Chapter 4

Energy-transport model

This chapter is devoted to showing Theorem 2.4, which asserts that the time global solution for the energy-transport model converges to that for the drift-diffusion model as the parameter ζ tends to zero. The proof is discussed in several sections. We firstly prove in Sections 4.1–4.3 the existence of the time global solution for the energy-transport model with the large initial data $(\rho_0, \theta_0) \in H^1(\Omega)$, which is summarized in Theorem 4.2. The relaxation limit from the energy-transport model to the drift-diffusion model is justified in Section 4.4. These discussion complete the proof of Theorem 2.4.

The unique existence of the time global solution $(\rho_0^0, j_0^0, \phi_0^0)$ for the drift-diffusion model with the initial data $\rho_0 \in H^2(\Omega)$ has been shown in Theorem 2.4 in the authors' previous paper [33]. This result is, however, insufficient in the present paper as we take the initial data $(\rho_0, \theta_0) \in H^1(\Omega)$ to consider the relaxation limit. Hence we show the time global solvability of the model for $\rho_0 \in H^1(\Omega)$ in the next lemma by applying Theorem 2.4 in [33]. Here and hereafter, we use the function spaces

$$
\begin{aligned}
\mathfrak{Y}([0,T]) &= C^1([0,T]; L^2(\Omega)) \cap C([0,T]; H^2(\Omega)) \cap H^1(0,T; H^1(\Omega)),\\
\mathfrak{Y}_{loc}((0,T)) &:= C^1((0,T); L^2(\Omega)) \cap C((0,T); H^2(\Omega)) \cap H^1_{loc}(0,T; H^1(\Omega)),\\
\mathfrak{Z}([0,T]) &:= C([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega)),\\
\mathfrak{Z}_{loc}((0,T)) &:= C((0,T); H^1(\Omega)) \cap L^2_{loc}(0,T; H^2(\Omega)) \cap H^1_{loc}(0,T; L^2(\Omega)),
\end{aligned}
$$

where $\mathfrak{Y}$ is defined in Section 2.3.

Lemma 4.1. *Let $(\tilde{\rho}_0^0, \tilde{j}_0^0, \tilde{\phi}_0^0)$ be the stationary solution to* (2.18)*,* (2.20) *and* (3.2)*. Suppose that the initial data $\rho_0 \in H^1(\Omega)$ and the boundary data ρ_l, ρ_r and ϕ_r satisfy* (2.4)*,* (2.6)*,* (2.7a) *and* (2.10a)*. Then there exists a positive constant δ_0 such that if $\delta \leq \delta_0$, the initial boundary value problem* (2.15)*,* (2.12a)*,* (2.4) *and* (2.6) *has a unique solution $(\rho_0^0, j_0^0, \phi_0^0)$ satisfying $\rho_0^0 - \tilde{\rho}_0^0 \in \mathfrak{Z}([0,\infty)) \cap \mathfrak{Y}_{loc}((0,\infty))$, $j_0^0 - \tilde{j}_0^0 \in C([0,\infty); L^2(\Omega))$, $\phi_0^0 - \tilde{\phi}_0^0 \in C([0,\infty); H^3(\Omega)) \cap$*

$H^1(0,\infty;H^2(\Omega))$ and the positivity (2.10a). *Moreover it verifies the estimates*

$$\min\{B_m,\ \inf\rho_0\}\le\rho_0^0(t,x)\le\max\{B_M,\ \sup\rho_0\},\tag{4.1a}$$

$$\|(\rho_0^0-\tilde\rho_0^0)(t)\|_1^2+\|(j_0^0-\tilde j_0^0)(t)\|^2+\|(\phi_0^0-\tilde\phi_0^0)(t)\|_3^2\le Ce^{-\alpha t},\tag{4.1b}$$

$$t\|(\{\rho_0^0\}_t,\{\rho_0^0\}_{xx})(t)\|^2+\int_0^t\|(\{\rho_0^0\}_t,\{\rho_0^0\}_{xx})(\tau)\|+\tau\|(\rho_0^0)_{xt}(\tau)\|^2\,d\tau\le C(1+t)\tag{4.1c}$$

for $x\in\overline{\Omega}$ and $t\ge 0$, where C and α are positive constants independent of t and δ.

Proof. In order to apply Theorem 2.4 in [33], we take an approximation sequence $\{\rho_{0i}\}_{i=1}^\infty\subset H^2(\Omega)$ such that $\{\rho_{0i}\}_{i=1}^\infty$ converges to the initial data ρ_0 strongly in $H^1(\Omega)$ and each ρ_{0i} satisfies the compatibility condition $\rho_{0i}(0)=\rho_l$ and $\rho_{0i}(1)=\rho_r$. Theorem 2.4 in [33] shows that the problem (2.15), (2.12a), (2.4) and (2.6) has a unique solution (ρ_i,j_i,ϕ_i) in the space $\mathfrak{Y}([0,\infty))\times\mathfrak{Z}([0,\infty))\times C^1([0,\infty);H^2(\Omega))$ for the initial data ρ_{0i}. It is shown by a similar computation as in [33] with using the maximum principle and the energy method that the sequence $\{\rho_i\}_{i=1}^\infty$ is bounded in the space $\mathfrak{Z}([0,T])\cap\mathfrak{Y}([s,T])$ for arbitrary positive constants s and T. Applying the energy method again to the equation for the difference $\rho^n-\rho^m$, we show that $\{\rho_i\}_{i=1}^\infty$ is the Cauchy sequence in $\mathfrak{Z}([0,T])\cap\mathfrak{Y}([s,T])$. Hence, there exists a function ρ in $\mathfrak{Z}([0,T])\cap\mathfrak{Y}_{loc}((0,T))$ such that ρ_i converges to ρ in $\mathfrak{Z}([0,T])\cap\mathfrak{Y}([s,T])$. Let $j:=\rho\phi_x-\rho_x$ and $\phi:=\Phi[\rho]$, where $\Phi[\,\cdot\,]$ is defined in (4.17). It is easily to see that (ρ,j,ϕ) is the desired solution with the initial data $\rho_0\in H^1(\Omega)$. The estimates (4.1) are also shown similarly as in [33]. □

The stability theorem for the energy-transport model is summarized as

Theorem 4.2. *Let $(\tilde\rho,\tilde j,\tilde\theta,\tilde\phi)$ be the stationary solution of* (2.18)–(2.20) *and* (3.1)*, which is constructed in Theorem 3.5. Suppose that the initial data $(\rho_0,\theta_0)\in H^1(\Omega)$ and the boundary data ρ_l, ρ_r and ϕ_r satisfy* (2.4), (2.6), (2.7a), (2.10a) *and* (2.10b). *Then there exist positive constants δ_0 and ζ_0 such that if $\delta\le\delta_0$ and $\zeta\le\zeta_0$, the initial boundary value problem* (2.14), (2.12a), (2.12c) *and* (2.4)–(2.6) *has a unique solution (ρ,j,θ,ϕ) satisfying $\rho-\tilde\rho,\theta-\tilde\theta\in\mathfrak{Z}([0,\infty))\cap\mathfrak{Y}_{loc}((0,\infty))$, $j-\tilde j\in C([0,\infty);L^2(\Omega))\cap\mathfrak{Z}_{loc}((0,\infty))$, $\phi-\tilde\phi\in C([0,\infty);H^3(\Omega))\cap H^1([0,\infty);H^2(\Omega))$; the positivity* (2.10a) *and* (2.10b). *Moreover, it verifies $\sqrt{t}\rho_{xt},\sqrt{t}\theta_{xt}\in L^2(0,\infty:L^2(\Omega))$ and the decay estimates*

$$\|(j-\tilde j)(t)\|^2+\|(\rho-\tilde\rho,\theta-\tilde\theta)(t)\|_1^2+\|(\phi-\tilde\phi)(t)\|_3^2\le Ce^{-\alpha t},\tag{4.2a}$$

$$t\|(j_x-\tilde j_x)(t)\|^2+\frac{t}{\zeta}\|(\theta-\tilde\theta)(t)\|_1^2+t\|(\rho-\tilde\rho,\theta-\tilde\theta)(t)\|_2^2\le Ce^{-\alpha t},\tag{4.2b}$$

where C and α are positive constants independent of ζ, δ and t.

To study the initial boundary value problem (2.14), (2.12a), (2.12c) and (2.4)–(2.6) with the positivity (2.10a) and (2.10b), it is convenient to employ new unknown functions

$$v := \log \rho, \quad w := \log \theta$$

and rewrite the system of the equations (2.14) as

$$\begin{pmatrix} v \\ 3w/2 \end{pmatrix}_t - A[v,w] \begin{pmatrix} v \\ w \end{pmatrix}_{xx} + \begin{pmatrix} 0 \\ 3(1-e^{-w})/2\zeta \end{pmatrix} = G[v,w], \tag{4.3a}$$

$$\begin{aligned} A[v,w] &:= \begin{pmatrix} e^w & e^w \\ e^w & e^w + \kappa_0 e^{-v} \end{pmatrix}, \quad G[v,w] := \begin{pmatrix} g_1[v,w] \\ g_1[v,w] + 3g_2[v,w]/2 \end{pmatrix}, \\ g_1[v,w] &:= e^w (v_x + w_x)^2 - e^v + D - v_x (\Phi[e^v])_x, \\ g_2[v,w] &:= -\left(\frac{2}{3}(\Phi[e^v])_x - \frac{5}{3} e^w w_x \right) \left\{ v_x + w_x - e^{-w}(\Phi[e^v])_x \right\} + \frac{2\kappa_0}{3e^v}(w_x)^2, \end{aligned} \tag{4.3b}$$

where we have used (2.8) and (2.14d). Note that the matrix $A[v,w]$ is symmetric and positive definite. The initial and the boundary data for (v,w) are also derived from (2.4)–(2.6), (2.12a) and (2.12c) as

$$v(0,x) = v_0(x) := \log \rho_0(x), \quad w(0,x) = w_0(x) := \log \theta_0(x), \tag{4.4}$$

$$v(t,0) = \log \rho_l, \quad v(t,1) = \log \rho_r, \tag{4.5}$$

$$w_x(t,0) = w_x(t,1) = 0. \tag{4.6}$$

Apparently, (4.3)–(4.6) is equivalent to (2.4)–(2.6), (2.12a), (2.12c) and (2.14) if the density ρ and the temperature θ are positive. Namely, once it is shown that the problem (4.3)–(4.6) has a solution (v,w), the existence of the solution to the problem (2.14), (2.12a), (2.12c) and (2.4)–(2.6) immediately follows. In fact, letting

$$\rho := e^v, \quad j := -(e^v e^w)_x + e^v (\Phi[e^v])_x, \quad \theta := e^w, \quad \phi := \Phi[e^v], \tag{4.7}$$

we see that (ρ, j, θ, ϕ) is the solution to the problem (2.14), (2.12a), (2.12c) and (2.4)–(2.6). We also rewrite the stationary solution $(\tilde{\rho}^0_\zeta, \tilde{j}^0_\zeta, \tilde{\theta}^0_\zeta, \tilde{\phi}^0_\zeta)$ to the energy-transport model, which is constructed in Theorem 3.5, as

$$\tilde{v} := \log \tilde{\rho}^0_\zeta, \quad \tilde{w} := \log \tilde{\theta}^0_\zeta.$$

It is obvious that $(\tilde{v},\tilde{w})$ satisfies the equation

$$-A[\tilde{v},\tilde{w}] \begin{pmatrix} \tilde{v} \\ \tilde{w} \end{pmatrix}_{xx} + \begin{pmatrix} 0 \\ 3(1-e^{-\tilde{w}})/2\zeta \end{pmatrix} = G[\tilde{v},\tilde{w}] \tag{4.8}$$

and boundary conditions (4.5) and (4.6), where A and G are defined in (4.3b).

We prove Theorem 4.2 in following steps. It is an essentially same procedure as in the authors' previous paper [33], where the isothermal hydrodynamic model is studied.

First step. We discuss the unique existence of the time local solution (v, w) to the problem (4.3)–(4.6) in Section 4.1. Here it is shown in Corollary 4.5 that there exists certain positive time T_* independent of ζ such that the solution for the energy-transport model uniquely exists until T_*. This independence is crucial in order to construct the time global solution by taking the parameter ζ sufficiently small. Here we can take the initial data (v_0, w_0) arbitrarily large as far as it belongs to $H^1(\Omega)$.

Second step. In Section 4.2, a "semi-global existence" of the solution (v, w) is established. Precisely, we prove in Theorem 4.9 that the solution with the arbitrary initial data (v_0, w_0) in $H^1(\Omega)$ exists until arbitrary time T by taking the parameter ζ_T is sufficiently small subject to T. Here we also show that the difference between the non-stationary solution $(v, w)(T, x)$ and the stationary solution $(\tilde{v}, \tilde{w})(x)$ becomes arbitrarily small if T is sufficiently large. This result is summarized in Corollary 4.10.

Third step. Owing to Second step, we see that the perturbation $(v-\tilde{v}, w-\tilde{w})(T, x)$ becomes arbitrarily small by taking T large (and thus ζ_T small). Hence, in order to complete the proof of Theorem 4.2, it suffices to show Theorem 4.11, which asserts that the asymptotic stability of the stationary solution for the energy-transport model with the small initial disturbance. Consequently, the proof of Theorem 4.2 follows from Theorem 4.11, Corollaries 4.5 and 4.10 in Sections 4.1–4.3.

4.1 Uniform estimate of local solution

We show in this section that there exists a certain positive time T_*, independent of the parameter ζ, such that the solution for the energy-transport model uniquely exists until T_*. This argument is essentially same as in [33]. We firstly state the unique existence of the solution to the problem (4.3)–(4.6), where the existence time T_ζ may depend on the parameter ζ. The proof is postponed until the Appendix.

Lemma 4.3. *Suppose the initial data $(v_0, w_0) \in H^1(\Omega)$ and the boundary data ρ_l, ρ_r and ϕ_r satisfy* (2.4)–(2.6) *and* (2.7a). *Let N be a certain positive constant satisfying $\|(v_0, w_0)\|_1 \leq N$. Then there exists a positive constant T_ζ, depending on ζ and N, such that the initial boundary value problem* (4.3)–(4.6) *has a unique solution $(v, w) \in \mathfrak{Z}([0, T_\zeta]) \cap \mathfrak{Y}_{loc}((0, T_\zeta))$. Moreover, it satisfies $\sqrt{t}v_{xt}, \sqrt{t}w_{xt} \in L^2(0, T_\zeta : L^2(\Omega))$ and the convergence*

$$t\|(v_t, w_t, v_{xx}, w_{xx})(t)\|^2 \to 0 \quad as \quad t \to 0. \tag{4.9}$$

In the above lemma, the existence time of the solution is denoted by T_ζ for the clarity of its dependence on ζ. This existence theorem is insufficient in the following discussion, which require that the existence time is independent of ζ. The independence is shown in Corollary 4.5. For this purposes, we derive the estimates (4.11) and (4.12) below. For positive constant T and M, define $X(T;M)$ by a set of the functions

$$(v,w) \in \mathfrak{Z}([0,T]) \cap \mathfrak{Y}_{loc}((0,T))$$

satisfying

$$\|(v,w)(t)\|_1^2 \le M \tag{4.10}$$

for $t \in [0,T]$.

Lemma 4.4. *There exists a positive constant M, depending on $\|(v_0,w_0)\|_1$ but independent of ζ, such that if the solution (v,w) to the problem (4.3)–(4.6) belongs to $X(T;2M)$, then it satisfies*

$$\|(v,w)(t)\|_1^2 \le M + C[M]t, \tag{4.11}$$

$$\int_0^t \frac{1}{\zeta}\|w(\tau)\|_1^2 + \|(v_{xx},w_{xx})(\tau)\|^2 \, d\tau \le C[M](1+t) \tag{4.12}$$

for $t \in [0,T]$, where $C[M]$ is a positive constant depending on M but independent of ζ and t.

Proof. Taking the inner product of (4.3a) with the vector (v,w) in $L^2(0,t;L^2(\Omega))$ and applying the integration by part yield

$$\begin{aligned}
&\frac{1}{2}\|v(t)\|^2 + \frac{3}{4}\|w(t)\|^2 + \int_0^t\int_0^1 \frac{3}{2\zeta e^w}(e^w-1)w \, dxd\tau \\
&= \frac{1}{2}\|v_0\|^2 + \frac{3}{4}\|w_0\|^2 + \int_0^t\int_0^1 (v,w)\left(A[v,w](v_{xx},w_{xx})^\top + G[v,w]\right) dxd\tau \\
&\le \frac{1}{2}\|v_0\|^2 + \frac{3}{4}\|w_0\|^2 + \mu\int_0^t \|(v_{xx},w_{xx})(\tau)\|^2 \, d\tau + C[\mu,M]t,
\end{aligned} \tag{4.13}$$

where μ is an arbitrary positive constant to be determined. In deriving the above inequality, we have also applied the Sobolev and the Young inequalities to the right hand side with using the inequality

$$\|\Phi[e^v](t)\|_2 \le C[M], \tag{4.14}$$

which holds due to the formula (2.8) and $(v,w) \in X(T;2M)$. Next, take the inner product of (4.3a) with the vector $(-v_{xx},-w_{xx})$ in $L^2(0,t;L^2(\Omega))$, apply the integration by part and use the boundary conditions $v_t(t,0) = v_t(t,1) = w_x(t,0) = w_x(t,1) = 0$. Then estimate the resulting equality by using (4.14) as well as the Sobolev and the Young inequalities to get

$$\begin{aligned}
&\frac{1}{2}\|v_x(t)\|^2 + \frac{3}{4}\|w_x(t)\|^2 + \int_0^t\int_0^1 (v_{xx}, w_{xx})A[v,w](v_{xx}, w_{xx})^\top \, dxd\tau + \int_0^t\int_0^1 \frac{3}{2\zeta e^w} w_x^2 \, dxd\tau \\
&= \frac{1}{2}\|v_{0x}\|^2 + \frac{3}{4}\|w_{0x}\|^2 + \int_0^t\int_0^1 (v_{xx}, w_{xx})G[v,w] \, dxd\tau \\
&\leq \frac{1}{2}\|v_{0x}\|^2 + \frac{3}{4}\|w_{0x}\|^2 + \mu\int_0^t \|(v_{xx}, w_{xx})(\tau)\|^2 \, d\tau + C[\mu, M]t. \qquad (4.15)
\end{aligned}$$

Notice that the third term in the left hand side of (4.15) is estimated from below as

$$c[M]\int_0^t \|(v_{xx}, w_{xx})(\tau)\|^2 \, d\tau \leq \int_0^t\int_0^1 (v_{xx}, w_{xx})A[v,w](v_{xx}, w_{xx})^\top \, dxd\tau \qquad (4.16)$$

since the matrix $A[v,w]$ is symmetric and positive definite. Thus, by adding (4.13) to (4.15), taking μ sufficiently small and then using the estimate (4.16), we have

$$\begin{aligned}
&\frac{1}{2}\|v(t)\|_1^2 + \frac{3}{4}\|w(t)\|_1^2 + \int_0^t\int_0^1 \frac{3}{2\zeta e^w}(e^w - 1)w + \frac{3}{2\zeta e^w} w_x^2 \, dxd\tau \\
&\qquad + c[M]\int_0^t \|(v_{xx}, w_{xx})(\tau)\|^2 \, d\tau \leq \frac{1}{2}\|v_0\|_1^2 + \frac{3}{4}\|w_0\|_1^2 + C[M]t. \qquad (4.17)
\end{aligned}$$

Now determine the constant M by

$$M := \|v_0\|_1^2 + \frac{3}{2}\|w_0\|_1^2,$$

which is apparently independent of ζ. Then the estimate (4.17) immediately means the desired estimate (4.11). It also implies the estimate (4.12) due to the mean value theorem. □

Lemma 4.4 yields that the existence time of the solution (v, w) in Lemma 4.3 can be taken independently of ζ. In addition, it gives the estimate of the time local solution uniformly in ζ. These results are proven in the next corollary.

Corollary 4.5. *Suppose the initial data $(v_0, w_0) \in H^1(\Omega)$ and the boundary data ρ_l, ρ_r and ϕ_r satisfy* (2.4)–(2.6) *and* (2.7a). *Let N_0 be a certain positive constant satisfying $\|(v_0, w_0)\|_1 \leq N_0$. Then there exists a positive constant T_*, depending on N_0 but independent of ζ, such that the initial boundary value problem* (4.3)–(4.6) *has a unique solution $(v, w) \in \mathfrak{Z}([0, T_*]) \cap$*

$\mathfrak{Y}_{loc}((0,T_*))$. Moreover, it satisfies $\sqrt{t}v_{xt}, \sqrt{t}w_{xt} \in L^2(0,T_*;L^2(\Omega))$, the convergence (4.9) and the estimates

$$\|(v,w)(t)\|_1^2 \le C, \tag{4.18a}$$

$$\int_0^t \frac{1}{\zeta}\|w(\tau)\|_1^2 + \|(v_{xx},w_{xx})(\tau)\|^2\,d\tau \le C, \tag{4.18b}$$

$$\int_0^t \|v_t(\tau)\|^2\,d\tau \le C \tag{4.18c}$$

for $t \in [0,T_*]$, where C is a positive constant independent of ζ and t.

Proof. Take a positive constant T_* so small that the right hand side of (4.11) is less than $2M$ for an arbitrary $t \in [0,T_*]$. Here T_* is apparently independent of ζ. On the other hand, define T_s by the supermum of time T until which the solution (v,w) to (4.3)–(4.6) exists in the set $X(T;2M)$. The existence of T_s is ensured in the Lemma 4.3 even though it may depend on ζ. We show that the solution exists in the time interval $[0,T_s]$ and belongs to $X(T_s;2M)$ as follows. For arbitrary t_0 in $[0,T_s)$, $\|(v,w)(t_0)\| \le 2M$ holds owing to $(v,w) \in X(t_0;2M)$. Regarding t_0 as the initial time and $(v,w)(t_0)$ as the initial data, and letting $N := 2M$, we apply Lemma 4.3. Hence, there exists a positive constant T_0, depending only on N and ζ, such that the solution (v,w) exists in $X(t_0+T_0;2M)$. Since t_0 is arbitrary in $[0,T_s)$, the solution exists in $\mathfrak{Z}([0,T_s+t_0)) \cap \mathfrak{Y}_{loc}((0,T_s+t_0))$. Consequently, the solution (v,w) belongs to $X(T_s;2M)$.

To show T_* is the desired existence time, it suffices to prove the inequality $T_* \le T_s$. This inequality is proven by contradiction as follows. Suppose that $T_s < T_*$. Lemma 4.4 means that the solution contained in $X(T_s;2M)$ satisfies the estimate (4.11) for an arbitrary $t \in [0,T_s]$. Applying Lemma 4.3 with regarding T_s as initial time, we see that there exists a positive constant t_0 such that the solution exists until the time T_s+t_0 and belongs to $X(T_s+t_0;2M)$. Apparently it contradicts the definition of T_s. Hence we have $T_* \le T_s$, which means that the solution (v,w) belongs to $X(T_*;2M)$.

In constructing the time local solution in Lemma 4.3, we have already proven the convergence (4.9). The estimates (4.18a) and (4.18b) apparently hold owing to Lemma 4.4. Moreover, solve the first component of the system (4.3a) with respect to v_t and take the L^2-norm of the result. Then, by using the inequalities (4.18a) and (4.18b), we have the desired estimate (4.18c). □

4.2 Semi-global existence of solution

This section is devoted to proving the semi-global existence of the solution in Theorem 4.9, which asserts the solution to (2.1) exists until arbitrary positive time T provided that ζ is sufficiently small. It is proven by the essentially same argument as in the proof of Corollary 4.5 together with a-priori estimates in Lemmas 4.6 and 4.8.

Hereafter in this section, (v_ζ, w_ζ) denotes the solution to the problem (4.3)–(4.6). For the solution $(\rho_0^0, j_0^0, \phi_0^0)$ to the problem (2.15), (2.12a), (2.4) and (2.6) define

$$v_0^0 := \log \rho_0^0.$$

Then we have

$$(v_0^0)_t - (v_0^0)_{xx} = g_1[v_0^0, 0], \tag{4.19}$$

where g_1 is given in (4.3b). The notations

$$R_\zeta := v_\zeta - v_0^0, \quad Q_\zeta := w_\zeta,$$
$$L_\zeta(t) := \sup_{T_* \le \tau \le t} \|(R_\zeta, Q_\zeta)(\tau)\|_1$$

are frequently used in the following discussions. Here and hereafter in this section, the constant T_* means the one defined in Corollary 4.4 with $N_0 := \|(v_0, w_0)\|_1$. Subtracting the equation (4.19) from the first component of the system (4.3a) gives

$$(R_\zeta)_t - (R_\zeta)_{xx} = e^{w_\zeta}(Q_\zeta)_{xx} - (e^{w_\zeta} - 1)(v_\zeta)_{xx} + g_1[v_\zeta, w_\zeta] - g_1[v_0^0, 0]. \tag{4.20}$$

Subtract the first component of the system (4.3a) from the second component of the system (4.3a) to obtain

$$(Q_\zeta)_t - \frac{2}{3}(v_\zeta)_t - \frac{2\kappa_0}{3e^{v_\zeta}}(Q_\zeta)_{xx} + \frac{1}{\zeta}\left(1 - \frac{1}{e^{w_\zeta}}\right) = g_2[v_\zeta, w_\zeta]. \tag{4.21}$$

The boundary conditions for R_ζ and Q_ζ are derived from (2.4) and (2.5) as

$$R_\zeta(t,0) = R_\zeta(t,1) = (Q_\zeta)_x(t,0) = (Q_\zeta)_x(t,1) = 0. \tag{4.22}$$

Lemma 4.6. *Let T be an arbitrary positive constant greater than or equal to T_*, and $(v_\zeta, w_\zeta) \in \mathfrak{Z}([0,T]) \cap \mathfrak{Y}_{loc}((0,T))$ be a solution to (4.3)–(4.6). Then there exist positive constants δ_0 and δ_1 such that if $\delta + \zeta \le \delta_0$ and $L_\zeta(T) \le \delta_1$, then the estimates*

$$\|(v_\zeta, w_\zeta)(t)\|_1^2 + |\Phi[e^{v_\zeta}](t)|_2 \le C, \tag{4.23a}$$

$$\int_0^t \frac{1}{\zeta}\|w_\zeta(\tau)\|_1^2 + \|(\{v_\zeta\}_t, \{v_\zeta\}_{xx}, \{w_\zeta\}_{xx})(\tau)\|^2 \, d\tau \le C(1+t), \tag{4.23b}$$

$$\frac{t}{\zeta}\|w_\zeta(t)\|_1^2 + t\|(\{v_\zeta\}_t, \{v_\zeta\}_{xx}, \{w_\zeta\}_{xx})(t)\|^2 + \int_0^t \frac{\tau}{\zeta^2}\|w_\zeta(\tau)\|_1^2$$
$$+ \frac{\tau}{\zeta}\|(w_\zeta)_{xx}(\tau)\|^2 + \tau\|(\{w_\zeta\}_t, \{v_\zeta\}_{xt}, \{w_\zeta\}_{xt})(\tau)\|^2 \, d\tau \le Ce^{\beta t} \tag{4.23c}$$

hold for an arbitrary $t \in [0,T]$, where C and β are positive constants independent of t, δ and ζ.

Proof. The estimate (4.18a) and the definition of $L_\zeta(T)$ immediately give $\|(v_\zeta, w_\zeta)(t)\|_1^2 \leq C$, which together with the formula (2.8) shows $|\Phi[e^{v_\zeta}](t)|_2 \leq C$. Hence the estimate (4.23a) holds. The inequality (4.23b) is derived similarly as the derivations of (4.12) and (4.18c).

We derive the estimate (4.23c) as follows. Multiply the equation (4.21) by tw_ζ/ζ and integrate the result by part over $[0,T]\times\Omega$ to obtain

$$
\begin{aligned}
& t\int_0^1 \frac{1}{2\zeta}(w_\zeta)^2\,dx + \int_0^t\int_0^1 \frac{\tau}{\zeta^2 e^{w_\zeta}}(e^{w_\zeta}-1)w_\zeta\,dxd\tau \\
&= \int_0^t\int_0^1 \frac{1}{2\zeta}(w_\zeta)^2 + \frac{\tau}{\zeta}\left(\frac{2}{3}(v_\zeta)_t + \frac{2\kappa_0}{3e^{v_\zeta}}(w_\zeta)_{xx} + g_2[v_\zeta, w_\zeta]\right) w_\zeta\,dxd\tau \\
&\leq \mu \int_0^t \frac{\tau}{\zeta^2}\|(e^{w_\zeta}-1)(\tau)\|^2 + C[\mu](1+t^2).
\end{aligned}
$$

In deriving the last inequality, we have also used the estimates (4.23a) and (4.23b). Making μ in the above inequality so small that the inequality

$$
\frac{t}{\zeta}\|w_\zeta(t)\|^2 + \int_0^t \frac{\tau}{\zeta^2}\|w_\zeta(\tau)\|^2\,d\tau \leq C(1+t^2) \tag{4.24}
$$

holds. Taking the inner product of (4.3a) with the vector $(-t\{v_\zeta\}_{xxt}, -t\{w_\zeta\}_{xxt})$ in $L^2(0,t;L^2(\Omega))$ and applying the integration by parts, we have

$$
\begin{aligned}
& \frac{t}{2}\int_0^1 \frac{1}{\zeta e^{w_\zeta}}\{(w_\zeta)_x\}^2 + (\{v_\zeta\}_{xx}, \{w_\zeta\}_{xx})A[v_\zeta, w_\zeta](\{v_\zeta\}_{xx}, \{w_\zeta\}_{xx})^\top\,dx \\
&\quad + \int_0^t\int_0^1 \tau\{(v_\zeta)_{xt}\}^2 + \frac{3}{2}\tau\{(w_\zeta)_{xt}\}^2\,dxd\tau \\
&= -t\int_0^1 (\{v_\zeta\}_{xx}, \{w_\zeta\}_{xx})G[v_\zeta, w_\zeta]\,dx + \int_0^t\int_0^1 (\{v_\zeta\}_{xx}, \{w_\zeta\}_{xx})\left(\tau G[v_\zeta, w_\zeta]\right)_\tau\,dxd\tau \\
&\quad + \int_0^t\int_0^1 \frac{1}{2}(\{v_\zeta\}_{xx}, \{w_\zeta\}_{xx})(\tau A[v_\zeta, w_\zeta])_\tau(\{v_\zeta\}_{xx}, \{w_\zeta\}_{xx})^\top + \left(\frac{\tau}{2\zeta e^{w_\zeta}}\right)_\tau \{(w_\zeta)_x\}^2\,dxd\tau \\
&\leq \mu t\|(v_{xx}, w_{xx})(t)\|^2 + \mu\int_0^t \tau\|(v_{xt}, w_{xt})(\tau)\|^2\,d\tau \\
&\quad + C[\mu]\int_0^t \tau\left(1 + \left\|\left(\frac{w_x}{\sqrt{\zeta}}, v_{xx}, w_{xx}\right)\right\|^2\right)\left\|\left(\frac{w_x}{\sqrt{\zeta}}, v_{xx}, w_{xx}\right)\right\|^2\,d\tau + C[\mu](1+t^2).
\end{aligned}
$$

In deriving the last inequality, we have used the Sobolev and the Young inequalities as well as the estimates (4.23a), (4.23b), (4.24) and

$$
\|(w_\zeta)_t(t)\| \leq \frac{C}{\zeta}\|w_\zeta(t)\| + C\|(\{v_\zeta\}_t, \{w_\zeta\}_{xx})(t)\| + C, \tag{4.25}
$$

which follows from the equation (4.21). As the matrix A is positive definite, taking μ sufficiently small and using the Gronwall inequality yield that

$$\frac{t}{\zeta}\|(w_\zeta)_x(t)\|^2 + t\|(\{v_\zeta\}_{xx}, \{w_\zeta\}_{xx})(t)\|^2 + \int_0^t \tau\|(\{v_\zeta\}_{xt}, \{w_\zeta\}_{xt})(\tau)\|^2\, d\tau \le Ce^{\beta t}. \qquad (4.26)$$

Multiply the equation (4.21) by $-t(w_\zeta)_{xx}/\zeta$ and integrate the result by part over $[0,T]\times\Omega$. Then estimate the resulting equality by using the estimates (4.23a), (4.23b), (4.24) and (4.26). The result is

$$\int_0^t \frac{\tau}{\zeta^2}\|(w_\zeta)_x(\tau)\|^2 + \frac{\tau}{\zeta}\|(w_\zeta)_{xx}(\tau)\|^2\, d\tau \le Ce^{\beta t}. \qquad (4.27)$$

The estimate (4.23c) except the term $(v_\zeta)_t$ immediately holds with aid of (4.24), (4.25), (4.26) and (4.27). Solve the first component of the system (4.3) with respect to $(v_\zeta)_t$ and take the L^2-norm of the result. These computations yield the estimate of $(v_\zeta)_t$. Hence, the proof is completed. □

The next corollary immediately follows from the same computations as in the proof of Lemma 4.6.

Corollary 4.7. *Let* $(v_\zeta, w_\zeta) \in \mathfrak{Z}([0,T]) \cap \mathfrak{Y}_{loc}((0,T))$ *be a solution to* (4.3)–(4.6). *If the solution* (v_ζ, w_ζ) *verifies the estimate* (4.18) *uniformly in* ζ *for arbitrary* $t \in [0,T]$, *it also verifies the estimate* (4.23) *uniformly in* ζ *for arbitrary* $t \in [0,T]$.

The next lemma ensures that $L_\zeta(T)$ becomes arbitrarily small if ζ is taken sufficiently small.

Lemma 4.8. *Let* T *be an arbitrary positive constant greater than or equal to* T_*, *and* $(v_\zeta, w_\zeta) \in \mathfrak{Z}([0,T]) \cap \mathfrak{Y}_{loc}((0,T))$ *be a solution to* (4.3)–(4.6). *Suppose that the inequalities in* (4.23) *hold for* $t \in [0,T]$. *Then it holds that*

$$\|R_\zeta(t)\|^2 + \int_0^t \|(R_\zeta)_x(\tau)\|^2\, d\tau \le C\zeta e^{\beta t}, \qquad (4.28a)$$

$$\|Q_\zeta(t)\|^2 \le \|\log\theta_0\|^2 e^{-\nu t/\zeta} + C\zeta e^{\beta t}, \qquad (4.28b)$$

$$\|(\{R_\zeta\}_x, \{Q_\zeta\}_x)(t)\|^2 \le C\zeta\frac{e^{\beta t}}{t}, \qquad (4.28c)$$

$$L_\zeta(T) \le C\zeta\frac{e^{\beta T}}{T_*} \qquad (4.28d)$$

for $t \in (0,T]$, *where* ν, β *and* C *are positive constants independent of* t, δ *and* ζ.

Proof. Firstly, we show the estimate (4.28a). The straight forward computation leads to the estimate

$$\|\Phi[e^{v_\zeta}] - \Phi[e^{v_0^0}]\|_2 \le C\|R_\zeta\|. \tag{4.29}$$

Multiply the equation (4.20) by R_ζ and integrate the resulting equality by part over the domain Ω. Then apply the Sobolev and the Young inequalities to the resultant equality with using (4.1), (4.23a) and (4.29). These computations give

$$\begin{aligned}
&\int_0^1 \frac{1}{2}(R_\zeta)^2(t)\,dx + \int_0^t\!\!\int_0^1 \{(R_\zeta)_x\}^2\,dxd\tau \\
&= -\int_0^t\!\!\int_0^1 (Q_\zeta)_x(e^{w_\zeta}R_\zeta)_x + \left\{(e^{w_\zeta}-1)\,(v_\zeta)_{xx} - g_1[v_\zeta,w_\zeta] + g_1[v_0^0,0]\right\} R_\zeta\,dxd\tau \\
&\le \int_0^t \mu\|(R_\zeta)_x(\tau)\|^2 + C[\mu]\left\{(1+\|(v_\zeta)_{xx}(\tau)\|^2)\|R_\zeta(\tau)\|^2 + \|Q_\zeta(\tau)\|_1^2\right\}\,d\tau,
\end{aligned}$$

where μ is an arbitrary positive constant. Then taking μ small enough and using the estimate of Q_ζ in (4.23b), we have

$$\|R_\zeta(t)\|^2 + \int_0^t \|(R_\zeta)_x(\tau)\|^2\,d\tau \le C\int_0^t (1+\|(v_\zeta)_{xx}(\tau)\|^2)\|R_\zeta(\tau)\|^2\,d\tau + C\zeta(1+t). \tag{4.30}$$

The estimate (4.28a) is derived by the application of the Gronwall inequality to (4.30) with aid of (4.23b).

Secondly, the estimate (4.28b) is shown. Multiplying the equation (4.21) by $e^{\nu t/\zeta}Q_\zeta$ where ν is a positive constant to be determined and integrating the resultant equality by part over the domain Ω yield

$$\begin{aligned}
e^{\nu t/\zeta}\int_0^1 \frac{1}{2}(Q_\zeta)^2(t)\,dx + \int_0^t\!\!\int_0^1 \frac{e^{\nu\tau/\zeta}}{\zeta e^{w_\zeta}}(e^{w_\zeta}-1)Q_\zeta\,dxd\tau - \int_0^t\!\!\int_0^1 \frac{\nu e^{\nu\tau/\zeta}}{2\zeta}(Q_\zeta)^2\,dxd\tau \\
+ \int_0^t\!\!\int_0^1 \frac{2\kappa_0 e^{\nu\tau/\zeta}}{3e^{v_\zeta}}\{(Q_\zeta)_x\}^2\,dxd\tau = \int_0^1 \frac{1}{2}(\log\theta_0)^2\,dx \\
+ \int_0^t\!\!\int_0^1 e^{\nu\tau/\zeta}\left\{\frac{2\kappa_0}{3e^{v_\zeta}}(v_\zeta)_x(Q_\zeta)_x + \frac{2}{3}(v_\zeta)_t + g_2[v_\zeta,w_\zeta]\right\} Q_\zeta\,dxd\tau.
\end{aligned} \tag{4.31}$$

Use the estimate (4.23a) and the mean value theorem to handle the second term in the left hand side of (4.31) as

$$\int_0^t\!\!\int_0^1 \frac{e^{\nu\tau/\zeta}}{\zeta e^{w_\zeta}}(e^{w_\zeta}-1)(Q_\zeta)\,dxd\tau \ge c\int_0^t \frac{e^{\nu\tau/\zeta}}{\zeta}\|Q_\zeta(\tau)\|^2 d\tau, \tag{4.32}$$

where c is a positive constant independent of ζ. Moreover, by the estimates (4.23a) and (4.23b) as well as the Sobolev and the Young inequalities, the last term in the right hand side of (4.31) is estimated as

$$\text{(last term)} \leq \mu \int_0^t \frac{e^{\nu\tau/\zeta}}{\zeta} \|Q_\zeta(\tau)\|^2 \, d\tau + C[\mu]\zeta e^{\nu t/\zeta}(1+t). \tag{4.33}$$

Substituting (4.32) and (4.33) in (4.31), making μ and ν so small that $c > \mu + \nu/2$ and then dividing the result by $e^{\nu t/\zeta}$, we obtain (4.28b).

Thirdly, we derive the estimate (4.28c). For this purpose, it suffices to show the estimate of R_ζ since the estimate of Q_ζ have been already shown in (4.23c). By the Poincaré and the Sobolev inequalities as well as (4.1), (4.23a) and (4.29), the L^2-norm of the right hand side of the equation (4.20) is handled as

$$t \,\|\text{(right hand side)}\|^2 \leq Ce^{\beta t}\|(R_\zeta, Q_\zeta)(t)\|_1^2 + Ct\|\{Q_\zeta\}_{xx}(t)\|^2. \tag{4.34}$$

Multiplying the equation (4.21) by $-t(R_\zeta)_{xx}$, integrating the result by part over the domain $[0,t] \times \Omega$ and then estimating the resulting equality by the Sobolev and the Schwartz inequalities as well as (4.23b), (4.23c), (4.28a) and (4.34), we have

$$\begin{aligned}
&\frac{t}{2}\int_0^1 \{(R_\zeta)_x\}^2(t)\, dx + \int_0^t\int_0^1 \tau\{(R_\zeta)_{xx}\}^2 \, dxd\tau \\
&= \int_0^t\int_0^1 \frac{\{(R_\zeta)_x\}^2}{2} - \tau\left\{e^{w_\zeta}(Q_\zeta)_{xx} - \left(e^{w_\zeta}-1\right)(v_\zeta)_{xx} + g_1[v_\zeta, w_\zeta] - g_1[v_0^0, 0]\right\}(R_\zeta)_{xx}\, dxd\tau \\
&\leq \mu \int_0^t \tau\|(R_\zeta)_{xx}(\tau)\|^2\, d\tau + C[\mu]\zeta e^{\beta t}.
\end{aligned}$$

Then making μ sufficiently small yields the desired estimate (4.28c). Lastly, the estimate (4.28d) immediately follows from the estimates (4.28a)–(4.28c). □

Now we are at the position to prove the "semi-global existence" of the solution to the energy-transport model.

Theorem 4.9. *Suppose that the initial data $(v_0, w_0) \in H^1(\Omega)$ and the boundary data ρ_l, ρ_r and ϕ_r satisfy (2.4), (2.6) and (2.7a). For arbitrarily positive time T, there exist positive constants δ_0, independent of T, and ζ_T, depending on T, such that if $\delta \leq \delta_0$ and $\zeta \leq \zeta_T$, then the initial boundary value problem (4.3)–(4.6) has a unique solution $(v_\zeta, w_\zeta) \in \mathfrak{Z}([0,T]) \cap \mathfrak{Y}_{loc}((0,T))$. Moreover, it satisfies $\sqrt{t}v_{xt}, \sqrt{t}w_{xt} \in L^2(0,T;L^2(\Omega))$ as well as the estimates (4.23) and (4.28).*

Proof. Corollary 4.5 ensures the solution (v_ζ, w_ζ) exists until time T_* independent of ζ. Moreover, (v_ζ, w_ζ) satisfies (4.18), which immediately means (4.23) owing to Corollary 4.7. Then we can apply Lemma 4.8 and see that the estimate (4.28d) holds. Hence $L_\zeta(T_*)$ becomes arbitrarily small by taking ζ small enough in (4.28d). Here it is crucial that the existence time T_* is independent of ζ.

To construct the solution (v_ζ, w_ζ) until the time T, take δ and ζ so small that $\delta + \zeta \le \delta_0$,

$$L_\zeta(T_*) < \delta_1, \tag{4.35}$$

$$\zeta < \delta_1 T_*/2Ce^{\beta T}, \tag{4.36}$$

where δ_0 and δ_1 are defined in Lemma 4.6 as well as T_*, C and β are given in (4.28d). The condition (4.36) makes the right hand side of (4.28d) be less than $\delta_1/2$. Let T^* be the supermum of time t until which the solution exists and satisfies $L_\zeta(t) \le \delta_1$, that is,

$$T_*^s := \sup_t \{t > 0; L_\zeta(t) \le \delta_1\}$$

It is obvious that $T_* < T_*^s$ owing to (4.35). Since $L_\zeta(t_0) \le \delta_1$ holds for arbitrary t_0 in $[T_*, T_*^s)$, we have

$$\|(v_\zeta, w_\zeta)(t_0)\|_1 \le \delta_1 + \sup_{0 \le T_*} \|v_0^0(t)\|_1.$$

Regarding the right hand side above as N_0 in Corollary 4.5, t_0 as the initial time and $(v_\zeta, w_\zeta)(t_0)$ as the initial data, we see that the solution (v_ζ, w_ζ) exists in the time interval $[T_*, T_*^s]$ and satisfies $L_\zeta(T_*^s) \le \delta_1$.

We show $T \le T_*^s$ by contradiction. Suppose that $T_*^s < T$. As $L_\zeta(T_*^s) \le \delta_1$ which means the assumptions in Lemma 4.6 hold, the solution satisfies the estimate (4.23) for $t \in [0, T_*^s]$. Thus it is possible to apply Lemma 4.8 and get $L_\zeta(T_*^s) \le \delta_1/2$ due to (4.28d) and (4.36). Applying Lemma 4.3 with regarding T_*^s as initial time, we see that there exists a positive constant T_0 such that the solution exists until $T_*^s + T_0$ and satisfies $L_\zeta(T_*^s + T_0) \le \delta_1$. It contradicts the definition of T_*^s. Consequently, we have $T \le T_*^s$, that is, the solution exists until time T. $\square$

The difference between the solution to the non-stationary problem and the stationary solution becomes arbitrarily small as the time T is taken large enough, and thus ζ is small enough, in Theorem 4.9. This property is shown in the next corollary.

Corollary 4.10. *Let $(\tilde{v}_\zeta, \tilde{w}_\zeta)$ be the stationary solution to the problem* (4.8), (4.5) *and* (4.6). *Suppose the same assumptions as in Theorem 4.9. For an arbitrary positive number Λ, there exist positive constants T_Λ and ζ_Λ such that if $\zeta \le \zeta_\Lambda$, the solution (v_ζ, w_ζ) to the problem* (4.3)–(4.6) *exists in the function space $\mathfrak{Z}([0, T_\Lambda]) \cap \mathfrak{Y}_{loc}((0, T_\Lambda))$ and verifies*

$$\|(v_\zeta - \tilde{v}_\zeta, w_\zeta - \tilde{w}_\zeta)(T_\Lambda)\|_1 \le \Lambda. \tag{4.37}$$

Moreover, it satisfies $\sqrt{t}v_{xt}, \sqrt{t}w_{xt} \in L^2(0, T_\Lambda; L^2(\Omega))$ as well as the estimates (4.23).

Proof. It is sufficient to show the inequality (4.37) as the other assertions are proven in Theorem 4.9. Use the inequalities (3.49), (4.1b) and (4.28a), and then take T_Λ sufficiently large to obtain

$$\begin{aligned} \|(v_\zeta - \tilde{v}_\zeta)(T_\Lambda)\| &\leq \|R_\zeta(T_\Lambda)\| + \|(v_0^0 - \tilde{v}_0^0)(T_\Lambda)\| + \|\tilde{v}_0^0 - \tilde{v}_\zeta\| \\ &< C\left\{\zeta^{1/2}e^{\beta T_\Lambda/2} + \zeta\right\} + \Lambda/8, \end{aligned} \tag{4.38}$$

where $\tilde{v}_0^0 := \log \tilde{\rho}_0^0$. We take ζ_Λ so small that the right hand side of (4.38) is smaller than $\Lambda/4$ for an arbitrary $\zeta \in (0, \zeta_\Lambda]$. As the other estimates in (4.37) are shown similarly, the proof is completed. □

4.3 Global existence of solution

In this section, we prove the time global existence of the solution and the asymptotic stability of the stationary solution for the large initial data. For this purpose, it suffices to show the stability theorem with the small initial disturbance by virtue of Corollary 4.10.

Theorem 4.11. *Let $(\tilde{v}, \tilde{w})$ be the stationary solution for (4.8). Suppose that the initial data $(v_0, w_0) \in H^1(\Omega)$ and the boundary data ρ_l, ρ_r and ϕ_r satisfy (2.4), (2.6) and (2.7a). Then there exists a positive constant δ_*, independent of ζ, such that if*

$$\delta + \zeta + \|(v_0 - \tilde{v}, w_0 - \tilde{w})\|_1 \leq \delta_*, \tag{4.39}$$

then the initial boundary value problem (4.3)–(4.6) has a unique solution (v, w) satisfying $(v-\tilde{v}, w-\tilde{w}) \in \mathfrak{Z}([0,\infty)) \cap \mathfrak{Y}_{loc}((0,\infty))$. Moreover, the solution (v, w) verifies $\sqrt{t}v_{xt}, \sqrt{t}w_{xt} \in L^2(0,\infty; L^2(\Omega))$ and the convergence (4.9). It also satisfies the decay estimates

$$\|(v - \tilde{v}, w - \tilde{w}, \Phi[e^v] - \Phi[e^{\tilde{v}}])(t)\|_1^2 \leq C\|(v_0 - \tilde{v}, w_0 - \tilde{w})\|_1^2 e^{-\alpha t}, \tag{4.40a}$$

$$\frac{t}{\zeta}\|(w - \tilde{w})(t)\|_1^2 + t\|(v - \tilde{v}, w - \tilde{w})(t)\|_2^2 \leq C\|(v_0 - \tilde{v}, w_0 - \tilde{w})\|_1^2 e^{-\alpha t}, \tag{4.40b}$$

where C and α are positive constants independent of t, δ and ζ.

To show Theorem 4.11, we regard the solution (v, w) to the non-stationary problem (4.3)–(4.6) as a perturbation from the stationary solution $(\tilde{v}, \tilde{w})$ to (4.8):

$$u(t,x) := v(t,x) - \tilde{v}(x), \quad \varpi(t,x) := w(t,x) - \tilde{w}(x).$$

Subtracting (4.8) from (4.3a), we see that (u, ϖ) verifies the equation

$$\begin{pmatrix} u \\ 3\varpi/2 \end{pmatrix}_t - A[\tilde{v} + u, \tilde{w} + \varpi]\begin{pmatrix} u \\ \varpi \end{pmatrix}_{xx} + \begin{pmatrix} 0 \\ -3(e^{-\varpi-\tilde{w}} - e^{-\tilde{w}})/2\zeta \end{pmatrix} = H, \tag{4.41a}$$

$$H := \{A[\tilde{v} + u, \tilde{w} + \varpi] - A[\tilde{v}, \tilde{w}]\}\begin{pmatrix} \tilde{v} \\ \tilde{w} \end{pmatrix}_{xx} + G[\tilde{v} + u, \tilde{w} + \varpi] - G[\tilde{v}, \tilde{w}]. \tag{4.41b}$$

The initial and the boundary conditions to the system (4.41) follow from (4.4)–(4.6) as

$$u(x,0) = u_0(x) := v_0(x) - \tilde{v}(x), \quad \varpi(x,0) = \varpi_0(x) := w_0(x) - \tilde{w}(x), \tag{4.42}$$

$$u(t,0) = u(t,1) = \varpi_x(t,0) = \varpi_x(t,1) = 0. \tag{4.43}$$

Theorem 3.5 and Corollary 4.5 apparently mean the local existence of the solution (u,ϖ) to the initial boundary value problem (4.41)–(4.43).

Lemma 4.12. *Suppose that the initial data (u_0,ϖ_0) belongs to $H^1(\Omega)$. Then there exists a positive constant T_*, independent of ζ, such that the initial boundary value problem (4.41)–(4.43) has a unique local solution $(u,\varpi) \in \mathfrak{Z}([0,T_*]) \cap \mathfrak{Y}_{loc}((0,T_*))$. Moreover, it verifies $\sqrt{t}u_{xt}, \sqrt{t}\varpi_{xt} \in L^2(0,T_*;L^2(\Omega))$ and the convergence (4.9).*

The standard continuation argument with the local existence in Lemma 4.12 and an a-priori estimate (4.44) in Proposition 4.13 yields the existence of the solution globally in time to the problem (4.41)–(4.43), stated in Theorem 4.11. To show the a-priori estimate (4.44), we use a notation

$$N_\zeta(t) := \sup_{0\le\tau\le t}\left\{\|(u,\varpi)(\tau)\|_1 + \sqrt{\frac{t}{\zeta}}\|\varpi(\tau)\|_1 + \sqrt{t}\|(u,\varpi)(\tau)\|_2\right\}.$$

Proposition 4.13. *Let $T>0$ and let $(u,\varpi) \in \mathfrak{Z}([0,T]) \cap \mathfrak{Y}_{loc}((0,T))$ be a solution to (4.41)–(4.43) satisfying $\sqrt{t}u_{xt}, \sqrt{t}\varpi_{xt} \in L^2(0,T;L^2(\Omega))$ and the convergence (4.9). Then there exists a positive constant δ_0, independent of T and ζ, such that if $N_\zeta(T)+\delta+\zeta \le \delta_0$, then the estimate*

$$\begin{aligned}(1+t)\left(\|(u,\varpi)(t)\|_1^2 + \|\{\Phi[e^{\tilde{v}+u}]-\Phi[e^{\tilde{v}}]\}_x(t)\|^2\right) + t\left(\frac{1}{\zeta}\|\varpi(\tau)\|_1^2 + \|(u_{xx},\varpi_{xx})(t)\|^2\right)&\\ + \int_0^t (1+\tau)\left(\frac{1}{\zeta}\|\varpi(\tau)\|_1^2 + \|\{\Phi[e^{\tilde{v}+u}]-\Phi[e^{\tilde{v}}]\}_x(\tau)\|^2 + \|(u,\varpi)(\tau)\|_2^2\right)d\tau&\\ + \int_0^t \tau\left(\frac{1}{\zeta^2}\|\varpi(\tau)\|^2 + \|(u_{xt},\varpi_{xt})(\tau)\|^2\right)d\tau \le C\|(u_0,\varpi_0)\|_1^2&\end{aligned} \tag{4.44}$$

holds for $t\in[0,T]$, where C is a positive constant independent of T, δ and ζ.

Proof. The proof is divided into the three parts studied in Lemmas 4.14–4.16. Multiply the estimate (4.57) by α, the estimate (4.58) by α^2, and the estimate (4.68) by α^3, respectively. Summing up these three resulting inequalities and the estimate (4.54), using the estimate (4.66a) and making α and $N_\zeta(T)+\delta+\zeta^{1/4}$ sufficiently small, we obtain the a-priori estimate (4.44). □

We begin detailed discussions with deriving the basic estimate (4.50) in Lemma 4.14. For this purpose, an energy form

$$\mathcal{E}_1 := \rho\tilde{\theta}\Psi\left(\frac{\tilde{\rho}}{\rho}\right) + \frac{1}{2}\left\{(\phi - \tilde{\phi})_x\right\}^2 + \frac{3}{2}\rho\tilde{\theta}\Psi\left(\frac{\theta}{\tilde{\theta}}\right), \tag{4.45}$$
$$\Psi(s) := s - 1 - \log s.$$

is employed. Here $\mathcal{E}_1$ is equivalent to $|(u, \varpi, \{\Phi[e^{\tilde{v}+u}] - \Phi[e^{\tilde{v}}]\}_x)|^2$ if $|(u, \varpi)|$ is sufficiently small since $\Psi(s)$ is equivalent to $|s-1|^2$ if $s \geq c > 0$. Namely, there exist positive constants δ_0, c_1 and C_1 such that if $|(u, \varpi)| \leq \delta_0$, then the next inequality holds:

$$c_1 \left|\left(u, \varpi, \{\Phi[e^{\tilde{v}+u}] - \Phi[e^{\tilde{v}}]\}_x\right)\right|^2 \leq \mathcal{E}_1 \leq C_1 \left|\left(u, \varpi, \{\Phi[e^{\tilde{v}+u}] - \Phi[e^{\tilde{v}}]\}_x\right)\right|^2. \tag{4.46}$$

Moreover, the energy form $\mathcal{E}_1$ verifies the equation

$$(\mathcal{E}_1)_t + \frac{1}{\rho}(j - \tilde{j})^2 + \frac{3\rho}{2\zeta\theta}(\theta - \tilde{\theta})^2 + \frac{\kappa_0}{\theta}\{(\theta - \tilde{\theta})_x\}^2 = -\{(\theta - \tilde{\theta})(j - \tilde{j})\}_x + (\mathcal{R}_1)_x + \mathcal{R}_2, \tag{4.47}$$

$$\mathcal{R}_1 := (\phi - \tilde{\phi})(\phi - \tilde{\phi})_{xt} + (\phi - \tilde{\phi})(j - \tilde{j}) - \tilde{\theta}u(j - \tilde{j}) + \frac{\kappa_0}{\theta}(\theta - \tilde{\theta})(\theta - \tilde{\theta})_x,$$
$$\mathcal{R}_2 := -\tilde{j}\left(\frac{1}{\rho} - \frac{1}{\tilde{\rho}}\right)(j - \tilde{j}) + \tilde{\theta}_x u(j - \tilde{j}) - \frac{3}{2}\tilde{\theta}(j - \tilde{j})_x\Psi\left(\frac{\theta}{\tilde{\theta}}\right) + \frac{\kappa_0\theta_x}{\theta^2}(\theta - \tilde{\theta})(\theta - \tilde{\theta})_x$$
$$- \left\{\frac{3}{2}(j\theta_x - \tilde{j}\tilde{\theta}_x) + \tilde{j}(v_x\theta - \tilde{v}_x\tilde{\theta}) - \left(\frac{j^2}{\rho^2} - \frac{\tilde{j}^2}{\tilde{\rho}^2}\right) + \frac{3(\tilde{\theta} - 1)}{2\zeta}(\rho - \tilde{\rho})\right\}\frac{(\theta - \tilde{\theta})}{\theta}.$$

Here the potentials are given by the formula (2.8):

$$\phi := \Phi[e^v], \quad \tilde{\phi} := \Phi[e^{\tilde{v}}].$$

Owing to the boundary condition (2.6), we have

$$(\phi - \tilde{\phi})(t, 0) = (\phi - \tilde{\phi})(t, 1) = 0. \tag{4.48}$$

The equation (4.47) is seen as a special case of the equation for an energy form (5.63) to the hydrodynamic model (2.11). Actually, it is derived similarly as (5.63). See Section 5.3 for the derivation.

In the proof of the following lemma, we use the estimates

$$c\|u(t)\|_i \leq \|(\rho - \tilde{\rho})(t)\|_i = \|(e^v - e^{\tilde{v}})(t)\|_i \leq C\|u(t)\|_i, \tag{4.49a}$$
$$c\|\varpi(t)\|_i \leq \|(\theta - \tilde{\theta})(t)\|_i = \|(e^w - e^{\tilde{w}})(t)\|_i \leq C\|\varpi(t)\|_i, \tag{4.49b}$$
$$\|(\Phi[e^{\tilde{v}+u}] - \Phi[e^{\tilde{v}}])_x(t)\|_{1+i} \leq C\|u(t)\|_i, \tag{4.49c}$$
$$\|(j_x - \tilde{j}_x)(t)\| \leq C\|u_t(t)\| \tag{4.49d}$$

for $i = 0, 1, 2$, which immediately follow from the equation (2.14a).

Lemma 4.14. *Under the same conditions as in Proposition 4.13, the following estimate holds for* $t \in [0,T]$.

$$(1+t)\left(\|(u,\varpi)(t)\|^2 + \|(\Phi[e^{\tilde{v}+u}] - \Phi[e^{\tilde{v}}])_x(t)\|^2\right) + \int_0^t (1+\tau)\left(\frac{1}{\zeta}\|\varpi(\tau)\|^2 + \|(u,\varpi)(\tau)\|_1^2 + \|(\Phi[e^{\tilde{v}+u}] - \Phi[e^{\tilde{v}}])_x(\tau)\|^2\right) d\tau \le C\|(u_0,\varpi_0)\|^2 + C(N_\zeta(T)+\delta+\zeta^{1/4})\int_0^t (1+\tau)\|u_t(\tau)\|^2\,d\tau, \quad (4.50)$$

where C *is a positive constant independent of* T, δ *and* ζ.

Proof. Multiplying the equation (4.45) by t^k for $k=0,1$ and integrating the resulting equality by part over Ω give

$$\frac{d}{dt}\left(t^k \int_0^1 \mathcal{E}_1\,dx\right) + t^k \int_0^1 \frac{1}{\rho}(j-\tilde{j})^2 + \frac{3\rho}{2\zeta\theta}(\theta-\tilde{\theta})^2 + \frac{\kappa_0}{\theta}\{(\theta-\tilde{\theta})_x\}^2\,dx = kt^{k-1}\int_0^1 \mathcal{E}_1\,dx - t^k(\theta-\tilde{\theta})(j-\tilde{j})(t,1) + t^k(\theta-\tilde{\theta})(j-\tilde{j})(t,0) + t^k\int_0^1 \mathcal{R}_2\,dx \quad (4.51)$$

since the integration of $(\mathcal{R}_1)_x$ over Ω is zero owing to the boundary conditions (4.43) and (4.48). Applying the Sobolev and the Young inequalities with using (4.49b) yields

$$|(\theta-\tilde{\theta})(j-\tilde{j})(t)|_0 \le C\|(\theta-\tilde{\theta})(t)\|^{1/2}(\|(\theta-\tilde{\theta})(t)\| + \|(\theta-\tilde{\theta})_x(t)\|)^{1/2}\|(j-\tilde{j})(t)\|_1 \le C\zeta^{1/4}\left(\frac{1}{\zeta}\|\varpi(t)\|^2 + \|(\varpi, j-\tilde{j})(t)\|_1^2\right). \quad (4.52)$$

Moreover, using the inequalities (3.11a), (3.25) and (4.49), we estimate the last term of (4.51) as

$$\int_0^1 \mathcal{R}_2\,dx \le C(N_\zeta(T)+\delta)\|(u,\varpi,j-\tilde{j})(t)\|_1^2. \quad (4.53)$$

Substituting (4.46), (4.52) and (4.53) in (4.51) and then using (4.49), we have

$$\frac{d}{dt}\left(t^k\int_0^1 \mathcal{E}_1\,dx\right) + ct^k\left(\|(j-\tilde{j})(t)\|^2 + \frac{1}{\zeta}\|\varpi(t)\|^2 + \|\varpi_x(t)\|^2\right) \le kCt^{k-1}\|(u,\varpi)(t)\|^2 + C(N_\zeta(T)+\delta+\zeta^{1/4})t^k\left(\frac{1}{\zeta}\|\varpi(t)\|^2 + \|(u, j-\tilde{j}, u_x, \varpi_x, u_t)(t)\|^2\right). \quad (4.54)$$

Divide the equation (2.14d) by $e^{\tilde{v}+u}$ and the equation (3.1d) by $e^{\tilde{v}}$, respectively. Take the difference between the two results and multiply the resulting equation by u_x. Then integrate the resultant equality by parts over Ω and use the equations (2.14c) and (3.1c) as well as the boundary condition (4.43) to get

$$\begin{aligned}
& t^k \int_0^1 e^{\tilde{w}}(u_x)^2 + u(e^{\tilde{v}+u} - e^{\tilde{v}})\,dx \\
&= t^k \int_0^1 \left\{ (e^{\tilde{w}+\varpi} - e^{\tilde{w}})_x + v_x(e^{\tilde{w}+\varpi} - e^{\tilde{w}}) + \left(\frac{j}{e^{\tilde{v}+u}} - \frac{\tilde{j}}{e^{\tilde{v}}} \right) \right\} u_x\,dx \\
&\le t^k \left\{ \mu \|u_x(t)\|^2 + C(N_\zeta(T)+\delta)\|u(t)\|_1^2 + C[\mu]\|(j-\tilde{j}, \varpi, \varpi_x)(t)\|^2 \right\}, \qquad (4.55)
\end{aligned}$$

where μ is an arbitrary positive constant to be determined. In deriving the above inequality, we have also used (3.25) and (4.49) as well as the Schwarz and the Sobolev inequalities. The left hand side of (4.55) is estimated from below by $ct^k\|u\|_1^2$ for a positive constant c due to the mean value theorem. Using this fact with (4.49c) and letting μ sufficiently small, we obtain

$$\begin{aligned}
& t^k\|u(t)\|_1^2 + t^k\|(\Phi[e^{\tilde{v}+u}] - \Phi[e^{\tilde{v}}])_x(t)\|^2 \\
&\qquad \le Ct^k \left\{ (N_\zeta(T)+\delta)\|u(t)\|_1^2 + \|(j-\tilde{j}, \varpi, \varpi_x)(t)\|^2 \right\}. \qquad (4.56)
\end{aligned}$$

Multiply (4.56) with $k = 1$ by α^3, (4.54) with $k = 1$ by α^2 and (4.56) with $k = 0$ by α, respectively, where α is an arbitrary positive constant. Then sum up these results and (4.54) with $k = 0$, let α and $N_\zeta(T) + \delta + \zeta^{1/4}$ small enough, and then integrate the resulting inequality with respect to t. These computations give the desired estimate (4.50). □

Lemma 4.15. *Under the same conditions as in Proposition 4.13, the following estimates hold for $t \in [0, T]$.*

$$\begin{aligned}
& (1+t)\|(u_x, \varpi_x)(t)\|^2 + \int_0^t (1+\tau) \left(\frac{1}{\zeta}\|\varpi_x(\tau)\|^2 + \|(u_{xx}, \varpi_{xx})(\tau)\|^2 \right) d\tau \\
&\qquad \le C\|(u_0, \varpi_0)\|_1^2 + C\int_0^t (1+\tau)\|(u,\varpi)(\tau)\|_1^2\,d\tau, \qquad (4.57)
\end{aligned}$$

$$\frac{t}{\zeta}\|\varpi(t)\|^2 + \int_0^t \frac{\tau}{\zeta^2}\|\varpi(\tau)\|^2\,d\tau \le C\int_0^t \frac{1}{\zeta}\|\varpi(\tau)\|^2 + \tau\|(u,\varpi)(\tau)\|_2^2\,d\tau, \qquad (4.58)$$

where C is a positive constant independent of T, δ and ζ.

Proof. Take the inner product of the equation (4.41a) with the vector $(-t^k u_{xx}, -t^k \varpi_{xx})$ for

$k = 0, 1$ in $L^2(\Omega)$ and apply the integration by part to get

$$\frac{d}{dt}\left(t^k \int_0^1 \frac{1}{2}(u_x)^2 + \frac{3}{4}(\varpi_x)^2\, dx\right) + t^k \int_0^1 (u_{xx}, \varpi_{xx}) A[v,w](u_{xx}, \varpi_{xx})^\top\, dx$$
$$+ t^k \int_0^1 \frac{3}{2\zeta e^w}(\varpi_x)^2\, dx = kt^{k-1}\int_0^1 \frac{1}{2}(u_x)^2 + \frac{3}{4}(\varpi_x)^2\, dx$$
$$- t^k \int_0^1 \frac{3\tilde{w}_x}{2\zeta}\left(\frac{1}{e^{\tilde{w}+\varpi}} - \frac{1}{e^{\tilde{w}}}\right)\varpi_x\, dx + t^k \int_0^1 (u_{xx}, \varpi_{xx}) H\, dx. \quad (4.59)$$

Notice that the L^2-norm of H, which is defined in (4.41b), is estimated as

$$\|H\| \le C\|(u,\varpi)\|_1 + C\|(u,\varpi)\|_1^{1/2}\|(u_{xx}, \varpi_{xx})\|^{1/2}. \quad (4.60)$$

This inequality together with (3.25d) gives the estimate of the right hand side of (4.59) as

$$\text{(right hand side)} \le \mu t^k\|(u_{xx}, \varpi_{xx})(t)\|^2 + C[\mu]t^k\|(u,\varpi)(t)\|_1^2 + Ckt^{k-1}\|(u_x, \varpi_x)(t)\|^2, \quad (4.61)$$

where μ is an arbitrary positive constant. Since $A[v,w]$ is symmetric and positive definite, substituting (4.61) in (4.59) and making μ sufficiently small yield the inequality

$$\frac{d}{dt}\left(\frac{t^k}{2}\|u_x(t)\|^2 + \frac{3t^k}{4}\|\varpi_x(t)\|^2\, dx\right) + ct^k\left(\|(u_{xx}, \varpi_{xx})(t)\| + \frac{1}{\zeta}\|\varpi_x(t)\|^2\right)$$
$$\le Ct^k\|(u,\varpi)(t)\|_1^2 + Ckt^{k-1}\|(u_x, \varpi_x)(t)\|^2. \quad (4.62)$$

Summing up the estimates (4.62) with $k = 0, 1$ and integrating the result with respect to t, we have the desired estimate (4.57).

Taking the inner product of the equation (4.41a) with the vector $(0, -t\varpi/\zeta)$ in $L^2(\Omega)$ and applying the integration by part yield

$$\frac{d}{dt}\left(\frac{3t}{4\zeta}\int_0^1 \varpi^2\, dx\right) - \frac{3t}{2\zeta^2}\int_0^1 \left(\frac{1}{e^{\tilde{w}+\varpi}} - \frac{1}{e^{\tilde{w}}}\right)\varpi\, dx$$
$$= \frac{3}{4\zeta}\int_0^1 \varpi^2\, dx + \frac{t}{\zeta}\int_0^1 (0,\varpi)\left(A[v,w](u_{xx}, \varpi_{xx})^\top + H\right)\, dx. \quad (4.63)$$

Due to the mean value theorem, the second term on the left hand side is estimated as

$$\frac{ct}{\zeta^2}\|\varpi(t)\|^2 \le -\frac{3t}{2\zeta^2}\int_0^1 \left(\frac{1}{e^{\tilde{w}+\varpi}} - \frac{1}{e^{\tilde{w}}}\right)\varpi\, dx \quad (4.64)$$

from below. Substituting (4.64) in (4.63) and computing similarly as in the derivation of (4.62), we have

$$\frac{d}{dt}\left(\frac{3t}{4\zeta}\|\varpi(t)\|^2\right) + \frac{ct}{\zeta^2}\|\varpi(t)\|^2 \le \frac{C}{\zeta}\|\varpi(t)\|^2 + tC\|(u,\varpi)(t)\|_2^2. \quad (4.65)$$

Integration of (4.65) with respect to t gives the desired estimate (4.58). □

To derive the estimates for the second derivatives, we use

$$\|u_t\| \le C\|(u,\varpi)\|_2, \tag{4.66a}$$
$$\|\varpi_t\| \le C\|\varpi\|/\zeta + C\|(u,\varpi)\|_2, \tag{4.66b}$$
$$\|H_t\| \le C(1+\|(u,\varpi)\|_2)\|(u_t,\varpi_t)\|_1. \tag{4.66c}$$

They follow from the equation (4.41a), the estimates (3.25), (4.60) and

$$\|(\Phi[e^{\tilde{v}+u}] - \Phi[e^{\tilde{v}}])_{xt}(t)\|_{1+i} \le C\|u_t(t)\|_i \tag{4.67}$$

for $i = 0, 1$.

Lemma 4.16. *Under the same conditions as in Proposition 4.13, the estimate*

$$t\|(u_{xx},\varpi_{xx})(t)\|^2 + \frac{t}{\zeta}\|\varpi_x(t)\|^2 + \int_0^t \tau\|(u_{xt},\varpi_{xt})(\tau)\|^2\,d\tau \le Ct\|(u,\varpi)(t)\|_1^2$$
$$+ C\int_0^t \frac{\tau}{\zeta^2}\|\varpi(\tau)\|^2 + (1+\tau)\left(\frac{1}{\zeta}\|\varpi(\tau)\|_1^2 + \|(u,\varpi)(\tau)\|_2^2\right)d\tau \tag{4.68}$$

holds for $t \in [0,T]$, where C is a positive constant independent of T, δ and ζ.

Proof. Taking the inner product of the equation (4.41a) with the vector $(-tu_{xxt}, -t\varpi_{xxt})$ in $L^2(\Omega)$ and applying the integration by part yield

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{t}{2}\int_0^1 (u_{xx},\varpi_{xx})A[v,w](u_{xx},\varpi_{xx})^\top + \frac{3}{4\zeta e^{\tilde{w}}}\varpi_x^2\,dx\right) + t\int_0^1 (u_{xt})^2 + \frac{3}{2}(\varpi_{xt})^2\,dx\\
&= \frac{d}{dt}\left(t\int_0^1 (u_{xx},\varpi_{xx})H\,dx\right) + \int_0^1 \frac{3}{4\zeta e^{\tilde{w}}}\varpi_x^2\,dx + \frac{3t}{2\zeta}\int_0^1 w_x\left(\frac{1}{e^{\tilde{w}+\varpi}} - \frac{1}{e^{\tilde{w}}}\right)\varpi_{xt}\,dx\\
&\quad + \int_0^1 (u_{xx},\varpi_{xx})\left\{\frac{1}{2}(tA[v,w])_t(u_{xx},\varpi_{xx})^\top + (tH)_t\right\}dx\\
&\le \frac{d}{dt}\left(t\int_0^1 (u_{xx},\varpi_{xx})H\,dx\right) + \mu t\|(u_{xt},\varpi_{xt})(t)\|^2 + C[\mu]\frac{t}{\zeta^2}\|\varpi(t)\|^2\\
&\quad + C[\mu](1+t)\left(\frac{1}{\zeta}\|\varpi(t)\|_1^2 + \|(u,\varpi)(t)\|_2^2\right),
\end{aligned} \tag{4.69}$$

where μ is an arbitrary positive constant. In deriving the last inequality, we have used the Sobolev and the Young inequalities as well as the estimates (3.25d), (4.60), (4.66) and

$$\sqrt{t}(\|(u,\varpi)(t)\|_2 + \|\varpi(t)\|_1/\zeta) \le CN_\zeta(T) \le C.$$

Making μ small enough, we obtain

$$\frac{d}{dt}\left(\frac{t}{2}\int_0^1 (u_{xx},\varpi_{xx})A[v,w](u_{xx},\varpi_{xx})^\top + \frac{3}{2\zeta e^{\tilde{w}}}\varpi_x^2\,dx\right) + \frac{t}{2}\int_0^1 (u_{xt})^2 + (\varpi_{xt})^2\,dx$$
$$\leq \frac{d}{dt}\left(t\int_0^1 (u_{xx},\varpi_{xx})H\,dx\right) + C\frac{t}{\zeta^2}\|\varpi(t)\|^2 + C(1+t)\left(\frac{1}{\zeta}\|\varpi(t)\|_1^2 + \|(u,\varpi)(t)\|_2^2\right). \tag{4.70}$$

The first term of the right hand side of (4.70) is handled by using

$$t\int_0^1 (u_{xx},\varpi_{xx})H\,dx \leq \mu' t\|(u_{xx},\varpi_{xx})(t)\|^2 + C[\mu']t\|(u,\varpi)(t)\|_1^2, \tag{4.71}$$

where μ' is an arbitrary positive constant. The inequality (4.71) follows from the estimate (4.60). On the other hand, the first term in the left hand side of (4.70) is handled by

$$ct\|(u_{xx},\varpi_{xx})(t)\|^2 \leq t\int_0^1 (u_{xx},\varpi_{xx})A[v,w](u_{xx},\varpi_{xx})^\top\,dx \leq Ct\|(u_{xx},\varpi_{xx})(t)\|^2, \tag{4.72}$$

which holds as $A[v,w]$ is positive definite. Integrate (4.70) over $[\varepsilon,t]$, substitute (4.71) and (4.72) in the result, and then let μ' sufficiently small. Finally, letting $\varepsilon \downarrow 0$ yields the desired estimate (4.68) since the right hand side both of (4.71) and (4.72) converge to zero due to (4.9). □

Proof of Theorem 4.11. The existence of the time global solution is established by the standard continuation argument with the local existence in Corollary 4.12 and the a-priori estimate in Proposition 4.13. Hence, to complete the proof of Theorem 4.11, it suffices to show the decay estimates in (4.40).

Multiply (4.56) with $k=0$ by β, (4.54) with $k=1$ by β^2, (4.56) with $k=1$ by β^3, (4.62) with $k=0$ by β^4, (4.62) with $k=1$ by β^5, (4.65) by β^6, (4.70) by β^7, respectively. Here $\beta \in (0,1]$ is a constant, to be determined. Summing up these results and the estimate (4.54) with $k=0$, we have an ordinary differential inequality

$$\frac{d}{dt}E(t) + c_1 D(t) \leq C_1\beta D(t)$$
$$+ C(N_\zeta(T) + \delta + \zeta^{1/4})(1+t)\left(\frac{1}{\zeta}\|\varpi(t)\|^2 + \|(j-\tilde{j})(t)\|^2 + \|(u,\varpi)(t)\|_2^2\right), \tag{4.73}$$

$$E(t) := \int_0^1 (1+\beta^2 t)\mathcal{E}_1 + (\beta^4+\beta^5 t)\left(\frac{1}{2}(u_x)^2 + \frac{3}{4}(\varpi_x)^2\right) + \beta^6\frac{3t}{4\zeta}\varpi^2 + \beta^7\frac{3t}{4\zeta e^{\tilde{w}}}q_x^2$$
$$+ \beta^7\frac{t}{2}(u_{xx},\varpi_{xx})A[v,w](u_{xx},\varpi_{xx})^\top - \beta^7 t(u_{xx},\varpi_{xx})H\,dx,$$

$$D(t) :=(1+\beta^2 t)\left(\|(j-\tilde{j},\varpi_x)(t)\|^2+\frac{1}{\zeta}\|\varpi(t)\|^2\right)+(\beta+\beta^3 t)\|(\Phi[e^{\tilde{v}+u}]-\Phi[e^{\tilde{v}}])_x(t)\|^2$$
$$+(\beta+\beta^3 t)\|u(t)\|_1^2+(\beta^4+\beta^5 t)\left(\|(u_{xx},\varpi_{xx})(t)\|^2+\frac{1}{\zeta}\|\varpi_x(t)\|^2\right)+\beta^6\frac{t}{\zeta^2}\|\varpi(t)\|^2.$$

If the constant β is sufficiently small, we see from (4.71) and the Poincaré inequality that $E(t)$ is estimated as

$$c\|(u,\varpi)(t)\|_1^2+c(1+t)\|(\Phi[e^{\tilde{v}+u}]-\Phi[e^{\tilde{v}}])(t)\|_1^2+c\frac{t}{\zeta}\|\varpi(\tau)\|_1^2+ct\|(u,\varpi)(t)\|_2^2\le E(t), \quad (4.74)$$

where c is a positive constant. Let β so small that both (4.74) and $c_1-C_1\beta>0$ hold. Then take $N_\zeta(T)+\delta+\zeta^{1/4}$ small enough in (4.73) and use $\bar{c}E(t)\le D(t)$, which holds for a suitably chosen small positive constant $\bar{c}$, to get an ordinary differential inequality

$$\frac{d}{dt}E(t)+\alpha E(t)\le 0, \quad (4.75)$$

where α is a positive constant. Solving (4.75), we have the inequality

$$E(t)\le E(0)e^{-\alpha t}\le C\|(u_0,\varpi_0)\|_1^2 e^{-\alpha t}.$$

This inequality together with (4.74) yields the decay estimates in (4.40). □

We are now at the position to complete the proof of Theorem 4.2, which shows the time global existence of the solution for the energy transport model (2.14) with the large initial data.

Proof of Theorem 4.2. Determine the constant Λ in Corollary 4.10 so small that the assumption (4.39) in Theorem 4.11 holds. Applying Theorem 4.11 with regarding the time T_Λ in Corollary 4.10 as the initial time, we see that the initial boundary value problem (4.3)–(4.6) has a unique time global solution (v,w) satisfying $(v-\tilde{v},w-\tilde{w})\in\mathfrak{Z}([0,\infty))\cap\mathfrak{Y}_{loc}((0,\infty))$ without any restriction on the norm of the initial data. The decay estimates in (4.40) immediately means

$$\|(v-\tilde{v},w-\tilde{w},\Phi[e^v]-\Phi[e^{\tilde{v}}])(t)\|_1^2\le Ce^{-\alpha t}, \quad (4.76a)$$
$$\frac{t}{\zeta}\|(w-\tilde{w})(t)\|_1^2+t\|(v-\tilde{v},w-\tilde{w})(t)\|_2^2\le Ce^{-\alpha t}, \quad (4.76b)$$

for $t\in[0,\infty)$. Owing to (4.66b), it also verifies

$$\int_0^\infty \frac{1}{\zeta}\|(w-\tilde{w})(\tau)\|_1^2+\|(v_t,v_{xx}-\tilde{v}_{xx},w_{xx}-\tilde{w}_{xx})(\tau)\|^2+\tau\|(w_t,v_{xt},w_{xt})(\tau)\|^2\,d\tau\le C. \quad (4.77)$$

In computing (4.77), we have divided the integral interval $[0,\infty)$ into two parts $[0,T_\Lambda]$ and $[T_\Lambda,\infty]$, and then used (4.23) and (4.44), respectively. The estimate (4.77) shows the solution verifies $\sqrt{t}v_{xt}, \sqrt{t}w_{xt} \in L^2(0,\infty;L^2(\Omega))$.

Letting

$$\rho := e^v, \quad j := -(e^v e^w)_x + e^v(\Phi[e^v])_x, \quad \theta := e^w, \quad \phi := \Phi[e^v],$$

we see that (ρ, j, θ, ϕ) is the desired time global solution. Moreover, the estimates (4.2) follow from (4.76). □

4.4 Energy relaxation limit

In this section, we justify the relaxation limit of the energy-transport model to the drift-diffusion model. Since we have already constructed the time global solutions to the both models, it suffices to show the estimates (2.25)–(2.27) in order to complete the proof.

Proof of Theorem 2.4. By virtue of Corollary 4.10, the time global solution (v,w), constructed in the proof of Theorem 4.2, satisfies the estimates in (4.28) for arbitrary time $t \in [0,T_\Lambda]$. We show that (4.28) holds for $t \in [0,\infty)$. As the solution (v,w) verifies (4.76a), the formula (2.8) gives (4.23a). Moreover, the estimates (4.23b) and (4.23c) are shown for $t \in [0,\infty)$ by the same manner as in the proof of Lemma 4.6. Consequently, since the assumption (4.23) in Lemma 4.8 holds, the estimates in (4.28) follow for $t \in [0,\infty)$.

Secondly, we show the estimates (2.25)–(2.27). Let $\lambda \in (0,1)$ be an arbitrarily fixed constant and define a constant $T_1 := (\log 1/\zeta^\lambda)/\beta$. By the estimates in (4.28), the difference $(\rho^0_\zeta - \rho^0_0, \theta^0_\zeta - \theta^0_0)$ between the solutions of both models is estimated as

$$\|(\rho^0_\zeta - \rho^0_0)(t)\|^2 \le C\|R_\zeta(t)\|^2 \le Ce^{\beta T_1} \le C\zeta^{1-\lambda}, \tag{4.78a}$$

$$\|(\theta^0_\zeta - \theta^0_0)(t)\|^2 \le C\|Q_\zeta(t)\|^2 \le C\|\theta_0 - 1\|^2 e^{-\nu t/\zeta} + C\zeta^{1-\lambda}, \tag{4.78b}$$

$$\begin{aligned}\|(\{\rho^0_\zeta - \rho^0_0\}_x, \{\theta^0_\zeta - \theta^0_0\}_x)(t)\|^2 &\le C\|(R_\zeta, Q_\zeta)(t)\|_1^2 \\ &\le C\|\theta_0 - 1\|^2 e^{-\nu t/\zeta} + C\zeta^{1-\lambda}(1+t^{-1}) \le C\zeta^{1-\lambda}(1+t^{-1})\end{aligned} \tag{4.78c}$$

for $t \le T_1$. If $t \ge T_1$, it holds from the estimates (3.54), (4.1b) and (4.2a) that

$$\begin{aligned}\|(\rho^0_\zeta - \rho^0_0, \theta^0_\zeta - 1)(t)\|_1^2 &\le C\|(\rho^0_\zeta - \tilde\rho^0_\zeta, \rho^0_0 - \tilde\rho^0_0, \tilde\rho^0_\zeta - \tilde\rho^0_0)(t)|_1^2 + C\|(\theta^0_\zeta - \tilde\theta^0_\zeta, \tilde\theta^0_\zeta - 1)(t)\|_1^2 \\ &\le C\left(e^{-\alpha T_1} + \zeta\right) \le C\left(\zeta^{\alpha\lambda/\beta} + \zeta\right).\end{aligned} \tag{4.79}$$

Let $\gamma := \min\{1-\lambda, \alpha\lambda/\beta\}$. Then the estimates (2.25)–(2.27) follow from (4.78) and (4.79) together with (2.8), (2.14d) and (2.15c). □

4.5 Additional regularity

We improve, in this section, the regularity of the solution for the energy-transport model by assuming additional regularity of the initial data. This discussion is necessary since the regularity, shown in the previous sections, are insufficient to justify the relaxation limit, which is discussed in Section 5.4. Precisely, we study the regularity of the solution (ρ, j, θ, ϕ) for the energy-transport model with the initial data $(\rho_0, \theta_0) \in H^2(\Omega)$ in place of $(\rho_0, \theta_0) \in H^1(\Omega)$ in Theorem 4.2.

Corollary 4.17. *Let* $(\tilde{\rho}, \tilde{j}, \tilde{\theta}, \tilde{\phi})$ *be the stationary solution of* (2.18)–(2.20) *and* (3.1)*, which is constructed in Theorem 3.5. Suppose that the initial data* $(\rho_0, \theta_0) \in H^2(\Omega)$ *and the boundary data* ρ_l*,* ρ_r *and* ϕ_r *satisfy* (2.4), (2.6), (2.7a), (2.7b), (2.10a) *and* (2.10b). *Then there exist positive constants* δ_0 *and* ζ_0 *such that if* $\delta \le \delta_0$ *and* $\zeta \le \zeta_0$ *the initial boundary value problem* (2.14), (2.12a), (2.12c) *and* (2.4)–(2.6) *has a unique solution* (ρ, j, θ, ϕ) *satisfying* $(\rho - \tilde{\rho}, \theta - \tilde{\theta}) \in \mathfrak{Y}([0, \infty))$*,* $j - \tilde{j} \in C([0, \infty); H^1(\Omega)) \cap H^1(0, \infty; L^2(\Omega))$*,* $\phi - \tilde{\phi} \in C^1([0, \infty); H^2(\Omega))$*; the positivity* (2.10a) *and* (2.10b). *Moreover, it verifies the additional regularity* $\rho_t, \theta_t \in \mathfrak{Y}_{loc}((0, \infty))$*,* $\rho_{tt}, \theta_{tt} \in \mathfrak{Z}_{loc}((0, \infty))$ *and* $\theta_{xxx} \in L^2_{loc}(0, \infty; L^2(\Omega))$*, the convergence*

$$t\|(\rho_{xt}, \theta_{xt})(t)\| + t^2\|\rho_{tt}(t)\|^2 \to 0 \quad \textit{as} \quad t \to 0 \tag{4.80}$$

and the estimates

$$\inf_{x \in \Omega} \rho, \quad \inf_{x \in \Omega} \theta \ge c, \tag{4.81a}$$

$$\|(j - \tilde{j})(t)\|_1^2 + \frac{1}{\zeta}\|(\theta - \tilde{\theta})(t)\|_1^2 + \|(\rho - \tilde{\rho}, \theta - \tilde{\theta})(t)\|_2^2 + \|\rho_t(t)\|^2 \le Ce^{-\alpha t}, \tag{4.81b}$$

$$t\|(\rho_{xt}, \theta_{xt})(t)\|^2 \le C(1 + t), \quad t^2\|\rho_{tt}(t)\|^2 \le C(1 + t^2), \tag{4.81c}$$

$$\int_0^t \|(\rho_t, \theta_t)(\tau)\|_1^2 + \|j_t(\tau)\|^2 \, d\tau \le C, \tag{4.81d}$$

$$\int_0^t \tau\|(\rho_{tt}, \rho_{xxt}, \theta_{xxt})(\tau)\|^2 + \|\theta_{xxx}(\tau)\|^2 \, d\tau \le C(1 + t), \tag{4.81e}$$

$$\int_0^t \tau^2\|(\theta_{tt}, \rho_{xtt}(\tau)\|^2 \, d\tau \le C(1 + t^2), \quad \int_0^t \tau^3\|\rho_{ttt}(\tau)\|^2 \, d\tau \le C(1 + t^3) \tag{4.81f}$$

for $t \in [0, T]$*, where* C *and* c *are positive constants independent of* δ *and* t*.*

Proof. The proof of Corollary 4.17 is divided into the five steps, which are stated in Lemmas 4.19–4.23. Once they are proven, Corollary 4.17 immediately follows from the relations in (4.7) with aid of the estimates (4.76a) and (4.77). □

Remark 4.18. *For the special case* $\theta_0 = 1$*, the constant* C *in* (4.81b)–(4.81f) *is taken independently of* ζ*. It is shown similarly as in the proofs of Lemmas 4.19–4.23. This fact is utilized in the proof of Remark 2.6.*

Lemma 4.19. *Let $(\tilde{v}, \tilde{w})$ be the stationary solution for* (4.8)*. Suppose that the initial data $(v_0, w_0) \in H^2(\Omega)$ and the boundary data ρ_l, ρ_r and ϕ_r satisfy* (2.4)*,* (2.6)*,* (2.7a) *and* (2.7b)*. Then there exists a positive constant δ_0 and ζ_0 such that if $\delta \le \delta_0$ and $\zeta \le \zeta_0$, then the initial boundary value problem* (4.3)*–*(4.6) *has a unique solution (v, w) satisfying $(v - \tilde{v}, w - \tilde{w}) \in \mathfrak{Y}([0,\infty))$ and $w_{xxx} \in L^2_{loc}(0,\infty; L^2(\Omega))$. Moreover, the solution (v, w) verifies the additional regularity*

$$v_t, w_t \in \mathfrak{Y}_{loc}((0,\infty)), \quad v_{tt}, w_{tt} \in \mathfrak{Z}_{loc}((0,\infty)), \tag{4.82}$$

the convergence

$$t\|(v_{xt}, w_{xt})(t)\|^2 + t^2\|(v_{tt}, w_{tt}, v_{xxt}, w_{xxt})(t)\|^2 + t^3\|(v_{xtt}, w_{xtt})(t)\|^2 \to 0 \quad as \quad t \to 0 \tag{4.83}$$

and the estimates

$$\|v_t(t)\|^2 + \frac{1}{\zeta}\|(w - \tilde{w})(t)\|_1^2 + \|(v_{xx} - \tilde{v}_{xx}, w_{xx} - \tilde{w}_{xx})(t)\|^2 \le Ce^{-\alpha t}, \tag{4.84a}$$

$$\int_0^t \|(w_t, v_{xt}, w_{xt})(\tau)\|^2 \, d\tau \le C, \tag{4.84b}$$

$$\int_0^t \|w_{xxx}(\tau)\|^2 \, d\tau \le C(1+t) \tag{4.84c}$$

for $t \in [0,\infty]$, where C and α are positive constants independent of δ and t.

Proof. Theorem 4.2 ensures the existence of the time global solution (v, w) for the initial data $(v_0, w_0) \in H^1(\Omega)$. As the initial data (v_0, w_0) belongs to $H^2(\Omega)$, it is obvious that the solution verifies $(v-\tilde{v}, w-\tilde{w}) \in \mathfrak{Y}([0,\infty))$. Moreover, the regularity $w_{xxx} \in L^2_{loc}(0,\infty; L^2(\Omega))$ is shown by the straight forward computation with using the equation (4.21).

We derive the estimate (4.84). It is shown that the estimates (4.84a) and (4.84b) hold for $t \in [0,1]$ by the essentially same computation as in the derivation of (4.23c). On the other hand, the estimates (4.84a) and (4.84b) apparently hold for $t \in [1,\infty)$ thanks to the estimates (4.66a), (4.76b) and (4.77). To show (4.84c), differentiate the equation (4.21) with respect to x, multiply the result by $-w_{xxx}$ and integrate by part over the domain Ω. Then apply the Sobolev and the Young inequalities to the resulting equality with using the estimates (3.25d), (4.84a) and (4.84b). The result is

$$\begin{aligned}\int_0^t\int_0^1 \frac{2\kappa_0}{3e^v}(w_{xxx})^2 + \frac{(w_{xx})^2}{\zeta e^w}\, dxd\tau &= \int_0^t\int_0^1 \frac{(w_x)^2}{\zeta e^w} w_{xx} + \left(w_t - \frac{2}{3}v_t - g_2[v_\zeta, w_\zeta]\right)_x w_{xxx}\, dxd\tau \\ &\le \mu\int_0^t \|w_{xxx}(\tau)\|^2 + \frac{1}{\zeta}\|w_{xx}(\tau)\|\, d\tau + C[\mu](1+t),\end{aligned}$$

where μ is an arbitrary positive constant. Taking μ sufficiently small gives the desired estimate (4.84c). Finally, the solution (v, w) verifies the regularity (4.82) and the convergence (4.83) by the standard theory of the parabolic systems. □

The assertion on the regularity and the convergence in Corollary 4.17 follows from Lemma 4.19. In order to complete the proof, it suffice to derive the estimates of higher derivatives. Differentiating the equation (4.3a) yields

$$\begin{pmatrix} v \\ 3w/2 \end{pmatrix}_{tt} - A[v,w]\begin{pmatrix} v \\ w \end{pmatrix}_{xxt} + \begin{pmatrix} 0 \\ 3w_t/2\zeta e^w \end{pmatrix} = (A[v,w])_t \begin{pmatrix} v \\ w \end{pmatrix}_{xx} + (G[v,w])_t. \quad (4.85)$$

The L^2-norm of the derivatives of A and G in t are estimated as

$$\|(A[v,w])_t\|_1^2 + \|(G([v,w])_t\|^2 \leq C\|(v_t,w_t)(t)\|_1^2 \quad (4.86)$$

with aid of the estimates (4.76a) and (4.84a). Moreover, differentiate (4.85) in t again to obtain

$$\begin{aligned}\begin{pmatrix} v \\ 3w/2 \end{pmatrix}_{ttt} - A[v,w]\begin{pmatrix} v \\ w \end{pmatrix}_{xxtt} + \begin{pmatrix} 0 \\ 3w_t/2\zeta e^w \end{pmatrix}_t& \\ = (A[v,w])_t \begin{pmatrix} v \\ w \end{pmatrix}_{xxt} + \left\{(A[v,w])_t \begin{pmatrix} v \\ w \end{pmatrix}_{xx} + (G[v,w])_t\right\}_t.& \quad (4.87)\end{aligned}$$

Lemma 4.20. *Under the same conditions as in Lemma 4.19, it holds that*

$$t\|(v_{xt},w_{xt})(t)\|^2 + \int_0^t \tau\|(v_{xxt},w_{xxt})(\tau)\|^2 + \frac{\tau}{\zeta}\|w_{xt}(\tau)\|^2\,d\tau \leq C(1+t), \quad (4.88a)$$

$$\int_0^t \tau\|v_{tt}(\tau)\|^2\,d\tau \leq C(1+t) \quad (4.88b)$$

for $t \in [0,\infty)$, where C is a positive constant independent of δ and t.

Proof. Take the inner product of the equation (4.85) with $(-tv_{xxt}, -tw_{xxt})$ in $L^2(\Omega)$ and apply the integration by part to obtain

$$\begin{aligned}\frac{d}{dt}\left(t\int_0^1 \frac{1}{2}(v_{xt})^2 + \frac{3}{4}(w_{xt})^2\,dx\right) + t\int_0^1 (v_{xxt},w_{xxt})A[v,w](v_{xxt},w_{xxt})^\top\,dx& \\ + t\int_0^1 \frac{3}{2\zeta e^w}(w_{xt})^2\,dx = \int_0^1 \frac{1}{2}(v_{xt})^2 + \frac{3}{4}(w_{xt})^2\,dx + t\int_0^1 \frac{3w_x}{2\zeta e^w}w_t w_{xt}\,dx& \\ - t\int_0^1 (v_{xxt},w_{xxt})\left\{(A[v,w])_t(v_{xx},w_{xx})^\top + (G[v,w])_t\right\}dx.& \quad (4.89)\end{aligned}$$

Applying the Schwarz and the Sobolev inequalities to the right hand side of (4.89) with using the estimates (4.76a), (4.84a), (4.86) and $\|w_x\|^2/\zeta \leq C$, which follows from (3.25d) and (4.84a), we have

$$(\text{right hand side}) \leq \mu t\left(\|(v_{xxt},w_{xxt})(t)\|^2 + \frac{1}{\zeta}\|w_{xt}(t)\|^2\right) + C[\mu](1+t)\|(v_t,w_t)(t)\|_1^2, \quad (4.90)$$

where μ is an arbitrary positive constant. On the other hand, the second term on the left hand side is estimated by $c\|(v_{xxt}, w_{xxt})\|^2$ from below since A is positive definite. Substitute (4.90) in (4.89), integrate the resultant inequality with respect to t and successively let μ small enough. The estimates (4.77) and (4.84b) as well as the convergence (4.83) give the desired estimate (4.88a).

By solving the first component of the system (4.85) with respect to v_{tt} and then taking the L^2-norm, we obtain

$$\|v_{tt}(t)\|^2 \leq C\|(v_t, w_t)(t)\|_2^2, \tag{4.91}$$

which immediately yields the estimate (4.88b) with aid of (4.77), (4.84b) and (4.88a). □

Lemma 4.21. *Under the same conditions as in Lemma 4.19, it holds that*

$$\frac{t^{k+1}}{\zeta^k}\|w_t(t)\|^2 + \int_0^t \frac{\tau^{k+1}}{\zeta^{k+1}}\|w_t(\tau)\|^2\,d\tau \leq C(1+t^{k+1}), \tag{4.92a}$$

$$\int_0^t \tau^2\|w_{tt}(\tau)\|^2\,d\tau \leq C(1+t^2) \tag{4.92b}$$

for $k = 0, 1$ and $t \in [0, \infty)$, where C is a positive constant independent of δ and t.

Proof. Taking the inner product in $L^2(\Omega)$ of the equation (4.85) with $(0, -t^{k+1}w_t/\zeta^k)$ for $k = 0, 1$ and applying the integration by part lead to

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{t^{k+1}}{\zeta^k}\int_0^1 \frac{3}{4}(w_t)^2\,dx\right) + \frac{t^{k+1}}{\zeta^{k+1}}\int_0^1 \frac{3}{2e^w}(w_t)^2\,dx \\
&= (k+1)\frac{t^k}{\zeta^k}\int_0^1 \frac{3}{4}(w_t)^2\,dx + \frac{t^{k+1}}{\zeta^k}\int_0^1 (0, w_t)\left\{A[v,w](v_{xx}, w_{xx})^\top + G[v,w]\right\}_t\,dx \\
&\leq \mu\frac{t^{k+1}}{\zeta^{2k}}\|w_t(t)\|^2 + C[\mu]t^{k+1}\|(v_t, w_t)(t)\|_2^2 + C\frac{t^k}{\zeta^k}\|w_t(t)\|^2,
\end{aligned} \tag{4.93}$$

where μ is an arbitrary positive constant. In deriving the last inequality, we have used the Schwarz and the Sobolev inequalities with the estimates (4.76a), (4.84a) and (4.86). Integrating (4.93) with $k = 0$ in t, making μ sufficiently small, and then using the estimates (4.77), (4.84b) and (4.88a), we have the estimate (4.92a) with $k = 0$. The estimate (4.92a) with $k = 1$ follows from the similar computation as above, where we have to utilize (4.92a) with $k = 0$. Finally, solve the second component of the system (4.85) with w_{tt}, take L^2-norm and use (4.86), (4.88a) and (4.92a) to get the estimate (4.92b). □

Owing to the estimates (4.76a), (4.84a), (4.88a) and (4.92a), the estimate (4.86) is rewritten to

$$t\|(A[v,w])_t\|_1^2 + t\|(G[v,w])_t\|^2 \leq C(1+t). \tag{4.94}$$

Similarly, the second derivatives of A and G are estimated as

$$t^2\|(A[v,w])_{tt}\|_1^2 + t^2\|(G[v,w])_{tt}\|^2 \le C(1+t)t\|(v_t,w_t)(t)\|_2^2 + Ct^2\|(v_{tt},w_{tt})(t)\|_1^2. \tag{4.95}$$

Lemma 4.22. *Under the same conditions as in Lemma 4.19, it holds that*

$$t^2\|(v_{tt},v_{xxt},w_{xxt})(t)\|^2 + \frac{t^2}{\zeta}\|w_{xt}(t)\|^2 + \int_0^t \tau^2\|(v_{xtt},w_{xtt})(\tau)\|^2\,d\tau \le C(1+t^2) \tag{4.96}$$

for $t \in [0,\infty)$, where C is a positive constant independent of δ and t.

Proof. Take the inner product of the equation (4.85) with the vector $(-t^2 v_{xxtt}, -t^2 w_{xxtt})$ in $L^2(\Omega)$ and apply the integration by part to get

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{t^2}{2}\int_0^1 (v_{xxt},w_{xxt})A[v,w](v_{xxt},w_{xxt})^\top + \frac{3}{2\zeta e^w}w_{xt}^2\,dx\right) + t^2\int_0^1 (v_{xtt})^2 + \frac{3}{2}(w_{xtt})^2\,dx\\
&= -\frac{d}{dt}\left(t^2\int_0^1 (v_{xxt},w_{xxt})\left\{(A[v,w])_t(v_{xx},w_{xx})^\top + (G[v,w])_t\right\}\,dx\right)\\
&\quad + \int_0^1 \frac{1}{2}(v_{xxt},w_{xxt})\left(t^2A[v,w]\right)_t(v_{xxt},w_{xxt})^\top + \frac{3t^2}{2\zeta e^w}w_x w_{xtt} w_t\,dx\\
&\quad + \int_0^1 \left(\frac{3t^2}{4\zeta e^w}\right)_t w_{xt}^2 + (v_{xxt},w_{xxt})\left\{t^2(A[v,w])_t(v_{xx},w_{xx})^\top + t^2(G[v,w])_t\right\}_t\,dx.
\end{aligned} \tag{4.97}$$

We integrate (4.97) with respect to t. The left hand side gives the positive terms appearing in (4.96) since A is positive definite. We handle the right hand side by applying the Sobolev and the Schwarz inequalities with using using the estimates (4.76a), (4.84a), (4.88a), (4.92a), (4.94) and (4.95) as

$$\begin{aligned}
&\text{(integration of right hand side in } t)\\
&\le \mu t^2\|(v_{xxt},w_{xxt})(t)\|^2 + \mu\int_0^t \tau^2\|(v_{xtt},w_{xtt})(\tau)\|^2\,d\tau + C[\mu](1+t^2)\\
&\quad + C[\mu]\int_0^t \tau^2\|(v_{tt},w_{tt})(\tau)\|^2\,d\tau + C[\mu](1+t)\int_0^t \frac{\tau}{\zeta}\|w_t(\tau)\|_1^2 + \tau\|(v_t,w_t)(\tau)\|_2^2\,d\tau\\
&\le \mu t^2\|(v_{xxt},w_{xxt})(t)\|^2 + \mu\int_0^t \tau^2\|(v_{xtt},w_{xtt})(\tau)\|^2\,d\tau + C[\mu](1+t^2),
\end{aligned} \tag{4.98}$$

where μ is an arbitrary positive constant. In deriving the second inequality, we have also used the estimates (4.77) (4.84b), (4.88) and (4.92). Letting μ sufficiently small and then using (4.91) yields the estimate (4.96). □

Lemma 4.23. *Under the same conditions as in Lemma 4.19, it holds that*

$$t^3\|(v_{xtt}, w_{xtt})(t)\|^2 + \int_0^t \tau^3\|(v_{xxtt}, w_{xxtt})(\tau)\|^2 + \frac{\tau^3}{\zeta}\|w_{xtt}(\tau)\|^2\, d\tau \le C(1+t^3), \tag{4.99a}$$

$$\int_0^t \tau^3\|v_{ttt}(\tau)\|^2\, d\tau \le C(1+t^3) \tag{4.99b}$$

for $t \in [0, \infty)$*, where* C *is a positive constant independent of* δ *and* t*.*

Proof. Taking the inner product of the equation (4.87) with the vector $(-t^3 v_{xxtt}, -t^3 w_{xxtt})$ in $L^2(\Omega)$ and applying the integration by part, we have

$$\begin{aligned}
&\frac{d}{dt}\left(t^3 \int_0^1 \frac{1}{2}(v_{xtt})^2 + \frac{3}{4}(w_{xtt})^2\, dx\right) + t^3 \int_0^1 (v_{xxtt}, w_{xxtt})A[v,w](v_{xxtt}, w_{xxtt})^\top\, dx \\
&\quad + t^3 \int_0^1 \frac{3}{2\zeta e^w}(w_{xtt})^2\, dx = t^2 \int_0^1 \frac{3}{2}(v_{xtt})^2 + \frac{9}{4}(w_{xtt})^2\, dx + t^3 \int_0^1 \frac{3w_x}{2\zeta e^w} w_{tt} w_{xtt}\, dx \\
&\qquad - t^3 \int_0^1 \frac{3}{2\zeta e^w}(w_t)^2 w_{xtt} + (v_{xxt}, w_{xxt})(A[v,w])_t (v_{xxt}, w_{xxt})^\top\, dx \\
&\qquad\quad - t^3 \int_0^1 (v_{xxt}, w_{xxt}) \left\{(A[v,w])_t (v_{xx}, w_{xx})^\top + (G[v,w])_t\right\}_t\, dx.
\end{aligned} \tag{4.100}$$

Integrating (4.100) with respect to t, we have the positive terms appearing in (4.99a) from the left hand side. On the other hand, the right hand side is estimated as

$$\begin{aligned}
&\text{(integration of right hand side)} \\
&\le \mu \int_0^t \tau^3\|(v_{xxtt}, w_{xxtt})(\tau)\|^2 + \frac{\tau^3}{\zeta}\|w_{xtt}(\tau)\|^2\, d\tau + C[\mu](1+t^3),
\end{aligned} \tag{4.101}$$

where μ is an arbitrary positive constant. Hence, integrating (4.100) in t and making μ small enough give the desired estimate (4.99a). Moreover, solving the first component of the system (4.87) with respect to v_{ttt}, taking the L^2-norm and using the estimates (4.94)–(4.96) and (4.99a), we obtain (4.99b). □