condition $\lambda_{0,\nu}+i \ne 0$ for any ν and i such that there exists an integer ℓ with $0 \le i \le m_{0,\nu} - \ell - 1$ and $N - m_{1,1} + 1 \le \ell \le N$. This condition is valid if (5.6) is valid, namely, $m_{1,1} = 0$ or

$$\lambda_{0,\nu} \notin \{0, -1, \dots, m_{0,1} - m_{0,\nu} - d + 2\}$$

for ν satisfying $m_{0,\nu} \ge m_{0,1} - d + 2$. Under this condition we have

$$Q = \sum_{\ell=0}^{N} \partial^\ell s_\ell(\vartheta) \prod_{1\le i\le N-m_{1,1}-\ell} (\vartheta + i) \cdot \prod_{\substack{1\le\nu\le n_0\\ 0\le i<m_{0,\nu}-\ell}} (\vartheta + \lambda_{0,\nu} + i),$$

$$\begin{aligned}\mathrm{Ad}(\partial^{-\mu})Q = \sum_{\ell=0}^{N} \partial^\ell s_\ell(\vartheta-\mu) &\prod_{1\le i\le N-m_{1,1}-\ell} (\vartheta - \mu + i)\\ \cdot \prod_{1\le i\le m_{0,1}-\ell} (\vartheta + i) \cdot &\prod_{\substack{2\le\nu\le n_0\\ 0\le i<m_{0,\nu}-\ell}} (\vartheta - \mu + \lambda_{0,\nu} + i)\end{aligned}$$

since $\mu = \lambda_{0,1} - 1$. Hence $\partial^{-m_{0,1}}\mathrm{Ad}(\partial^{-\mu})Q$ equals

$$\sum_{\ell=0}^{m_{0,1}-1} x^{m_{0,1}-\ell} s_\ell(\vartheta-\mu) \prod_{1\le i\le N-m_{1,1}-\ell} (\vartheta - \mu + i) \prod_{\substack{2\le\nu\le n_0\\ 0\le i<m_{0,\nu}-\ell}} (\vartheta - \mu + \lambda_{0,\nu} + i)$$

$$+ \sum_{\ell=m_{0,1}}^{N} \partial^{\ell-m_{0,1}} s_\ell(\vartheta-\mu) \prod_{1\le i\le N-m_{1,1}-\ell} (\vartheta - \mu + i) \prod_{\substack{2\le\nu\le n_0\\ 0\le i<m_{0,\nu}-\ell}} (\vartheta - \mu + \lambda_{0,\nu} + i)$$

and then the set of characteristic exponents of this operator at ∞ is

$$\{[1-\mu]_{(m_{0,1}-d)}, [\lambda_{0,2}-\mu]_{(m_{0,2})}, \ldots, [\lambda_{0,n_0}-\mu]_{(m_{0,n_0})}\}.$$

Moreover $\partial^{-m_{0,1}-1}\mathrm{Ad}(\partial^{-\mu})Q \notin W[x]$ if $\lambda_{0,1}+m_{0,1}$ is not a characteristic exponent of P at ∞ and $-\lambda_{0,1}+1+i \ne m_{0,1}+1$ for $1 \le i \le N - m_{1,1} = m_{0,1} - d$, which assures $x^{m_{0,1}} s_0 \prod_{1\le i\le N-m_{1,1}} (\vartheta - \mu + i) \prod_{\substack{2\le\nu\le n_0\\ 0\le i<m_{0,\nu}}} (\vartheta - \mu + \lambda_{1,\nu} + i) \notin \partial W[x]$.

Similarly we have

$$\begin{aligned}P' = \sum_{\ell=0}^{m_{1,1}} \partial^{m_{1,1}-\ell} q_\ell(\vartheta) &\prod_{\substack{2\le\nu\le n_1\\ 0\le i<m_{1,\nu}-\ell}} (\vartheta - \lambda_{1,\nu} - i)\\ + \sum_{\ell=m_{1,1}+1}^{N} x^{\ell-m_{1,1}} q_\ell(\vartheta) &\prod_{\substack{2\le\nu\le n_1\\ 0\le i<m_{1,\nu}-\ell}} (\vartheta - \lambda_{1,\nu} - i),\end{aligned}$$

$$\begin{aligned}Q = \sum_{\ell=0}^{m_{1,1}} \partial^{N-\ell} q_\ell(\vartheta) &\prod_{\substack{2\le\nu\le n_1\\ 0\le i<m_{1,\nu}-\ell}} (\vartheta + \lambda_{1,\nu} - i)\\ + \sum_{\ell=m_{1,1}+1}^{N} \partial^{N-\ell} q_\ell(\vartheta) &\prod_{i=1}^{\ell-m_{1,1}} (\vartheta + i) \prod_{\substack{2\le\nu\le n_1\\ 0\le i<m_{1,\nu}-\ell}} (\vartheta - \lambda_{1,\nu} - i).\end{aligned}$$

$$\begin{aligned}\mathrm{Ad}(\partial^{-\mu})Q = \sum_{\ell=0}^{N} \partial^{N-\ell} q_\ell(\vartheta-\mu) &\prod_{1\le i\le \ell-m_{1,1}} (\vartheta - \mu + i)\\ \cdot \prod_{\substack{2\le\nu\le n_1\\ 0\le i<m_{1,\nu}-\ell}} &(\vartheta - \mu - \lambda_{1,\nu} - i)\end{aligned}$$

with $q_0 \in \mathbb{C}^\times$. Then the set of characteristic exponents of $\partial^{-m_{0,1}}\mathrm{Ad}(\partial^{-\mu})Q$ equals

$$\{[0]_{(m_{1,1}-d)}, [\lambda_{1,2}+\mu]_{(m_{1,2})}, \ldots, [\lambda_{1,n_1}+\mu]_{(m_{1,n_1})}\}$$

if

$$\prod_{\substack{2\le\nu\le n_1\\0\le i<m_{1,\nu}-\ell}}(\vartheta-\mu-\lambda_{1,\nu}-i)\notin\partial W[x]$$

for any integers ℓ satisfying $0\le\ell\le N$ and $N-\ell<m_{0,1}$. This condition is satisfied if (5.7) is valid, namely, $m_{0,1}=0$ or

$$\lambda_{0,1}+\lambda_{1,\nu}\notin\{0,-1,\dots,m_{1,1}-m_{1,\nu}-d+2\}$$
$$\text{for}\ \ \nu\ge2\ \ \text{satisfying}\ \ m_{1,\nu}\ge m_{1,1}-d+2$$

because $m_{1,\nu}-\ell-1\le m_{1,\nu}+m_{0,1}-N-2=m_{1,\nu}-m_{1,1}+d-2$ and the condition $\vartheta-\mu-\lambda_{1,\nu}-i\in\partial W[x]$ means $-1=\mu+\lambda_{1,\nu}+i=\lambda_{0,1}-1+\lambda_{1,\nu}+i$.

Now we will prove (5.11). Under the conditions, it follows from (5.18) that

$$\begin{aligned}
\tilde P&:=x^{m_{0,1}-N}\operatorname{Ad}\bigl(x^{\lambda_{0,1}}\bigr)\prod_{j=2}^{p}(x-c_j)^{-m_{j,1}}P\\
&=x^{m_{0,1}+m_{1,1}-N}\operatorname{Ad}\bigl(x^{\lambda_{0,1}}\bigr)P'\\
&=\sum_{\ell=0}^{N}x^{m_{0,1}-\ell}\operatorname{Ad}\bigl(x^{\lambda_{0,1}}\bigr)s_\ell(\vartheta)\prod_{0\le\nu<\ell-N+m_{1,1}}(\vartheta-\nu)\prod_{\substack{1\le\nu\le n_0\\0\le i<m_{0,\nu}-\ell}}(\vartheta+\lambda_{0,\nu}+i),\\
\tilde Q&:=(-\partial)^{N-m_{0,1}}T^*_{\frac1x}\tilde P\\
&=(-\partial)^{N-m_{0,1}}\sum_{\ell=0}^{N}x^{\ell-m_{0,1}}s_\ell(-\vartheta-\lambda_{0,1})\prod_{0\le\nu<\ell-N+m_{1,1}}(-\vartheta-\lambda_{0,1}-\nu)\\
&\quad\cdot\prod_{\substack{2\le\nu\le n_0\\0\le i<m_{0,\nu}-\ell}}(-\vartheta+\lambda_{0,\nu}-\lambda_{0,1}+i)\prod_{0\le i\le m_{0,1}-\ell}(-\vartheta+i)\\
&=\sum_{\ell=0}^{N}(-\partial)^{N-\ell}s_\ell(-\vartheta-\lambda_{0,1})\prod_{1\le i\le\ell-m_{0,1}}(-\vartheta-i)\\
&\quad\cdot\prod_{0\le\nu<\ell-N+m_{1,1}}(-\vartheta-\lambda_{0,1}-\nu)\prod_{\substack{2\le\nu\le n_0\\0\le i<m_{0,\nu}-\ell}}(-\vartheta+\lambda_{0,\nu}-\lambda_{0,1}+i)
\end{aligned}$$

and therefore

$$\begin{aligned}
\operatorname{Ad}(\partial^{-\mu})\tilde Q=\sum_{\ell=0}^{N}&(-\partial)^{N-\ell}s_\ell(-\vartheta-1)\prod_{1\le i\le\ell-m_{0,1}}(-\vartheta+\lambda_{0,1}-1-i)\\
&\cdot\prod_{0\le\nu<\ell-N+m_{1,1}}(-\vartheta-1-\nu)\prod_{\substack{2\le\nu\le n_0\\0\le i<m_{0,\nu}-\ell}}(-\vartheta+\lambda_{0,\nu}-1+i).
\end{aligned}$$

Since

$$\begin{aligned}
(-\partial)^{N-\ell-m_{1,1}}\prod_{0\le\nu<\ell-N+m_{1,1}}(-\vartheta-1-\nu)&=\begin{cases}x^{\ell-N+m_{1,1}}&(N-\ell<m_{1,1})\\(-\partial)^{N-\ell-m_{1,1}}&(N-\ell\ge m_{1,1})\end{cases}\\
&=x^{\ell-N+m_{1,1}}\prod_{0\le\nu<N-\ell-m_{1,1}}(-\vartheta+\nu),
\end{aligned}$$

we have

$$\begin{aligned}\tilde{Q}' := (-\partial)^{-m_{1,1}} \operatorname{Ad}(\partial^{-\mu})\tilde{Q} = \sum_{\ell=0}^{N} x^{\ell-N+m_{1,1}} \prod_{0\le\nu<N-\ell-m_{1,1}} (-\vartheta+\nu) \\ \cdot s_\ell(-\vartheta-1) \prod_{0\le\nu<\ell-m_{0,1}} (-\vartheta+\lambda_{0,1}-2-\nu) \prod_{\substack{2\le\nu\le n_0\\ 0\le i<m_{0,\nu}-\ell}} (-\vartheta+\lambda_{0,\nu}-1+i)\end{aligned}$$

and

$$\begin{aligned}x^{m_{0,1}+m_{1,1}-N} \operatorname{Ad}(x^{\lambda_{0,1}-2}) T^*_{\frac{1}{x}} \tilde{Q}' = \sum_{\ell=0}^{N} x^{m_{0,1}-\ell} \prod_{0\le\nu<\ell-m_{0,1}} (\vartheta-\nu)\cdot s_\ell(\vartheta-\lambda_{0,1}+1) \\ \cdot \prod_{0\le\nu<N-m_{1,1}-\ell} (\vartheta-\lambda_{0,1}+2+\nu) \prod_{\substack{2\le\nu\le n_0\\ 0\le i<m_{0,\nu}-\ell}} (\vartheta+\lambda_{0,\nu}-\lambda_{0,1}+1+i),\end{aligned}$$

which equals $\partial^{-m_{0,1}}\operatorname{Ad}(\partial^{-\mu})Q$ because $\prod_{0\le\nu<k}(\vartheta-\nu) = x^k\partial^k$ for $k\in\mathbb{Z}_{\ge0}$.

iv) (Cf. Remark 7.4 ii) for another proof.) Since

$$\begin{aligned}\operatorname{idx}\mathbf{m} - \operatorname{idx}\mathbf{m}' &= \sum_{j=0}^{p} m_{j,1}^2 - (p-1)n^2 - \sum_{j=0}^{p}(m_{j,1}-d)^2 + (p-1)(n-d)^2 \\ &= 2d\sum_{j=0}^{p} m_{j,1} - (p+1)d^2 - 2(p-1)nd + (p-1)d^2 \\ &= d\Big(2\sum_{j=0}^{p} m_{j,1} - 2d - 2(p-1)n\Big) = 0\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}\lambda_{j,\nu} - \sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m'_{j,\nu}\lambda'_{j,\nu} \\ &\qquad = m_{0,1}(\mu+1) - (m_{0,1}-d)(1-\mu) + \mu\Big(n - m_{0,1} - \sum_{j=1}^{p}(n-m_{j,1})\Big) \\ &\qquad = \Big(\sum_{j=0}^{p} m_{j,1} - d - (p-1)n\Big)\mu - m_{0,1}d - (m_{0,1}-d) = d,\end{aligned}$$

we have the claim.

The claim iii) follows from the following lemma when P is irreducible.

Suppose $\lambda_{j,\nu}$ are generic in the sense of the claim iii). Put $\mathbf{m} = \gcd(\mathbf{m})\overline{\mathbf{m}}$. Then an irreducible subspace of the solutions of $Pu = 0$ has the spectral type $\ell'\overline{\mathbf{m}}$ with $1 \le \ell' \le \gcd(\mathbf{m})$ and the same argument as in the proof of the following lemma shows iii). □

The following lemma is known which follows from Scott's lemma (cf. §9.2).

Lemma 5.3. *Let P be a Fuchsian differential operator with the Riemann scheme* (4.15)*. Suppose P is irreducible. Then*

$$\operatorname{idx}\mathbf{m} \le 2. \tag{5.19}$$

Fix $\ell = (\ell_0,\dots,\ell_p)\in\mathbb{Z}_{>0}^{p+1}$ and suppose $\operatorname{ord} P > 1$*. Then*

$$m_{0,\ell_0} + m_{1,\ell_1} + \cdots + m_{p,\ell_p} - (p-1)\operatorname{ord}\mathbf{m} \le m_{k,\ell_k} \quad \textit{for} \quad k = 0,\dots,p. \tag{5.20}$$

Moreover the condition

$$\lambda_{0,\ell_0} + \lambda_{1,\ell_1} + \cdots + \lambda_{p,\ell_p} \in \mathbb{Z} \tag{5.21}$$

implies

$$m_{0,\ell_0} + m_{1,\ell_1} + \cdots + m_{p,\ell_p} \le (p-1)\operatorname{ord}\mathbf{m}. \tag{5.22}$$

PROOF. Let M_j be the monodromy generators of the solutions of $Pu = 0$ at c_j, respectively. Then $\dim Z(M_j) \ge \sum_{\nu=1}^{n_j} m_{j,\nu}^2$ and therefore $\sum_{j=0}^{p}\operatorname{codim} Z(M_j) \le (p+1)n^2 - \big(\operatorname{idx}\mathbf{m} + (p-1)n^2\big) = 2n^2 - \operatorname{idx}\mathbf{m}$. Hence Corollary 9.12 (cf. (9.47)) proves (5.19).

We may assume $\ell_j = 1$ for $j = 0, \dots, p$ and $k = 0$ to prove the lemma. By the map $u(x) \mapsto \prod_{j=1}^{p}(x - c_j)^{-\lambda_{j,1}}u(x)$ we may moreover assume $\lambda_{j,\ell_j} = 0$ for $j = 1, \dots, p$. Suppose $\lambda_{0,1} \in \mathbb{Z}$. We may assume $M_p \cdots M_1 M_0 = I_n$. Since $\dim\ker M_j \ge m_{j,1}$, Scott's lemma (Lemma 9.11) assures (5.22).

The condition (5.20) is reduced to (5.22) by putting $m_{0,\ell_0} = 0$ and $\lambda_{0,\ell_0} = -\lambda_{1,\ell_1} - \cdots - \lambda_{p,\ell_p}$ because we may assume $k = 0$ and $\ell_0 = n_0 + 1$. □

REMARK 5.4. i) Retain the notation in Theorem 5.2. The operation in Theorem 5.2 i) corresponds to the *addition* and the operation in Theorem 5.2 ii) corresponds to Katz's *middle convolution* (cf. [**Kz**]), which are studied by [**DR**] for the systems of Schlesinger canonical form.

The operation $c(P) := \operatorname{Ad}(\partial^{-\mu})\partial^{(p-1)n}P$ is always well-defined for the Fuchsian differential operator of the normal form which has $p+1$ singular points including ∞. This corresponds to the *convolution* defined by Katz. Note that the equation $Sv = 0$ is a quotient of the equation $c(P)\tilde{u} = 0$.

ii) Retain the notation in the previous theorem. Suppose the equation $Pu = 0$ is irreducible and $\lambda_{j,\nu}$ are generic complex numbers satisfying the assumption in Theorem 5.2. Let $u(x)$ be a local solution of the equation $Pu = 0$ corresponding to the characteristic exponent $\lambda_{i,\nu}$ at $x = c_i$. Assume $0 \le i \le p$ and $1 < \nu \le n_i$. Then the irreducible equations $\big(\operatorname{Ad}\big((x - c_j)^r\big)P\big)u_1 = 0$ and $\big(\operatorname{RAd}(\partial^{-\mu}) \circ \operatorname{R} P\big)u_2 = 0$ are characterized by the equations satisfied by $u_1(x) = (x - c_j)^r u(x)$ and $u_2(x) = I_{c_i}^{\mu}(u(x))$, respectively.

Moreover for any integers $k_0, k_1, \dots, k_p$ the irreducible equation $Qu_3 = 0$ satisfied by $u_3(x) = I_{c_i}^{\mu+k_0}\big(\prod_{j=1}^{p}(x - c_j)^{k_j}u(x)\big)$ is isomorphic to the equation $\big(\operatorname{RAd}(\partial^{-\mu}) \circ \operatorname{R} P\big)u_2 = 0$ as $W(x)$-modules (cf. §1.4 and §3.2).

EXAMPLE 5.5 (Okubo type). Suppose $\bar{P}_{\mathbf{m}}(\lambda) \in W[x]$ is of the form (11.35). Moreover suppose $\bar{P}_{\mathbf{m}}(\lambda)$ has the the Riemann scheme (11.34) satisfying (11.33) and $\lambda_{j,\nu} \notin \mathbb{Z}$. Then for any $\mu \in \mathbb{C}$, the Riemann scheme of $\operatorname{Ad}\big(\partial^{-\mu}\big)\bar{P}_{\mathbf{m}}(\lambda)$ equals

$$\begin{Bmatrix} x = c_0 = \infty & c_1 & \cdots & c_p \\ [\lambda_{0,1} - \mu]_{(m_{0,1})} & [0]_{(m_{1,1})} & \cdots & [0]_{(m_{p,1})} \\ [\lambda_{0,2} - \mu]_{(m_{0,2})} & [\lambda_{1,2} + \mu]_{(m_{1,2})} & \cdots & [\lambda_{p,2} + \mu]_{(m_{p,2})} \\ \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0} - \mu]_{(m_{0,n_0})} & [\lambda_{1,n_1} + \mu]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p} + \mu]_{(m_{p,n_p})} \end{Bmatrix}. \tag{5.23}$$

In particular we have $\operatorname{Ad}\big(\partial^{1-\lambda_{0,1}}\big)\bar{P}_{\mathbf{m}}(\lambda) \in \partial^{m_{0,1}}W[x]$.

EXAMPLE 5.6 (exceptional parameters). The Fuchsian differential equation with the Riemann scheme

$$\begin{Bmatrix} x = \infty & 0 & 1 & c \\ [\delta]_{(2)} & [0]_{(2)} & [0]_{(2)} & [0]_{(2)} \\ 2 - \alpha - \beta - \gamma - 2\delta & \alpha & \beta & \gamma \end{Bmatrix}$$

is a Jordan-Pochhammer equation (cf. Example 1.8 ii)) if $\delta \neq 0$, which is proved by the reduction using the operation $\mathrm{RAd}(\partial^{1-\delta})\,\mathrm{R}$ given in Theorem 5.2 ii).

The Riemann scheme of the operator

$$\begin{aligned}P_r &= x(x-1)(x-c)\partial^3\\ &\quad - \big((\alpha+\beta+\gamma-6)x^2 - ((\alpha+\beta-4)c+\alpha+\gamma-4)x + (\alpha-2)c\big)\partial^2\\ &\quad - \big(2(\alpha+\beta+\gamma-3)x - (\alpha+\beta-2)c - (\alpha+\gamma-2) - r\big)\partial\end{aligned}$$

equals

$$\begin{Bmatrix} x=\infty & 0 & 1 & c\\ [0]_{(2)} & [0]_{(2)} & [0]_{(2)} & [0]_{(2)}\\ 2-\alpha-\beta-\gamma & \alpha & \beta & \gamma \end{Bmatrix},$$

which corresponds to a Jordan-Pochhammer operator when $r=0$. If the parameters are generic, $\mathrm{RAd}(\partial)P_r$ is Heun's operator (6.19) with the Riemann scheme

$$\begin{Bmatrix} x=\infty & 0 & 1 & c\\ 2 & 0 & 0 & 0\\ 3-\alpha-\beta-\gamma & \alpha-1 & \beta-1 & \gamma-1 \end{Bmatrix},$$

which contains the accessory parameter r. This transformation doesn't satisfy (5.6) for $\nu=1$.

The operator $\mathrm{RAd}(\partial^{1-\alpha-\beta-\gamma})P_r$ has the Riemann scheme

$$\begin{Bmatrix} x=\infty & 0 & 1 & c\\ \alpha+\beta+\gamma-1 & 0 & 0 & 0\\ \alpha+\beta+\gamma & 1-\beta-\gamma & 1-\gamma-\alpha & 1-\alpha-\beta \end{Bmatrix}$$

and the monodromy generator at ∞ is semisimple if and only if $r=0$. This transformation doesn't satisfy (5.6) for $\nu=2$.

Definition 5.7. Let

$$P = a_n(x)\partial^n + a_{n-1}(x)\partial^{n-1} + \cdots + a_0(x)$$

be a Fuchsian differential operator with the Riemann scheme (4.15). Here some $m_{j,\nu}$ may be 0. Fix $\ell = (\ell_0,\dots,\ell_p) \in \mathbb{Z}_{>0}^{p+1}$ with $1 \le \ell_j \le n_j$. Suppose

$$\#\{j\,;\, m_{j,\ell_j} \neq n \text{ and } 0 \le j \le p\} \ge 2. \tag{5.24}$$

Put

$$d_\ell(\mathbf{m}) := m_{0,\ell_0} + \cdots + m_{p,\ell_p} - (p-1)\operatorname{ord}\mathbf{m} \tag{5.25}$$

and

(5.26)

$$\begin{aligned}\partial_\ell P := {}&\mathrm{Ad}\Big(\prod_{j=1}^p (x-c_j)^{\lambda_{j,\ell_j}}\Big)\prod_{j=1}^p (x-c_j)^{m_{j,\ell_j}-d_\ell(\mathbf{m})}\partial^{-m_{0,\ell_0}}\,\mathrm{Ad}(\partial^{1-\lambda_{0,\ell_0}-\cdots-\lambda_{p,\ell_p}})\\ &\cdot \partial^{(p-1)n-m_{1,\ell_1}-\cdots-m_{p,\ell_p}} a_n^{-1}(x)\prod_{j=1}^p (x-c_j)^{n-m_{j,\ell_j}}\,\mathrm{Ad}\Big(\prod_{j=1}^p (x-c_j)^{-\lambda_{j,\ell_j}}\Big)P.\end{aligned}$$

If $\lambda_{j,\nu}$ are generic under the Fuchs relation or P is irreducible, $\partial_\ell P$ is well-defined as an element of $W[x]$ and

$$\partial_\ell^2 P = P \quad \text{with } P \text{ of the form (4.43)}, \tag{5.27}$$

$$\begin{aligned}\partial_\ell P \in W(x)\,\mathrm{RAd}\Big(\prod_{j=1}^{p}(x-c_j)^{\lambda_{j,\ell_j}}\Big)\,\mathrm{RAd}\big(\partial^{1-\lambda_{0,\ell_0}-\cdots-\lambda_{p,\ell_p}}\big)\\ \cdot\,\mathrm{RAd}\Big(\prod_{j=1}^{p}(x-c_j)^{-\lambda_{j,\ell_j}}\Big)P\end{aligned} \tag{5.28}$$

and ∂_ℓ gives a correspondence between differential operators of normal form (4.43). Here the spectral type $\partial_\ell \mathbf{m}$ of $\partial_\ell P$ is given by

$$\partial_\ell \mathbf{m} := \big(m'_{j,\nu}\big)_{\substack{0\le j\le p\\ 1\le\nu\le n_j}} \quad\text{and}\quad m'_{j,\nu} = m_{j,\nu} - \delta_{\ell_j,\nu}\cdot d_\ell(\mathbf{m}) \tag{5.29}$$

and the Riemann scheme of $\partial_\ell P$ equals

$$\partial_\ell\big\{\lambda_{\mathbf{m}}\big\} := \big\{\lambda'_{\mathbf{m}'}\big\} \quad\text{with}\quad \lambda'_{j,\nu} = \begin{cases}\lambda_{0,\nu}-2\mu_\ell & (j=0,\ \nu=\ell_0)\\ \lambda_{0,\nu}-\mu_\ell & (j=0,\ \nu\neq\ell_0)\\ \lambda_{j,\nu} & (1\le j\le p,\ \nu=\ell_j)\\ \lambda_{0,\nu}+\mu_\ell & (1\le j\le p,\ \nu\neq\ell_j)\end{cases} \tag{5.30}$$

by putting

$$\mu_\ell := \sum_{j=0}^{p}\lambda_{j,\ell_j} - 1. \tag{5.31}$$

It follows from Theorem 5.2 that the above assumption is satisfied if

$$m_{j,\ell_j} \ge d_\ell(\mathbf{m}) \qquad (j=0,\dots,p) \tag{5.32}$$

and

$$\begin{aligned}\sum_{j=0}^{p}\lambda_{j,\ell_j+(\nu-\ell_j)\delta_{j,k}} \notin \Big\{i\in\mathbb{Z}\,;\,(p-1)n-\sum_{j=0}^{p}m_{j,\ell_j+(\nu-\ell_j)\delta_{j,k}}+2\le i\le 0\Big\}\\ \text{for } k=0,\dots,p \text{ and } \nu=1,\dots,n_k.\end{aligned} \tag{5.33}$$

Note that $\partial_\ell\mathbf{m}\in\mathcal{P}_{p+1}$ is *well-defined* for a given $\mathbf{m}\in\mathcal{P}_{p+1}$ if (5.32) is valid. Moreover we define

$$\partial\mathbf{m} := \partial_{(1,1,\dots)}\mathbf{m}, \tag{5.34}$$

$$\begin{aligned}\partial_{max}\mathbf{m} := \partial_{\ell_{max}(\mathbf{m})}\mathbf{m} \quad\text{with}\\ \ell_{max}(\mathbf{m})_j := \min\big\{\nu\,;\, m_{j,\nu}=\max\{m_{j,1},m_{j,2},\dots\}\big\},\end{aligned} \tag{5.35}$$

$$d_{max}(\mathbf{m}) := \sum_{j=0}^{p}\max\{m_{j,1},m_{j,2},\dots,m_{j,n_j}\} - (p-1)\operatorname{ord}\mathbf{m}. \tag{5.36}$$

For a Fuchsian differential operator P with the Riemann scheme (4.15) we define

$$\partial_{max}P := \partial_{\ell_{max}(\mathbf{m})}P \quad\text{and}\quad \partial_{max}\big\{\lambda_{\mathbf{m}}\big\} = \partial_{\ell_{max}(\mathbf{m})}\big\{\lambda_{\mathbf{m}}\big\}. \tag{5.37}$$

A tuple $\mathbf{m}\in\mathcal{P}$ is called *basic* if $\mathbf{m}$ is indivisible and $d_{max}(\mathbf{m})\le 0$.

PROPOSITION 5.8 (linear fractional transformation). *Let ϕ be a linear fractional transformation of $\mathbb{P}^1(\mathbb{C})$, namely there exists $\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\in GL(2,\mathbb{C})$ such that $\phi(x)=\frac{\alpha x+\beta}{\gamma x+\delta}$. Let P be a Fuchsian differential operator with the Riemann scheme* (4.15). *We may assume $-\frac{\delta}{\gamma}=c_j$ with a suitable j by putting $c_{p+1}=-\frac{\delta}{\gamma}$, $\lambda_{p+1,1}=0$ and*

$m_{p+1,1} = n$ *if necessary. Fix* $\ell = (\ell_0, \cdots \ell_p) \in \mathbb{Z}^{p+1}_{>0}$. *If* (5.32) *and* (5.33) *are valid, we have*

$$\begin{aligned}&\partial_\ell P \in W(x)\,\mathrm{Ad}\big((\gamma x+\delta)^{2\mu}\big)T^*_{\phi^{-1}}\partial_\ell T^*_\phi P,\\ &\mu = \lambda_{0,\ell_0} + \cdots + \lambda_{p,\ell_p} - 1.\end{aligned} \tag{5.38}$$

PROOF. The claim is clear if $\gamma = 0$. Hence we may assume $\phi(x) = \frac{1}{x}$ and the claim follows from (5.11). □

REMARK 5.9. i) Fix $\lambda_{j,\nu} \in \mathbb{C}$. If P has the Riemann scheme $\{\lambda_{\mathbf{m}}\}$ with $d_{max}(\mathbf{m}) = 1$, $\partial_\ell P$ is well-defined and $\partial_{max}P$ has the Riemann scheme $\partial_{max}\{\lambda_{\mathbf{m}}\}$. This follows from the fact that the conditions (5.5), (5.6) and (5.7) are valid when we apply Theorem 5.2 to the operation $\partial_{max} : P \mapsto \partial_{max}P$.

ii) We remark that

$$\operatorname{idx}\mathbf{m} = \operatorname{idx}\partial_\ell\mathbf{m}, \tag{5.39}$$

$$\operatorname{ord}\partial_{max}\mathbf{m} = \operatorname{ord}\mathbf{m} - d_{max}(\mathbf{m}). \tag{5.40}$$

Moreover if $\operatorname{idx}\mathbf{m} > 0$, we have

$$d_{max}(\mathbf{m}) > 0 \tag{5.41}$$

because of the identity

$$\Big(\sum_{j=0}^{p} m_{j,\ell_j} - (p-1)\operatorname{ord}\mathbf{m}\Big)\cdot\operatorname{ord}\mathbf{m} = \operatorname{idx}\mathbf{m} + \sum_{j=0}^{p}\sum_{\nu=1}^{n_j}(m_{j,\ell_j} - m_{j,\nu})\cdot m_{j,\nu}. \tag{5.42}$$

If $\operatorname{idx}\mathbf{m} = 0$, then $d_{max}(\mathbf{m}) \ge 0$ and the condition $d_{max}(\mathbf{m}) = 0$ implies $m_{j,\nu} = m_{j,1}$ for $\nu = 2, \ldots, n_j$ and $j = 0, 1, \ldots, p$ (cf. Corollary 6.3).

iii) The set of indices $\ell_{max}(\mathbf{m})$ is defined in (5.35) so that it is uniquely determined. It is sufficient to impose only the condition

$$m_{j,\ell_{max}(\mathbf{m})_j} = \max\{m_{j,1}, m_{j,2}, \ldots\} \qquad (j = 0, \ldots, p) \tag{5.43}$$

on $\ell_{max}(\mathbf{m})$ for the arguments in this paper.

Thus we have the following result.

THEOREM 5.10. *A tuple* $\mathbf{m} \in \mathcal{P}$ *is realizable if and only if* $s\mathbf{m}$ *is trivial* (cf. Definitions 4.10 and 4.11) *or* $\partial_{max}\mathbf{m}$ *is well-defined and realizable.*

PROOF. We may assume $\mathbf{m} \in \mathcal{P}^{(n)}_{p+1}$ is monotone.

Suppose $\#\{j\,;\, m_{j,1} < n\} < 2$. Then $\partial_{max}\mathbf{m}$ is not well-defined. We may assume $p = 0$ and the corresponding equation $Pu = 0$ has no singularities in $\mathbb{C}$ by applying a suitable addition to the equation and then $P \in W(x)\partial^n$. Hence $\mathbf{m}$ is realizable if and only if $\#\{j\,;\, m_{j,1} < n\} = 0$, namely, $\mathbf{m}$ is trivial.

Suppose $\#\{j\,;\, m_{j,1} < n\} \ge 2$. Then Theorem 5.2 assures that $\partial_{max}\mathbf{m}$ is realizable if and only if $\partial_{max}\mathbf{m}$ is realizable. □

In the next chapter we will prove that $\mathbf{m}$ is realizable if $d_{max}(\mathbf{m}) \le 0$. Thus we will have a criterion whether a given $\mathbf{m} \in \mathcal{P}$ is realizable or not by successive applications of ∂_{max}.

EXAMPLE 5.11. There are examples of successive applications of $s \circ \partial$ to monotone elements of $\mathcal{P}$:

$\underline{4}11, \underline{4}11, \underline{4}2, \underline{3}3 \xrightarrow{15-2\cdot 6=3} \underline{1}11, \underline{1}11, \underline{2}1 \xrightarrow{4-3=1} \underline{1}1, \underline{1}1, \underline{1}1 \xrightarrow{3-2=1} 1, 1, 1$ (rigid)

$\underline{2}11, \underline{2}11, \underline{1}111 \xrightarrow{5-4=1} \underline{1}11, \underline{1}11, \underline{1}11 \xrightarrow{3-3=0} 111, 111, 111$ (realizable, not rigid)

$\underline{2}11, \underline{2}11, \underline{2}11, \underline{3}1 \xrightarrow{9-8=1} \underline{1}11, \underline{1}11, \underline{1}11, \underline{2}1 \xrightarrow{5-6=-1}$ (realizable, not rigid)

$\underline{2}2, \underline{2}2, \underline{1}111 \xrightarrow{5-4=1} \underline{2}1, \underline{2}1, \underline{1}11 \xrightarrow{5-3=2} \times$ (not realizable)

The numbers on the above arrows are $d_{(1,1,\ldots)}(\mathbf{m})$. We sometimes delete the trivial partition as above.

The transformation of the generalized Riemann scheme of the application of ∂_{max}^k is described in the following definition.

DEFINITION 5.12 (Reduction of Riemann schemes). Let $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{j=0,\dots,p\\ \nu=1,\dots,n_j}} \in \mathcal{P}_{p+1}$ and $\lambda_{j,\nu} \in \mathbb{C}$ for $j = 0,\dots,p$ and $\nu = 1,\dots,n_j$. Suppose $\mathbf{m}$ is realizable. Then there exists a positive integer K such that

$$\begin{aligned}&\operatorname{ord}\mathbf{m} > \operatorname{ord}\partial_{max}\mathbf{m} > \operatorname{ord}\partial_{max}^2\mathbf{m} > \cdots > \operatorname{ord}\partial_{max}^K\mathbf{m}\\ &\text{and } s\partial_{\max}^K\mathbf{m} \text{ is trivial or } d_{max}\big(\partial_{max}^K\mathbf{m}\big) \le 0.\end{aligned} \tag{5.44}$$

Define $\mathbf{m}(k) \in \mathcal{P}_{p+1}$, $\ell(k) \in \mathbb{Z}$, $\mu(k) \in \mathbb{C}$ and $\lambda(k)_{j,\nu\in\mathbb{C}}$ for $k = 0,\dots,K$ by

$$\mathbf{m}(0) = \mathbf{m} \quad\text{and}\quad \mathbf{m}(k) = \partial_{max}\mathbf{m}(k-1) \quad (k = 1,\dots,K), \tag{5.45}$$

$$\ell(k) = \ell_{max}\big(\mathbf{m}(k)\big) \quad\text{and}\quad d(k) = d_{max}\big(\mathbf{m}(k)\big), \tag{5.46}$$

$$\big\{\lambda(k)_{\mathbf{m}(k)}\big\} = \partial_{max}^k\big\{\lambda_{\mathbf{m}}\big\} \quad\text{and}\quad \mu(k) = \lambda(k+1)_{1,\nu} - \lambda(k)_{1,\nu} \quad (\nu \ne \ell(k)_1). \tag{5.47}$$

Namely, we have

$$\lambda(0)_{j,\nu} = \lambda_{j,\nu} \quad (j = 0,\dots,p,\ \nu = 1,\dots,n_j), \tag{5.48}$$

$$\mu(k) = \sum_{j=0}^{p} \lambda(k)_{j,\ell(k)_j} - 1, \tag{5.49}$$

$$\begin{aligned}\lambda(k+1)_{j,\nu} &= \begin{cases}\lambda(k)_{0,\nu} - 2\mu(k) & (j = 0,\ \nu = \ell(k)_0),\\ \lambda(k)_{0,\nu} - \mu(k) & (j = 0,\ 1 \le \nu \le n_0,\ \nu \ne \ell(k)_0),\\ \lambda(k)_{j,\nu} & (1 \le j \le p,\ \nu = \ell(k)_j),\\ \lambda(k)_{j,\nu} + \mu(k) & (1 \le j \le p,\ 1 \le \nu \le n_j,\ \nu \ne \ell(k)_j)\end{cases}\\ &= \lambda(k)_{j,\nu} + \big((-1)^{\delta_{j,0}} - \delta_{\nu,\ell(k)_j}\big)\mu(k),\end{aligned} \tag{5.50}$$

$$\big\{\lambda_{\mathbf{m}}\big\} \xrightarrow{\partial_{\ell(0)}} \cdots \longrightarrow \big\{\lambda(k)_{\mathbf{m}(k)}\big\} \xrightarrow{\partial_{\ell(k)}} \big\{\lambda(k+1)_{\mathbf{m}(k+1)}\big\} \xrightarrow{\partial_{\ell(k+1)}} \cdots. \tag{5.51}$$