Part V

QUANTIZATION OF LOOP SPACES

Chapter 15

Quantization of the loop space of a vector space

In this Chapter we solve the geometric quantization problem for the classical system $(\Omega\mathbb{R}^d, \mathcal{A}_d)$, where the phase space $\Omega\mathbb{R}^d$ consists of smooth loops in the d-dimensional vector space $\mathbb{R}^d$, and the algebra of observables $\mathcal{A}_d$ is the Lie algebra of the Frechet Lie group $\mathcal{G}_d$, being the semi-direct product of the loop group $\widetilde{L\mathbb{R}^d}$ and the diffeomorphism group $\mathrm{Diff}_+(S^1)$ of the circle.

We start from the quantization of the "enlarged" system, obtained from $(\Omega\mathbb{R}^d, \mathcal{A}_d)$ by enlarging both the phase space and the algebra of observables. More precisely, we enlarge the phase space $\Omega\mathbb{R}^d$ to the Sobolev space V^d of half-differentiable vector-functions (a vector analogue of the Sobolev space V, introduced in Sec. 9.1), and the algebra of observables $\mathcal{A}_d$ to the Lie algebra $\mathcal{A}$ of the Hilbert Lie group $\mathcal{G}$, being the semi-direct product of the Heisenberg group $\mathrm{Heis}(V^d)$ and the symplectic Hilbert–Schmidt group $\mathrm{Sp}_{\mathrm{HS}}(V^d)$. The group $\mathcal{G}$ may be considered as a Hilbert-space (symplectic) analogue of the standard group of motions of the d-dimensional vector space $\mathbb{R}^d$. The latter group is the semi-direct product of the group of translations of $\mathbb{R}^d$ and the group of rotations of $\mathbb{R}^d$. In the case of the Hilbert space V the role of translation group is played by the Heisenberg group, and the group of rotations is replaced by the symplectic group $\mathrm{Sp}_{\mathrm{HS}}(V)$.

To simplify the formulas, we set $d = 1$ in the most part of this Chapter, replacing it with a general d only in Sec. 15.6, where the quantization of $\Omega\mathbb{R}^d$ is completed. The last Sec. 15.7 is devoted to the quantization of the universal Teichmüller space.

15.1 Heisenberg representation

15.1.1 Fock space

Consider the *Sobolev space*

$$V := H_0^{1/2}(S^1, \mathbb{R})$$

of half-differentiable functions on the circle S^1 (cf. Sec. 9.1) and its complexification

$$V^{\mathbb{C}} = H_0^{1/2}(S^1, \mathbb{C}) \ .$$

A natural complex structure operator J_0 on $V^{\mathbb{C}}$, introduced in Sec. 9.1, generates a decomposition of $V^{\mathbb{C}}$ into the direct sum of subspaces

$$V^{\mathbb{C}} = W_+ \oplus W_- =: W_0 \oplus \overline{W_0} \ , \tag{15.1}$$

where $W_\pm$ is the $(\mp i)$-eigenspace of the operator $J^0 \in \operatorname{End} V^{\mathbb{C}}$. The subspaces $W_\pm$ are isotropic with respect to the symplectic form ω on $V^{\mathbb{C}}$. Moreover, the splitting (15.1) is an orthogonal direct sum with respect to the Hermitian inner product on $V^{\mathbb{C}}$, defined by

$$< z, w >= i\omega(z_+, \bar{w}_+) - i\omega(z_-, \bar{w}_-) \ ,$$

where $z_\pm$ (resp. $w_\pm$) denotes the projection of $z \in V^{\mathbb{C}}$ (resp. $w \in V^{\mathbb{C}}$) onto the subspace $W_\pm$.

We introduce the *Fock space* $F_0 \equiv F(V^{\mathbb{C}}, J_0)$ as the completion of the algebra of symmetric polynomials on W_0 with respect to a natural norm.

In more detail, denote by $S(W_0)$ the algebra of symmetric polynomials in variables $z \in W_0 \equiv W_+$ and introduce an inner product on $S(W_0)$, induced by the Hermitian product $< \cdot, \cdot >$ on $V^{\mathbb{C}}$. This inner product on monomials is given by the formula

$$< z_1 \cdot \ldots \cdot z_n, z_1' \cdot \ldots \cdot z_n' >= \sum_{\{i_1,\ldots,i_n\}} < z_1, z_{i_1}' > \cdot \ldots \cdot < z_n, z_{i_n}' > \ ,$$

where the summation is taken over all permutations $\{i_1, \ldots, i_n\}$ of the set $\{1, \ldots, n\}$ (the inner product of monomials of different degrees is set to 0 by definition). This inner product is extended by linearity to the whole algebra $S(W_0)$. The completion $\widehat{S(W_0)}$ of $S(W_0)$ with respect to the introduced norm is called the *Fock space* $F_0 \equiv F(V^{\mathbb{C}}, J_0)$ over $V^{\mathbb{C}}$ with respect to the complex structure J^0

$$F_0 = F(V^{\mathbb{C}}, J^0) := \widehat{S(W)} \ .$$

If $\{w_n\}$, $n = 1, 2, \ldots$, is an orthonormal base of W_0, then one can take for an orthonormal base of F_0 the family of polynomials of the form

$$P_K(z) = \frac{1}{\sqrt{K!}} < z, w_1 >^{k_1} \cdot \ldots \cdot < z, w_n >^{k_n} \ , \quad z \in W_0 \ , \tag{15.2}$$

where $K = (k_1, \ldots, k_n)$, $k_i \in \mathbb{N}$, and $K! = k_1! \cdot \ldots \cdot k_n!$.

Recall that, according to Sec. 11.4, any complex structure J on V, compatible with ω, determines a decomposition

$$V^{\mathbb{C}} = W_J \oplus \overline{W}_J =: W \oplus \overline{W} \tag{15.3}$$

into the direct sum of subspaces W and $\overline{W}$, isotropic with respect to ω. The subspaces W and $\overline{W}$ are identified, respectively, with the $(-i)$- and $(+i)$-eigenspaces of the operator J on $V^{\mathbb{C}}$. The complex structure J and the symplectic form ω determine together a Kähler metric g_J and the associated inner product $< \cdot, \cdot >_J$ on $V^{\mathbb{C}}$. The decomposition (15.3) is orthogonal with respect to the Kähler metric g_J on $V^{\mathbb{C}}$.

Using the decomposition (15.3), we can define the *Fock space* $F_J \equiv F(V^{\mathbb{C}}, J)$ as the completion of the algebra of symmetric polynomials on W with respect to the norm, generated by $< \cdot, \cdot >_J$:

$$F_J = F(V^{\mathbb{C}}, J) := \text{ completion of } S(W) \text{ with respect to } < \cdot, \cdot >_J \ .$$

15.1.2 Heisenberg algebra and Heisenberg group

The *Heisenberg algebra* $\mathrm{heis}(V)$ of the Hilbert space V is a central extension of the Abelian Lie algebra V, generated by the coordinate functions. In other words, it coincides, as a vector space, with

$$\mathrm{heis}(V) = V \oplus \mathbb{R} ,$$

and is provided with the Lie bracket

$$[(x,s),(y,t)] := (0, \omega(x,y)) \ , \quad x, y \in V, \ s, t, \in \mathbb{R} .$$

The Heisenberg algebra $\mathrm{heis}(V)$ is the Lie algebra of the *Heisenberg group* $\mathrm{Heis}(V)$, which coincides with a central extension of the Abelian group V. In other words, $\mathrm{Heis}(V)$ is the direct product

$$\mathrm{Heis}(V) = V \times S^1 ,$$

provided with the group operation, given by

$$(x,\lambda)\cdot(y,\mu) := \left(x+y, \lambda\mu e^{i\omega(x,y)}\right) .$$

15.1.3 Heisenberg representation

Representation of the Heisenberg algebra. We are going to construct an irreducible representation of the Heisenberg algebra $\mathrm{heis}(V)$ in the Fock space $F_J = F(V^{\mathbb{C}}, J)$, where $V^{\mathbb{C}} = W \oplus \overline{W}$ and F_J is the completion of the symmetric algebra $S(W)$ with respect to the norm, generated by $< \cdot , \cdot >_J$. We can consider elements of $S(W)$ as holomorphic functions on $\overline{W}$ by identifying $z \in W$ with a holomorphic function $\bar{w} \mapsto < w, z >$ on $\overline{W}$. Accordingly, F_J may be considered as a subspace of the space $\mathcal{O}(\overline{W})$ of functions, holomorphic on $\overline{W}$ (provided with the topology of uniform convergence on compact subsets).

With this convention we can define the *Heisenberg representation*

$$r_J : \mathrm{heis}(V) \longrightarrow \mathrm{End}\, F_J$$

of the Heisenberg algebra $\mathrm{heis}(V)$ in the Fock space $F_J = F(V^{\mathbb{C}}, J)$ by the formula

$$v \longmapsto r_J(v) f(\bar{w}) := -\partial_v f(\bar{w}) + < w, v >_J f(\bar{w}) , \tag{15.4}$$

where ∂_v is the derivation operator in the direction of $v \in V^{\mathbb{C}}$. Extending r_J to the complexified algebra $\mathrm{heis}^{\mathbb{C}}(V)$ by the same formula (15.4), we'll have for $v = \bar{z} \in \overline{W}$

$$r_J(\bar{z}) f(\bar{w}) := -\partial_{\bar{z}} f(\bar{w}) ,$$

and for $z \in W$

$$r_J(z) f(\bar{w}) := < w, z >_J f(\bar{w}) .$$

For the central element $c \in \mathrm{heis}(V)$ we set

$$c \longmapsto r_J(c) := \lambda \cdot I ,$$

where λ is an arbitrary fixed non-zero constant.

Introduce *creation* and *annihilation operators* on F_J, defined for $v \in V^{\mathbb{C}}$ by

$$a_J^*(v) := \frac{r_J(v) - ir_J(Jv)}{2} \ , \quad a_J(v) := \frac{r_J(v) + ir_J(Jv)}{2} \ . \tag{15.5}$$

In particular, for $z \in W$

$$a_J^*(z)f(\bar{w}) = < w, z >_J f(\bar{w}) \ , \tag{15.6}$$

and for $\bar{z} \in \overline{W}$

$$a_J(\bar{z})f(\bar{w}) = -\partial_{\bar{z}} f(\bar{w}) \ . \tag{15.7}$$

Choosing an orthonormal basis $\{w_n\}$ of W, we can introduce the operators

$$a_n^* := a^*(w_n) \ , \quad a_n := a(\bar{w}_n) \ , \quad n = 1, 2, \ldots ,$$

and $a_0 := \lambda \cdot I$.

A vector $f_J \in F_J \setminus \{0\}$ is called the *vacuum*, if $a_n f_J = 0$ for $n = 1, 2, \ldots$. In other words, the vacuum is a non-zero vector, annihilated by all operators a_n. It is uniquely defined by r_J (up to a multiplicative constant) and in the case of the initial Fock space $F_0 = F(V, J_0)$ we take $f_0 \equiv 1$. By acting on the vacuum f_J by creation operators a_n^*, we can define the action of the representation r_J on any polynomial, which implies the irreducubility of r_J.

Moreover, any irreducible representation $r : \mathrm{heis}^{\mathbb{C}}(V) \to \mathrm{End}\, F$ of the algebra $\mathrm{heis}^{\mathbb{C}}(V)$, having a vacuum f, is equivalent to the Heisenberg representation r_0. Indeed, vectors of the form $(a_1^*)^{k_1} \cdot \dots \cdot (a_n^*)^{k_n} f$, obtained from the vacuum by the action of creation operators, are linearly independent and generate the whole representation space F. Assigning to a polynomial $P(z) = P(z_1, \ldots, z_n)$ in the Fock space F_0 the vector of the form $P(a_1^*, \ldots, a_n^*)f$ in the space F, we obtain an intertwining map from F_0 into F. This map can be made unitary by introducing a Hermitian inner product on F, for which the vectors $(a_1^*)^{k_1} \cdot \dots \cdot (a_n^*)^{k_n} f$ form an orthogonal base.

Representation of the Heisenberg group. The Heisenberg representation r_J of the algebra $\mathrm{heis}^{\mathbb{C}}(V)$ may be integrated to an *irreducible unitary representation R_J of the Heisenberg group* $\mathrm{Heis}^{\mathbb{C}}(V)$ in the Fock space F_J. The integrated representation is given by the formula

$$R_J(\bar{z})f(\bar{w}) = f(\bar{w} - \bar{z})$$

for $\bar{z} \in \overline{W}$, and by

$$R_J(z)f(\bar{w}) = e^{<w,z>_J} f(\bar{w})$$

for $z \in W$. In particular, the creation operator $a^*(z)$ generates the multiplication operator $f(\bar{w}) \mapsto e^{<w,z>_J} f(\bar{w})$ and the annihilation operator $a(\bar{z})$ generates the translation operator $f(\bar{w}) \mapsto f(\bar{w} - \bar{z})$.

The constructed representation of the group $\mathrm{Heis}^{\mathbb{C}}(V)$ in F_J may be conveniently described in terms of the so called *coherent states*, given by the functions in F_J of the form

$$\epsilon_z(\bar{w}) := e^{<z,w>_J} \ ,$$

parameterized by vectors $z \in W$. The action of the representation of $\mathrm{Heis}^{\mathbb{C}}(V)$ on coherent states is given by the formula

$$v \in V \longmapsto R_J(v)\epsilon_z = e^{-<w,z>_J - \frac{1}{2}<w,w>_J}\epsilon_{z+w}$$

for $v = w + \bar{w}$. We have

$$< \epsilon_z, \epsilon_{z'} >_{F_J} = e^{<z,z'>_J} \tag{15.8}$$

and

$$< R_J(v)\epsilon_z, R_J(v)\epsilon_{z'} >_{F_J} = < \epsilon_z, \epsilon_{z'} >_{F_J} \quad .$$

The Fock space F_J may be defined in terms of coherent states as the completion of the complex vector space, generated by vectors $\{\epsilon_z\}$, $z \in W$, with respect to the norm, given by the inner product (15.8).

Using these properties of coherent states, it may be proved (cf. [65], Sec. 9.5) that the defined representation of the Heisenberg group in the Fock space F_J is unitary and irreducible.

15.2 Action of Hilbert–Schmidt symplectic group on Fock spaces

Recall the definition of the symplectic Hilbert–Schmidt group $\mathrm{Sp}_{\mathrm{HS}}(V)$ from Sec. 11.5. In terms of the block representation, generated by the decomposition

$$V^{\mathbb{C}} = W_+ \oplus W_- = W_0 \oplus \overline{W_0} \ ,$$

the elements A of $\mathrm{Sp}_{\mathrm{HS}}(V)$ are written in the form

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \ ,$$

where

$$\bar{a}^t a - b^t \bar{b} = 1 \ , \ \bar{a}^t b = b^t \bar{a} \ ,$$

and the operator b is Hilbert–Schmidt. The unitary group $\mathrm{U}(W_+)$ is embedded into $\mathrm{Sp}_{\mathrm{HS}}(V)$ as a subgroup of operators of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \ .$$

In Subsec. 15.1.3 we have constructed the Heisenberg representations r_J of the Heisenberg algebra $\mathrm{heis}^{\mathbb{C}}(V)$ in Fock spaces F_J. A general *theorem of Shale* (cf. [69]) asserts that the *representations r_0 in F_0 and r_J in F_J are unitary equivalent if and only if $J \in Sp_{HS}(V)$*. In other words, for $J \in \mathrm{Sp}_{\mathrm{HS}}(V)$ there exists a unitary intertwining operator $U_J : F_0 \to F_J$ such that

$$r_J = U_J \circ r_0 \circ U_J^{-1} \ .$$

The $\mathrm{Sp}_{\mathrm{HS}}(V)$-action, defined by

$$\mathrm{Sp}_{\mathrm{HS}}(V) \ni A \longmapsto U_J : F_0 \to F_J \quad \text{with } J = A \cdot J^0 \ ,$$

defines a projective (unitary) action of the group $\mathrm{Sp}_{\mathrm{HS}}(V)$ on the Fock bundle

$$\mathcal{F} := \bigcup_{J\in\mathcal{D}_{\mathrm{HS}}} F_J \longrightarrow \mathcal{D}_{\mathrm{HS}} = \frac{\mathrm{Sp}_{\mathrm{HS}}(V)}{\mathrm{U}(W_0)} ,$$

covering the $\mathrm{Sp}_{\mathrm{HS}}(V)$-action on the Siegel disc D_{HS} (cf. Sec. 11.5). An explicit description of this projective action is given in [66].

15.3 Hilbert–Schmidt symplectic algebra representation

The algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ is the Lie algebra of symplectic Hilbert–Schmidt group $\mathrm{Sp}_{\mathrm{HS}}(V)$. It follows from the definition of this group (cf. Sec. 15.2) that $\mathrm{sp}_{\mathrm{HS}}(V)$ consists of linear operators A in $V^{\mathbb{C}}$, which have the following block representation (with respect to the decomposition $V^{\mathbb{C}} = W_0 \oplus \overline{W_0}$)

$$A = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} ,$$

where α is a bounded skew-Hermitian operator and β is a symmetric Hilbert–Schmidt operator. The complexified Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)^{\mathbb{C}}$ consists of operators of the form

$$A = \begin{pmatrix} \alpha & \beta \\ \bar{\gamma} & -\alpha^t \end{pmatrix} ,$$

where α is a bounded operator, while β and $\bar{\gamma}$ are symmetric Hilbert–Schmidt operators.

The infinitesimalization of the projective $\mathrm{Sp}_{\mathrm{HS}}(V)$-action on the Fock bundle $\mathcal{F}$, described in the previous Sec. 15.2, yields a projective representation of $\mathrm{sp}_{\mathrm{HS}}(V)$ in the Fock space $F_0 \equiv F_{J_0}$. Its complexified version is given by the formula (cf. [66])

$$\mathrm{sp}_{\mathrm{HS}}(V^{\mathbb{C}}) \ni A = \begin{pmatrix} \alpha & \beta \\ \bar{\gamma} & -\alpha^t \end{pmatrix} \longmapsto \rho(A) = D_\alpha + \frac{1}{2}M_\beta + \frac{1}{2}M_\gamma^* .$$

Here, D_α for $\alpha : W_0 \to W_0$ is the derivation of F_0 in the α-direction, defined by

$$D_\alpha f(\bar{w}) =< \alpha w, \partial_{\bar{w}} > f(\bar{w}) .$$

The operator M_β for $\beta : \overline{W_0} \to W_0$ is the multiplication operator on F_0, defined by

$$M_\beta f(\bar{w}) =< \bar{\beta} w, \bar{w} > f(\bar{w}) ,$$

and the operator M_γ^* is the adjoint of M_γ:

$$M_\gamma^* f(\bar{w}) =< \gamma \partial_w, \partial_{\bar{w}} > f(\bar{w}) .$$

This is a projective representation with the cocycle

$$[\rho(A_1), \rho(A_2)] - \rho([A_1, A_2]) = \frac{1}{2}\mathrm{tr}(\bar{\gamma}_2\beta_1 - \bar{\gamma}_1\beta_2) . \tag{15.9}$$

Note that the constructed Lie-algebra representation of $\mathrm{sp}_{\mathrm{HS}}(V)$ is intertwined with the Heisenberg representation r_0 of $\mathrm{heis}(V)$ on F_0 (cf. [66]).

15.4 Twistor interpretation

15.4.1 Twistor bundle

Let us call a complex structure J on V *admissible*, if it can be obtained from a reference complex structure J_0 by the action of the $\mathrm{Sp_{HS}}(V)$ group. Such structures are parameterized by points of the Siegel disc

$$\mathcal{D}_{\mathrm{HS}} = \mathrm{Sp_{HS}}(V)/\mathrm{U}(W_0) .$$

The *twistor bundle* $\pi : \mathcal{Z} \to V$ is, by definition, the vector bundle of admissible complex structures on V. Its fibre $Z_x \cong \mathcal{D}_{\mathrm{HS}}$ at $x \in V$ is formed by the restrictions J_x of admissible complex structures J to the tangent space $T_xV \cong V$. The twistor bundle is a trivial bundle on V, and the admissible complex structures on V may be considered as its translation-invariant sections. In particular, we have a natural projection $p : \mathcal{Z} \to \mathcal{D}_{\mathrm{HS}}$, assigning to a point $z = (x, J_x)$ the translation-invariant complex structure $J = J_x$ on V. The fibre $p^{-1}(J)$ of this projection is identified with the space (V, J), i.e. with the space V, provided with the complex structure J. The introduced maps may be united into the following *twistor diagram*

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{p} & \mathcal{D}_{\mathrm{HS}} \\ {\scriptstyle\pi}\big\downarrow & & \\ V & & \end{array} \quad .$$

The *twistor space* $\mathcal{Z}$ has a natural complex structure. To define it, consider a decomposition of the tangent bundle $T\mathcal{Z}$ into the direct sum

$$T\mathcal{Z} = \mathcal{V} \oplus \mathcal{H} \tag{15.10}$$

of the *vertical subbundle* $\mathcal{V}$, identified with the tangent bundle to the fibres of π, and the *horizontal subbundle* $\mathcal{H}$, identified with the tangent bundle to the fibres of p. The complex structure $\mathcal{J}$ at $z \in \mathcal{Z}$ is the direct sum

$$\mathcal{J}_z = \mathcal{J}_z^v \oplus \mathcal{J}_z^h$$

of the natural complex structure $\mathcal{J}_z^v$ on the vertical space $\mathcal{V}_z$, identified (by p_*) with the tangent space $T_{p(z)}\mathcal{D}_{\mathrm{HS}}$ to the Siegel disc $\mathcal{D}_{\mathrm{HS}}$, and the complex structure $\mathcal{J}_z^h = J_{\pi(z)}$ on the horizontal space $\mathcal{H}_z$, identified (by π_*) with the tangent space $T_{\pi(z)}V$. Note that the map p is holomorphic with respect to the introduced complex structure (while π is not!).

We note that with respect to the decomposition (15.10) the Heisenberg group $\mathrm{Heis}(V)$ acts on the twistor space $\mathcal{Z}$ horizontally, preserving the fibres of p, and the symplectic group $\mathrm{Sp_{HS}}(V)$ acts on $\mathcal{Z}$ vertically (this action is induced by the action of $\mathrm{Sp_{HS}}(V)$ on the Siegel disc $\mathcal{D}_{\mathrm{HS}}$).

15.4.2 Fock bundle

The Fock space $F_J = F(V, J)$ can be characterized in terms of the twistor diagram as the Fock space $F(p^{-1}(J))$ of holomorphic functions on the fibre $p^{-1}(J)$ (in variables

$\bar{w} \in \overline{W}_J$) with respect to the complex structure on $\mathcal{Z}$, introduced above. The *Fock bundle*

$$\mathcal{F} = \bigcup_{J\in\mathcal{D}_{\mathrm{HS}}} F_J \longrightarrow \mathcal{D}_{\mathrm{HS}}$$

is a Hermitian holomorphic Hilbert-space bundle over $\mathcal{D}_{\mathrm{HS}}$. Since $\mathcal{D}_{\mathrm{HS}}$ is contractible (even convex), it is trivial on $\mathcal{D}_{\mathrm{HS}}$. Moreover, the holomorphic map $U_J : F_0 \to F_J$, defined in Sec. 15.2, establishes an explicit holomorphic trivialization of $\mathcal{F}$. Note that the trivialization map $U_J : F_0 \to F_J$ is equivariant with respect to the action of the $\mathrm{Sp}_{\mathrm{HS}}(V)$ group.

In Sec. 15.3 a projective representation ρ of the Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ in the Fock space F_0 was constructed. Using this representation, we can define a linear connection on the Fock bundle $\mathcal{F}$, whose curvature coincides with the cocycle of the representation ρ.

Using the description of the Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$, given in Sec. 15.3, we can decompose it into the direct sum

$$\mathrm{sp}_{\mathrm{HS}}(V) = \mathfrak{u}(W_0) \oplus \mathfrak{m} \ . \tag{15.11}$$

Here, $\mathfrak{u}(W_0)$ is the Lie algebra of the unitary group $\mathrm{U}(W_0)$, identified with the set of matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha^t \end{pmatrix} ,$$

where α is a bounded skew-Hermitian operator. The linear subspace $\mathfrak{m} \cong T_0\mathcal{D}_{\mathrm{HS}}$ is identified with the set of matrices

$$\begin{pmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{pmatrix} ,$$

where β is a symmetric Hilbert–Schmidt operator. Note that the adjoint action of $\mathrm{U}(W_0)$ on $\mathrm{sp}_{\mathrm{HS}}(V)$ preserves the subspace $\mathfrak{m}$.

According to the general theory of invariant connections (cf. [45], Ch. II.11), the decomposition (15.11) together with the projective representation ρ determine an $\mathrm{Sp}_{\mathrm{HS}}(V)$-invariant connection $\mathbf{A}$ on the Fock bundle $\mathcal{F}$ with the curvature, given by the cocycle of ρ.

The original quantization problem from Sec. 12.2 can be reformulated in twistor terms as follows: *construct a quantization Hilbert-space bundle $\mathcal{H} \to \mathcal{D}_{HS}$ together with a flat unitary connection on it.* The connection in this definition may be considered as an infinitesimal analogue of the BKS-operator from Sec. 14.4. In the next Sec. 15.5 we consider in more detail a relation between the twistor and Dirac quantizations of the system $(V, \mathcal{A})$, where $\mathcal{A}$ is the semi-direct product of the Heisenberg algebra $\mathrm{heis}(V)$ and the symplectic Hilbert–Schmidt algebra $\mathrm{sp}_{\mathrm{HS}}(V)$.

15.5 Quantization bundle

In this Section we construct a quantization bundle $\mathcal{H} \to \mathcal{D}_{\mathrm{HS}}$ over $\mathcal{D}_{\mathrm{HS}}$. From finite-dimensional considerations in Ch. 14, it is clear that a good candidate for $\mathcal{H}$ should be the Fock bundle of half-forms, which we are going to define next.

15.5.1 Bundle of half-forms

We define first a *bundle of half-forms*

$$K^{-1/2} \longrightarrow \mathcal{D}_{\mathrm{HS}}$$

on the Siegel disc $\mathcal{D}_{\mathrm{HS}}$.

Namely, consider on $\mathcal{D}_{\mathrm{HS}}$ the following analogue of the Poincaré metric:

$$g_Z(\xi,\eta) = \mathrm{tr}\left\{(1-\bar{Z}Z)^{-2}\xi\bar{\eta}\right\}$$

for $Z \in \mathcal{D}_{\mathrm{HS}}$, $\xi,\eta \in T_Z^{1,0}\mathcal{D}_{\mathrm{HS}} \cong E_{\mathrm{HS}}$. It is a correctly defined Kähler metric on $\mathcal{D}_{\mathrm{HS}}$ with Kähler potential $K(Z,\bar{Z}) := -\mathrm{tr}\,\log(1-\bar{Z}Z)$. Moreover, it is invariant under the action of the group $\mathrm{Sp}_{\mathrm{HS}}(V)$ on the Siegel disc (cf. Sec. 11.4).

The canonical bundle $K \to \mathcal{D}_{\mathrm{HS}}$ is the restriction of the determinant bundle $\mathrm{Det} \to \mathrm{Gr}_{\mathrm{HS}}(V)$, defined in Sec. 5.3, to the Siegel disc $\mathcal{D}_{\mathrm{HS}}$. The metric g on $\mathcal{D}_{\mathrm{HS}}$ induces a Hermitian metric $\tilde{g}$ on K, given by the formula

$$\|(\lambda, Z)\|^2 = |\lambda|^2 \det(1-\bar{Z}Z)^2 \tag{15.12}$$

for $\lambda \in \mathbb{C}$, $Z \in \mathcal{D}_{\mathrm{HS}}$.

There is a natural action of a central extension $\widetilde{\mathrm{Sp}_{\mathrm{HS}}(V)}$ of symplectic group $\mathrm{Sp}_{\mathrm{HS}}(V)$ on the canonical bundle K, covering the action of $\mathrm{Sp}_{\mathrm{HS}}(V)$ on the Siegel disc $\mathcal{D}_{\mathrm{HS}}$. If $\tilde{A} \in \widetilde{\mathrm{Sp}_{\mathrm{HS}}(V)}$ projects to

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathrm{Sp}_{\mathrm{HS}}(V)\ ,$$

then $\tilde{A}$ acts on K by the formula

$$\tilde{A}\cdot(\lambda, Z) = \left(\lambda \det(1+\bar{a}^{-1}\bar{b}Z)^2, A\cdot Z\right)\ ,$$

where $A\cdot Z = (aZ+b)(\bar{b}Z+\bar{a})^{-1}$. The canonical connection on K, determined by the metric (15.12), is invariant under this $\widetilde{\mathrm{Sp}_{\mathrm{HS}}(V)}$-action on K.

The anticanonical bundle $K^{-1} \to \mathcal{D}_{\mathrm{HS}}$ of $\mathcal{D}_{\mathrm{HS}}$ coincides with the restriction of the dual determinant bundle $\mathrm{Det}^* \to \mathrm{Gr}_{\mathrm{HS}}(V)$, defined in Sec. 5.3, to $\mathcal{D}_{\mathrm{HS}}$. Since the Siegel disc $\mathcal{D}_{\mathrm{HS}}$ is contractible, the anticanonical bundle K^{-1} has a square root $K^{-1/2} \to \mathcal{D}_{\mathrm{HS}}$. The metric $\tilde{g}$ on K induces a Hermitian metric on $K^{-1/2}$, given by the formula

$$\|(\lambda, Z)\|^2 = |\lambda|^2 \det(1-\bar{Z}Z)^{-1}\ . \tag{15.13}$$

The group $\widetilde{\mathrm{Sp}_{\mathrm{HS}}(V)}$ acts on $K^{-1/2}$ by the formula

$$\tilde{A}\cdot(\lambda, Z) = \left(\lambda \det(1+\bar{a}^{-1}\bar{b}Z)^{-1}, A\cdot Z\right)\ .$$

The canonical connection $\mathbf{B}$ on $K^{-1/2} \to \mathcal{D}_{\mathrm{HS}}$, generated by Hermitian metric (15.13), is invariant under the action of $\widetilde{\mathrm{Sp}_{\mathrm{HS}}(V)}$ on $K^{-1/2}$.

15.5.2 Quantization bundle

By definition, the *quantization bundle* $\mathcal{H}$ coincides with the *Fock bundle of half-forms* on $\mathcal{D}_{\mathrm{HS}}$, given by the tensor product of the Fock bundle $\mathcal{F}$ and the bundle of half-forms $K^{-1/2}$:

$$\mathcal{H} := \mathcal{F} \otimes K^{-1/2} \longrightarrow \mathcal{D}_{\mathrm{HS}} \ .$$

We provide it with the tensor product connection

$$\mathbf{C} := \mathbf{A} \otimes 1 + 1 \otimes \mathbf{B} \ .$$

15.6 Twistor quantization of the loop space $\Omega\mathbb{R}^d$

In this Section we apply the construction of quantization bundle, described in Sec. 15.5, to the original system $(\Omega\mathbb{R}^d, \mathcal{A}_d)$. As in Sec. 9.2, we can embed the phase space $\Omega\mathbb{R}^d$ into the Sobolev space V^d of half-differentiable loops in $\mathbb{R}^d$. The space V^d coincides with the Sobolev space of half-differentiable vector-functions $S^1 \to \mathbb{R}^d$, defined in the same way, as its scalar analogue V (cf. also [17], Sec. VI.5.1). The embedding of $\Omega\mathbb{R}^d$ into V^d realizes the loop algebra $\widetilde{L\mathbb{R}^d}$ as a subalgebra of the Heisenberg algebra $\mathrm{heis}(V^d)$ and the Lie algebra $\mathrm{Vect}(S^1)$ as a subalgebra of the symplectic Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V^d)$. Moreover, under the above embedding the diffeomorphism group $\mathrm{Diff}_+(S^1)$ is realized as a subgroup of $\mathrm{Sp}_{\mathrm{HS}}(V^d)$. We have also, according to Sec. 11.5, a holomorphic embedding

$$\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1) \hookrightarrow \mathrm{Sp}_{\mathrm{HS}}(V^d)/\mathrm{U}(W_+^d) = \mathcal{D}_{\mathrm{HS}}$$

of the space $\mathcal{S}$ into the Siegel disc $\mathcal{D}_{\mathrm{HS}}$.

Denote by

$$\mathcal{F} \longrightarrow \mathcal{S}$$

the Fock bundle over $\mathcal{S}$, obtained from the Fock bundle $\mathcal{F} \to \mathcal{D}_{\mathrm{HS}}$ by restricting it to $\mathcal{S}$. We still have the Heisenberg representations

$$r_J : \widetilde{L\mathbb{R}^d} \longrightarrow \mathrm{End}^* F_J$$

for $J \in \mathcal{S}$, defined by the same formulas, as in Sec. 15.1. The projective $\mathrm{Sp}_{\mathrm{HS}}(V^d)$-action on the Fock bundle yields a projective $\mathrm{Diff}_+(S^1)$-action on $\mathcal{F} \to \mathcal{S}$. This action of $\mathrm{Diff}_+(S^1)$ on $\mathcal{F} \to \mathcal{S}$ was constructed in [27]. Its infinitesimal version is a projective representation

$$\rho : \mathrm{Vect}(S^1) \longrightarrow \mathrm{End}^* F_0 \ .$$

It can be described explicitly in terms of the basis $\{e_n\}$ of the complexified algebra $\mathrm{Vect}^{\mathbb{C}}(S^1)$ (cf. Sec. 2.2).

Denote by

$$L_n := \rho(e_n)$$

the operators in F_0, corresponding to the basis elements of $\mathrm{Vect}^{\mathbb{C}}(S^1)$. They are called otherwise the *Virasoro operators* and can be computed explicitly, using the formulas, given in Sec. 15.3. The cocycle of representation ρ in the basis $\{e_n\}$ is equal to (cf. [14])

$$[\rho(e_m), \rho(e_n)] - \rho([e_m, e_n]) = \frac{d}{12}(m^3 - m)\delta_{m,-n} . \tag{15.14}$$

This cocycle coincides with the curvature of the connection $\mathbf{A}$ on the Fock bundle $\mathcal{F} \to \mathcal{S}$, defined in Sec. 15.4.2.

Consider the anticanonical bundle $K^{-1/2} \to \mathcal{S}$, obtained by the restriction of the bundle $K^{-1/2} \to \mathcal{D}_{\mathrm{HS}}$ (cf. Sec. 15.5.1) to $\mathcal{S}$. The curvature of the canonical connection $\mathbf{B}$ on $K^{-1/2} \to \mathcal{S}$ in the basis $\{e_n\}$ was computed in [13]. It is equal to

$$R_{\mathbf{B}}(e_m, e_n) = -\frac{26}{12}(m^3 - m)\delta_{m,-n} . \tag{15.15}$$

We define the quantization bundle, as in Sec. 15.5.2, to be the Fock bundle of half-forms

$$\mathcal{H} := \mathcal{F} \otimes K^{-1/2} \longrightarrow \mathcal{S}$$

and provide it with the tensor product connection

$$\mathbf{C} := \mathbf{A} \otimes 1 + 1 \otimes \mathbf{B} .$$

The curvature of $\mathbf{C}$ is equal to the sum of the curvatures of connections $\mathbf{A}$ and $\mathbf{B}$, i.e.

$$R_{\mathbf{C}}(e_m, e_n) = \frac{d-26}{12}(m^3 - m)\delta_{m,-n} . \tag{15.16}$$

It vanishes precisely, when $d = 26$. For this dimension our system $(\Omega\mathbb{R}^d, \mathcal{A}_d)$, where the algebra of observables $\mathcal{A}_d$ is the semi-direct product of the loop algebra $\widetilde{L\mathbb{R}^d}$ and $\mathrm{Vect}(S^1)$, admits the twistor quantization.

To derive from an obtained solution of the twistor quantization problem a solution of the original quantization problem , i.e. a representation of the algebra of observables $\mathcal{A}_d$ in the Fock space of half-forms $\mathcal{H}_0 = F_0 \otimes K_0^{-1/2}$, identified with the fibre of the quantization bundle at the origin $o \in \mathcal{S}$, we should proceed along the same lines, as in the BKS-quantization method in Sec. 14.5. Namely, the representations of the Heisenberg algebra in the fibres of the Fock bundle $\mathcal{F}$ extend to representations in the fibres of the quantization bundle $\mathcal{H}$. The group $\mathrm{Diff}_+(S^1)$ acts projectively on the bundle $\mathcal{H}$ and this action intertwines with representations of the Heisenberg algebra in the fibres. The Kostant–Souriau operators L_n, corresponding to the basis elements of the algebra $\mathrm{Vect}(S^1)$, do not preserve, in general, the spaces F_0 and $\mathcal{H}_0$, since the symplectic diffeomorphisms φ^t, corresponding to L_n, transform the spaces F_0 and $\mathcal{H}_0$ into the spaces F_t and $\mathcal{H}_t$, associated with the complex structure $J^t = \varphi_*^t \circ J^0 \circ (\varphi_*^t)^{-1}$. However, by integrating the flat Hermitian connection on the quantization bundle $\mathcal{H}$, one can construct a unitary operator U_t, identifying $\mathcal{H}_t$ with $\mathcal{H}_0$. The composition $U_t \circ L_n$ acts now in $\mathcal{H}_0$, and, after the differentiation, yields the required representation of the algebra $\mathrm{Vect}(S^1)$ in $\mathcal{H}_0$.

15.7 Quantization of the universal Teichmüller space

In the previous Section we have defined the Fock bundle

$$\mathcal{F} \longrightarrow \mathcal{S}$$

over the smooth part $\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{M\ddot{o}b}(S^1)$ of the universal Teichmüller space $\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1)$. This bundle is provided with a projective action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$, covering the natural action of $\mathrm{Diff}_+(S^1)$ on the base $\mathcal{S}$. The infinitesimal version of this action yields a projective representation of the Lie algebra $\mathrm{Vect}(S^1)$ in the Fock space $\mathcal{H}_0$. We can consider this construction as a geometric quantization of the phase space $\mathcal{S}$ with the algebra of observables, given by the Virasoro algebra vir, the quantization being given by the projective representation of $\mathrm{Vect}(S^1)$ in $\mathcal{H}_0$. As we have pointed out in Sec. 15.6 it can be obtained by restriction to $\mathcal{S}$ of the analogous construction over the Hilbert–Schmidt Siegel disc $\mathcal{D}_{\mathrm{HS}} = \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+)$. Recall that in Subsec. 15.4.2 we have constructed the Fock bundle

$$\mathcal{F} \longrightarrow \mathcal{D}_{\mathrm{HS}}$$

over $\mathcal{D}_{\mathrm{HS}}$, provided with the projective action of the symplectic group $\mathrm{Sp}_{\mathrm{HS}}(V)$, covering the natural action of $\mathrm{Sp}_{\mathrm{HS}}(V)$ on $\mathcal{D}_{\mathrm{HS}}$. The infinitesimal version of this action yielded the projective representation of the symplectic algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ in the Fock space $\mathcal{H}_0$, described in Sec. 15.3. This construction may be considered as a geometric quantization of the phase space $\mathcal{D}_{\mathrm{HS}} = \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+)$ with the algebra of observables, given by a central extension of the Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$, the quantization being given by the projective representation of $\mathrm{sp}_{\mathrm{HS}}(V)$ in $\mathcal{H}_0$.

Unfortunately, the described quantization procedure does not apply to the whole universal Teichmüller space $\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1)$. According to Prop. 25 from Sec. 11.4, we can still embed this space into the infinite-dimensional Siegel disc $\mathcal{D} = \mathrm{Sp}(V)/\mathrm{U}(W_+)$, but we cannot construct a Fock bundle over $\mathcal{D} = \mathrm{Sp}(V)/\mathrm{U}(W_+)$ with a projective action of the whole symplectic group $\mathrm{Sp}(V)$. The reason is that, according to the theorem of Shale (cf. Sec. 15.2), it is possible only for the Hilbert–Schmidt symplectic subgroup $\mathrm{Sp}_{\mathrm{HS}}(V)$ of $\mathrm{Sp}(V)$. So one should look for another approach to the quantization of universal Teichmüller space $\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1)$. It seems that a natural way to do that is to use the quantized calculus of A.Connes and D.Sullivan. We now present briefly the idea of this approach in application to our problem, borrowed from Ch.IV of the Connes' book [16].

Recall that in Dirac's approach (cf. Sec. 12.2), we quantize a classical system $(M, \mathcal{A})$, consisting of the phase space M, which is a symplectic manifold, and the algebra of observables $\mathcal{A}$, which is a Poisson Lie algebra, consisting of smooth functions on M. The quantization of this system is given by a representation π of $\mathcal{A}$ in a Hilbert space H, sending the Poisson bracket $\{f, g\}$ of two functions $f, g \in \mathcal{A}$ into the commutator $[\pi(f), \pi(g)]$ (times $1/i$) of the corresponding operators. In Connes' approach the algebra of observables $\mathcal{A}$ is an associative involutive algebra, provided with an exterior differential d. Its quantization is, by definition, a representation of $\mathcal{A}$ in H, sending the differential df of a function $f \in \mathcal{A}$ into the commutator $[S, \pi(f)]$ of the operator $\pi(f)$ with a symmetry operator S, which is self-adjoint and of square 1. We can reformulate the notion of Connes quantization also in terms of Lie algebras. For that recall that a *derivation* of the algebra $\mathfrak{A}$ is a linear map of $\mathfrak{A}$ into itself, satisfying the Leibnitz rule. Derivations of $\mathfrak{A}$ form a Lie algebra $\mathrm{Der}(\mathfrak{A})$ and Connes quantization of the algebra of observables $\mathfrak{A}$ is given in these terms by a representation of the Lie algebra $\mathrm{Der}(\mathfrak{A})$ in the Lie algebra $\mathrm{End}\, H$.

If the algebra of observables $\mathcal{A}$ consists of smooth functions on the phase manifold M, this new formulation is essentially equivalent to that of Dirac. Indeed, the differential df of an observable $f \in \mathcal{A}$ is symplectically dual to the Hamiltonian vector field X_f, so we can reproduce the Poisson Lie algebra from the associative algebra with the exterior differential. On the other hand, a symmetry operator S on the polarized quantization space $H = H_+ \oplus H_-$ is given by the rule: $S = \pm I$ on $H_\pm$. (By this reason we do not make difference between the symmetry and complex structure operators.) But in the case, when $\mathcal{A}$ contains non-smooth functions, the Dirac definition does not work, while Connes quantization still makes sense, as we shall demonstrate on examples below.

Consider an example, in which $\mathcal{A}$ coincides with the algebra $L^\infty(S^1)$ of bounded functions on the circle S^1. Any function $f \in \mathcal{A}$ defines a bounded multiplication operator in the Hilbert space $H = L^2(S^1)$:

$$M_f : v \in H \longmapsto fv \in H \ .$$

The operator S in this case is given by the Hilbert transform $S : L^2(S^1) \to L^2(S^1)$. The differential of a general function $f \in \mathcal{A}$ is not defined in the classical sense, but we can still consider its quantum analogue by setting

$$d^q f := [S, M_f] \ .$$

The correspondence between functions $f \in \mathcal{A}$ and operators M_f on H has the following remarkable properties (cf. [64]):

1. The differential $d^q f$ is a finite rank operator if and only if f is a rational function.

2. The differential $d^q f$ is a compact operator if and only if the function f has a vanishing mean oscillation.

3. The differential $d^q f$ is a bounded operator if and only if the function f has a bounded mean oscillation.

This list may be supplemented by further function-theoretic properties of functions in $\mathcal{A}$, which have nice operator-theoretic characterizations (cf. [16], Ch.IV).

How this idea can be applied to the quantization of the universal Teichmüller space $\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$? Let us switch for convenience from S^1 to the real line $\mathbb{R}$, so that $\mathcal{T}$ will be identified with the space $\mathrm{QS}(\mathbb{R})/\mathrm{Möb}(\mathbb{R})$ of normalized quasisymmetric homeomorphisms of $\mathbb{R}$. Our main Sobolev space $H^{1/2}(\mathbb{R}) := H^{1/2}(\mathbb{R}, \mathbb{R})$ of half-differentiable functions on the real line $\mathbb{R}$ has a simple description in terms of the quantum differential. Namely, the symmetry operator S is again given by the Hilbert transform

$$(Sf)(s) = \frac{1}{\pi i} \ \mathrm{P.V.} \int \frac{f(t)}{s-t}\, dt \ , \quad f \in L^2(\mathbb{R}) \ , \tag{15.17}$$

where the integral is taken in the principal value sense.

The quantum differential $d^q f = [S, M_f]$ of a function $f \in L^\infty(\mathbb{R})$ is an operator on $L^2(\mathbb{R})$, given by

$$(d^q f)v(s) = \frac{1}{\pi i}\int k(s,t)v(t)\,dt \tag{15.18}$$

with the kernel, equal to

$$k(s,t) = \frac{f(s) - f(t)}{s - t}\ , \quad s,t \in \mathbb{R}\ .$$

Note that the quasiclassical limit of this operator, defined by taking the value of the kernel on the diagonal, i.e. for $s \to t$, coincides with the multiplication operator $v \mapsto f'v$, and the quantization means in this case the replacement of the derivative by its finite-difference analogue.

Then $f \in H^{1/2}(\mathbb{R})$ *if and only if its quantum differential $d^q f$ is a Hilbert–Schmidt operator on* $L^2(\mathbb{R})$; moreover, the Hilbert–Schmidt norm of $d^q f$ coincides with the $H^{1/2}(\mathbb{R})$-norm of f (cf. [58], Prop. 6.1).

As in Sec. 11.4 we can define a natural action of the group $\mathrm{QS}(\mathbb{R})$ of quasisymmetric homeomorphisms on the Sobolev space $H_R = H^{1/2}(\mathbb{R})$ by change of a variable. However, this action does admit the differentiation so there is no classical Lie algebra, associated with the group $\mathrm{QS}(\mathbb{R})$, or, in other words, there is no classical algebra of observables, associated with $\mathcal{T}$. (The situation is similar to the one, considered in the example above.) We shall construct a quantum algebra of observables, associated with $\mathcal{T}$.

The quantum infinitesimal version of $\mathrm{QS}(\mathbb{R})$-action on H_R is given by the integral operator $d^q f$, defined by formula (15.18). We extend this operator $d^q f$ to the Fock space F_0 by defining it first on elements of the basis (15.2) of F_0 with the help of Leibnitz rule, and then extending to the whole symmetric algebra $S(W_0)$ by linearity. The completion of the obtained operator yields an operator $d^q f$ on F_0. The operators $d^q f$ with $f \in \mathrm{QS}(\mathbb{R})$, constructed in this way, generate a *quantum Lie algebra* $\mathrm{Der}^q(\mathrm{QS})$, associated with $\mathcal{T}$. We consider it as a quantum Lie algebra of observables, associated with $\mathcal{T}$. We can also consider the constructed Lie algebra $\mathrm{Der}^q(\mathrm{QS})$ as a replacement of the (non-existing) classical Lie algebra of the group $\mathrm{QS}(\mathbb{R})$.,

Compare now the main steps of Connes quantization of $\mathcal{T}$ with the analogous steps in Dirac quantization of $\mathcal{D}_{\mathrm{HS}}$ (returning again to the case of S^1).

In the case of $\mathcal{D}_{\mathrm{HS}}$:

1. we start with the $\mathrm{Sp}_{\mathrm{HS}}(V)$-action on $\mathcal{D}_{\mathrm{HS}}$;

2. then, using Shale theorem, extend this action to a projective unitary action of $\mathrm{Sp}_{\mathrm{HS}}(V)$ on Fock spaces $F(V,J)$;

3. an infinitesimal version of this action yields a projective unitary representation of symplectic Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ in the Fock space F_0.

In the case of $\mathcal{T}$:

1. we have an action of $\mathrm{QS}(S^1)$ on the space V; however, in contrast with Dirac quantization of $\mathcal{D}_{\mathrm{HS}}$, the step (2) in case of $\mathcal{T}$ is impossible, since by Shale theorem we cannot extend the action of $\mathrm{QS}(S^1)$ to Fock spaces $F(V,S)$;

2. we define instead a quantized infinitesimal action of $\mathrm{QS}(S^1)$ on V, given by quantum differentials $d^q f$;

3. extending operators $d^q f$ to the Fock space F_0, we obtain a quantum Lie algebra $\mathrm{Der}^q(\mathrm{QS})$, generated by extended operators $d^q f$ on F_0.

So, the Connes quantization of the universal Teichmüller space $\mathcal{T}$ consists of two steps:

1. The first step ("the first quantization") is the construction of quantized infinitesimal $\mathrm{QS}(S^1)$-action on V, given by quantum differentials $d^q f$ with $f \in \mathrm{QS}(S^1)$.

2. The second step ("the second quantization") is the extension of quantum differentials $d^q f$ to the Fock space F_0. The extended operators $d^q f$ with $f \in \mathrm{QS}(S^1)$ generate the quantum algebra of observables $\mathrm{Der}^q(\mathrm{QS})$, associated with $\mathcal{T}$.

Note that the correspondence principle for the constructed Connes quantization of $\mathcal{T}$ means that this quantization, being restricted to $\mathcal{S}$, coincides with Dirac quantization of $\mathcal{S}$.

Bibliographic comments

In Sec. 15.1 we have collected well known facts about the Fock spaces and Heisenberg representations. They can be found in a number of books and papers, starting from Berezin's book [7]. In Sec. 15.2 we study the projective action of the Hilbert–Schmidt symplectic group on Fock spaces. This study was initiated by Shale [69] (cf. also [66, 65, 75]). The projective representation of the Hilbert–Schmidt symplectic algebra in the Fock space was computed by Segal [66]. The Section 15.4, devoted to the twistor interpretation of our construction, is based on [63, 17]. The twistor quantization of the loop space $\Omega\mathbb{R}^d$ was initiated by Bowick–Rajeev [14]. In particular, they have found in [14] that the twistor quantization problem for $\Omega\mathbb{R}^d$ can be solved in the critical dimension $d = 26$. The last Section 15.7 is based mainly on the paper [67].

Chapter 16

Quantization of the loop space $\Omega_T G$

In this Chapter we solve the geometric quantization problem for the phase space, represented by the Kähler-Frechet manifold $\Omega_T G$. The role of the algebra of observables $\mathcal{A}$ is played by the Lie algebra $\widehat{L\mathfrak{g}} \rtimes \mathrm{vir}$, an extension of the Lie algebra $L\mathfrak{g} \rtimes \mathrm{Vect}(S^1)$. The latter is the Lie algebra of the Frechet Lie group $LG \rtimes \mathrm{Diff}_+(S^1)$, the semi-direct product of the loop group LG and the diffeomorphism group $\mathrm{Diff}_+(S^1)$ of the circle.

In the most part of this Chapter we assume that G is a simply connected and simple Lie group.

16.1 Representations of loop algebras

In the loop space case the role of the Heisenberg algebra and its Heisenberg representation from Ch. 15 is played by central extensions $\widetilde{L\mathfrak{g}}$ of the loop algebras $L\mathfrak{g}$ and its lowest weight representations.

16.1.1 Affine algebras

The S^1-action plays a central role in the representation theory of the loop algebras and groups. To take care of this action, it is convenient to extend the loop algebra $L\mathfrak{g}$ to the *extended loop algebra* $\mathbb{C} \oplus L\mathfrak{g}$, the generator of $\mathrm{U}(1)$-action being denoted by e_0 in accordance with Sec. 10.1. In the same way we extend the loop group LG to the *extended loop group* $\mathrm{U}(1) \ltimes LG$ by taking the semi-direct product of LG with the circle group $S^1 \equiv \mathrm{U}(1)$.

Suppose that $\mathfrak{g}_{\mathbb{C}}$ is a complex simple Lie algebra and fix a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. The corresponding *root decomposition* of the extended Lie algebra $\mathbb{C}e_0 \oplus L\mathfrak{g}_{\mathbb{C}}$ with respect to the Cartan subalgebra $\mathbb{C}e_0 \oplus \mathfrak{h}_{\mathbb{C}}$ has the form

$$\mathbb{C}e_0 \oplus L\mathfrak{g}_{\mathbb{C}} = \mathbb{C}e_0 \oplus \left[\bigoplus_{n\in\mathbb{Z}} \mathfrak{h}_{\mathbb{C}} z^n\right] \oplus \left[\bigoplus_{(n,\alpha)} \mathfrak{g}_\alpha z^n\right] , \qquad (16.1)$$

where $\mathfrak{g}_\alpha$ are the root subspaces of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$. The pairs $a = (n, \alpha)$, where $n \in \mathbb{Z}$ and α is a root of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$, are called the *roots* of the algebra

$L\mathfrak{g}_{\mathbb{C}}$. They can be considered as linear functionals on the Lie algebra $\mathbb{C}e_0 \oplus \mathfrak{h}_{\mathbb{C}}$. If, in particular, we introduce a functional $\delta \in (\mathbb{C}e_0 \oplus \mathfrak{h}_{\mathbb{C}})^*$ by setting:

$$\delta(e_0) = 1 \ , \quad \delta(\mathfrak{h}_{\mathbb{C}}) = 0 \ ,$$

then the whole set of roots of $\mathbb{C}e_0 \oplus L\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathbb{C}e_0 \oplus \mathfrak{h}_{\mathbb{C}}$ will be described as

$$\hat{\Delta} = \{\alpha + n\delta : \alpha \in \Delta,\, n \in \mathbb{Z}\} \cup \{n\delta : n \in \mathbb{Z}\} \ ,$$

where Δ is the set of roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$. Accordingly, the set of positive roots of $\mathbb{C}e_0 \oplus L\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathbb{C}e_0 \oplus \mathfrak{h}_{\mathbb{C}}$ is identified with

$$\hat{\Delta}^+ = \{\alpha + n\delta : \alpha \in \Delta,\, n > 0\} \cup \{n\delta : n > 0\} \cup \Delta^+ \ ,$$

where Δ^+ is the set of positive roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$. If $\{\alpha_1, \dots, \alpha_l\}$ is a system of simple roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$, and A is the highest root in Δ^+, then any root in $\hat{\Delta}^+$ may be written in the form

$$n_0\alpha_0 + n_1\alpha_1 + \cdots + n_l\alpha_l$$

with non-negative integer coefficients $n_0, n_1, \dots, n_l$, where $\alpha_0 := \delta - A$. We call $\{\alpha_0, \alpha_1, \dots, \alpha_l\}$ a system of *affine simple roots* in $\hat{\Delta}$.

We associate with any root $a = (n, \alpha)$ the *root subspace* $\mathfrak{g}_{(n,\alpha)}$ in $L\mathfrak{g}_{\mathbb{C}}$, defined by

$$\begin{aligned} \mathfrak{g}_{(n,\alpha)} &= \mathfrak{g}_\alpha z^n \quad \text{for } \alpha \neq 0 \ , \\ \mathfrak{g}_{(n,0)} &= \mathfrak{h}_{\mathbb{C}} z^n \quad \text{for } \alpha = 0 \ . \end{aligned}$$

The loop analogue of the decomposition of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^- \ ,$$

where $\mathfrak{n}^\pm$ are nilpotent subalgebras of $\mathfrak{g}_{\mathbb{C}}$ of the form

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \quad , \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha \ ,$$

has the form

$$L\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus N^+\mathfrak{g}_{\mathbb{C}} \oplus N^-\mathfrak{g}_{\mathbb{C}} \ ,$$

where

$$N^+\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}^+ \oplus \left[\bigoplus_{n>0} \mathfrak{g}_{\mathbb{C}} \cdot z^n\right] \quad , \quad N^-\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}^- \oplus \left[\bigoplus_{n<0} \mathfrak{g}_{\mathbb{C}} \cdot z^n\right] \ .$$

The loop analogues of the Borel subalgebras have the form

$$B^\pm\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus N^\pm\mathfrak{g}_{\mathbb{C}} \ .$$

We introduce now a central extension $\widetilde{L\mathfrak{g}_{\mathbb{C}}}$ of the loop algebra $L\mathfrak{g}_{\mathbb{C}}$. Recall (cf. Sec. 8.2) that such an extension is determined by a 2-cocycle on $L\mathfrak{g}_{\mathbb{C}}$, given by the formula

$$\omega(\xi, \eta) = \omega_0(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} < \xi(e^{i\theta}), \eta'(e^{i\theta}) > d\theta \ , \quad \xi, \eta \in L\mathfrak{g}_{\mathbb{C}} \ ,$$

where $< \cdot , \cdot >$ is an invariant inner product on the Lie algebra $\mathfrak{g}_{\mathbb{C}}$. As a vector space,

$$\widetilde{L\mathfrak{g}_{\mathbb{C}}} = L\mathfrak{g}_{\mathbb{C}} \oplus \mathbb{C}\, c \ ,$$

with commutation relations

$$[\xi + s\, c, \eta + t\, c] = [\xi, \eta] + \omega(\xi, \eta)\, c$$

for $\xi, \eta \in L\mathfrak{g}_{\mathbb{C}}$, $s, t \in \mathbb{C}$. We denote the corresponding central extension of the loop group $LG_{\mathbb{C}}$ (cf. Sec. 8.2) by $\widetilde{LG_{\mathbb{C}}}$.

The representations of the loop algebra $L\mathfrak{g}$ and the loop group LG, which we consider here, are projective and intertwine with the S^1-action. It means that they arise, in fact, from representations of the *affine algebra*

$$\widehat{L\mathfrak{g}_{\mathbb{C}}} = \mathbb{C}e_0 \oplus \widetilde{L\mathfrak{g}_{\mathbb{C}}} = \mathbb{C}e_0 \oplus L\mathfrak{g}_{\mathbb{C}} \oplus \mathbb{C}\, c$$

and the *affine group*

$$\widehat{LG_{\mathbb{C}}} := \mathbb{C}^* \ltimes \widetilde{LG_{\mathbb{C}}} \ .$$

The root decomposition of the affine algebra $\widehat{L\mathfrak{g}_{\mathbb{C}}}$ has the form

$$\widehat{L\mathfrak{g}_{\mathbb{C}}} = \widehat{\mathfrak{h}_{\mathbb{C}}} \oplus N^+\mathfrak{g}_{\mathbb{C}} \oplus N^-\mathfrak{g}_{\mathbb{C}} \ ,$$

where

$$\widehat{\mathfrak{h}_{\mathbb{C}}} = \mathbb{C}e_0 \oplus \widetilde{\mathfrak{h}_{\mathbb{C}}} = \mathbb{C}e_0 \oplus \mathfrak{h}_{\mathbb{C}} \oplus \mathbb{C}\, c \ .$$

Accordingly,

$$\widehat{B^{\pm}\mathfrak{g}_{\mathbb{C}}} = \widehat{\mathfrak{h}_{\mathbb{C}}} \oplus N^{\pm}\mathfrak{g}_{\mathbb{C}} \ .$$

Having a root $\alpha \in \mathfrak{h}^*_{\mathbb{C}}$, we extend it to $\widehat{\mathfrak{h}_{\mathbb{C}}}$ by setting $\alpha(\, c) = \alpha(e_0) = 0$. We also extend the functional $\delta \in (\mathbb{C}e_0 \oplus \mathfrak{h}_{\mathbb{C}})^*$ to $\widehat{\mathfrak{h}_{\mathbb{C}}}$ by setting $\delta(\, c) = 0$. It's also useful to introduce a functional $\beta \in (\widehat{\mathfrak{h}_{\mathbb{C}}})^*$, defined by

$$\beta(\, c) = 1 \ , \quad \beta(e_0) = 0 \ , \quad \beta(\mathfrak{h}_{\mathbb{C}}) = 0 \ .$$

With any system $\alpha_0, \alpha_1, \ldots, \alpha_l$ of affine simple roots we can associate a corresponding system of co-roots $\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_l^\vee$, where $\alpha_j^\vee$, $j = 1, \ldots, l$, are the co-roots, associated with simple roots α_j of the algebra $\mathfrak{g}_{\mathbb{C}}$, and

$$\alpha_0^\vee = -A^\vee + \frac{2\, c}{< A, A >}$$

is the affine co-root, associated with the highest root $A \in \Delta^+$.

Denote by $\{\omega_1, \ldots, \omega_l\}$ the system of fundamental weights of the algebra $\mathfrak{g}_{\mathbb{C}}$, dual to the simple root system $\alpha_1, \ldots, \alpha_l$. We can introduce the corresponding system $\{\hat{\omega}_0, \hat{\omega}_1, \ldots, \hat{\omega}_l\}$ of fundamental weights of $\widehat{L\mathfrak{g}_{\mathbb{C}}}$, dual to the system $\alpha_0, \alpha_1, \ldots, \alpha_l$ of affine simple roots, defined by

$$\hat{\omega}_i(\alpha_k^\vee) = \delta_{ik} \quad \text{for} \quad 0 \le i, k \le l \ , \quad \hat{\omega}_i(e_0) = 0 \ .$$

Then

$$\hat{\omega}_0 = \frac{1}{2} < A, A > \beta \ , \quad \hat{\omega}_j = \omega_j + < \omega_j, A > \beta \ , \quad 1 \le j \le l \ .$$

16.1.2 Highest weight representations of affine algebras

Suppose that $\rho : \widehat{L\mathfrak{g}_\mathbb{C}} \to V$ is a representation of the loop algebra $\widehat{L\mathfrak{g}_\mathbb{C}}$, i.e. an $\widehat{L\mathfrak{g}_\mathbb{C}}$-module. Consider for any linear form on $\widehat{\mathfrak{h}_\mathbb{C}}$, i.e. an element $\lambda \in (\widehat{\mathfrak{h}_\mathbb{C}})^*$, the subspace
$$V_\lambda = \{v \in V : \rho(h)v = \lambda(h)v \quad \text{for } h \in \widehat{\mathfrak{h}_\mathbb{C}}\} \ .$$
If $V_\lambda \neq 0$, then λ is called the *weight* of ρ, and the subspace V_λ is the *weight subspace* of ρ, corresponding to λ. Any vector $v \in V_\lambda \setminus \{0\}$ is called the *weight vector* of ρ.

A weight $\lambda \in (\widehat{\mathfrak{h}_\mathbb{C}})^*$ is *dominant integral*, if $\lambda(\alpha_i^\vee)$ is a non-negative integer for any affine co-root $\alpha_i^\vee$, $0 \le i \le l$. Any such weight can be written in the form

$$\lambda = n_0\hat{\omega}_0 + \ldots + n_l\hat{\omega}_l + s\delta \ , \tag{16.2}$$

where $n_i = \lambda(\alpha_i^\vee)$, $0 \le i \le l$, and $s = \lambda(e_0) \in \mathbb{C}$. Respectively, an *anti-dominant integral weight* $\lambda \in (\widehat{\mathfrak{h}_\mathbb{C}})^*$ takes non-positive integer values on affine co-roots $\alpha_i^\vee$, $0 \le i \le l$, and can be written in the same form (16.2) with non-positive integer coefficients n_i, $0 \le i \le l$.

Given a weight $\lambda \in (\widehat{\mathfrak{h}_\mathbb{C}})^*$, we can extend it to the Borel subalgebra $\widehat{B^+\mathfrak{g}_\mathbb{C}}$ by setting it equal to zero on $N^+\mathfrak{g}_\mathbb{C}$. Consider an $\widehat{L\mathfrak{g}_\mathbb{C}}$-module of the form

$$\hat{V} \equiv \hat{V}_\lambda = \mathcal{U}(\widehat{L\mathfrak{g}_\mathbb{C}}) \otimes_{\mathcal{U}(\widehat{B^+\mathfrak{g}_\mathbb{C}})} \mathbb{C}_\lambda \ ,$$

where the symbol "$\mathcal{U}$" stands for the universal enveloping algebra, and $\mathbb{C}_\lambda$ denotes the 1-dimensional $\widehat{B^+\mathfrak{g}_\mathbb{C}}$-module, i.e. the complex line $\mathbb{C}$, provided with an action of the Borel subalgebra $\widehat{B^+\mathfrak{g}_\mathbb{C}}$, given by: $z \longmapsto \lambda(b)z$ for $b \in \widehat{B^+\mathfrak{g}^\mathbb{C}}$, $z \in \mathbb{C}$. Since

$$\widehat{L\mathfrak{g}_\mathbb{C}} = N^-\mathfrak{g}_\mathbb{C} \oplus \widehat{B^+\mathfrak{g}_\mathbb{C}} \ ,$$

the Poincaré–Birkhoff–Witt theorem implies that

$$\mathcal{U}(\widehat{L\mathfrak{g}_\mathbb{C}}) \cong \mathcal{U}(N^-\mathfrak{g}_\mathbb{C}) \otimes \mathcal{U}(\widehat{B^+\mathfrak{g}_\mathbb{C}}) \ .$$

So we have a natural isomorphism

$$\hat{V}_\lambda \cong \mathcal{U}(N^-\mathfrak{g}_\mathbb{C}) \otimes \mathbb{C}_\lambda \ .$$

Denote by $V \equiv V_\lambda$ the quotient of $\hat{V}$ modulo the maximal submodule in $\hat{V}$, strictly contained in $\hat{V}$ (in other words, the maximal submodule, not containing $1 \otimes 1$). This V, together with the natural action of $\widehat{L\mathfrak{g}_\mathbb{C}}$, is called the *standard representation of the Lie algebra* $\widehat{L\mathfrak{g}_\mathbb{C}}$ *with the highest weight* λ and the weight vector $1 \otimes 1$.

More generally, we shall say that a representation $\rho : \widehat{L\mathfrak{g}_\mathbb{C}} \to \operatorname{End} V_\lambda$ of the affine algebra $\widehat{L\mathfrak{g}_\mathbb{C}}$ is the *highest weight representation* with weight $\lambda \in (\widehat{\mathfrak{h}_\mathbb{C}})^*$, if there exists a *highest weight vector* $v_\lambda \in V_\lambda$ such that:

1. $\rho(h)v_\lambda = \lambda(h)v_\lambda \quad$ for any $h \in \widehat{\mathfrak{h}_\mathbb{C}}$;
2. $\rho(n)v_\lambda = 0 \quad$ for any $n \in N^+\mathfrak{g}_\mathbb{C}$;

3. V_λ is spanned by vectors $\rho(b)v_\lambda$ with $b \in \widehat{B^-\mathfrak{g}_\mathbb{C}}$.

The highest weight vector v_λ plays the role, analogous to that of the vacuum in the Heisenberg representation.

In analogous way one can define the *lowest weight representation* of the affine algebra $\widehat{L\mathfrak{g}_\mathbb{C}}$. For that one should replace in the above definition the nilpotent subalgebra $N^+\mathfrak{g}_\mathbb{C}$ by the nilpotent subalgebra $N^-\mathfrak{g}_\mathbb{C}$ and the Borel subalgebra $\widehat{B^-\mathfrak{g}_\mathbb{C}}$ by the Borel subalgebra $\widehat{B^+\mathfrak{g}_\mathbb{C}}$.

The standard $\widehat{L\mathfrak{g}_\mathbb{C}}$-module V_λ, defined above, is an irreducible highest weight representation of $\widehat{L\mathfrak{g}_\mathbb{C}}$, if $\lambda \in (\widehat{\mathfrak{h}_\mathbb{C}})^*$ is an integral dominant weight. Moreover, it was proved in [23], that if $\lambda(e_0)$ is real, then V_λ admits a positive-definite (contravariant) Hermitian inner product. We denote by $H \equiv H_\lambda$ the completion of $V \equiv V_\lambda$ with respect to this inner product. The space H_λ will play the role of the Fock space, associated with the weight λ.

16.2 Representations of loop groups

We present here some general properties of irreducible representations of the affine group $\widehat{LG^\mathbb{C}}$ and the Borel–Weil construction for $\widehat{LG^\mathbb{C}}$.

16.2.1 Irreducible representations of affine groups

Consider the affine group

$$\widehat{LG} := \mathrm{U}(1) \ltimes \widetilde{LG}$$

and fix a *maximal torus* $\widehat{T}$ in $\widehat{LG}$, given by

$$\widehat{T} := \mathrm{U}(1) \times T \times S$$

Here, the first factor $\mathrm{U}(1) = S^1$ is the group of rotations, the second factor T is a maximal torus in G, and the third one $S = S^1$ is a central subgroup in $\widetilde{LG}$.

Any irreducible representation of the affine group $\widehat{LG}$ has a unique highest weight λ, which is a character of the maximal torus $\widehat{T}$. This character has the form

$$\lambda = (n, \lambda_0, h) ,$$

where $n \in \mathbb{Z}$ is an eigenvalue of the S^1-rotation operator e_0, called the *energy* of the representation, λ_0 is a character of T, and $h \in \mathbb{Z}$ is an eigenvalue of the central subgroup action, called the *level* of the representation. The highest weights of $\widehat{LG}$ are integral and dominant and the isomorphism classes of irreducible representations of $\widehat{LG}$ are in 1:1 correspondence with the set of integral dominant weights.

There is a similar characterization of irreducible representations of the affine group $\widehat{LG}$ in terms of lowest weights.

16.2.2 Borel–Weil construction

Consider the full flag loop space (cf. Sec. 7.6)

$$\Omega_T G = LG/T = LG^\mathbb{C}/B^+G^\mathbb{C} .$$

In terms of central extensions, $\Omega_T G$ may be written in the form

$$\Omega_T G = \widetilde{LG^{\mathbb{C}}}/\widetilde{B^+G^{\mathbb{C}}} \ .$$

Suppose that λ is a lowest weight of the maximal torus $\widetilde{T} = T \times S$. We extend it to $\widetilde{B^+G^{\mathbb{C}}}$ by setting $\lambda = 1$ on the nilpotent subgroup $N^+G^{\mathbb{C}}$ in

$$\widetilde{B^+G^{\mathbb{C}}} = \widetilde{T^{\mathbb{C}}} \times N^+G^{\mathbb{C}} \ .$$

Define a holomorphic line bundle $L = L_\lambda$ over $\Omega_T G$ by

$$L = \widetilde{LG^{\mathbb{C}}} \times_{\widetilde{B^+G^{\mathbb{C}}}} \mathbb{C} \longrightarrow \Omega_T G = \widetilde{LG^{\mathbb{C}}}/\widetilde{B^+G^{\mathbb{C}}} \ ,$$

where $\widetilde{B^+G^{\mathbb{C}}}$ acts on the complex line $\mathbb{C}$ by the character λ:

$$\widetilde{B^+G^{\mathbb{C}}} \ni b : z \longmapsto \lambda(b)z \ .$$

Denote by $\Gamma = \Gamma_\lambda$ the vector space of holomorphic sections of $L = L_\lambda$. Sections $s \in \Gamma$ can be identified with holomorphic functions $\dot{s} : \widetilde{LG^{\mathbb{C}}} \to \mathbb{C}$, satisfying the condition

$$\dot{s}(\gamma b^{-1}) = \lambda(b)\dot{s}(\gamma)$$

for any $b \in \widetilde{B^+G^{\mathbb{C}}}$, $\gamma \in \widetilde{LG^{\mathbb{C}}}$. The group $\widetilde{LG^{\mathbb{C}}}$ acts in a natural way on L and on Γ, and this action defines a holomorphic representation of $\widetilde{LG^{\mathbb{C}}}$ on Γ. We note that Γ *is non-trivial* (i.e. contains non-zero holomorphic sections of L) *if and only if the weight λ is anti-dominant* (cf. [65], Prop. 11.3.1). Under this condition it may be proved (cf. [65], Prop. 11.1.1) that the corresponding representation of the loop group $\widetilde{LG}$ is an irreducible lowest weight representation of $\widetilde{LG}$ with the lowest weight λ. Moreover, it can be proved (cf. [65], Prop. 11.2.3) that any irreducible representation of the group $\widetilde{LG}$ is essentially equivalent to some Γ_λ.

Note that Γ contains a 1-dimensional subspace of sections, invariant under the action of the nilpotent subgroup $N^-G^{\mathbb{C}}$. Indeed, it follows from the representation (7.18) in Sec. 7.6 that $\Omega_T G$ contains a dense open orbit, containing the origin $o \in \Omega_T G$, which can be identified with the subgroup $N^-G^{\mathbb{C}}$. Hence, any $N^-G^{\mathbb{C}}$-invariant section in Γ is uniquely determined by its value at o. We take for the vacuum the lowest weight vector $v = v_\lambda$, which is an $N^-G^{\mathbb{C}}$-invariant section in Γ, equal to 1 at the origin o.

There is a Hermitian inner product, defined on a dense subspace of Γ. Namely, consider the anti-dual space $\overline{\Gamma}^*$ and introduce a complex-linear map $\beta : \overline{\Gamma}^* \to \Gamma$, which value on the element $\xi \in \overline{\Gamma}^*$ is a section $\beta(\xi) \in \Gamma$, identified with the function $\dot{\beta}(\xi)$ on $\widetilde{LG^{\mathbb{C}}}$, defined by

$$\dot{\beta}(\xi)(\gamma) := \xi(\gamma \cdot v) \quad \text{for } \gamma \in \widetilde{LG^{\mathbb{C}}} \ .$$

Using this map, we define a Hermitian inner product of two elements $\xi, \eta \in \overline{\Gamma}^*$ by

$$< \xi, \eta > := \eta\left(\overline{\beta(\xi)}\right) \ .$$

The constructed inner product on $\overline{\Gamma}^*$ is positive definite and we denote by $H = H_\lambda$ the completion of $\overline{\Gamma}^*$ with respect to this inner product, so that $\overline{\Gamma}^* \subset H \subset \Gamma$. The space H plays the role of the Fock space, associated with the lowest weight λ.

The elements ϵ_γ of $\overline{\Gamma}^*$ with $\gamma \in \widetilde{LG}$, defined by

$$\epsilon_\gamma(s) := \overline{\dot{s}(\bar{\gamma}^{-1})} \ , \quad s \in \Gamma \ ,$$

play the role of the coherent states. They have the inner product, equal to

$$< \epsilon_{\gamma_1}, \epsilon_{\gamma_2} >= \upsilon(\gamma_2 \bar{\gamma}_1^{-1}) \ ,$$

and generate a dense subset in $\overline{\Gamma}^*$.

16.3 Twistor quantization of $\Omega_T G$

There are two different approaches to the geometric quantization of the loop space $\Omega_T G$. One method is to replace the original classical system $(\Omega_T G, \widehat{L\mathfrak{g}} \rtimes \mathrm{vir})$ by an enlarged system. One can do it by enlarging first the phase space $\Omega_T G$ to the Sobolev space HG of half-differentiable loops in G (cf. Sec. 9.1), and then embedding HG into the space $VG := H^{1/2}(S^1, \mathrm{GL}\,(V))$, using a faithful representation V of the group G. Accordingly, the algebra of observables $\widehat{L\mathfrak{g}} \rtimes \mathrm{vir}$ should be enlarged to an algebra $\mathcal{A}$, which is an extension of the semi-direct product of the algebra $H\mathfrak{g}$, embedded into $V\mathfrak{g} := H^{1/2}(S^1, \mathrm{End}(V))$, and the Lie algebra of the symplectic Hilbert–Schmidt group $\mathrm{Sp}_{\mathrm{HS}}(V)$, acting on VG and $V\mathfrak{g}$ by change of variables. We obtain the quantization of the original system by first quantizing the enlarged system and then by restricting this quantization to the original system. The described method was used in Ch. 15 for the quantization of $\Omega\mathbb{R}^d$. In this Chapter we follow a more direct approach, based on the Goodman–Wallach construction of a projective action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ on representations of the affine algebra $\widehat{L\mathfrak{g}^{\mathbb{C}}}$ and affine group $\widehat{LG^{\mathbb{C}}}$.

16.3.1 Projective representation of Vect(S^1)

The projective action of $\mathrm{Diff}_+(S^1)$, mentioned in the introduction to this Section, can be generated by exponentiating a projective representation of the Lie algebra $\mathrm{Vect}(S^1)$, constructed in this Subsection.

Choose an orthonormal base $\{e_\alpha\}$, $\alpha = 1, \dots, N$, of the Lie algebra $\mathfrak{g}$ with respect to an invariant inner product $< \cdot, \cdot >$ on $\mathfrak{g}$. Then the elements

$$e_\alpha(n) := e_\alpha z^n \ , \quad z = e^{i\theta},\ \alpha = 1, \dots, N,\ n \in \mathbb{Z} \ ,$$

form a basis in the vector space $L\mathfrak{g}^{\mathbb{C}}$.

Introduce for $k \in \mathbb{Z}$ the operators, given by the formal series

$$\Delta_k := \frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{\alpha=1}^{N} :\ e_\alpha(n)\, e_\alpha(k-n))\ : \ ,$$

where the *normally ordered product* $: \cdot :$ is defined by the rule

$$: e(m)e(n) \; := \begin{cases} e(m)e(n) & \text{for } m \le n, \\ e(n)e(m) & \text{for } m > n \; . \end{cases}$$

The operators Δ_k are correctly defined, when applied to any element $v \in V$, since in this case the series reduces to a finite sum (cf. [23]). In other words, the operators Δ_k determine endomorphisms of V. The operator Δ_k is homogeneous of order k with respect to the action of the operator e_0 in the sense that

$$e_0 \Delta_k v = \Delta_k (e_0 + k) v \quad \text{for any } v \in V \; .$$

Moreover, for any $\xi \in \mathfrak{g}^{\mathbb{C}}$ and any $n \in \mathbb{Z}$ the following relation between operators on V holds

$$[\xi(n), \Delta_k] = n \left(c + \frac{1}{2} \right) \xi(n+m) \; .$$

Given a $\lambda \in (\widehat{L\mathfrak{h}^{\mathbb{C}}})^*$, denote by λ_0 its restriction to the Cartan subalgebra $\mathfrak{h}$, and set $\rho = \sum_{j=1}^{l} \omega_j$. Then we have the following

Proposition 30. *([26]) The operators* $\Delta_0 + \left(c + \frac{1}{2}\right) e_0$ *and*

$$[\Delta_m, \Delta_n] + \left(c + \frac{1}{2} \right) (n-m) \Delta_{m+n}$$

commute with the action of $\widetilde{L\mathfrak{g}^{\mathbb{C}}}$ *on* V*. Moreover,*

$$\Delta_0 = -\mu e_0 + \left(\frac{1}{2} < \lambda_0, \lambda_0 + 2\rho > + \mu \lambda(e_0) \right) I \; ,$$
$$[\Delta_m, \Delta_n] = \mu(m-n)\Delta_{m+n} + \delta_{m,-n} \nu m (m^2 - 1) \; ,$$

where $\mu := \lambda(c) + \frac{1}{2}$, $\nu := \frac{\dim \mathfrak{g}}{12} \lambda(c) \mu$.

Using the introduced operators Δ_k, we construct a projective action of $\mathrm{Vect}(S^1)$ on V. More precisely, recall (cf. Sec. 10.1) that the Virasoro algebra vir is a central extension of the Lie algebra $\mathrm{Vect}(S^1)$. As a vector space, $\mathrm{vir} = \mathrm{Vect}(S^1) \oplus \mathbb{R}\kappa$, and the Lie bracket is given by

$$[\xi + s\kappa, \eta + t\kappa] = [\xi, \eta] + \omega(\xi, \eta)\kappa \; ,$$

where $\xi, \eta \in \mathrm{Vect}(S^1)$, $s, t \in \mathbb{R}$, and ω is the Gelfand–Fuks cocycle, defined on the basis elements $\{e_n\}$ by

$$\omega(e_m, e_n) = \delta_{m,-n} \frac{n(n^2-1)}{12} \; .$$

Then the following Theorem is true.

Theorem 13. *([26]) Let $(V,\pi) \equiv (V_\lambda, \pi_\lambda)$ be a highest weight representation of $\widehat{L\mathfrak{g}^{\mathbb{C}}}$ with the dominant integral weight λ. Introduce the operators*

$$D_k := -\frac{1}{\mu}\Delta_k \quad \text{for } k \in \mathbb{Z} \ .$$

Then the representation π of $\widetilde{L\mathfrak{g}^{\mathbb{C}}}$ on V can be extended to a representation $\hat{\pi}$ of the algebra $\widetilde{L\mathfrak{g}^{\mathbb{C}}} \rtimes vir$ on V by setting

$$\hat{\pi}(e_k) = D_k \ , \quad \hat{\pi}(\kappa) = \frac{\dim \mathfrak{g}}{12\mu}\lambda(c)I \ .$$

Moreover, V can be provided with a positive definite Hermitian form, contravariant with respect to $\widetilde{L\mathfrak{g}^{\mathbb{C}}} \rtimes vir$.

The operator $D_0 = \hat{\pi}(e_0)$ from Theor. 13, which is given by the formula

$$D_0 = \pi(e_0) - \lambda(e_0) - \frac{< \lambda_0, \lambda_0 + 2\rho >}{2\lambda(c) + 1} \ ,$$

is diagonalizable on V with eigenvalues

$$\mu_i = -i - \frac{< \lambda_0, \lambda_0 + 2\rho >}{2\lambda(c) + 1} \ , \quad i = 0, 1, \ldots, \ .$$

The eigenspaces of D_0 are finite-dimensional and mutually orthogonal. Denote by T the closure of $I - D_0$, then T is a self-adjoint operator, bounded from below by I and having a compact inverse T^{-1}. So by spectral theorem, all its powers T^t with $t \in \mathbb{R}$ are correctly defined and we can set

$$\|v\|_t := \|T^t v\| \quad \text{for any } v \in V \ .$$

Denote by $H^t \equiv H^t_\lambda$ the completion of $V \equiv V_\lambda$ with respect to the norm $\|\cdot\|_t$ and set

$$H^\infty \equiv H^\infty_\lambda = \bigcap_{t\in\mathbb{R}} H^t_\lambda \ , \quad H^{-\infty} \equiv H^{-\infty}_\lambda = \bigcup_{t\in\mathbb{R}} H^t_\lambda \ .$$

The inner product on H defines a sesquilinear pairing between H^∞ and $H^{-\infty}$, and the operator T^t yields an isomorphism between H^s and H^{t-s}, defining a pairing between them, given by

$$(u, v) := (T^t u, T^{-t} v) \quad \text{for } u \in H^t, v \in H^{-t} \ ,$$

where the inner product on the right is taken in H.

16.3.2 Goodman–Wallach construction

We extend a natural right action of $\mathrm{Diff}_+(S^1)$ on $L\mathfrak{g}^{\mathbb{C}}$ by change of variables to $\widetilde{L\mathfrak{g}^{\mathbb{C}}}$, demanding that $\mathrm{Diff}_+(S^1)$ acts trivially on the central subalgebra in $\widetilde{L\mathfrak{g}^{\mathbb{C}}}$. For $f \in \mathrm{Diff}_+(S^1)$ we denote the action of f on $\widetilde{L\mathfrak{g}^{\mathbb{C}}}$ by: $\xi \mapsto \xi_f$, $\xi \in \widetilde{L\mathfrak{g}^{\mathbb{C}}}$.

Given a highest weight representation $(V,\pi) \equiv (V_\lambda, \pi_\lambda)$ of $\widetilde{L\mathfrak{g}^{\mathbb{C}}}$ we define an action of $f \in \mathrm{Diff}_+(S^1)$ on (V,π) by setting

$$f : \pi \longmapsto \pi_f \ , \quad \text{where } \pi_f(\xi)v := \pi(\xi_f)v$$

for $\xi \in \widetilde{L\mathfrak{g}^{\mathbb{C}}}$, $v \in V$. Note that for $v \in H^\infty$ the image $\pi(\xi_f)v$ is again in H^∞. The main result of [26] asserts that representations π and π_f are unitary equivalent. More precisely, we have the following

Theorem 14. *(Goodman–Wallach [26]) There is a unitary projective action σ of Diff$_+(S^1)$ on $H \equiv H_\lambda$ such that the map*

$$\mathit{Diff}_+(S^1) \times H^n \longrightarrow H^n \ , \quad (f,v) \longmapsto \sigma(f)v \ ,$$

is continuous for any $n \geq 0$, and

$$\sigma(f)\pi_f(\xi)v = \pi(\xi)\sigma(f)v$$

for any $v \in H^\infty$, $f \in$ Diff$_+(S^1)$, $\xi \in \widetilde{L\mathfrak{g}^{\mathbb{C}}}$.

Moreover, in [26] it is proved that this $\mathrm{Diff}_+(S^1)$-action on H is uniquely defined up to projective equivalence. More precisely, suppose that τ is another projective action of $\mathrm{Diff}_+(S^1)$ on H, such that $\tau_f H^\infty \subset H^\infty$ for any $f \in \mathrm{Diff}_+(S^1)$, which intertwines π with π_f, i.e.

$$\tau_f \pi_f(\xi) = \pi(\xi)\tau_f$$

for any $f \in \mathrm{Diff}_+(S^1)$, $\xi \in \widetilde{L\mathfrak{g}^{\mathbb{C}}}$. Then there exists a continuous map $\mu : \mathrm{Diff}_+(S^1) \to S^1$, such that $\tau_f = \mu(f)\sigma_f$.

16.3.3 Twistor quantization of $\Omega_T G$

In Subsec. 16.2.2 we have constructed for any lowest weight λ of the loop algebra $\widetilde{L\mathfrak{g}}$ a holomorphic line bundle $L \equiv L_\lambda \to \Omega_T G$ and the space $\Gamma \equiv \Gamma_\lambda$ of its holomorphic sections, on which the representation of $\widetilde{LG}$ with lowest weight λ is realized. We denoted by $H \equiv H_\lambda$ the completion of $\overline{\Gamma}^*$ with respect to the natural norm on $\overline{\Gamma}^*$.

This construction depends on the complex structure on $\Omega_T G$, which is provided by the complex representation

$$\Omega_T G = LG^{\mathbb{C}}/B^+G^{\mathbb{C}} \ .$$

Denote this complex structure by J^0 and the corresponding spaces of sections Γ_λ and H_λ respectively by Γ_0 and H_0, so that we have a representation π_0 of $\widetilde{LG}$ in Γ_0.

If we change this complex structure to J_f by the action of a diffeomorphism $f \in \mathrm{Diff}_+(S^1)$, then we can again, using the Borel–Weil construction, realize the lowest weight representation π_f of the group $\widetilde{LG}$, corresponding to the lowest weight λ, in the space Γ_f of sections of L, holomorphic with respect to the complex structure J_f on $\Omega_T G$. Denote the corresponding completion of $\overline{\Gamma}^*_f$ by H_f.

By the Goodman–Wallach construction, there is a projective unitary action

$$U_f : \Gamma_0 \longrightarrow \Gamma_f$$

of the group $\text{Diff}_+(S^1)$, intertwining the representations π_0 and π_f:

$$\pi_f U_f(v) = U_f \pi_0(v) \quad \text{for } v \in H_0 \ .$$

It is uniquely defined by the normalization condition on the lowest weight vectors: $U_f v_0 = v_f$, and defines a continuous unitary operator

$$U_f : H^0 \longrightarrow H^f \ .$$

So we have again, as in Sec. 15.4, a holomorphic Hilbert space bundle

$$H = \bigcup_{f \in \mathcal{S}} H_f$$

and a projective unitary action of the group $\text{Diff}_+(S^1)$ on H, given by $f \mapsto U_f$, which covers the natural $\text{Diff}_+(S^1)$-action on $\mathcal{S}$. The infinitesimalization of this action yields a projective unitary representation ρ of lowest weight λ of the Lie algebra $\text{Vect}(S^1)$ in the space H_0, constructed in Subsec. 16.3.1.

Having a projective representation ρ of $\text{Vect}(S^1)$, we can construct a $\text{Diff}_+(S^1)$-invariant connection $\mathbf{A}$ on the bundle $H \to \mathcal{S}$, whose curvature at the origin $o \in \mathcal{S}$ coincides with the cocycle of ρ, given in the basis $\{e_k\}$ by (cf. [53, 54])

$$[\rho(e_m), \rho(e_n)] - \rho([e_m, e_n]) = \frac{c(\mathfrak{g})}{12}(m^3 - m)\delta_{m,-n} \ ,$$

where

$$c(\mathfrak{g}) = \frac{h \dim \mathfrak{g}}{h + \kappa(\mathfrak{g})} \ ,$$

and $\kappa(\mathfrak{g})$ is the *dual Coxeter number* of $\mathfrak{g}$ (cf., e.g., [77]).

The construction of the connection $\mathbf{A}$ is similar to that in Subsec. 15.4.2. Namely, we have again a splitting of the Lie algebra $\text{Vect}(S^1)$ into the direct sum

$$\text{Vect}(S^1) = \text{sl}(2, \mathbb{R}) \oplus \mathfrak{m} \ ,$$

where $\text{sl}(2, \mathbb{R})$ is the Lie algebra of $\text{Möb}(S^1)$ and $\mathfrak{m} \cong T_0\mathcal{S}$. This splitting is, in fact, induced by the splitting (15.11) from Subsec. 15.4.2, under the embedding of $\text{Vect}(S^1)$ into $\text{sp}_{\text{HS}}(H_0)$. The above splitting together with the projective representation $\rho : \text{Vect}(S^1) \to \text{End}\,(H_0)$ determine, as in Subsec. 15.4.2, a $\text{Diff}_+(S^1)$-invariant connection $\mathbf{A}$ on the bundle $H \to \mathcal{S}$, whose curvature at the origin $o \in \mathcal{S}$ coincides with the cocycle of ρ.

Consider now, as in Sec. 15.5.2, the quantization bundle

$$\mathcal{H} := H \otimes K^{-1/2} \to \mathcal{S}$$

and provide it with the tensor-product connection $\mathbf{C}$:

$$\mathbf{C} := \mathbf{A} \otimes 1 + 1 \otimes \mathbf{B} \ ,$$

where $\mathbf{B}$ is the connection on $K^{-1/2}$, defined in Subsec. 15.5.1. The curvature of $\mathbf{C}$ in the basis $\{e_k\}$ is equal to

$$R_{\mathbf{C}}(e_m, e_n) = \frac{c(\mathfrak{g}) - 26}{12}(m^3 - m)\delta_{m,-n} \ ,$$

which vanishes precisely for $c(\mathfrak{g}) = 26$. Under this condition we get a flat unitary connection on $\mathcal{H}$. By integrating it, we obtain a unitary action of $\text{Diff}_+(S^1)$ on $\mathcal{H}$, yielding the geometric quantization of the system $(\Omega_T G, \mathcal{A})$ in H_0.

Bibliographic comments

In Sec. 16.1 we follow mostly the papers [23, 26]. The Borel–Weil construction of the lowest weight representations of the loop group is explained in Chap. 11 of Pressley–Segal's book [65]. The projective action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ on the lowest weight representations of the loop algebra is studied in detail in Goodman–Wallach's paper [26]. Its infinitesimal version, i.e. the projective representation of the Lie algebra $\mathrm{Vect}(S^1)$, given by operators D_k from Subsec. 16.3.1, is well known and may be found, for example, in the books [38],[65]. The geometric quantization of loop spaces of compact Lie groups was first considered by Mickelsson [53, 54].