Chapter 3

Geometry and discrete groups

In this section, we will introduce basic materials in the Lie group theory and geometry and discrete group actions on the geometric spaces.

Geometry will be introduced as in the Erlangen program of Klein. We discuss projective geometry in some depth. Hyperbolic geometry will be given an emphasis by detailed descriptions of models. Finally, we discuss the discrete group actions, the Poincaré polyhedron theorem and the crystallographic group theory.

We will not go into details as these are somewhat elementary topics. A good source of the classical geometry is carefully written down in the book [Berger (2009)]. The rest of material is heavily influenced by the books [Ratcliffe (2006); Thurston (1997)]; however, we sketch the material.

3.1 Geometries

We will now describe classical geometries from Lie group action perspectives, as expounded in the Erlangen program of Felix Klein submerging all classical geometries under the theory of Lie group actions: We think of an (G, X)-*geometry* as the invariant properties of a manifold X under a group G acting on it transitively and effectively. Formally, the (G, X)-geometry is simply the pair (G, X) and we should know everything about the (G, X)-geometry from this pair.

Of course, there are many particular hidden treasures under this pair which should surface when we try to study them.

3.1.1 *Euclidean geometry*

The Euclidean space is $\mathbb{R}^n$ (or denoted $\mathbb{E}^n$) and the group $\mathbf{Isom}(\mathbb{R}^n)$ of rigid motions is generated by $\mathbb{O}(n)$ and T_n the translation group. In fact, we have an inner-product giving us a metric.

A system of linear equations gives us a subspace (affine or linear). Hence, we have a notion of points, lines, planes, and angles. Notice that these notions are invariantly defined under the group of rigid motions. These give us the set theoretical model for the axioms of the Euclidean geometry. Very nice elementary introductions can be found in the books [Berger (2009); Ryan (1987)] for example.

3.1.2 *Spherical geometry*

Let us consider the unit sphere $\mathbf{S}^n$ in the Euclidean space $\mathbb{R}^{n+1}$. The transformation group is $\mathbb{O}(n+1, \mathbb{R})$.

Many great spheres exist and they are subspaces as they are given by a homogeneous system of linear equations in $\mathbb{R}^{n+1}$. The lines are replaced by arcs in great circles and lengths and angles are also replaced by arc lengths and angles in the tangent space of $\mathbf{S}^n$.

A *triangle* is a disk bounded by three geodesic arcs meeting transversally in acute angles. Such a triangle up to the action of $\mathbb{O}(n+1, \mathbb{R})$ is classified by their angles $\theta_0, \theta_1, \theta_2$ satisfying

$$0 < \theta_i < \pi \tag{3.1}$$

$$\theta_0 + \theta_1 + \theta_2 > \pi \tag{3.2}$$

$$\theta_i < \theta_{i+1} + \theta_{i+2} - \pi, i \in \mathbb{Z}_3. \tag{3.3}$$

(See Figure 3.2.)

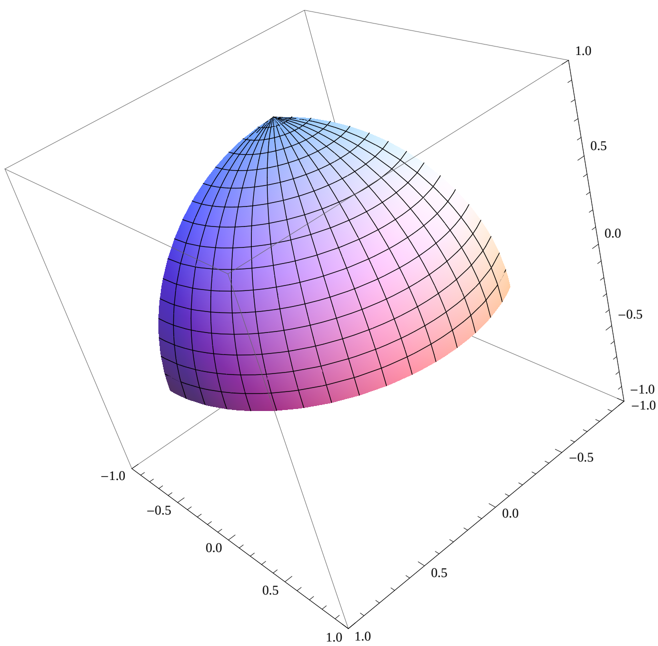

Fig. 3.1 An example of a spherical triangle of angles $2\pi/3, \pi/2, \pi/2$.

Many spherical triangle theorems exist. Given a triangle with angles $\theta_0, \theta_1, \theta_2$

and opposite side lengths l_0, l_1, l_2, we obtain

$$\begin{aligned}
\cos l_i &= \cos l_{i+1} \cos l_{i+2} + \sin l_{i+1} \sin l_{i+2} \cos\theta_i, \\
\cos\theta_i &= -\cos\theta_{i+1}\cos\theta_{i+2} + \sin\theta_{i+1}\sin\theta_{i+2}\cos l_i, \\
\frac{\sin\theta_0}{\sin l_0} &= \frac{\sin\theta_1}{\sin l_1} = \frac{\sin\theta_2}{\sin l_2}, i \in \mathbb{Z}_3.
\end{aligned} \tag{3.4}$$

(See `http://mathworld.wolfram.com/SphericalTrigonometry.html` for more details and proofs.) This shows for example that a triple of angles detemines the isometry classes of spherical triangles. Also, so does the triples of lengths.

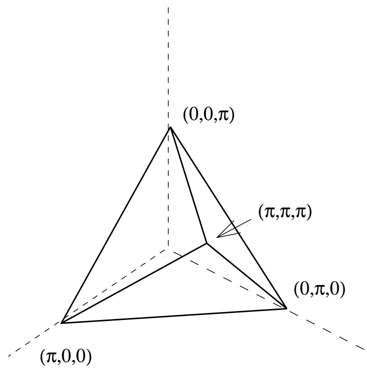

Fig. 3.2 The space of isometric spherical triangles in terms of angle coordinates. See the article [Choi (2011)].

3.1.3 *Affine geometry*

A vector space $\mathbb{R}^n$ becomes an affine space by forgetting about the privileges of the origin. An affine transformation of $\mathbb{R}^n$ is one given by $x \mapsto Ax + b$ for $A \in \mathbb{GL}(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. This notion is more general than that of rigid motions.

The Euclidean space $\mathbb{R}^n$ with the group $\mathbb{A}(\mathbb{R}^n) = \mathbb{GL}(n, \mathbb{R}) \cdot \mathbb{R}^n$ of affine transformations forms the affine geometry. Of course, angles and lengths do not make sense. But the notion of lines exists. Also, the affine subspaces that are linear subspaces translated by vectors make sense.

The set of three points in a line has an invariant based on ratios of lengths.

3.1.4 *Projective geometry*

Projective geometry was first considered from fine art. Desargues (and Kepler) first considered points at infinity from the mathematical point of view. Poncelet first added ideal points to the euclidean plane.

A transformation of projecting one plane to another plane by light rays from a point source which may or may not be at infinity is called a *perspectivity*. Projective transformations are compositions of perspectivities. Often, they send finite points to ideal points and vice versa, e.g., perspectivity between two planes that are not parallel. For example, some landscape paintings will have horizons that are from the "infinity" from vertical perspectives. Therefore, we need to add ideal points while the added points are same as ordinary points up to projective transformations.

Lines have well-defined ideal points and are really circles topologically because we added an ideal point at each pair of a direction and its opposite direction. Some notions such as angles and lengths lose meanings. However, many interesting theorems can be proved. Also, theorems always come in dual pairs by switching lines to points and vice versa. Duality of theorems plays an interesting role (Busemann and Kelly, 1953).

A formal definition with topology was given by Felix Klein using homogeneous coordinates. The projective space $\mathbb{RP}^n$ is defined as the quotient space $\mathbb{R}^{n+1} - \{O\}/\sim$ where $\sim$ is given by $v \sim w$ if $v = sw$ for $s \in \mathbb{R} - \{O\}$. Each point is given a homogeneous coordinate: $[v] = [x_0, x_1, \dots, x_n]$ where two homogeneous coordinates are equal if they differ only by a nonzero scalar. That is $[x_0, x_1, \dots, x_n] = [\lambda x_0, \lambda x_1, \dots, \lambda x_n]$ for $\lambda \in \mathbb{R} - \{0\}$. The projective transformation group $\mathbb{PGL}(n+1, \mathbb{R})$ is defined as $\mathbb{GL}(n+1, \mathbb{R})/\sim$ where $g \sim h$ for $g, h \in \mathbb{GL}(n+1, \mathbb{R})$ if $g = ch$ for a nonzero constant c. The group equals the quotient group $\mathbb{SL}_{\pm}(n+1, \mathbb{R})/\{\mathrm{I}, -\mathrm{I}\}$ of the group $\mathbb{SL}_{\pm}(n+1, \mathbb{R})$ of determinant ± 1. Now $\mathbb{PGL}(n+1, \mathbb{R})$ acts on $\mathbb{RP}^n$ where each element sends each ray to a ray by the corresponding general linear map. Each element of g of $\mathbb{PGL}(n+1, \mathbb{R})$ acts by $[v] \mapsto [g'(v)]$ for a representative g' in $\mathbb{GL}(n+1, \mathbb{R})$ of g and is said to be a *projective automorphism*.

Given a basis B of $n+1$ vectors $v_0, v_1, \dots, v_n$ of $\mathbb{R}^{n+1}$, we let $[v]_B = [\lambda_0, \lambda_1, \dots, \lambda_n]_B$ for a point v if we write $v = \lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_n v_n$. Here, $[\lambda_0, \dots, \lambda_n]_B = [c\lambda_0, c\lambda_1, \dots, c\lambda_n]_B$ for $c \in \mathbb{R} - \{0\}$.

Also any homogeneous coordinate change is viewed as induced by a linear map: That is, $[v]_B$ has the same homogeneous coordinate as $[Mv]$ where M is the coordinate change linear map so that $Mv_i = e_i$ for $i = 0, 1, \dots, n$.

- For $n = 1$, $\mathbb{RP}^1$ is homeomorphic to a circle. One considers this as a real line union an infinity.
- A set of points in $\mathbb{RP}^n$ is *independent* if the corresponding vectors in $\mathbb{R}^{n+1}$ are independent. The dimension of a subspace is the maximal cardinality of an independent set minus 1.
- A subspace is the set of points whose representative vectors satisfy a homogeneous system of linear equations. The subspace in $\mathbb{R}^{n+1}$ corresponds to a projective subspace in $\mathbb{RP}^n$ in a one-to-one manner while the dimension drops by 1.
- The affine geometry can be "embedded": $\mathbb{R}^n$ can be identified with the set of points in $\mathbb{RP}^n$ where x_0 is not zero, i.e., the set of points $\{[1, x_1, x_2, \dots, x_n]\}$.

This is called an affine subspace. The subgroup of $\mathbb{PGL}(n+1,\mathbb{R})$ fixing $\mathbb{R}^n$ is precisely $\mathbb{A}(\mathbb{R}^n) = \mathbb{GL}(n,\mathbb{R}) \cdot \mathbb{R}^n$ as can be seen by computations.

- The subspace of points $\{[0, x_1, x_2, \dots, x_n]\}$ is the complement homeomorphic to $\mathbb{RP}^{n-1}$. This is the set of ideal points, i.e., directions in the affine space $\mathbb{R}^n$.
- From affine geometry, one could construct a unique projective geometry and conversely using this idea. (See the book [Berger (2009)] for the complete abstract approach.)
- A hyperspace is given by a single linear equation. The complement of a hyperspace can be identified with an affine space since we can put this into the subspace in the third item.
- A line is the set of points $[v]$ where $v = sv_1 + tv_2$ for $s, t \in \mathbb{R}$ for the independent pair v_1, v_2. Actually a line is $\mathbb{RP}^1$ or a line $\mathbb{R}^1$ with a unique infinity. A point on a line is given a homogeneous coordinate $[s, t]$ where $[s, t] \sim [\lambda s, \lambda t]$ for $\lambda \in \mathbb{R} - \{O\}$.
- $\mathbb{RP}^i$ can be identified to the subspace of points given by $x_0 = 0, \dots, x_{n-i-1} = 0$.
- A subspace is always diffeomorphic to $\mathbb{RP}^i$ for some i, $i = 0, 1, \dots, n$, by a projective automorphism.

The projective geometry has well-known invariants called cross ratios even though lengths of immersed geodesics and angles between smooth arcs are not invariants. (However, we do note that the properties of angles or lengths being $< \pi, = \pi$, or $> \pi$ are invariant properties.)

A *line* is either a subspace of dimension one or a connected subset of it. A *complete affine line* is a complement of a point in a subspace of dimension-one or sometimes we say it is a line of spherical length π. Since they are subsets of a subspace isomorphic to $\mathbb{RP}^1$, we can give it a homogeneous coordinate system $[x_0, x_1]$ regarding it as quotient space of $\mathbb{R}^2 - \{O\}$.

- The cross ratio of four points x, y, z, and t on a one-dimensional subspace $\mathbb{RP}^1$ is defined as follows. There is a unique coordinate system so that $x = [1, 0], y = [0, 1], z = [1, 1], t = [b, 1]$. $b = b(x, y, z, t)$ is defined as the *cross-ratio*. Thus, it is necessary that at least three points x, y, z are mutually distinct.
- If the four points are on a complete affine line, the cross ratio is given as
$$[x, y; z, t] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$
if we find a two-variable coordinate system where
$$x = [1, z_1], y = [1, z_2], z = [1, z_3], t = [1, z_4]$$
by some coordinate change. That is, if x, y, z, and t have coordinates z_1, z_2, z_3, and z_4 respectively in some affine coordinate system of an affine subspace of dimension 1, then the above expression is valid.

- One can define cross ratios of four hyperplanes meeting in a projective subspace of codimension 2. By duality, they correspond to four points on a line.

3.1.4.1 *The $\mathbb{RP}^2$-geometry*

Let us consider $\mathbb{RP}^2$ as an example. We proceed with basic definitions and facts, which can be found in the book [Coxeter (1994)]: We recall that the plane projective geometry is a geometry based on the pair consisting of the projective plane $\mathbb{RP}^2$, the space of lines passing through the origin in $\mathbb{R}^3$ with the group $\mathbb{PGL}(3, \mathbb{R})$, the projectivized general linear group acting on it. $\mathbb{RP}^2$ is considered as the quotient space of $\mathbb{R}^3 - \{O\}$ by the equivalence relation $v \sim w$ iff $v = sw$ for a scalar s.

Here we have a familiar projective plane as topological type of $\mathbb{RP}^2$, which is a Mobiüs band with a disk filled in at the boundary. See `http://www.geom.uiuc.edu/zoo/toptype/pplane/cap/`.

A *point* is an element of $\mathbb{RP}^2$ and a *line* is a codimension-one subspace of $\mathbb{RP}^2$, i.e., the image of a two-dimensional vector subspace of $\mathbb{R}^3$ with the origin removed under the quotient map. Two points are contained in a unique line, and two lines meet at a unique point. Points are *collinear* if they lie on a common line. Lines are *concurrent* if they pass through a common point. A pair of points and/or lines are *incident* if the elements meet with each other.

The dual projective plane $\mathbb{RP}^{2\dagger}$ is given as the space of lines in $\mathbb{RP}^2$. We can identify it as the quotient of the dual vector space $\mathbb{R}^{3,\dagger}$ of $\mathbb{R}^3$ with the origin removed by the scalar equivalence relations as above:

$$\alpha \sim \beta \text{ if } \alpha = s\beta, s \in \mathbb{R} - \{0\}, \alpha, \beta \in \mathbb{R}^{3,\dagger} - \{O\}.$$

An element of $\mathbb{PGL}(3, \mathbb{R})$ acting on $\mathbb{RP}^2$ is said to be a *collineation* or *projective automorphism*. The elements are uniquely represented by matrices of determinant equal to 1. The set of their conjugacy classes is in a one-to-one correspondence with the set of topological conjugacy classes of their actions on $\mathbb{RP}^2$. (Sometimes, we will use matrices of determinant -1 for convenience.)

Among collineations, an order-two element is said to be a *reflection*. It has a unique line of fixed points and an isolated fixed point. Actually, any pair of reflections are conjugate to each other, and given a line and a point not on the line, we can find a unique reflection with these fixed point sets. A reflection will often be represented by a matrix of determinant equal to -1 and the isolated fixed point corresponds to the eigenvector of eigenvalue -1.

Given two lines, we say that a map between the points in one line l_1 to the other l_2 is a *projectivity* or *projective isomorphism* if the map is induced from a rank-two linear map from the vector subspace corresponding to l_1 to that corresponding to l_2.

By *duality*, we mean the one-to-one correspondence between the set of lines in $\mathbb{RP}^2$ with the set of points in $\mathbb{RP}^{2\dagger}$ and one between the points in $\mathbb{RP}^2$ with the lines in $\mathbb{RP}^{2\dagger}$. The correspondence preserves the incidence relationships.

Under duality, a line in $\mathbb{RP}^{2\dagger}$ corresponds to the set of all lines through a point in $\mathbb{RP}^2$, so-called a pencil of lines, and vice-versa.

By duality, given the pencil of lines through a point p and the pencil of lines through another point q, we define that a *projectivity* between the two pencils is a one-to-one correspondence that is the projectivity from the dual line of p to that of q.

Let l_1 and l_2 be two lines and let p_1^1, p_2^1, p_3^1 be three distinct points of l_1 and let p_1^2, p_2^2, p_3^2 be three distinct points in l_2. Then there is a projectivity sending p_i^1 to p_i^2 for $i = 1, 2, 3$.

A quadruple of points in $\mathbb{RP}^2$ in a general position are always equivalent by a collineation. (By a general position, we mean that no three of them are in a line.)

A nonzero vector v in $\mathbb{R}^3$ *represents* a point p of $\mathbb{RP}^2$ if v is in the equivalence class of p or in the ray p. We often label a point of $\mathbb{RP}^2$ by a vector representing it and vice versa by an abuse of notation.

We have another definition.

Definition 3.1. Let y, z, u, v be four distinct collinear points in $\mathbb{RP}^n$ with $u = \lambda_1 y + \lambda_2 z$ and $v = \mu_1 y + \mu_2 z$. The cross-ratio $[y, z; u, v]$ is defined to be $\lambda_2\mu_1/\lambda_1\mu_2$.

Given a set of four mutually distinct points $p_1^1, p_2^1, p_3^1, p_4^1$ on a line l_1 and another such set $p_1^2, p_2^2, p_3^2, p_4^2$ on a line l_2, we obtain a projectivity $l_1 \to l_2$ sending p_i^1 to p_i^2 iff

$$[p_1^1, p_2^1; p_3^1, p_4^1] = [p_1^2, p_2^2; p_3^2, p_4^2].$$

For example, if the coordinates y, z, u, v of four points are $y = 1, z = 0$, and $1 > u > v > 0$ in some affine coordinate system of an affine line, then the cross ratio $[1, 0, u, v]$ equals

$$\frac{1-u}{u}\frac{v}{1-v}$$

which is positive and realizes any values in the open interval $(0, 1)$.

The cross-ratio of four concurrent lines in $\mathbb{RP}^2$ is also defined similarly (see the book [Busemann and Kelly (1953)]) using the dual projective plane where they become four collinear points.

Given a notation $[y, z; u, v]$ with four points y, z, u, v, we usually assume that they are to be on an image of a segment under a projective map where y, z the endpoints and y, v separates u from z. This is the standard position of the four points in this paper.

However, if we exchange y, z or u, v, we obtain a reciprocal. If we exchange y, z and u, v at the same time, we do not change the cross ratios. The symmetry properties of cross ratios are well-known and we skip the discussion here.

3.1.4.2 *Oriented projective geometry*

Note that $\mathbf{S}^n$ double-covers $\mathbb{RP}^n$. Moreover, the group $\mathbb{SL}_{\pm}(n+1,\mathbb{R})$, i.e., linear maps of $\mathbb{R}^{n+1}$ with determinant ± 1, maps to $\mathbb{PGL}(n+1,\mathbb{R})$ with discrete kernels in the center. Then $(\mathbf{S}^n, \mathbb{SL}_{\pm}(n+1,\mathbb{R}))$ defines a geometry called an *oriented projective geometry.*

This is an old idea actually, and there are several advantages working in this space.

Each point is given a homogeneous coordinate: $[v] = [x_0, x_1, \dots, x_n]$ where two homogeneous coordinates are equal if they differ only by a positive scalar; i.e., $[x_0, x_1, \dots, x_n] = [\lambda x_0, \lambda x_1, \dots, \lambda x_n]$ for $\lambda \in \mathbb{R}, \lambda > 0$.

Two points are *antipodal* if their homogeneous coordinates are negatives of the other.

Subspaces are defined by linear equations as above. A great circle is a subspace of dimension 1. A set of a point is not a subspace. A pair of antipodal points is a subspace. The independence is defined as above.

Again a great circle has a homogeneous coordinate system: A great circle is the set of points $[v]$ where $v = sv_1 + tv_2$ for $s, t \in \mathbb{R}$ for the independent pair v_1, v_2. A point on a great circle is given a homogeneous coordinate $[s, t]$ where $[s, t] \sim [\lambda s, \lambda t]$ for $\lambda \in \mathbb{R}, \lambda > 0$. Cross ratios can be defined on four distinct points (x, y, z, t) on a great circle with the first homogeneous coordinates positive.

A *hemisphere* is a subset defined by

$$\{[x_0, x_1, \dots, x_n] | f(x_0, x_1, \dots, x_n) \geq 0\}$$

for a linear function f on $\mathbb{R}^{n+1}$. A *convex subset* of $\mathbf{S}^n$ is a subset such that any two points can be connected by a segment in the subset of length $\leq \pi$. A convex subset is always a subset of a hemisphere of dimension n or $\mathbf{S}^n$ itself. (Under this definition, the intersection of two convex subsets may not be convex. However, if they intersect in their interiors, this problem does not happen.) See the article [Choi (1994a)] for this point of view.

3.1.5 ***Conformal geometry***

We can introduce two classes of symmetries of $\mathbb{R}^n$. The first class is the set of reflections of $\mathbb{R}^n$. Let the hyperplane $P(a,t)$ given by $a \cdot x = t$ for a unit vector a. Then the reflection about $P(a,t)$ is given by $\rho(x) = x + 2(t - a \cdot x)a$. The second class is the set of inversions. Let the hypersphere $S(a, r)$ be given by $|x - a| = r$. Then the inversion about $S(a, r)$ is given by $\sigma(x) = a + (\frac{r}{|x-a|})^2 (x - a)$.

We compactify $\mathbb{R}^n$ to $\hat{\mathbb{R}}^n = \mathbf{S}^n$ by adding infinity. This is to be accomplished as follows: Let $\mathbf{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$ and identify $\mathbb{R}^n$ with the subspace $x_{n+1} = -1$. Consider the stereographic projection from the point $(0, 0, \dots, 1)$. Taking the inverse image of $\mathbb{R}^n$ in $\mathbf{S}^n$, we obtain a copy of $\mathbb{R}^n$ in $\mathbf{S}^n$. The usual differentiable structure of $\mathbf{S}^n$ extends that of embedded $\mathbb{R}^n$. Since the stereographic map preserves angles, the angles of $\mathbb{R}^n$ agree with those of the copy in $\mathbf{S}^n$ with the standard metric. The reflections and inversions of $\mathbb{R}^n$ become diffeomorphisms of the copy in $\mathbf{S}^n$, which extend uniquely to real analytic diffeomorphisms of $\mathbf{S}^n$ respectively; that is, their Jacobians are nowhere zero. Since the maps preserve angles almost everywhere, they do so everywhere by a limiting argument. Thus, these reflections and inversions induce conformal homeomorphisms of $\hat{\mathbb{R}}^n = \mathbf{S}^n$; that is, they preserve angles.

- The group of transformations generated by these homeomorphisms is called the *Mobiüs transformation group.*
- They form the conformal transformation group of $\hat{\mathbb{R}}^n = \mathbf{S}^n$.
- For $n = 2$, $\hat{\mathbb{R}}^2$ is the Riemann sphere $\hat{\mathbb{C}}$ and a Mobiüs transformation is a either a fractional linear transformation of form

$$z \mapsto \frac{az+b}{cz+d}, ad - bc \neq 0, a, b, c, d \in \mathbb{C}$$

  or a fractional linear transformation pre-composed with the conjugation map $z \mapsto \bar{z}$.
- In higher-dimensions, a description as a sphere of positive null-lines and the special Lorentzian group exists in the Lorentzian space $\mathbb{R}^{1,n+1}$.

3.1.5.1 *Poincaré extensions*

We can identify $\mathbb{E}^{n-1}$ with $\mathbb{E}^{n-1} \times \{O\}$ in $\mathbb{E}^n$ and extend each Mobiüs transformation of $\hat{\mathbb{E}}^{n-1}$ to one of $\hat{\mathbb{E}}^n$ that preserves the upper half space U^n. That is, we extend reflections and inversions in the obvious way: by extending a reflection in $\mathbb{E}^{n-1}$ about a hyperplane to a reflection in $\mathbb{E}^n$ about a hyperplane containing the hyperplane and perpendicular to $\mathbb{E}^{n-1}$, and extending the inversion in $\mathbb{E}^{n-1}$ about a sphere of radius r with center $x \in \mathbb{E}^{n-1}$ to the inversion in $\mathbb{E}^n$ with the same radius and center.

Each Mobiüs transformation m of $\hat{\mathbb{E}}^{n-1}$ is a composition of reflections and inversions, say $r_1 r_2 \dots r_n$. Denoting $\hat{r}_i$ the extension, we let the extension $\hat{m}$ of m be given by $\hat{r}_1 \hat{r}_2 \dots \hat{r}_n$.

- The Mobiüs transformations of $\hat{\mathbb{E}}^n$ that preserve the open upper half space are exactly the extensions of the Mobiüs transformations of $\hat{\mathbb{E}}^{n-1}$. Therefore, $M(U^n)$ is identical with $M(\hat{\mathbb{E}}^{n-1})$.
- We put the pair $(U^n, \hat{\mathbb{E}}^{n-1})$ to $(B^n, \mathbf{S}^{n-1})$ by a Mobiüs transformation η of $\hat{\mathbb{E}}^n$. Thus, $M(U^n)$ is isomorphic to $M(\mathbf{S}^{n-1})$ for the boundary sphere by a conjugation by η.
- By a similar reason to the above, $M(B^n)$ is identical with $M(\mathbf{S}^{n-1})$ by considering the Poincaré extension of reflections and inversions on hyperplanes and spheres orthogonal to $\mathbf{S}^{n-1}$.

3.1.6 *Hyperbolic geometry*

A hyperbolic space $\mathbb{H}^n$ is defined as a complete Riemannian manifold of constant curvature equal to -1. Such a space cannot be realized as a submanifold in a Euclidean space of even very large dimensions. But it is realized as a "sphere" in a Lorentzian space as we will see soon. A *Lorentzian space* is the vector space $\mathbb{R}^{1+n}$ with an inner product

$$x \cdot y = -x_0 y_0 + x_1 y_1 + \cdots + x_{n-1} y_{n-1} + x_n y_n.$$

We will denote it by $\mathbb{R}^{1,n}$.

- A Lorentzian norm $||x|| = (x \cdot x)^{1/2}$ is a positive number, a positive imaginary number, or zero. The vector is said to be *space-like*, *null*, or *time-like* depending on its norm being positive, zero, or a positive imaginary number.
- The *null* vectors form a light cone divided into a cone of positive null vectors, a cone of negative null vectors, and $\{O\}$.
- The subspace of time-like vectors also has two components where $x_0 > 0$ and $x_0 < 0$ respectively. A time-like vector is also *positive* or *negative* depending on which component it lies in.
- Ordinary notions such as orthogonality can be defined by the Lorentzian inner product. A basis is *orthonormal* if its vectors have norms of 1 or i and they mutually orthogonal. The Gram-Schmidt orthogonalization is possible also for a set of vectors starting with a positive time-like vector.
- A subspace of $\mathbb{R}^{1,n}$ is either *space-like* where all vectors in it are space-like, is *null* where at least one nonzero-vector is null and no vector is time-like, or finally *time-like* where at least one vector is time-like: This can be seen by looking at the restriction of the Lorentzian inner product on the subspace where it could be either positive-definite, semi-definite, or definite with at least one vector with an imaginary norm.
- A pair of space-like vectors v and w spanning a space-like subspace have an angle between them given by the formula $\cos\theta = v \cdot w/||v||||w||$. This can be generalized to the situations where they do not span a space-like subspace and span a null subspace or a time-like subspace. (For details, see the book [Ratcliffe (2006)]).

3.1.6.1 *The Lorentz group*

A *Lorentzian transformation* is a linear map preserving the inner-product. A *Lorentzian matrix* is a matrix corresponding to a Lorentzian transformation under a standard coordinate system. For the diagonal matrix J with entries $-1, 1, \ldots, 1$, $A^t JA = J$ if and only if A is a Lorentzian matrix.

The set of Lorentzian transformations forms a Lie group $\mathbb{O}(1, n)$ given by

$$\{A \in \mathbb{GL}(n+1, \mathbb{R}) | A^t JA = J\},$$

which is a subgroup of $\mathbb{GL}(n+1,\mathbb{R})$. A Lorentzian transformation sends time-like vectors to time-like vectors. Thus, by continuity, it either preserves both components of the subspace of positive time-like vectors or switches the components. It is either *positive* or *negative* if it sends positive time-like vectors to positive time-like ones or negative time-like ones. The set of positive Lorentzian transformations forms a Lie subgroup $\mathbb{PO}(1,n)$.

The quotient map

$$\mathbb{GL}(n+1,\mathbb{R}) \to \mathbb{PGL}(n+1,\mathbb{R})$$

maps the subgroup diffeomorphic to its image subgroup. Hence, there is an inclusion map

$$\mathbb{PO}(1,n) \to \mathbb{PGL}(n+1,\mathbb{R}).$$

We regard the first group as the subgroup of the next.

3.1.6.2 *The hyperbolic space*

For two positive time-like vectors, the subspace spanned by them is time-like and the Lorentzian inner product restricts to an inner product of signature $-1, 1$. In a new coordinate system with coordinate functions s, t, the inner product becomes $-s^2 + t^2$. Since the vectors are positive time-like, the absolute values of second components of the two vectors are smaller than those of the first components. Thus, the Lorentzian inner-product of the two vectors is a negative number. Their norms are positive imaginary numbers, and the absolute value of the inner-product is greater than the product of the absolute values of their norms as can be verified by simple computations. Given $(s_1, t_1), (s_2, t_2), s_i > 0, s_i > t_i$, we can show

$$(-s_1 s_2 + t_1 t_2)^2 > (-s_1^2 + t_1^2)(-s_2^2 + t_2^2),$$

which follows from

$$2 s_1 s_2 t_1 t_2 < s_1^2 t_2^2 + s_2^2 t_1^2.$$

Therefore, there is a *time-like angle* $\eta(x, y)$ for two time-like vectors x and y defined by

$$x \cdot y = |||x||||||y||| \cosh \eta(x, y)$$

where $|||v|||$ for a vector v denotes the absolute value of the norm $||v||$ of v.

A hyperbolic space $\mathbb{H}^n$ is an upper component of the submanifold defined by $||x||^2 = -1$ or $x_0^2 = 1 + x_1^2 + \cdots + x_n^2$. This is a subset of a positive cone, the upper sheet of a hyperboloid. Topologically, it is homeomorphic to $\mathbb{R}^n$ since one realizes it as a graph of the function. Sometimes, this object is called a *hyperboloid model* of the hyperbolic space. (See also `http://www.geom.uiuc.edu/~crobles/hyperbolic/hypr/modl/mnkw/`.)

One induces a metric from the Lorentzian space: for two tangent vectors x, y to the hyperboloid, we define $x \cdot y$ by the Lorentzian inner product. Since the tangent vectors at a point u of the hyperboloid is orthogonal to u, the tangent space is space-like and the norms are always positive. This gives us a Riemannian metric of constant curvature -1. (The computation of curvature is very similar to the computations for the sphere.)

A *hyperbolic line* is an intersection of $\mathbb{H}^n$ with a time-like two-dimensional vector subspace. A *triangle* is given by three segments meeting at three vertices. Denote the vertices by A, B, and C and the opposite segments by a, b, and c. By denoting their angles and lengths again by A, B, C, a, b, and c respectively, we obtain

- Hyperbolic law of sines:
$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}.$$
- Hyperbolic law of cosines:
$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C, \tag{3.5}$$
$$\cosh c = \frac{\cosh A \cosh B + \cos C}{\sin A \sin B}. \tag{3.6}$$

One can assign any interior angles to a hyperbolic triangle as long as the sum is less than π. One can assign any lengths to a hyperbolic triangle as long as the lengths satisfy the triangle inequality.

We note that the triangle formula can be generalized to formulas for quadrilaterals, pentagons, hexagons with some right angles. Basic philosophy here is that one can push the vertex outside and the angles become distances between lines. (See the book [Ratcliffe (2006)] or `http://online.redwoods.cc.ca.us/instruct/darnold/staffdev/Assignments/sinandcos.pdf`)

Since $\mathbb{PO}(1, n)$ includes $\mathbb{O}(n, \mathbb{R})$ acting on the subspace given by $x_0 = 0$ and $\mathbb{PO}(1, 1)$ acting transitively on the hyperbolic line through e_0 and $\sqrt{2}e_0 + e_1$, it follows that $\mathbb{PO}(1, n)$ acts transitively on $\mathbb{H}^n$. Given any isometry k, we can find an element $g \in \mathbb{PO}(1, n)$ so that $g \circ k$ fixes e_0 and every vector at the tangent space at e_0. By analyticity of the isometry group, it follows that $k = g^{-1}$. Therefore, the Lie group $\mathbb{PO}(1, n)$ is the isometry group of $\mathbb{H}^n$ and acts faithfully and transitively.

3.1.7 *Models of hyperbolic geometry*

3.1.7.1 *Beltrami-Klein models of hyperbolic geometry*

The hyperboloid model $\mathbb{H}^n$ is a bit complicated in that we have to see a one-dimension higher space to realize its meaning. We will give more intrinsic definitions which are obtainable from the hyperboloid model easily.

The *Klein model* is directly obtained from the hyperboloid model. Recall that an affine patch $\mathbb{R}^n$ in $\mathbb{RP}^n$ is identifiable with a complement of a subspace. A standard one is given by $x_0 \neq 0$. The standard affine patch has coordinate functions $x_1, \ldots, x_n$. There is an embedding from $\mathbb{H}^n$ onto an open unit ball B in the standard affine patch $\mathbb{R}^n$ of $\mathbb{RP}^n$:

$$[x_0, x_1, x_2, \dots, x_n] \to (x_1/x_0, x_2/x_0, \dots, x_n/x_0)$$

induced from a standard radial projection $\mathbb{R}^{n+1} - \{O\} \to \mathbb{RP}^n$.

We regard B as a ball of radius 1 with the center at O in $\mathbb{R}^n$. The hyperboloid has a distance metric induced from the Riemannian metric. By the projection, we obtain a distance metric d_k on B. We compute that $d_k(P, Q) = 1/2|\log(ab, PQ)|$ where a, P, Q, b are on a segment with endpoints a, b and

$$[ab, PQ] = \left| \frac{aP}{bP} \frac{bQ}{aQ} \right| \tag{3.7}$$

where aP, bP, bQ, and aQ denote the Euclidean distances between the designated points respectively.

We can verify this formula as follows: The metric is induced on B by the radial projection

$$\pi_{\mathbb{RP}^n} : \mathbb{H}^n \subset \mathbb{R}^{n+1} - \{O\} \to B \subset \mathbb{RP}^n.$$

Since $\lambda(t) = (\cosh t, \sinh t, 0, \dots, 0)$ define a unit speed geodesic in $\mathbb{H}^n$, we have $d_k([e_1], [(\cosh t, \sinh t, 0, \dots, 0)]) = t$ for t positive under the Riemannian metric d_k. On the right side of equation 3.7, we compute the same. Since any geodesic segment of same length is congruent under the isometry, we see that the two metrics coincide.

The isometry group $\mathbb{PO}(1, n)$ also maps injectively to a subgroup of $\mathbb{PGL}(n+1, \mathbb{R})$ that preserves B. Since the isometry corresponds to a linear map in $\mathbb{R}^{1+n}$ and it preserves $\mathbb{H}^n$, it follows that an isometry corresponds to a projective automorphism of B. Conversely, we see that a projective automorphism of B preserves d_k because it preserves the cross-ratios and hence, it must come from the isometry. The projective automorphism group of B is precisely $\mathbb{PO}(1, n)$.

- The Beltrami-Klein model is "nice" because you can see outside in $\mathbb{RP}^n$. The outside has the natural structure of the anti-de Sitter space. (See `http://en.wikipedia.org/wiki/Anti_de_Sitter_space`.) We can treat points outside and inside together.
- Each hyperplane in the model is dual (i.e., orthogonal by the Lorentzian inner-product) to a point outside. A point in the model is dual to a hyperplane outside. In fact, any subspace of dimension i is dual to a subspace of dimension $n - i - 1$ by orthogonality.
- For $n = 2$, the dual of a line is given by taking tangent lines to the disk at the endpoints and taking the intersection.
- The distance between two hyperplanes can be obtained by two dual points. The two dual points span a 2-dimensional orthogonal subspace to the both hyperperplanes and hence provide the shortest geodesic.

3.1.7.2 *The conformal ball model (Poincaré ball model)*

We consider a stereo-graphic projection $\mathbb{H}^n$ to the subspace P in $\mathbb{R}^{1+n}$ given by $x_0 = 0$ from the point $(-1, 0, \ldots, 0)$. The formula for the map $\kappa : \mathbb{H}^n \to B_P$ is given by

$$\kappa(y) = \left(\frac{y_1}{1+y_0}, \ldots, \frac{y_n}{1+y_0} \right),$$

where the image is the open ball B_P of radius 1 with the center O in P. The inverse is given by

$$\zeta(x) = \left(\frac{1+|x|^2}{1-|x|^2}, \frac{2x_1}{1-|x|^2}, \ldots, \frac{2x_n}{1-|x|^2} \right).$$

Since this is a diffeomorphism, B_P has an induced Riemannian metric of constant curvature -1. We show by computations

$$\cosh d_{B_P}(x, y) = 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}.$$

This formula shows that all inversions acting on B_P preserve the metric, and so does the group $M(B_P)$ of Mobiüs transformations of B_P. The corresponding Riemannian metric is $g_{ij} = 2\delta_{ij}/(1-|x|^2)^2$. Note that for two points x, y of B_P, there exists a circle perpendicular to the topological boundary sphere $\mathrm{bd}B_P$ of B_P containing x and y. We can choose a hypersphere passing the midpoint of the segment between x and y. Also, a stabilizer of a point x of B_P is generated by reflections about hyperspheres containing x. Since $M(B_P)$ is generated by reflections about spheres orthogonal to $\mathrm{bd}B_P$, it follows that $M(B_P)$ is transitive on B_P and the stabilizer of a point is easily seen to be isomorphic to $\mathbb{O}(n)$. Since the isometry group of B_P has the same property, it follows that the group of Mobiüs transformations acting on B_P is precisely the isometry group of B_P.

Moreover, $\mathbf{Isom}(B_P)$ can be identified with $M(\mathbf{S}^{n-1})$ where $\mathbf{S}^{n-1}$ is the boundary sphere of B_P (see Section 3.1.5.1).

Geodesics would be lines through O or would be arcs on circles perpendicular to the sphere of radius 1. A *sphere* in $\mathbf{S}^n$ is a sphere in $\mathbb{R}^n$ or the closure of an affine subspace of $\mathbb{R}^n$ in the sphere $\hat{\mathbb{R}}^n$ compactified at ∞. A *horosphere* in B_P is a sphere S in $\mathrm{Cl}(B_P)$ tangent to a point x in $\mathrm{bd}B_P$ with the point $\{x\} = S \cap \mathrm{bd}B_P$ removed. Given a point x of $\mathrm{bd}B_P$, we obtain a one parameter family of horospheres whose closures meet x.

3.1.7.3 *The upper-half space model.*

Let U be the upper half-space in $\mathbb{R}^n$. Then U is homeomorphic to an open ball in the compactification $\hat{\mathbb{R}}^n = \mathbf{S}^n$. Since B_P is an open ball, we can find a Mobiüs transformation sending B_P to U by a composition of two reflections. Now we put B_P to U by the Mobiüs transformation. This gives a Riemannian metric of constant curvature -1 on U.

We have by computations that $\cosh d_U(x,y) = 1 + |x-y|^2/2x_n y_n$ and that the Riemannian metric is given by $g_{ij} = \delta_{ij}/x_n^2$. Then $I(U) = M(U) = M(\mathbb{E}^{n-1})$. Geodesics would be arcs on lines or circles perpendicular to $\mathbb{E}^{n-1}$.

3.1.7.4 *The classification of isometries*

Let U^2 denote the 2-dimensional upper-half space model of the hyperbolic plane and U^3 the 3-dimensional one of the hyperbolic space. The topological boundary bdU^2 in $\mathbf{S}^2$ can be identified with the compactification $\hat{\mathbb{E}}^1$ of the Euclidean line $\mathbb{E}^1$ and bdU^3 in $\mathbf{S}^3$ can be done with the compactification $\hat{\mathbb{E}}^2$ of the Euclidean plane $\mathbb{E}^2$. Since $\hat{\mathbb{E}}^1$ is a circle and $\hat{\mathbb{E}}^2$ equals the complex sphere $\hat{\mathbb{C}}$, we obtain $\mathbf{Isom}^+(U^2) = \mathbb{PSL}(2,\mathbb{R})$ and $\mathbf{Isom}^+(U^3) = \mathbb{PSL}(2,\mathbb{C})$ respectively. In this model, it is easier to classify isometries.

- Apart from the identity, orientation-preserving isometries of hyperbolic plane U^2 can have at most one fixed point. An *elliptic* isometry is one fixing a unique point. A *hyperbolic* isometry is one preserving a unique line. The remaining type one is a *parabolic* isometry. The elliptic, hyperbolic, and parabolic isometries are ones conjugate to

$$\begin{aligned}
&z \mapsto \frac{z\cos\theta - \sin\theta}{z\sin\theta + \cos\theta}, \theta \neq 0 \mod 2\pi,\\
&z \mapsto az, a \neq 1, a \in \mathbb{R}^+,\\
&z \mapsto z+1
\end{aligned}$$

  in $M(U^2)$ respectively.
- Orientation-preserving isometries of a hyperbolic space are classified as loxodromic, hyperbolic, elliptic, or parabolic. A *loxodromic* isometry is one acting on a geodesic translating and having a nonzero rotation angle about the geodesic and fixes two points in bdU^3 corresponding to the endpoints of the geodesic. A *hyperbolic* isometry is one acting on a geodesic translating and having a zero rotation angle about the geodesic and fixes two points in bdU^3 corresponding to the endpoints of the geodesic. An *elliptic* isometry is one acting on a geodesic fixing each points of it and its closure and having a nonzero rotation angle about the geodesic. Finally, a *parabolic* isometry is one fixing no point and acting on no geodesic in U^3 but fixing a unique point in bdU^3 and acts on each of the horsopheres at this point. Up to conjugations, they are represented as Mobiüs transformations on bdU^3 which have forms
  - $z \mapsto \alpha z, \mathrm{Im}\alpha \neq 0, |\alpha| \neq 1.$
  - $z \mapsto az, a \neq 1, a \in \mathbb{R}^+.$
  - $z \mapsto e^{i\theta}z, \theta \neq 0.$
  - $z \mapsto z+1.$

The proofs are omitted but can be found in standard textbooks such as [Ratcliffe (2006)].

3.2 Discrete groups and discrete group actions

Here, we let X be generally a manifold with some Lie group G acting on it transitively. In order for most of the developed theory to work, we need that X be a sphere $\mathbf{S}^n$ with Lie groups such as $\mathbb{O}(n+1,\mathbb{R})$, $\mathbb{GL}(n+1,\mathbb{R})$, and the Mobiüs transformation group acting on it; $\mathbb{RP}^n$ with $\mathbb{PGL}(n+1,\mathbb{R})$ acting on it; $\mathbb{R}^n$ with $\mathbb{O}(n,\mathbb{R})\cdot\mathbb{R}^n$ or $\mathbb{A}(\mathbb{R}^n) = \mathbb{GL}(n,\mathbb{R})\cdot\mathbb{R}^n$ acting on it; or $\mathbb{H}^n$ with $\mathbb{PO}(1,n)$ acting on it. Sometimes, we cannot let X be a symmetric space with its isometry group even or a complex hyperbolic space. The reason is that there seems to be no good notion of m-planes, i.e., m-dimensional subspaces with pleasant intersection properties. (See Section 3.2.1 for details) It is a hope of geometric topologists that we can overcome these difficulties.

We will present facts for X that will be useful in many cases with some additional assumptions on X. However, the reader may wish to see X as one of the above. These will be mostly sufficient.

Let X be a manifold. A *discrete group* is a group with a discrete topology. (It is usually a finitely generated subgroup of a Lie group.) Any group can be made into a discrete group. We have many notions of a group action $\Gamma\times X\rightarrow X$ which induces a homomorphism $\Gamma\rightarrow\mathbf{Diff}(X)$ where $\mathbf{Diff}(X)$ denotes the group of diffeomorphisms of X with the C^r-topology ($r\geq 1$):

- The action is *effective* if an element g of Γ corresponds to I_X if and only if g is the identity in Γ. The action is *free* if an element g fixes a point of X if and only if g is the identity in Γ.
- The action is *discrete* if Γ is discrete in the group of homeomorphisms of X with the compact open topology. (We used the fact that $\mathbf{Diff}(X)$ is a subgroup of the group of homeomorphisms.)
- The action has *discrete orbits* if every x has a neighborhood U so that the number of orbit points in U is finite.
- The action is *wandering* if every x has a neighborhood U so that the set of elements γ of Γ so that $\gamma(U)\cap U\neq\emptyset$ is finite.
- The action is *properly discontinuous* if for every compact subset K the set of γ such that $K\cap\gamma(K)\neq\emptyset$ is finite.

The conditions of discrete action, discrete orbit action, wandering action, and properly discontinuous are strictly stronger according to the order presented here as long as X is a manifold. The proof of this fact without the strictness is not very involved by showing that the later condition implies the given condition (see Section 3.5 of [Thurston (1997)]).

- If the action is wandering and free, then the action gives a manifold quotient which is possibly non-Hausdorff.
- The action of Γ is free and properly discontinuous if and only if X/Γ is a (Hausdorff) manifold quotient and $X \to X/\Gamma$ is a covering map.
- Suppose that Γ is a discrete subgroup of a Lie group G acting on X with a compact stabilizer. Then X has a G-invariant Riemannian metric. Any (G, X)-manifold now has an induced Riemannian metric. Suppose that Γ acts properly discontinuously on X. Let us call this the *standard discrete action*.
- A *complete* (G, X)*-manifold* is one isomorphic to X/Γ where Γ acts freely and properly discontinuously. (The notion of completeness agrees with that of the induced Riemannian metric for G acting with compact stabilizers. Hence, this is a natural generalization.)
- We define the *deformation space of complete* (G, X)*-structures* on M as the set of equivalence classes of diffeomorphisms $f : M \to X/\Gamma$ for a discrete subgroup Γ of G acting freely and properly discontinuously with the equivalence relation that $f_1 : M \to X/\Gamma_1 \sim f_2 : M \to X/\Gamma_2$ if there is an (G, X)-diffeomorphism $g : X/\Gamma_1 \to X/\Gamma_2$ where $g \circ f_1$ is isotopic to f_2.
- Suppose that X is simply-connected. For a manifold M, the deformation space of complete (G, X)-structures on M is in a one-to-one correspondence with the space of the conjugacy classes of discrete faithful representations h of $\pi_1(M) \to G$, each of which giving a diffeomorphism $M \to X/h(\pi_1(M))$.

We remark that if we allow G to act on X without the compact stabilizer condition, then we call this *standard flexible type* action.

As examples, we give:

- $\mathbb{R}^2 - \{O\}$ with the group generated by $g_1 : (x, y) \to (2x, y/2)$. This is a free wondering action but is not properly discontinuous.
- $\mathbb{R}^2 - \{O\}$ with the group generated by $g : (x, y) \to (2x, 2y)$. This is a free and properly discontinuous action.
- The modular group $\mathbb{PSL}(2, \mathbb{Z})$ is the group of Mobiüs transformations or isometries of the hyperbolic plane given by

  $$z \mapsto \frac{az + b}{cz + d}, a, b, c, d \in \mathbb{Z}, ad - bc = 1.$$

  This is not a free action but a properly discontinuous action on the upper-half space model U^2 of $\mathbb{H}^2$ as the action is a standard discrete one. (See `http://en.wikipedia.org/wiki/Modular_group`.)

3.2.1 *Convex polyhedrons*

For $\mathbf{S}^n$, a *geodesic* is the arc segment in a 1-plane not containing an antipodal pair except at the endpoints. It could be a singleton. For $\mathbb{RP}^n$, a geodesic is just an arc segment in a 1-plane.

Suppose that X is a space where a Lie group G acts effectively and transitively. Furthermore, suppose X has notions of m-planes. An *m-plane* is an element of a collection of submanifolds of X of dimension m so that given generic $m + 1$ points, we have a unique one containing them. We require also that every 1-plane contains geodesic between any two points in it if geodesics are defined for the (G, X)-geometry. Obviously, we assume that elements of G send m-planes to m-planes. (For complex hyperbolic spaces, such notions seem to be absent.)

We also need to assume that X satisfies the increasing property: if we are given an m-plane and every set of generic $m+1$-points in it lies in an n-plane for $n \geq m$, then the entire m-plane lies in the n-plane.

For example, any geometry with models in $\mathbb{RP}^n$ and G a subgroup of $\mathbb{PGL}(n + 1, \mathbb{R})$ has a notion of m-planes. Thus, hyperbolic, euclidean, spherical, and projective geometries have notions of m-planes. Conformal geometry may not have such notions since a generic pair of points have infinitely many circles through them.

Suppose that the (G, X)-geometry has notions of geodesics well-defined. A *convex subset* of X is a subset A such that for any pair of points of A, there exists a geodesic segment in A between them. (We caution the readers that the intersection of two convex subsets may not be convex under this definition.)

A *convex hull* of a subset A is a minimal convex subset in X containing A. This is usually a well-defined set.

Assume that X is $\mathbf{S}^n$, $\mathbb{R}^n$, $\mathbb{H}^n$, or $\mathbb{RP}^n$ with Lie groups acting on X. Let us state some facts about convex sets:

- The dimension of a convex set is the least integer m such that C is contained in a unique m-plane $\hat{C}$ in X.
- The interior C^o and the boundary ∂C are defined as the topological interior and the topological boundary in $\hat{C}$ respectively.
- The closure of C is in $\hat{C}$. If C is convex, then the interior and the closure are convex. They are domains with the dimensions equal to that of $\hat{C}$.
- A *side* of C is a nonempty maximal convex subset of ∂C.
- A *convex polyhedron* is a nonempty closed convex subset such that the set of sides is locally finite in X.

3.2.2 *Convex polytopes*

Using the Beltrami-Klein model, the open unit ball B, i.e., the hyperbolic space, is a subset of an affine patch $\mathbb{R}^n$. In $\mathbb{R}^n$, one can talk about convex hulls.

- A *convex polytope* in $B = \mathbb{H}^n$ is a convex polyhedron with finitely many vertices and is the convex hull of its vertices in $B = \mathbb{H}^n$.

- A polyhedron P in $B = \mathbb{H}^n$ is a *generalized convex polytope* if its closure is a polytope in the affine patch. A generalized polytope may have ideal vertices. An *ideal vertex* is a vertex in the boundary of B. The triangle with all three vertices at the boundary of B is said to be the *ideal triangle*.
- For $X = \mathbb{RP}^n$ or $\mathbf{S}^n$, a *convex polytope* is given as a convex polyhedron in an affine patch or an open hemisphere with finitely many vertices and is a convex hull of its vertices.
- In general, for X with notions of m-planes, we define a *convex polytope* as above.

Note here that these definitions do not depend on the model of the hyperbolic space almost by coincidence. Of course, one needs to verify this.

A compact simplex which is a convex hull of $n+1$ points in $B = \mathbb{H}^n$ is an example of a convex polytope.

Take the origin O in B, and its tangent space $T_O B$. (In fact, O could be any point.) Start from the origin O in $T_O B$ and expand the infinitesimal euclidean polytope from an interior point radially in $T_O B$ using linear expansion maps given by scalars. Now map the vertices of the convex polytope by an exponential map to B. The convex hull of the vertices is a convex polytope. Thus for any Euclidean polytope, we obtain a one parameter family of hyperbolic polytopes. (We caution that sometimes the combinatorial structures of the polytope might change. But in many cases, they do not.)

A regular hyperbolic dodecahedron with all dihedral angles $\pi/2$ as seen from inside is pictured in Figure 3.6. This is to be constructed by the above method. Actually, the dihedral angle changes from near 116.565 degrees which is realized by a very small regular hyperbolic dodecahedron, i.e., when s is very small, to 60 degrees which is realized by an ideal dodecahedron, i.e., when $s = +\infty$. Therefore, the regular hyperbolic dodecahedron of 90-degree dihedral angles is achievable. (See also `http://demonstrations.wolfram.com/HyperbolizationOfADodecahedron/`.)

3.2.3 *The fundamental domains of discrete group actions*

Recall $\mathbf{S}^n$ with spherical geometry, $\mathbb{E}^n$ with Euclidean geometry and $\mathbb{H}^n$ with a hyperbolic geometry. Let X be $\mathbf{S}^n, \mathbb{E}^n$ or $\mathbb{H}^n$ or more generally a geometrical space with m-planes. Let Γ be a group acting on X. A *fundamental domain* for Γ is an open domain F so that $\{gF | g \in \Gamma\}$ is a collection of disjoint sets and their closures cover X. The fundamental domain is *locally finite* if the above closures are locally finite.

Suppose that X is either a hyperbolic, euclidean, or spherical space. Then the *Dirichlet domain* $D(u)$ for $u \in X$ is the intersection of all

$$H_g(u) = \{x \in X | d(x,u) < d(x, gu)\}, g \in \Gamma - \{\mathrm{I}\}.$$

Then the closure of $D(u)$ is a convex fundamental polyhedron. If X/Γ is compact, and Γ acts properly discontinuously, then $D(u)$ is a convex polytope. (If X is some other types of geometries, this is somewhat only vaguely understood.)

The regular octagon example of a hyperbolic surface of genus 2 is an example of a Dirichlet domain $D(u)$ with the origin as u. (See Figure 3.3.)

3.2.4 *Side pairings and the Poincaré fundamental polyhedron theorem*

A *tessellation* of X is a locally finite collection of polyhedra covering X with mutually disjoint interiors.

If P is a convex fundamental polyhedron of a discrete group Γ of isometries acting on X, then Γ is generated by

$$\Phi = \{g \in \Gamma | P \cap g(P) \text{ is a side of } P\}:$$

To see this, let g be an element of Γ, and let us choose a point x of P^o and consider its image $g(x)$ in $g(P^o)$. Then we choose a path from the initial point x to the terminal point $g(x)$. We perturb the path so that it meets only the interiors of the sides of the tessellating polyhedrons. Each time the path crosses a side $h(S)$ for a translate $h(P)$ for an element h of Γ, we take the side-pairing g_S obtained as below. Then multiplying all such side-pairings in the reverse order to what occurred, we obtain an element $g' \in \Gamma$ so that $g'(P) = g(P)$ as hg_Sh^{-1} moves $h(P)$ to the image of P adjacent in the side $h(S)$ for every $h \in \Gamma$. Since P is a fundamental domain, $g^{-1}g'$ is the identity element of Γ.

- Given a side S of a convex fundamental domain P, there is a unique element g_S such that $S = P \cap g_S(P)$. And $S' = g_S^{-1}(S)$ is also a side of P.
- $g_{S'} = g_S^{-1}$ since $S' = P \cap g_S^{-1}(P)$.
- The Γ-side-pairing is the set of g_S for sides S of P.
- The equivalence class at P is generated by $x \cong x'$ if there is a side-pairing sending x to x' for $x, x' \in P$.
- The equivalence class $[x]$ is finite for $x \in P$ where $[x]$ equals $P \cap \Gamma(x)$.

- A cycle relation for each side S of P.
  - Let $S_1 = S$ for a given side S. Choose the side R of S_1. Obtain S_1'. Let S_2 be the side adjacent to S_1' so that $g_{S_1}(S_1' \cap S_2) = R$ and so on. We obtain $S_1, S_1', S_2, S_2', \ldots, S_i, S_i'$.
  - Let S_{i+1} be the side of P adjacent to S_i' such that
  $$g_{S_i}(S_i' \cap S_{i+1}) = S_{i-1}' \cap S_i.$$
- Then we obtain
  - There is an integer l such that $S_{i+l} = S_i$ for each i.
  - $\sum_{i=1}^{l} \theta(S_i', S_{i+1}) = 2\pi/k$ where θ is the dihedral angle measure on X.
  - $g_{S_1} g_{S_2} \cdots g_{S_l}$ has order k.

  – The period l is the number of sides of codimension one coming into the image of a given side R of codimension two in X/Γ. (Of course l depends on the side R.)
- If X does not have a G-invariant metric, we have instead of the angle condition that for each $x \in R^o$, there exists a neighborhood N_i in P of x_i identified to x by $g_{S_1} g_{S_2} \cdots g_{S_i}$ for each i, $1 \leq i \leq l$ so that we obtain a neighborhood of x in X of form

$$N \cup g(N) \cup \cdots \cup g^{k-1}(N) \text{ where } g := g_{S_1} g_{S_2} \cdots g_{S_l} \text{ and}$$
$$N := g_{S_1}(N_1) \cup g_{S_1} g_{S_2}(N_2) \cup \cdots \cup g_{S_1} g_{S_2} \cdots g_{S_l}(N_l).$$

Also, these are all the relations since we can push any relation disk occurring in the presentation to be transversal to the codimension 2-sides of the images of P under Γ. Thus, any such disk reduces to a union of disks meeting the codimension 2-sides once. Thus, if Γ has a convex fundamental polytope, Γ is finitely presented.

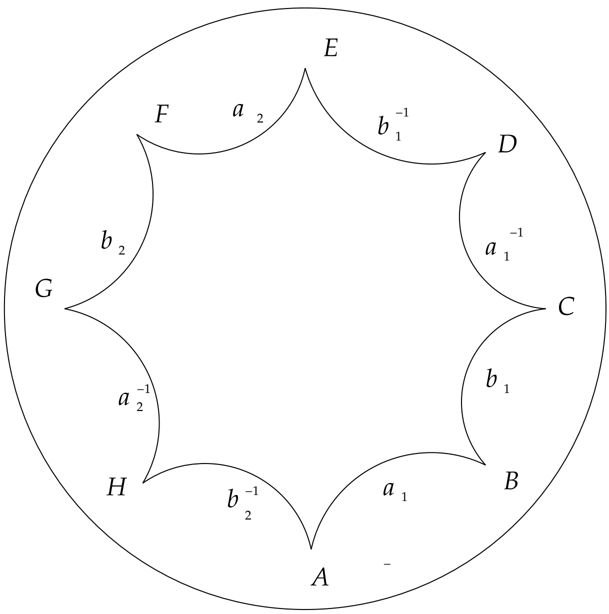

Fig. 3.3 Example: the octahedron in the hyperbolic plane identified to be a genus 2-surface. There is the cycle $(a_1, A), (a_1^{-1}, D), (b_1^{-1}, D), (b_1, C), (a_1^{-1}, C), (a_1, B), (b_1, B), (b_1^{-1}, E), (a_2, E), (a_2^{-1}, H), (b_2^{-1}, H), (b_2, G), (a_2^{-1}, G), (a_2, F), (b_2, F), (b_2^{-1}, A), (a_1, A), (a_1^{-1}, D), \ldots$.

The Poincaré fundamental polyhedron theorem is the converse. We claim that the theorem holds for geometries (G, X) with notions of m-planes. (See pp. 80–84 of the book [Kapovich (2009)].):

Theorem 3.2.1. *Let (G, X) be a geometry with notions of m-planes and geodesics and suppose that X has a G-invariant Riemannian metric. Given a convex polyhe-*

dron P in X with side-pairing automorphisms in G satisfying the above relations, then P is the fundamental domain for the discrete subgroup of G generated by the side-pairing isometries.

If every k equals 1, then the result of the face identification is a manifold. Otherwise, we obtain orbifolds. The results are always complete. (See Jeff Weeks `http://www.geometrygames.org/CurvedSpaces/index.html` for examples of hyperbolic or spherical manifolds as seen from "inside". There are more examples there such as Seifert-Weber manifolds and so on.)

When the side-pairing maps are not isometries or equivalently X has no G-invariant metrics, P is a fundamental domain of a manifold M with an immersion to X. The immersions are often embeddings to open domains. See Chapter 8 for some examples. (See the article [Sullivan and Thurston (1983)] for more details.)

We will be particularly interested in reflection groups. Suppose that X has notions of angles between m-planes. A *discrete reflection group* is a discrete subgroup in G generated by reflections in X about sides of a convex polyhedron. Assume that all the dihedral angles are submultiples of π. Then the side-pairing such that each face is side-paired to itself by a reflection satisfies the Poincaré fundamental theorem.

The reflection group has a presentation $\{S_i : (S_iS_j)^{k_{ij}}\}$ where $k_{ii} = 1$ and $k_{ij} = k_{ji}$ which are examples of Coxeter groups. Notice that k_{ij} is finite if and only if the faces corresponding to S_i and S_j meet in a codimension-two side of P.

The triangle groups are examples of discrete reflection groups.

- Find a triangle in X with angles $\pi/a, \pi/b, \pi/c$ submultiples of π where we assume $2 \leq a \leq b \leq c$. This exists always for $X = \mathbf{S}^2, \mathbb{E}^2$, or $\mathbb{H}^2$.
- We divide into three cases $\frac{\pi}{a} + \frac{\pi}{b} + \frac{\pi}{c} > \pi, = \pi, < \pi$. The triangles are then spherical, euclidean, or hyperbolic ones respectively. They exist and are uniquely determined up to isometry.

  - $> \pi$ cases: $(2, 2, c), (2, 3, 3), (2, 3, 4)$, and $(2, 3, 5)$ respectively corresponding to an index-two-extension of dihedral group of order $2c$, a tetrahedral group, an octahedral group, and an icosahedral group.
  - $= \pi$ cases: $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$. The reflections generate the corresponding wall paper groups.
  - $< \pi$ cases: Any other (p, q, r) gives a hyperbolic tessellation group. Thus, there are infinitely many such groups. (See Proposition 3.2.2.)

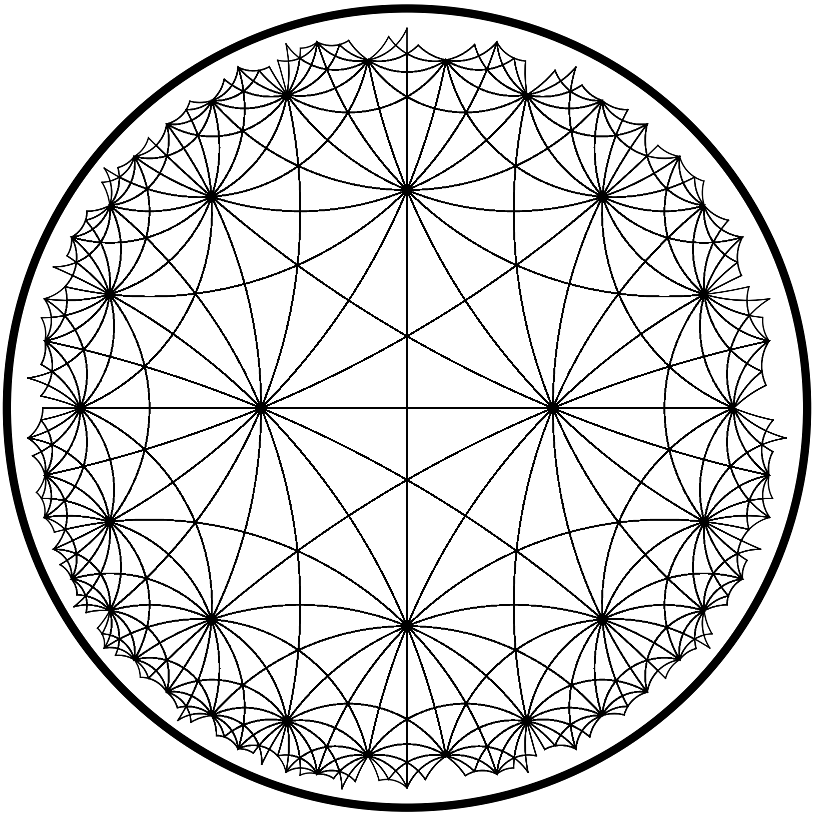

Fig. 3.4 The $(2, 3, 8)$-triangle reflection group in the Poincaré disk model. We used the package "PoincareModel" written by W. Goldman.

Proposition 3.2.2.

- *One can respectively construct a compact geodesic polygon P with angles $\pi/p_1, \pi/p_2, \ldots, \pi/p_n, n \geq 3, p_i \geq 2$ on a two-sphere, a Euclidean plane, or a hyperbolic plane depending on whether the sum of outer angles $\sum_{i=1}^{n} \pi(1 - 1/p_i)$ is smaller than 2π, equal to 2π, or greater than 2π.*
- *This is the necessary and sufficient condition also.*
- *The group generated by the reflection on the sides of P generates a discrete group.*

Proof. One shows that it is possible to construct all triangles in this way. Let us give arbitrary lengths $l, l_1, l_2, \ldots, l_5$.

- We show that a quadrilateral with angles π/p_1, $\pi/p_2, \pi/2, \pi/2$ at respective vertices v_1, v_2, v_3, v_4, and a distinguished edge $\overline{v_3 v_4}$ of length l, and
- a pentagon with angles $\pi/p_1, \pi/2, \pi/2, \pi/2, \pi/2$ at vertices $v_1, v_2, \ldots, v_5$

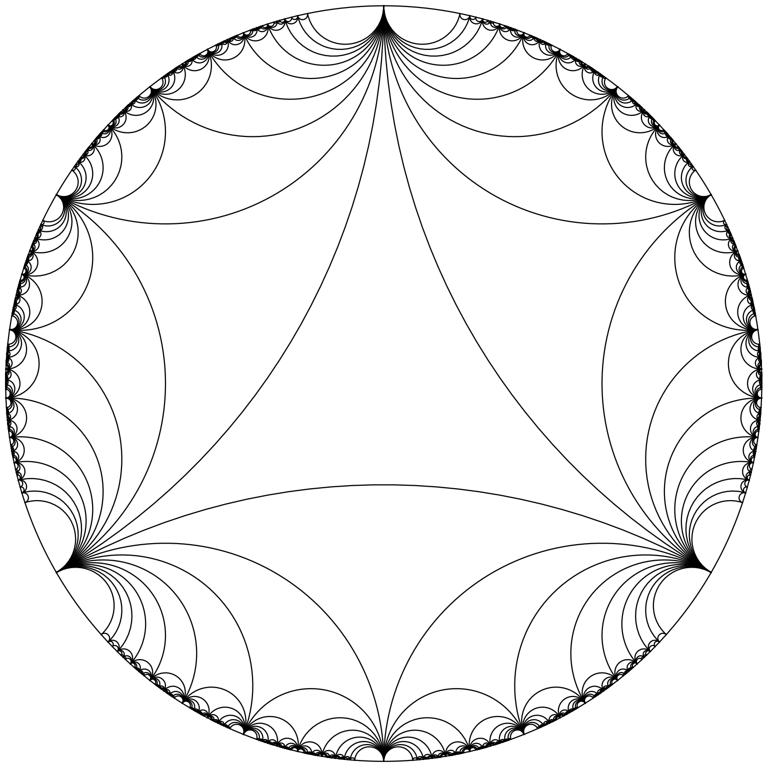

Fig. 3.5 The ideal-triangle reflection group: we use the group generated by the reflections on the sides of an ideal triangle on the hyperbolic plane. We used the package "PoincareModel", written by W. Goldman.

with distinguished edges $\overline{v_2v_3}, \overline{v_4v_5}$ of respective length l_1 and l_2, and
- a hexagon with all angles $\pi/2$ at vertices $v_1, v_2, \ldots, v_6$ with distinguished edges $\overline{v_1v_2}, \overline{v_3v_4}$, and $\overline{v_5v_6}$ of respective lengths l_3, l_4, and l_5 can be constructed.

These are accomplished in Chapter 3 of the book [Ratcliffe (2006)] for example.

Given a topological polygon, we can divide it into quadrilaterals, pentagons, and/or hexagons matching each other on edges of above types up to renaming vertices. Then desired polygon P can be constructed by matching lengths of the distinguished edges.

The necessary part comes from the Gauss-Bonnet theorem. □

3.2.4.1 *Higher-dimensional examples*

To construct a 3-dimensional example, we obtain a Euclidean regular dodecahedron in $T_O B$, put into the hyperbolic space, expand it, and decrease the dihedral angles until we achieve that all dihedral angles are $\pi/3$. (See Section 3.2.2.) There are pictures of these in Geometry Center archives including the Seifert-Weber manifold constructed in such a manner.

One can also achieve a regular octahedron with angles $\pi/2$. These are ideal polytope examples. Heard, Pervova, and Petronio (2008) for example found very many 3-manifolds obtained from an octahedron by side-paring constructions above.

Higher-dimensional examples were analyzed by Vinberg and so on. For example, there is no hyperbolic reflection group of compact type above dimension ≥ 30.

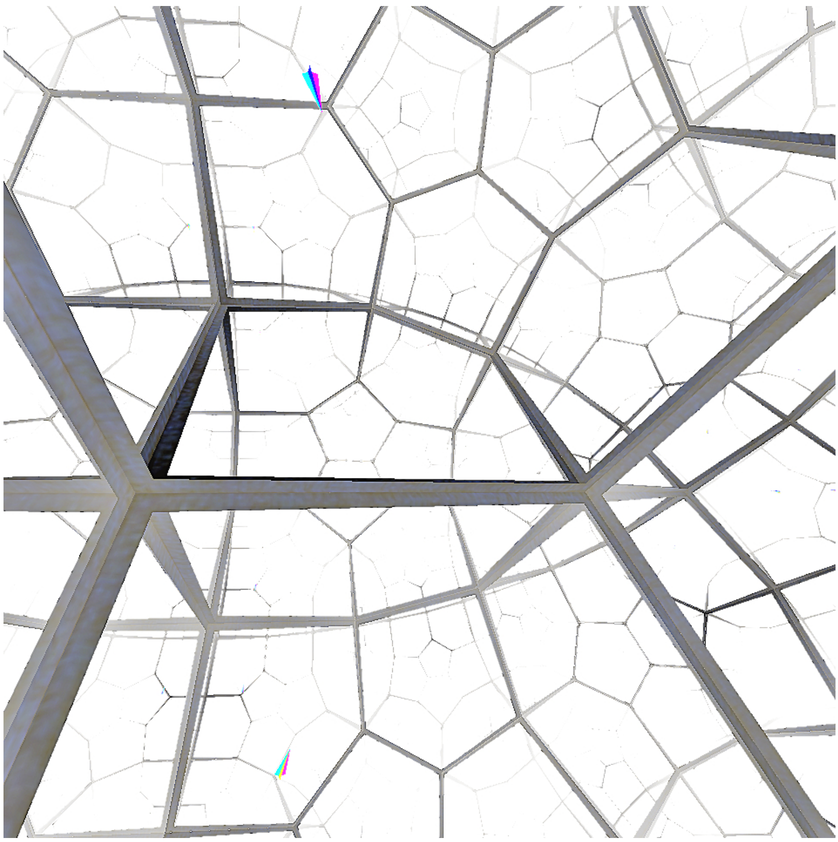

Fig. 3.6 The dodecahedral reflection group as seen by an insider: One has a regular dodecahedron with all edge angles $\pi/2$ and hence it is a fundamental domain of a hyperbolic reflection group. This figure is captured from the program CurvedSpace by J. Weeks.

3.2.5 *Crystallographic groups*

A *crystallographic group* is a discrete group of the rigid motions on $\mathbb{R}^n$ whose quotient space is compact.

The Bieberbach theorem states that

Theorem 3.2.3.

- *A group is isomorphic to a crystallographic group of $\mathbb{R}^n$ if and only if it contains a subgroup of finite index that is free abelian of rank equal to n.*
- *Two crystallographic groups are isomorphic as abstract groups if and only if they are conjugate by an affine transformation.*

Once we have this theorem, the classification of crystallographic groups is reduced to studying the finite group extensions of abelian crystallographic groups, which are lattices. There are only finitely many crystallographic groups for each dimension since once the abelian group action is determined, its symmetry group can be only finitely many. There are 17 wallpaper groups for dimension 2. (See `http://www.clarku.edu/~djoyce/wallpaper/` and see Kali by Weeks `http://www.geometrygames.org/Kali/index.html`.) There are 230 space groups for dimension 3 (Conway, Friedrichs, Huson, and Thurston, 2001). These groups have extensive applications in molecular chemistry. For further informations, see `http://www.ornl.gov/sci/ortep/topology.html`.

3.3 Notes

The figures 3.4 and 3.5 were drawn by packages developed by the Experimental Geometry Laboratory in University of Maryland, College Park. (See `http://egl.math.umd.edu/`.)

A good introduction to Euclidan, affine, and projective geometry can be found in the books [Berger (2009); Rosenbaum (1963)] and some early chapters of books [Thurston (1997); Goldman (1988)]. There are many interactive online courses and materials on projective geometry:

- `http://www.math.poly.edu/courses/projective_geometry/`
- `http://demonstrations.wolfram.com/TheoremeDePappusFrench/`,
- `http://demonstrations.wolfram.com/TheoremeDePascalFrench/`,

In fact, projective geometry is actively researched by engineers working in visions.

The book [Ratcliffe (2006)] gives us extensive descriptions of models of hyperbolic geometry. Discrete group actions and the Poincaré fundamental polyhedron theorems are described well in the books [Ratcliffe (2006); Kapovich (2009)]. In fact, this chapter is heavily influenced by the books [Ratcliffe (2006); Thurston (1997)]. There is also an elementary book [Ryan (1987)].