Part IV

QUANTIZATION OF FINITE-DIMENSIONAL KÄHLER MANIFOLDS

Chapter 12

Dirac quantization

This Chapter is devoted to the Dirac definition of the geometric quantization of classical mechanical systems. In Sec. 12.1 we recall the notion of classical systems from Hamiltonian mechanics. The geometric quantization of such systems is defined in Sec. 12.2.

12.1 Classical systems

We start from the definition of a classical (mechanical) system — an object to be quantized. A *classical (mechanical) system* is given by a pair $(M, \mathcal{A})$, consisting of the *phase space* M of the system and the *algebra of observables (Hamiltonians)* $\mathcal{A}$.

12.1.1 Phase spaces

Mathematically, the *phase manifold* M is a smooth symplectic manifold of an even dimension $2n$, provided with a symplectic 2-form ω. Locally, it is diffeomorphic (and, in fact, symplectomorphic) to the *standard model* $M_0 := (\mathbb{R}^{2n}, \omega_0)$, where ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$. In the conventional coordinates (p_i, q_i), $i = 1, \dots, n$, on $\mathbb{R}^{2n}$ this form is given by the expression

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i \ .$$

The corresponding local coordinates (p_i, q_i), $i = 1, \dots, n$, on M, in which the symplectic form ω takes on the above standard form, are called *canonical*. The coordinates q_i are interpreted as physical "coordinates", while p_i correspond to physical "momenta".

The standard examples of phase spaces, apart from the standard model $M_0 = (\mathbb{R}^{2n}, \omega_0)$, are given by the cotangent bundles and coadjoint orbits of Lie groups.

Example 30 (cotangent bundles)**.** Denote by M the cotangent bundle T^*N of a smooth n-dimensional manifold N, called the *configuration space*. Local canonical coordinates (p_i, q_i) on M have the following meaning: $q := (q_1, \dots, q_n)$ are local coordinates on N, and $p := (p_1, \dots, p_n)$ are coordinates in the fibre T_qN. A symplectic

2-form ω, given in the introduced local coordinates by the standard formula

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i \ ,$$

is a correctly defined (global) 2-form on M, as well as a 1-form θ, given in local coordinates by the expression

$$\theta = \sum_{i=1}^{n} p_i dq_i \ .$$

It follows that $\omega = d\theta$, that is ω in this case is exact. To show that θ is a correctly defined (global) 1-form, we note that it can be also defined in an invariant way. Namely, for any $p \in T_q^*N$ and any tangent vector $\xi \in T_{(p,q)}(T^*N)$ it can be given by

$$\theta(\xi) = p(\pi_*\xi) \ ,$$

where $\pi_* : T(T^*N) \to TN$ is the map, tangent to the projection $\pi : T^*N \to N$.

Example 31 (coadjoint orbits)**.** Consider the coadjoint representation of a Lie group G on the dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$ of G. It is given by the formula

$$K : G \longrightarrow \operatorname{End} \mathfrak{g}^* \ , \quad g \longmapsto \left(\operatorname{Ad} g^{-1}\right)^* \ .$$

The orbits of this action (when they are smooth) are symplectic manifolds with the symplectic structure, given by the *Kirillov form*, defined in the following way. Denote by ξ_* the vector field on $\mathfrak{g}^*$, generated by $\xi \in \mathfrak{g}$ via the coadjoint action K. More precisely,

$$\xi_*(x) = K_*(\xi)x \quad \text{for } x \in \mathfrak{g}^* \ ,$$

where $K_* : \mathfrak{g} \to \operatorname{End} \mathfrak{g}^*$ denotes the differential of $K : G \to \operatorname{End} \mathfrak{g}^*$. Then the Kirillov form is defined by the equality

$$\omega(\xi_*(x), \eta_*(x)) := x\big([\xi, \eta]\big) \quad \text{for } \xi, \eta \in \mathfrak{g},\ x \in \mathfrak{g}^* \ .$$

The restriction of this 2-form to a smooth K-orbit defines a symplectic structure on this orbit.

12.1.2 Algebras of observables

An *algebra of observables* $\mathcal{A}$, mathematically, is an arbitrary Lie subalgebra of the Poisson Lie algebra $C^\infty(M, \mathbb{R})$ of smooth real-valued functions on the phase space M with respect to the Poisson bracket, determined by the symplectic 2-form ω.

Recall the definition of this bracket. Given a smooth function $h \in C^\infty(M, \mathbb{R})$, denote by X_h the Hamiltonian vector field on M, associated with h. It is determined by the following relation

$$dh(\xi) = \omega(X_h, \xi) \ ,$$

fulfilled for any tangent vector field ξ on M. Then the Poisson bracket $\{f, g\}$ of two functions $f, g \in C^\infty(M, \mathbb{R})$ is uniquely defined by the relation

$$X_{\{f,g\}} = [X_f, X_g] \ .$$

Example 32 (Heisenberg algebra)**.** In the case of the standard model $M_0 = (\mathbb{R}^{2n}, \omega_0)$ we can take for the algebra of observables $\mathcal{A}$ the *Heisenberg algebra* $\mathrm{heis}(\mathbb{R}^{2n})$. It is the Lie algebra, generated by the coordinate functions p_i, q_i, $i = 1, \ldots, n$ and 1, satisfying the following commutation relations

$$\{p_i, p_j\} = \{q_i, q_j\} = 0 \ ,$$
$$\{p_i, q_j\} = \delta_{ij} \quad \text{for } i, j = 1, \ldots, n \ .$$

We consider $\mathrm{heis}(\mathbb{R}^{2n})$ as a "minimal" algebra of observables on M_0. The opposite extreme is the Poisson algebra $C^\infty(M_0, \mathbb{R})$. The Hamiltonian vector field X_f, corresponding to an observable $f \in C^\infty(M_0, \mathbb{R})$, is given in standard coordinates (p_i, q_i) on M_0 by the formula

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right) \ .$$

In particular, $X_{p_i} = \frac{\partial}{\partial q_i}$, $X_{q_i} = -\frac{\partial}{\partial p_i}$. The Poisson bracket on M_0 is given by the expression

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)$$

for $f, g \in C^\infty(M_0, \mathbb{R})$.

Example 33 (Hamiltonian algebra)**.** Let Γ be a Lie group of symplectomorphisms, acting on a phase space M, so that its Lie algebra $\mathrm{Lie}(\Gamma)$ can be regarded as a subalgebra of the Lie algebra of Hamiltonian vector fields on M. If M is simply connected, then $\mathrm{Lie}(\Gamma)$ may be also considered, in the dual way, as a subalgebra of the Poisson algebra $C^\infty(M, \mathbb{R})$. Namely, it can be identified with the algebra $\mathrm{Ham}(\Gamma)$ of Hamiltonians (smooth real functions) on M, generating symplectomorphisms from Γ.

If a Lie group Γ acts on M transitively, such a manifold M is called a *homogeneous symplectic* Γ*-manifold*. It is proved in [46] that any homogeneous symplectic Γ-manifold M is locally equivariantly symplectomorphic to a coadjoint orbit of Γ or its central extension $\tilde{\Gamma}$.

12.2 Quantization of classical systems

Definition 39. Let $(M, \mathcal{A})$ be a classical system. The *Dirac quantization* of $(M, \mathcal{A})$ is given by an irreducible Lie-algebra representation

$$r : \mathcal{A} \longrightarrow \mathrm{End}^* H$$

of the algebra of observables $\mathcal{A}$ in the algebra $\mathrm{End}^* H$ of linear self-adjoint operators, acting in a complex (separable) Hilbert space H, called the *quantization space*. The algebra $\mathrm{End}^* H$ is provided with the Lie bracket, given by the commutator of linear operators of the form

$$\frac{\hbar}{i}[A, B] = \frac{\hbar}{i}(AB - BA) \ .$$

In other words, it is required that

$$r(\{f,g\}) = \frac{\hbar}{i}\left(r(f)r(g) - r(g)r(f)\right)$$

for any $f, g \in \mathcal{A}$. We also assume the following normalization condition:

$$r(1) = \mathrm{id}\ .$$

If a representation r satisfies all these conditions, except for the irreducubility, it is called a *prequantization* of the system $(M, \mathcal{A})$.

We set $\hbar = 1$ in the sequel for the convenience.

Remark 19. Sometimes it is useful to deal with the *complexified algebra of observables* $\mathcal{A}^{\mathbb{C}}$ instead of $\mathcal{A}$. Its Dirac quantization is given by an irreducible Lie-algebra representation

$$r : \mathcal{A}^{\mathbb{C}} \longrightarrow \mathrm{End}\, H\ ,$$

satisfying the normalization condition and the *conjugation law*

$$r(\bar{f}) = r(f)^* \quad \text{for any } f \in \mathcal{A}\ .$$

In other words, the complex conjugation in $\mathcal{A}^{\mathbb{C}}$ should correspond to the Hermitian conjugation in $\mathrm{End}\, H$.

Remark 20. The quantization operators $r(f)$ in the Dirac definition are usually unbounded. In that case we require that all operators $r(f)$ for $f \in \mathcal{A}$ (or $f \in \mathcal{A}^{\mathbb{C}}$ in the complexified version) are densely defined and, moreover, have a common dense domain of definition in H.

Bibliographic comments

The Dirac definition of geometric quantization of classical systems is presented (with minor modifications) in all books on geometric quantization. A reader may look for a more detailed exposition [29, 37, 42, 71, 80].

Chapter 13

Kostant–Souriau prequantization

It is difficult (and, often, not possible) to construct the Dirac quantization, defined in the previous Chapter, for realistic classical systems. However, there exists a quite general prequantization construction, due to Kostant and Souriau, which is valid for a large class of phase spaces and the "maximal" algebra of observables $\mathcal{A} = C^\infty(M, \mathbb{R})$. We describe it in this Chapter, starting from the simple case of the cotangent bundle.

13.1 Prequantization of the cotangent bundle

Let N be a smooth n-dimensional manifold and $M = T^*N$ denotes its cotangent bundle. Recall (cf. Ex. 30) that the symplectic form ω on T^*N is given by the formula $\omega = d\theta$, where θ is a canonically defined 1-form on M with the local expression $\theta = \sum_{i=1}^n p_i dq_i$. We take for an algebra of observables $\mathcal{A}$ of our system the Poisson algebra $C^\infty(M, \mathbb{R})$ and for the Hilbert prequantization space H the space

$$H = L^2(M, \omega^n)$$

of square integrable functions on M with respect to the Liouville measure, given by ω^n. A representation of $\mathcal{A} = C^\infty(M, \mathbb{R})$ in H is given by the following formula

$$r : f \longmapsto r(f) = f - iX_f - \theta(X_f) \ , \tag{13.1}$$

where $f - \theta(X_f)$ is considered as the multiplication operator on H. Note that this operator, as well as the Hamiltonian vector field X_f, are correctly defined on the subspace $C_0^\infty(M, \mathbb{R})$ of $C^\infty(M, \mathbb{R})$, consisting of smooth functions with compact supports on M.

In particular, for the standard model $N = \mathbb{R}^n$, $M = T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ the representation (13.1) acts on the coordinate functions in the following way

$$r(p_j) = p_j - iX_{p_j} - \theta(X_{p_j}) = p_j - i\frac{\partial}{\partial q_j} - p_j = -i\frac{\partial}{\partial q_j} \ , \tag{13.2}$$

$$r(q_j) = q_j - iX_{q_j} - \theta(X_{q_j}) = q_j - i\left(-\frac{\partial}{\partial p_j}\right) = q_j + i\frac{\partial}{\partial p_j} \ , \tag{13.3}$$

since $X_{p_j} = \partial/\partial q_j$, $X_{q_j} = -\partial/\partial p_j$. Note that this representation is reducible, even if we restrict it to the "minimal" Heisenberg algebra heis$(\mathbb{R}^{2n})$. Indeed, the operators $i\frac{\partial}{\partial p_j}$ and $p_j + i\frac{\partial}{\partial q_j}$ commute with all operators $r(p_j)$, $r(q_j)$, being non scalar. However, we can make the representation of heis$(\mathbb{R}^{2n})$, defined by the above formulas (13.2),(13.3), irreducible by restricting it to the subspace of H, consisting of functions, depending only on (q_j). Then the representation (13.2),(13.3) will reduce to the well known *Heisenberg representation* of heis$(\mathbb{R}^{2n})$ in the space $H_{(q)} := L^2(\mathbb{R}^n_{(q)}, d^n q)$, given by

$$r(p_j) = -i\frac{\partial}{\partial q_j} \ , \quad r(q_j) = q_j \ .$$

We can also construct a *dual Heisenberg representation* of heis$(\mathbb{R}^{2n})$ in the space $H_{(p)} := L^2(\mathbb{R}^n_{(p)}, d^n p)$, given by

$$r(p_j) = p_j \ , \quad r(q_j) = i\frac{\partial}{\partial p_j} \ .$$

Remark 21. The "physical" explanation of the reducibility of the representation

$$r : \text{heis}(\mathbb{R}^{2n}) \longrightarrow \text{End}^* H \ ,$$

given by (13.1), is that, according to the Heisenberg uncertainty principle, the "physical" quantization space cannot contain the functions, depending on some pair of variables (p_j, q_j) simultaneously, as it occurs in the space $H = L^2(M, \omega^n)$.

13.2 Kostant–Souriau (KS) prequantization

13.2.1 Prequantization map

Suppose now that M is a general smooth symplectic manifold of dimension $2n$ with symplectic form ω. Take the Poisson algebra $C^\infty(M, \mathbb{R})$ as the algebra of observables. We are going to quantize the classical system, represented by the pair $(M, C^\infty(M, \mathbb{R}))$.

Let us begin with some heuristic considerations. Note that the symplectic 2-form ω, being closed, is locally exact, so we can find an open covering $\{U_\alpha\}$ of M, such that

$$\omega = d\theta_\alpha \quad \text{on } U_\alpha$$

for some smooth 1-forms θ_α, defined on U_α. Using these local forms θ_α, we can apply the idea, described in the previous Section 13.1, to construct local representation operators r_α in the spaces $L^2(U_\alpha, \omega^n)$ by the formula (13.1) with $\theta = \theta_\alpha$. It turns out that (under some topological restrictions) we can combine these local representation operators r_α into a unique operator r, which acts, however, not on functions, but on sections of a certain complex line bundle L over M. The structure of this line bundle $L \to M$ is, in fact, prescribed by the local formulas (13.1) with $\theta = \theta_\alpha$. Namely, the local expressions $X^\alpha - i\theta_\alpha(X^\alpha)$ (with X^α being a vector field on U_α) in the right hand sides of the local formulas (13.1) look like local expressions for the covariant derivative of a connection in a line bundle over M. If these expressions do arise from some connection ∇ on a line bundle $L \to M$ (i.e. if they match together on intersections $U_\alpha \cap U_\beta$ up to gauge transformations, given by the transition functions of L), then the local representation operators $r_\alpha(f)$ in the spaces $L^2(U_\alpha, \omega^n)$ will match into a global representation operator

$$r : f \longmapsto f - i\nabla_{X_f} \ , \quad f \in C^\infty(M, \mathbb{R}) \ ,$$

acting on sections of $L \to M$. In this case the curvature of such a connection would be equal to ω. In particular, the 2-form $\frac{1}{2\pi}\omega$, representing the first Chern class $c_1(L)$, should be integral, i.e.

$$\left[\frac{1}{2\pi}\omega\right] \in H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R}) \ .$$

From Sec. 8.1 we know that the integrality of $[\frac{1}{2\pi}\omega]$ is not only necessary, but also sufficient for the existence of a line bundle $L \to M$ with a connection ∇. Namely, rephrasing Prop. 15, we have the following

Proposition 29. *Suppose that the manifold M satisfies the following quantization condition: the cohomology class*

$$\left[\frac{1}{2\pi}\omega\right] \quad \textit{is integral in } H^2(M, \mathbb{R}) \ . \tag{13.4}$$

Then there exists a Hermitian line bundle $L \to M$, called the prequantization bundle*, having a Hermitian connection ∇, whose curvature is equal to ω.*

Proof. The only new assertion in this Proposition, compared to Prop. 15, is the Hermiticity of the connection ∇. Recall (cf. Rem. 16) that under the integrality condition (13.4) there exists a complex line bundle $L \to M$, such that $c_1(L) = [\omega/2\pi]$. We take now an arbitrary Hermitian metric and a Hermitian connection ∇' on L. Note that the curvature ω' of ∇' also represents the class $c_1(L)$. Hence,

$$\omega = 2\pi\omega' + d\beta$$

for some 1-form $\beta \in \Omega^1(M, \mathbb{R})$. If the connection ∇' is represented by a 1-form α', we introduce a connection ∇ on L, represented by the 1-form

$$\alpha = 2\pi\alpha' - i\beta \ .$$

This connection is Hermitian and its curvature is equal to ω. □

The Prop. 29 allows us to realize the scheme, described in the beginning of this Section. Namely, suppose that our phase space M satisfies the quantization condition, so that the assertion of Prop. 29 holds. In other words, there exists a Hermitian line bundle $L \to M$ together with a Hermitian connection ∇, having the curvature, equal to ω. We take for the algebra of observables the Poisson algebra $\mathcal{A} = C^\infty(M, \mathbb{R})$ and define the prequantization space as

$$H = L^2(M, L; \omega^n) \ ,$$

i.e. the Hilbert space of square integrable sections of $L \to M$ with respect to the inner product, given by

$$(s_1, s_2)_H := \int_M < s_1(x), s_2(x) > \omega^n \ ,$$

where $< s_1(x), s_2(x) >$ is the Hermitian product of sections s_1, s_2 of L at $x \in M$. Then the *Kostant-Souriau (KS) prequantization* of the algebra $\mathcal{A}$ in H will be given by the formula

$$r_{\mathrm{KS}} : \mathcal{A} \ni f \longmapsto r(f) = f - i\nabla_{X_f} \ . \tag{13.5}$$

It's easy to check directly (cf. also [29, 37, 42, 71, 73, 80]) that the formula (13.5) defines a representation of the algebra $\mathcal{A} = C^\infty(M, \mathbb{R})$ in the prequantization space H.

Remark 22. There is another interpretation of the Kostant–Souriau operator r_{KS} in terms of the automorphism group $\tilde{\mathcal{G}}$ of the prequantization bundle (L, ∇). An *automorphism* of (L, ∇) is a pair (φ, g), where $\varphi : L \to L$ is a fibrewise isomorphism, preserving the Hermitian metric on L and the connection ∇ (i.e. $\varphi^*\nabla = \nabla$). The projection of φ to M is a symplectomorphism $g : M \to M$, belonging to the group $\mathcal{G}$ of all symplectomorphisms of M. In other words, we have a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\ \varphi\ } & L \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow[\ g\ ]{} & M \end{array} \ .$$

According to Prop. 16, the automorphism group $\tilde{\mathcal{G}}$ of the prequantization bundle (L, ∇) can be identified with a central extension of the symplectomorphism group $\mathcal{G}$ by S^1, i.e. there is an exact sequence

$$1 \longrightarrow S^1 \longrightarrow \tilde{\mathcal{G}} \longrightarrow \mathcal{G} \longrightarrow 1 \ .$$

Note that (assuming that M is simply connected) the Lie algebra $\operatorname{Lie}\mathcal{G}$ of the group $\mathcal{G}$ can be identified with the Lie algebra of Hamiltonian vector fields on M, generated by Hamiltonians $f \in C^\infty(M, \mathbb{R})$, so that that the Lie algebra $\operatorname{Lie}\tilde{\mathcal{G}}$ of the group $\tilde{\mathcal{G}}$ is a central extension of $\operatorname{Lie}\mathcal{G}$ by $\mathbb{R}$.

The action of the symplectomorphism group $\mathcal{G}$ on M generates an action of its central extension $\tilde{\mathcal{G}}$ on L. Namely, if an action g on M is generated by a Hamiltonian vector field X_f with $f \in C^\infty(M, \mathbb{R}) = \operatorname{Lie}\mathcal{G}$, then the corresponding action $\varphi : C^\infty(M, L) \to C^\infty(M, L)$ on the space of sections of L is generated by

$$\tilde{X}_f(s) := fs - i\nabla_{X_f} s \ . \tag{13.6}$$

Remark 23. In conclusion of this Subsection, we give a description of the $\mathbb{C}^*$-bundle $\dot{L} \to M$, associated with the prequantization bundle $L \to M$. It is sometimes more convenient to use for computations this bundle, rather than $L \to M$. Denote by $\pi : \dot{L} \to M$ the bundle, obtained from the prequantization bundle $\pi : L \to M$ by deleting its zero section. It is a principal $\mathbb{C}^*$-bundle, associated with the line bundle

$\pi : L \to M$. The space $\Gamma(L) := C^\infty(M, L)$ of sections s of $L \to M$ can be identified with the space $\dot{\Gamma}(L)$ of complex-valued functions $\dot{s}$ on $\dot{L}$, subject to the condition

$$\dot{s}(zp) = \frac{1}{z}\dot{s}(p)$$

for any $p \in \dot{L}$ and any $z \in \mathbb{C}^*$. The correspondence between sections s of $L \to M$ and functions $\dot{s}$ on $\dot{L} \to M$ is established via the relation

$$s(\pi(p)) = \dot{s}(p)p \quad \text{for any } p \in \dot{L} \ .$$

Note that if a section s of $L \to M$ is non-vanishing at some point $x \in M$: $s(x) \neq 0$, then $s(x) \in \dot{L}$ and, applying the above relation for $p = s(x)$, we obtain that $s(x) = \dot{s}(s(x))s(x)$, i.e. $\dot{s} \circ s = 1$ at any point $x \in M$, where $s(x) \neq 0$.

We can introduce a connection $\dot{\nabla}$ on $\dot{L} \to M$, associated with the connection ∇ on $L \to M$. In terms of the local representatives θ_α of the connection ∇, the local representatives $\dot{\theta}_\alpha$ of $\dot{\nabla}$ are given by

$$\dot{\theta}_\alpha = \theta_\alpha + i\frac{dz}{z}$$

on $U_\alpha \times \mathbb{C}^*$. It's easy to check that these local forms define a global 1-form, which is the connection form of $\dot{\nabla}$. This connection generates the horizontal lifting of vector fields on M. Let ξ be such a vector field, then its horizontal lift is a vector field $\dot{\xi}$ on $\dot{L}$, such that $\pi_*(\dot{\xi}) = \xi$ and $\dot{\nabla}(\dot{\xi}) = 0$. A correspondence $\xi \leftrightarrow \dot{\xi}$ between vector fields ξ on M and their horizontal lifts $\dot{\xi}$ on $\dot{L}$ has the following properties

$$(\nabla_\xi s)^{\cdot} = \dot{\xi} \ , \quad (fs)^{\cdot} = f\dot{s}$$

for any vector field ξ on M, section s of L and function $f \in C^\infty(M, \mathbb{R})$.

We can also give an interpretation of the generator (13.6) in terms of the bundle $\dot{L}$ (cf. [73]). Given a Hamiltonian $f \in C^\infty(M, \mathbb{R})$, we define a vector field η_f on $\dot{L}$ by local representatives

$$\eta_{f,\alpha} := X_f + (\theta_\alpha(X_f) - f)\,\frac{\partial}{\partial\vartheta}$$

on $U_\alpha \times \mathbb{C}^*$. Here the vector field $\frac{\partial}{\partial\vartheta}$ is the differentiation with respect to the angle coordinate ϑ in the polar representation of the coordinate $z = re^{i\vartheta}$ on $\mathbb{C}^*$. It follows from this definition that the generator (13.6) can be written in terms of $\dot{L}$ as

$$\tilde{X}_f(s) = -i\eta_f\dot{s} \ . \tag{13.7}$$

Remark 24 (cf. [73]). Using the vector field η_f, introduced in Rem. 23, one can prove that the KS-operator $r_{\text{KS}}(f)$, given by the formula (13.5), is self-adjoint under the assumption that the Hamiltonian vector field X_f is complete. (In this case the vector field η_f is complete too.) Denote by $\dot{\varphi}^t_f$ the 1-parameter group of transformations of $\dot{L}$, generated by the vector field η_f. Consider the 1-parameter unitary group of transformations of $\dot{\Gamma}(L)$ (with respect to the inner product, induced from $\Gamma(L)$), generated by $\dot{\varphi}^t_f$. It acts by the formula: $\dot{s} \mapsto \dot{s} \circ \dot{\varphi}^t_f$ for $\dot{s} \in \dot{\Gamma}(L)$. The operator $r_{\text{KS}}(f)$, given by (13.5), coincides with the generator of this unitary group, according to (13.7). Hence, it is self-adjoint by Stone's theorem.

13.2.2 Polarizations

The representation of the algebra $\mathcal{A} = C^\infty(M, \mathbb{R})$ in the prequantization space H, defined by (13.5), is reducible by the same reasons, as in Sec. 13.1. According to the Heisenberg uncertainty principle, we can make this representation irreducible by restricting it to a "half" of the prequantization space H, i.e. to a subspace of H, containing the functions from H, which depend, in terms of the local canonical variables $(p_i, q_i)_{i=1}^n$, only on one variable from each pair (p_j, q_j). This naive idea may be formalized, using the notion of the polarization.

Let (M, ω) be a symplectic manifold of dimension $2n$. We extend its symplectic form ω complex linearly to the complexified tangent bundle $T^{\mathbb{C}}M$.

Definition 40. A *polarization* on M is an integrable involutive Lagrangian distribution P on M. In other words, P is a complex distribution $P : x \mapsto P_x \subset T^{\mathbb{C}}M$ of rank n, satisfying the following conditions: (a) P is involutive, i.e. $[P, P] \subset P$; (b) the restriction of ω to P is identically zero.

For a polarized phase space (M, P), satisfying the quantization condition (13.4), it's natural to choose for the quantization space H the space of *polarized* sections. It is defined as

$$H = L^2_P(M, L; \omega^n) := \{s \in L^2(M, L; \omega^n) : \ \nabla_\xi s = 0 \ \text{ for any } \xi \in P\} .$$

There are two distinguished classes of polarizations.

Example 34. A polarization P on a phase space M is called *real*, if $P = \bar{P}$, where "bar" denotes the complex conjugation in $T^{\mathbb{C}}M$. A standard example of such a polarization is the cotangent bundle $M = T^*N$ of a configuration manifold N with local canonical coordinates (p_i, q_i) and polarization P, given by the subbundle of TM, generated by the vector fields $\{\partial/\partial p_i\}$, $i = 1, \dots, n$. (One can take for P the subbundle of TM, generated by the vector fields $\{\partial/\partial q_i\}$, $i = 1, \dots, n$, as well.) The space $L^2_P(M, L; \omega^n)$ of polarized sections in this case consists of sections from $L^2(M, L; \omega^n)$, which do not depend on momenta $\{p_i\}$.

A polarization P is called *Kähler*, if $P \cap \bar{P} = 0$. To give an example of such a polarization, suppose that our phase space (M, ω) is Kähler, i.e. it is provided with a complex structure J, compatible with ω. Then we take for P the subbundle $T^{0,1}M$ of $(0, 1)$-vector fields in $T^{\mathbb{C}}M$. In this case the prequantization bundle L can be made holomorphic with the holomorphic structure, determined by the $\bar{\partial}$-operator, given by the $(0, 1)$-part $\nabla^{0,1}$ of the connection ∇. The space $L^2_P(M, L; \omega^n)$ of polarized sections for $P = T^{0,1}M$ coincides with the space $L^2_{\mathcal{O}}(M, L; \omega^n)$ of holomorphic sections of $L \to M$.

Given a polarized phase space (M, P), satisfying the quantization condition (13.4), we can hope to obtain an irreducible representation of the algebra of observables $\mathcal{A}$ by restricting the Kostant–Souriau prequantization map to the space $L^2_P(M, L; \omega^n)$ of polarized sections. Unfortunately, this straightforward idea works only for very special phase spaces and algebras of observables, since in most of the cases the space $L^2_P(M, L; \omega^n)$ of polarized sections is not invariant under the action of the Kostant–Souriau representation. In the next Section we shall demonstrate how the idea of restriction to the space of polarized sections can be realized for the flat space $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and the Heisenberg algebra of observables heis$(\mathbb{R}^{2n}) =:$ heis$(\mathbb{C}^n)$. In this case the restriction of Kostant–Souriau representation to the space $L^2_{\mathcal{O}}(\mathbb{C}^n; \omega^n)$ of holomorphic sections yields an irreducible Bargmann–Fock representation of the Heisenberg algebra in $L^2_{\mathcal{O}}(\mathbb{C}^n; \omega^n)$.

Bibliographic comments

The prequantization of the cotangent bundle was known long ago to physisists (cf., e.g., [35]). Its generalization to general manifolds, satisfying the quantization condition, due to B.Kostant and J.-M.Souriau, is presented in all books on geometric quantization (cf. [29, 37, 42, 71, 70, 80]). In these books a more detailed discussion of polarizations may be also found.

Chapter 14

Blattner–Kostant–Sternberg quantization

In this Chapter we present the Blattner–Kostant–Sternberg (BKS) quantization scheme for Kähler manifolds, provided with Kähler polarizations. We start from the simplest example of such a quantization, namely, the Bargmann–Fock quantization of the standard model $(\mathbb{R}^{2n}, \omega_0)$, provided with the Heisenberg algebra of observables. In Secs. 14.2-14.5 we explain how to construct the BKS-quantization of a quantizable Kähler manifold. In Sec. 14.2 we introduce the Fock spaces of half-forms and in Sec. 14.4 define the BKS-pairing between them, using the metaplectic structure, introduced in Sec. 14.3. In Sec. 14.5 we explain how to quantize Kähler phase manifolds, using the BKS-pairing.

14.1 Bargmann–Fock quantization

Let $M_0 = (\mathbb{R}^{2n}, \omega_0)$ be the standard model with standard coordinates (p_j, q_j), $j = 1, \ldots, n$. In these coordinates

$$\omega_0 = \sum_{j=1}^{n} dp_j \wedge dq_j \ ,$$

so that $\omega_0 = d\theta_0$ with $\theta_0 = \sum_{j=1}^{n} p_j \, dq_j$. We identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by introducing complex coordinates

$$z_j = \frac{p_j + iq_j}{\sqrt{2}} \ , \quad \bar{z}_j = \frac{p_j - iq_j}{\sqrt{2}} \ , \quad j = 1, \ldots, n,$$

(following [71], we have replaced the usual factor $1/2$ in these formulas by $1/\sqrt{2}$ to make the expression for KS-representation more symmetric). In these coordinates

$$\omega_0 = -i \sum_{j=1}^{n} d\bar{z}_j \wedge dz_j \ .$$

The Hamiltonian vector fields, corresponding to coordinates $z_j, \bar{z}_j$, have the form

$$X_{z_j} = -i\frac{\partial}{\partial \bar{z}_j} = \frac{1}{\sqrt{2}i}\left(\frac{\partial}{\partial p_j} + i\frac{\partial}{\partial q_j}\right) \ , \quad X_{\bar{z}_j} = i\frac{\partial}{\partial z_j} = \frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial p_j} - i\frac{\partial}{\partial q_j}\right) \ .$$

In particular, $i\omega(X_{z_j}, X_{\bar{z}_k}) = \delta_{jk}$. Evidently, the vector fields $\{X_{z_1}, \dots, X_{z_n}\}$ span the antiholomorphic tangent space $T^{0,1}(\mathbb{C}^n)$ (which is the Kähler polarization distribution in the sense of Ex. 34).

The prequantization bundle $L \to \mathbb{C}^n$ is the trivial bundle $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$. We fix a trivializing section $\lambda_0 : \mathbb{C}^n \to L$ with $< \lambda_0, \lambda_0 >= 1$. The connection ∇ on L is determined by the property

$$\nabla\lambda_0 = -i \sum_{j=1}^{n} p_j \, dq_j \otimes \lambda_0 \ .$$

Following [71], we replace the trivializing section λ_0 by another trivializing section λ_1, given by

$$\lambda_1 := \exp\left(-\frac{1}{4} \sum_{j=1}^{n} (q_j^2 + p_j^2 - 2ip_j q_j) \right) \lambda_0 \ .$$

Then

$$\nabla\lambda_1 = \theta_1 \otimes \lambda_1 \quad \text{with} \quad \theta_1 = -i \sum_{j=1}^{n} \bar{z}_j dz_j \ .$$

In particular, the section λ_1 is covariantly constant along the vector fields from $T^{0,1}(\mathbb{C}^n)$. Hence, any section of L, covariantly constant along $T^{0,1}(\mathbb{C}^n)$, have the form

$$\varphi(z)\lambda_1 \ ,$$

where $\varphi(z)$ is a holomorphic function of $z \in \mathbb{C}^n$. We also have

$$< \lambda_1, \lambda_1 >= \exp\left(-\frac{1}{2} \sum_{j=1}^{n} (q_j^2 + p_j^2) \right) = \exp\left(-|z|^2\right)$$

with $|z|^2 = \sum_j \bar{z}_j z_j$. The inner product in the prequantization space $H = L^2(\mathbb{C}^n, L; \omega_0^n)$ takes on the following form

$$(\varphi\lambda_1, \psi\lambda_1) = \int_{\mathbb{C}^n} \varphi(z)\bar{\psi}(z) e^{-|z|^2} \omega_0^n \ .$$

Following the idea, formulated at the end of Sec. 13.2, we define the quantization space to be the space of polarized sections $L^2_{\mathcal{O}}(\mathbb{C}^n, L; \omega_0^n)$. In our case it coincides with the *Bargmann–Fock space*

$$F(\mathbb{C}^n) = L^2_{\mathcal{O}}(\mathbb{C}^n, e^{-|z|^2/2})$$

of holomorphic functions on $\mathbb{C}^n$ which are square integrable with the Gaussian weight $e^{-|z|^2/2}$.

The Kostant–Souriau (KS)-operators, associated with observables from the Heisenberg algebra $\operatorname{heis}(\mathbb{R}^{2n}) = \operatorname{heis}(\mathbb{C}^n)$ by formula (13.5), leave the Bargmann–Fock space $F(\mathbb{C}^n)$ invariant and so admit a restriction to this space. To see that, we compute the KS-operators, corresponding to the coordinates $z_j, \bar{z}_j$:

$$r_{\text{KS}}(z_j)(\varphi\lambda_1) = z_j\varphi\lambda_1 \ , \quad r_{\text{KS}}(\bar{z}_j)(\varphi\lambda_1) = \frac{\partial\varphi}{\partial z_j}\lambda_1$$

for $j = 1, \ldots, n$. Using the expression for the basis Hamiltonian vector fields, corresponding to coordinates and momenta:

$$X_{p_j} = \frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial \bar{z}_j}\right) \ , \quad X_{q_j} = -\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \bar{z}_j}\right) \ ,$$

we get the expression for the KS-operators, corresponding to the generators of the Heisenberg algebra heis($\mathbb{R}^{2n}$):

$$r_{\mathrm{KS}}(p_j)(\varphi\lambda_1) = \frac{1}{\sqrt{2}}\left[\left(z_j + \frac{\partial}{\partial z_j}\right)\varphi\right]\lambda_1 \ , \quad r_{\mathrm{KS}}(q_j)(\varphi\lambda_1) = \frac{1}{\sqrt{2}i}\left[\left(z_j - \frac{\partial}{\partial z_j}\right)\varphi\right]\lambda_1 \ .$$

It is clear from this expression that these operators leave the Bargmann–Fock space invariant. So we can restrict our KS-representation to this space, obtaining a representation r_0 of the Heisenberg algebra heis($\mathbb{R}^{2n}$) = heis($\mathbb{C}^n$) in the Bargmann–Fock space $F(\mathbb{C}^n) = L^2_{\mathcal{O}}(\mathbb{C}^n, e^{-|z|^2/2})$.

This representation, which is called the *Bargmann–Fock representation*, is already irreducible. The easiest way to see that is to use the so called *creation* and *annihilation operators* , given in this case by the formulae

$$a^*_j = r_{\mathrm{KS}}(z_j) = \text{ multiplication by } z_j \ , \quad a_j = r_{\mathrm{KS}}(\bar{z}_j) = \partial/\partial z_j \ ,$$

acting in the Bargmann–Fock space $F(\mathbb{C}^n)$. Denote by $\varphi_0 \equiv 1$ the *vacuum vector* in $F(\mathbb{C}^n)$. Note that the Bargmann–Fock space $F(\mathbb{C}^n) = L^2_{\mathcal{O}}(\mathbb{C}^n, e^{-|z|^2/2})$ is generated by vectors, obtained from φ_0 by the action of creation operators a^*_j, i.e. by vectors of the form

$$a^*_{j_1} \ldots a^*_{j_k}\varphi_0 \ .$$

To show that the Bargmann–Fock representation r_0 is irreducible, suppose that we have an operator A in $F(\mathbb{C}^n)$, commuting with all creation and annihilation operators a^*_j, a_j of our representation. Then $A\varphi_0$ should be equal to $c\varphi_0$ for some constant c, since $A\varphi_0$ is annihilated by all annihilation operators $a_j = \partial/\partial z_j$. On the other hand,

$$A(a^*_{j_1} \ldots a^*_{j_k}\varphi_0) = a^*_{j_1} \ldots a^*_{j_k}(A\varphi_0) = c(a^*_{j_1} \ldots a^*_{j_k}\varphi_0) \ .$$

These two properties imply that $A = c \cdot \mathrm{id}$, so, by Schur's lemma, the Bargmann–Fock representation r_0 is irreducible.

Unfortunately, the described method of quantization of the standard model $M_0 = (\mathbb{R}^{2n}, \omega_0) = (\mathbb{C}^n, \omega_0)$, provided with the Heisenberg algebra heis($\mathbb{R}^{2n}$) = heis($\mathbb{C}^n$), does not apply to other Kähler phase spaces and algebras of observables, since the KS-prequantization operators do not preserve, in general, the Fock spaces of holomorphic sections. We describe this situation in more detail in the next Sec. 14.2.1.

14.2 Fock spaces of half-forms

14.2.1 KS-action on Fock spaces

Suppose that our phase space (M, ω) is a Kähler manifold, provided with a compatible complex structure J. Assume that (M, ω) satisfies the quantization condition

(13.4) and $L \to M$ is the prequantization bundle, provided with a Hermitian connection ∇. We introduce a holomorphic structure on L, which is determined by the $\bar\partial$-operator, given by the $(0,1)$-component $\nabla^{0,1}$ of the connection ∇ with respect to the complex structure J. The *Fock space*

$$F(M,J) := L^2_{\mathcal{O}}(M,L;\omega^n)$$

is the space of square integrable sections of $L \to M$, holomorphic with respect to the introduced holomorphic structure on L. Denote by $\mathcal{A}$ the Lie algebra of Hamiltonians, which can be identified (under the assumption that M is simply connected) with the Lie algebra of Hamiltonian vector fields on M. Any observable $f \in \mathcal{A}$ generates a (local) 1-parameter group Γ of symplectomorphisms of M, given by

$$\varphi_f^t := \exp(2\pi i t X_f) \ ,$$

where X_f is the Hamiltonian vector field, generated by f. As we have pointed out in Sec. 13.2 (cf. Rem. 22), the action of Γ can be lifted to the action of its central extension $\tilde\Gamma$ on L, and this lifted action is generated by the KS-operator $r(f) \equiv r_{\mathrm{KS}}(f)$. More precisely, the lifted action is given by

$$\Phi_f^t := \exp\left(2\pi i t r(f)\right) : L^2(M,L;\omega^n) \longrightarrow L^2(M,L;\omega^n) \ .$$

However, these operators do not preserve, in general, the Fock space $F(M,J)$, since Φ_f^t maps the Fock space $F(M,J)$ into the Fock space $F(M,J_f^t)$, associated with the transformed complex structure $J_f^t := \varphi_{f,\star}^t \circ J \circ \varphi_{f,\star}^{-t}$, which, in general, is not equivalent to J. When this happens, the corresponding KS-operator $r_{\mathrm{KS}}(f)$ does not admit a restriction to $F(M,J)$. If we still want in this case to construct a quantization of $(M,\mathcal{A})$, using the KS-operators, we need to find a method of canonical identification of Fock spaces $F(M,J)$ with different J. In other words, we are looking for a canonical unitary pairing between different Fock spaces $F(M,J)$.

A naive idea would be to have some sort of an integral pairing, given by

$$\int_M < s_1 \, , s_2 > \omega^n$$

for $s_1 \in F(M,J_1)$, $s_2 \in F(M,J_2)$. But this idea does not work already for the Bargmann–Fock quantization. In this case sections s_1 and s_2 belong to $L^2_{\mathcal{O}}$-spaces with different weights, more precisely, s_1 belongs to $F(\mathbb{C}^n,J_1) = L^2_{\mathcal{O}}(\mathbb{C}^n, e^{-K_1(z)/2})$ and s_2 belongs to $F(\mathbb{C}^n,J_1) = L^2_{\mathcal{O}}(\mathbb{C}^n, e^{-K_2(z)/2})$, where $K_1(z)$ and $K_2(z)$ denote the Kähler potentials of Kähler metrics, determined by J_1 and J_2. It is clear that the product of these two factors may be not integrable. A better idea is to replace square integrable sections s of $L \to M$ by square integrable "half-forms" $s \otimes \sqrt{\omega^n}$. Then the integral of their product will be finite by the Cauchy inequality. In the next Subsection we realize this approach by formalizing the notion of half-forms.

14.2.2 Half-forms

Bundle of J-frames. Let (M,ω,J) be a Kähler manifold of $\dim_{\mathbb{C}} M = n$. Its complexified tangent bundle $T^{\mathbb{C}}M$ splits into the direct sum

$$T^{\mathbb{C}}M = T_J^{1,0} \oplus T_J^{0,1}$$

of the subbundles, formed by the ($\pm i$)-eigenspaces of the operator J. The *bundle of J-frames*

$$\mathrm{Fr}_J \longrightarrow M$$

is the bundle of frames in $T_J^{0,1}$, i.e. its fibre at $x \in M$ consists of all frames in $T_{J,x}^{0,1}$. The change of frames in the fibre generates a right $\mathrm{GL}(n,\mathbb{C})$-action on Fr_J, making Fr_J a principal $\mathrm{GL}(n,\mathbb{C})$-bundle.

We denote by

$$\mathrm{Fr}_J^n = K_J^{-1} \longrightarrow M$$

the *anti-canonical bundle, associated with* Fr_J, which coincides with the maximal exterior power of Fr_J: $\mathrm{Fr}_J^n = \bigwedge^n(\mathrm{Fr}_J)$. This is a complex line bundle on M, associated to Fr_J by the homomorphism $\det : \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C}^*$. Its sections μ can be identified with functions $\dot{\mu}$ on Fr_J, satisfying the relation

$$\dot{\mu}(X \cdot C) = \det(C^{-1})\dot{\mu}(X) \tag{14.1}$$

for $X = (X_1, \dots, X_n) \in C^\infty(M, \mathrm{Fr}_J)$, $C \in \mathrm{GL}(n,\mathbb{C})$.

We can define a partial connection, acting on sections of the bundle Fr_J^n, following [71, 73]. Suppose that μ is a section of Fr_J^n, identified with the function $\dot{\mu}$ on Fr_J, and ξ is a $(0,1)$-vector field on M, i.e. a section of $T_J^{0,1}$. To define the value of $\nabla_\xi \dot{\mu}$ at a point $x^0 \in M$ on a frame $X^0 \in \mathrm{Fr}_{J,x_0}$, we extend X^0 to a local J-frame $X = (X_1, \dots, X_n)$ in a neighborhood U of x^0, formed by Hamiltonian vector fields $X_1, \dots, X_n$. Then we set

$$(\nabla_\xi \dot{\mu})(X^0) := \xi\, \dot{\mu}(X)|_{x^0} \ ,$$

i.e. the value of $\nabla_\xi \dot{\mu}$ on the frame X^0 at x^0 is equal to the value of the vector field ξ on the function $\dot{\mu}(X)$ at x^0. It can be checked that this definition is correct, i.e. $\nabla_\xi \dot{\mu}$ is again a function on Fr_J, satisfying (14.1), and does not depend on the choice of the local extension X of a J-frame X^0. So we can define $\nabla_\xi \mu$ as the section of Fr_J^n, identified with the function $\nabla_\xi \dot{\mu}$ on Fr_J.

The introduced derivative ∇ has the properties of a *partial connection* (cf. [18]). Namely, for any $(0,1)$-vector fields ξ, η, any functions $f, g \in C^\infty(M, \mathbb{R})$ and any sections μ, ν of Fr_J^n we have:

1. $\nabla_{f\xi + g\eta}\mu = f\nabla_\xi \mu + g\nabla_\eta \mu\,;$
2. $\nabla_\xi(\mu + \nu) = \nabla_\xi \mu + \nabla_\xi \nu\,;$
3. $\nabla_\xi(f\mu) = f\nabla_\xi \mu + (\xi f)\mu\,.$

Moreover, this partial connection satisfies the equality

$$\nabla_\xi \nabla_\eta \mu - \nabla_\eta \nabla_\xi \mu = \nabla_{[\xi,\eta]}\mu \ ,$$

which means that it is flat.

Bundle of half-forms. Denote by $\mathrm{ML}(n,\mathbb{C})$ the *metalinear group*, which is a double covering of $\mathrm{GL}(n,\mathbb{C})$:

$$\rho : \mathrm{ML}(n,\mathbb{C}) \xrightarrow{2:1} \mathrm{GL}(n,\mathbb{C}) \ .$$

Its elements can be identified with the square roots of $(n\times n)$-matrices from $\mathrm{GL}(n,\mathbb{C})$ in the sense that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{ML}(n,\mathbb{C}) & \xrightarrow{\quad\chi\quad} & \mathbb{C}^* \\ & {\scriptstyle\rho}\searrow \quad \nearrow{\scriptstyle\det} & \\ & \mathrm{GL}(n,\mathbb{C}) & \end{array} ,$$

where χ is a unique complex square root of det, such that $\chi(I) = 1$.

Suppose that the principal $\mathrm{GL}(n,\mathbb{C})$-bundle $\mathrm{Fr}_J \to M$ of J-frames can be extended to a principal $\mathrm{ML}(n,\mathbb{C})$-bundle over M. Note that such an extension, in general, may not exist, since there is a topological obstruction for its existence (cf. [80, 29, 71]). This obstruction is an element of the cohomology group $H^2(M,\mathbb{Z}_2)$, moreover, the different choices of such metalinear extensions (if there are any) are parameterized by the elements of $H^1(M,\mathbb{Z}_2)$. So we suppose that this topological obstruction vanishes for our J-frame bundle $\mathrm{Fr}_J \to M$ and it can be extended to a principal $\mathrm{ML}(n,\mathbb{C})$-bundle

$$\widetilde{\mathrm{Fr}}_J \longrightarrow M \ .$$

We call $\widetilde{\mathrm{Fr}}_J$ the *bundle of metalinear J-frames*. It is a principal $\mathrm{ML}(n,\mathbb{C})$-bundle over M together with a double covering bundle epimorphism τ, such that

$$\begin{array}{ccc} \widetilde{\mathrm{Fr}}_J & \xrightarrow{\quad\tau\quad} & \mathrm{Fr}_J \\ & {\scriptstyle\mathrm{ML}(n,\mathbb{C})}\searrow \quad \swarrow{\scriptstyle\mathrm{GL}(n,\mathbb{C})} & \\ & M & \end{array} .$$

We denote by

$$\widetilde{\mathrm{Fr}}^n_J = K_J^{-1/2} \longrightarrow M$$

a complex line bundle on M, associated to $\widetilde{\mathrm{Fr}}_J \to M$ by the homomorphism $\chi : \mathrm{ML}(n,\mathbb{C}) \to \mathbb{C}^*$. Its sections ν can be identified with functions $\tilde{\nu}$ on $\widetilde{\mathrm{Fr}}_J$, satisfying the relation

$$\tilde{\nu}(\tilde{X}\cdot\tilde{C}) = \chi(\tilde{C}^{-1})\tilde{\nu}(\tilde{X}) \tag{14.2}$$

for $\tilde{X} \in C^\infty(M,\widetilde{\mathrm{Fr}}_J)$, $\tilde{C} \in \mathrm{ML}(n,\mathbb{C})$.

We can define a *partial connection*, acting on sections of the bundle $\widetilde{\mathrm{Fr}}^n_J$, similar to the case of the bundle Fr^n_J. Suppose that ν is a section of $\widetilde{\mathrm{Fr}}^n_J$, identified with the function $\tilde{\nu}$ on $\widetilde{\mathrm{Fr}}_J$, and ξ is a $(0,1)$-vector field on M. To define the value of $\nabla_\xi\tilde{\nu}$ at a point $x^0 \in M$ on a metalinear frame $\tilde{X}^0 \in \widetilde{\mathrm{Fr}}_{J,x_0}$, we extend the corresponding J-frame $X^0 = \tau(\tilde{X}^0)$ to a local J-frame $X = (X_1,\dots,X_n)$ in a neighborhood of x^0, formed by Hamiltonian vector fields $X_1,\dots,X_n$. Since τ is a double covering, there exists a local metalinear J-frame $\tilde{X}$, defined (perhaps, on a smaller) neighborhood U of x^0, extending $\tilde{X}^0$ and covering X, i.e. $\tau(\tilde{X}) = X$. Then we set

$$(\nabla_\xi\tilde{\nu})(\tilde{X}^0) := \xi\,\tilde{\nu}(\tilde{X})|_{x^0} \ ,$$

i.e. the value of $\nabla_\xi\tilde{\nu}$ on the metalinear frame $\tilde{X}^0$ at x^0 is equal to the value of the vector field ξ on the function $\tilde{\nu}(\tilde{X})$ at x^0. This definition is correct, i.e. $\nabla_\xi\tilde{\nu}$ is again a function on $\widetilde{\mathrm{Fr}}^n_J$, satisfying (14.2), and does not depend on the choices of the extension X and its metalinear lift $\tilde{X}$. So we can define $\nabla_\xi \nu$ as the section of $\widetilde{\mathrm{Fr}}^n_J$, identified with the function $\nabla_\xi \tilde{\nu}$ on $\widetilde{\mathrm{Fr}}_J$. The defined partial connection ∇ on $\widetilde{\mathrm{Fr}}^n_J$ is again flat.

Fock space of half-forms. Consider a line bundle $L \otimes K_J^{-1/2} \to M$. It can be provided with a partial connection ∇, induced by the Hermitian connection on the prequantization bundle L and the partial connection on the anti-canonical bundle $K_J^{-1/2}$, defined above. More precisely, given a $(0,1)$-vector field ξ and a section $\sigma = \lambda \otimes \nu$ of $L \otimes K_J^{-1/2}$ we define

$$\nabla_\xi \sigma = (\nabla_\xi \lambda) \otimes \nu + \lambda \otimes (\nabla_\xi \nu) \ .$$

Denote by $\mathcal{O}_{1/2}(M,J)$ the space of holomorphic sections σ of $L \otimes K_J^{-1/2} \to M$. We want to define an inner product of two sections σ_1, σ_2 in $\mathcal{O}_{1/2}(M,J)$. Locally (in a neighborhood U of an arbitrary point $x \in M$) these sections may be written as

$$\sigma_1 = \lambda_1 \otimes \nu_1 \ , \quad \sigma_2 = \lambda_2 \otimes \nu_2$$

for $\lambda_1, \lambda_2 \in \mathcal{O}(U,L)$, $\nu_1, \nu_2 \in \mathcal{O}(U, K_J^{-1/2})$. We choose a local J-frame $X = (X_1, \dots, X_n)$ on U, so that $\{X_1, \dots, X_n, \bar{X}_1, \dots, \bar{X}_n\}$ form a basis of $T^{\mathbb{C}}M|_U$ and

$$i\omega(X_j, \bar{X}_k) = \delta_{jk} \ , \quad \omega(X_j, X_k) = \omega(\bar{X}_j, \bar{X}_k) = 0 \ .$$

Denote by $< \sigma_1, \sigma_2 >$ a density on U, defined by

$$< \sigma_1, \sigma_2 > := < \lambda_1(x), \lambda_2(x) > \tilde{\nu}_1(\tilde{X}) \overline{\tilde{\nu}_2(\tilde{X})}$$

for $x \in U$ and any metalinear lift $\tilde{X}$ of X (such a lift locally always exists). It may be checked (cf. [71, 73]) that this definition does not depend on the choice of the lift and correctly defines a density, linear in σ_1, anti-linear in σ_2 and positive definite in the sense that $\langle \sigma, \sigma \rangle > 0$ for non-vanishing σ.

Introduce a pre-Hilbert space

$$PF_{1/2}(M,J) := \{\sigma \in \mathcal{O}_{1/2}(M,J) : \int_M < \sigma, \sigma > < \infty\}$$

and provide it with the inner product, defined by

$$(\sigma_1, \sigma_2) := \int_M < \sigma_1, \sigma_2 > \ .$$

The *Fock space of half-forms* $F_{1/2}(M,J)$ is, by definition, the completion of $PF_{1/2}(M,J)$ with respect to this inner product.

Locally (in a neighborhood U of a point $x \in M$) we can write down the integrand $< \sigma_1, \sigma_2 >$ explicitly by choosing local trivializing holomorphic sections λ_0 of L and ν_0 of $K_J^{-1/2}$, subject to the conditions

$$< \lambda_0, \lambda_0 > \equiv 1 \ , \quad \tilde{\nu}_0(\tilde{X}) \equiv 1$$

in U. In terms of these trivializations, holomorphic sections σ_1, σ_2 of $L \otimes K_J^{-1/2}$ over U will be written as

$$\sigma_1 = f_1 \cdot \lambda_0 \otimes \tilde{\nu}_0 \ , \quad \sigma_2 = f_2 \cdot \lambda_0 \otimes \tilde{\nu}_0$$

for some holomorphic functions f_1, f_2 on U. Then in terms of J-holomorphic local coordinates $(z_1, \ldots, z_n)$ in U we'll have

$$< \sigma_1, \sigma_2 >= \left(\frac{i}{2}\right)^n f_1(z)\overline{f_2(z)} d^n z \wedge d^n \bar{z} \ .$$

14.3 Metaplectic structure

14.3.1 Bundle of metaplectic frames

Metaplectic group. The *metaplectic group* $\mathrm{Mp}(2n, \mathbb{R})$ is a connected double covering group of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$, i.e. there is a 2:1 group homomorphism

$$\rho : \mathrm{Mp}(2n, \mathbb{R}) \longrightarrow \mathrm{Sp}(2n, \mathbb{R}) \ .$$

Such a covering exists, because the fundamental group π_1 of $\mathrm{Sp}(2n, \mathbb{R})$ is equal to $\mathbb{Z}$. To see that, note that $\mathrm{Sp}(2n, \mathbb{R})$ is homeomorphic to

$$\mathrm{U}(n) \times \frac{\mathrm{Sp}(2n, \mathbb{R})}{\mathrm{U}(n)} \cong S^1 \times \mathrm{SU}(n) \times \{\text{Siegel disc}\} \ ,$$

and the second and third factors on the right are simply connected.

Metaplectic structure. Let (M, ω) be a symplectic manifold of dimension $2n$. Denote by $\mathrm{Fr}_\omega \to M$ the principal $\mathrm{Sp}(2n, \mathbb{R})$-*bundle of symplectic frames* on M. A *metaplectic structure* on M is an extension of the bundle $\mathrm{Fr}_\omega \to M$ to a principal $\mathrm{Mp}(2n, \mathbb{R})$-bundle $\widetilde{\mathrm{Fr}_\omega} \to M$, called the *bundle of metaplectic frames* on M. In other words, we have a double covering bundle epimorphism $\tau : \widetilde{\mathrm{Fr}_\omega} \to \mathrm{Fr}_\omega$, which may be included into the following commutative diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Fr}_\omega} & \xrightarrow{\ \tau\ } & \mathrm{Fr}_\omega \\ & {\scriptstyle \mathrm{Mp}(n,\mathbb{R})}\searrow \quad \swarrow{\scriptstyle \mathrm{Sp}(n,\mathbb{R})} & \\ & M & \end{array} \ .$$

There is a topological obstruction for the existence of the metaplectic structure on M, due to Kostant [46]. Namely, denote by J an almost complex structure on M, compatible with ω, so that $c_1(M)$ is the 1st Chern class of TM with respect to J. Then for the existence of a metaplectic structure on M it is necessary and sufficient that $c_1(M) \bmod 2 \equiv 0 \iff c_1(M)$ is even in $H^2(M, \mathbb{Z})$. If this condition is satisfied, then the *set of all metaplectic structures on M (up to a natural equivalence) is parameterized by* $H^1(M, \mathbb{Z}_2)$.

14.3.2 Bundle of Kähler frames

It is also convenient to introduce the *bundle* $\mathrm{Fr}_K \to M$ *of J-frames for all ω-compatible almost complex structures* J on M. It is a fibre bundle over M with the fibre at $x \in M$, parameterizing J_x-frames on T_xM for all ω_x-compatible complex structures J_x on T_xM. This fibre can be identified with

$$\frac{\mathrm{Sp}(2n, \mathbb{R})}{\mathrm{U}(n)} \times \mathrm{GL}(n, \mathbb{C}) \cong \{\text{Siegel disc}\} \times \mathrm{GL}(n, \mathbb{C})$$

in the following way. Given a symplectic frame $(\xi, \eta) := (\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_n)$ at $x \in M$, we can write down any J-frame $X = (X_1, \dots, X_n)$ at x uniquely as (cf. [71])

$$X = \xi U + \eta V \ ,$$

where U, V are complex $n \times n$-matrices, such that the rank of $(2n \times n)$-matrix ${}^t(U, V)$ equals n, ${}^tUV = {}^tVU$, and the matrix $i(V^*U - U^*V)$ is positive definite. The set of such matrices ${}^t(U, V)$ can be identified with the set: $\{\text{Siegel disc}\} \times \mathrm{GL}(n, \mathbb{C})$, by associating with a matrix ${}^t(U, V)$ a pair of matrices

$$W := (U + iV)(U - iV)^{-1} \ , \quad C := U - iV \ . \tag{14.3}$$

Then C belongs to $\mathrm{GL}(n, \mathbb{C})$ and W belongs to the Siegel disc

$$D := \{W \in \mathrm{L}(n, \mathbb{C}) : {}^tW = W, I - W^*W \text{ is positive definite}\} \ .$$

The structure group of the bundle $\mathrm{Fr}_K \to M$, acting on the left, coincides with $\mathrm{Sp}(2n, \mathbb{R})$. There is also a natural $\mathrm{GL}(n, \mathbb{C})$-action on $\mathrm{Fr}_K \to M$ from the right, given by the frame change. The bundle $\mathrm{Fr}_K \to M$ is associated to the bundle $\mathrm{Fr}_\omega \to M$ of symplectic frames by a natural $\mathrm{Sp}(2n, \mathbb{R})$-action on the fibre.

In a similar way, we introduce the bundle $\widetilde{\mathrm{Fr}}_K \to M$ of all metalinear J-frames on M for all ω-compatible J. It is a fibre bundle with the fibre at $x \in M$, given by

$$\frac{\mathrm{Sp}(2n, \mathbb{R})}{\mathrm{U}(n)} \times \mathrm{ML}(n, \mathbb{C}) \ , \tag{14.4}$$

and the structure group $\mathrm{Mp}(2n, \mathbb{R})$, acting by the homomorphism $\rho : \mathrm{Mp}(2n, \mathbb{R}) \to \mathrm{Sp}(2n, \mathbb{R})$ on the first factor. The bundle $\widetilde{\mathrm{Fr}}_K \to M$ is associated to the bundle $\widetilde{\mathrm{Fr}}_\omega \to M$ of metaplectic frames by the $\mathrm{Mp}(2n, \mathbb{R})$-action. There is a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Fr}}_K & \xrightarrow{\ \tau\ } & \mathrm{Fr}_K \\ & \searrow \quad \swarrow & \\ & M & \end{array} \ ,$$

where τ is a double covering.

Note that for a fixed ω-compatible almost complex structure J on M the bundle $\mathrm{Fr}_J \to M$ is a subbundle of $\mathrm{Fr}_K \to M$, invariant under the right $\mathrm{GL}(n, \mathbb{C})$-action. The bundle $\widetilde{\mathrm{Fr}}_J \to M$ is a $\mathrm{ML}(n, \mathbb{C})$-invariant subbundle of $\widetilde{\mathrm{Fr}}_K \to M$, which coincides with the inverse image of $\mathrm{Fr}_J \to M$ under the double covering map $\tau : \widetilde{\mathrm{Fr}}_K \to \mathrm{Fr}_K$. In other words, we can say that a metaplectic structure on M, given by the metaplectic frame bundle together with the double covering $\tau : \widetilde{\mathrm{Fr}}_\omega \to \mathrm{Fr}_\omega$, induces metalinear structures on all J-frame bundles simultaneously.

14.4 Blattner–Kostant–Sternberg (BKS) pairing

Lemma 5. *Suppose that J_1, J_2 are two ω-compatible almost complex structures on a symplectic manifold (M, ω). Then they are* transversal *in the sense that*

$$T^{1,0}_{J_1} \oplus T^{0,1}_{J_2} = T^{\mathbb{C}}M \ .$$

Proof. Suppose, on the contrary, that there exists a vector $\xi \neq 0$, such that

$$\xi \in T^{1,0}_{J_1,x} \oplus T^{0,1}_{J_2,x} \quad \text{for some } x \in M \ .$$

Then

$$0 < \omega(\xi, J_1\xi) = \omega(\xi, i\xi) = i\omega(\xi, \xi) \ ,$$

where the inequality on the left is implied by the ω-compatibility of J_1 and the first equality is provided by $\xi \in T^{1,0}_{J_1,x}$. Similarly,

$$0 < \omega(\xi, J_2\xi) = \omega(\xi, -i\xi) = -i\omega(\xi, \xi) \ .$$

So we have simultaneously the two following relations

$$i\omega(\xi, \xi) > 0 \quad \text{and} \quad -i\omega(\xi, \xi) > 0 \ ,$$

contradicting each other. Hence, $T^{1,0}_{J_1,x} \cap T^{0,1}_{J_2,x} = \{0\}$ for any $x \in M$. By dimension counting we obtain that

$$T^{1,0}_{J_1,x} \oplus T^{0,1}_{J_2,x} = T^{\mathbb{C}}_x M \quad \text{for any } x \in M \ .$$

□

Due to the above Lemma 5, we can always choose locally, in a neighborhood U of an arbitrary fixed point $x \in M$, a J_1-frame X_1 and J_2-frame X_2, so that

$$i\omega(X^j_1, \overline{X^k_2}) = \delta_{jk} \ . \tag{14.5}$$

Given two sections σ_1 of $L \otimes K^{-1/2}_{J_1}$ and σ_2 of $L \otimes K^{-1/2}_{J_2}$ on U, we can write them down in the form

$$\sigma_1 = \lambda_1 \otimes \nu_1 \ , \quad \sigma_2 = \lambda_2 \otimes \nu_2 \ .$$

We define a density, similar to that in Subsec. 14.2.2:

$$< \sigma_1, \sigma_2 > := < \lambda_1(x), \lambda_2(x) > \tilde{\nu}_1(\tilde{X}_1)\overline{\tilde{\nu}_2(\tilde{X}_2)} \tag{14.6}$$

where $\tilde{X}_1, \tilde{X}_2$ are metalinear lifts of X_1, X_2, satisfying a metalinear analogue of (14.5). We shall describe this analogue (formula (14.9)) in Rem. 25 below. Now we note only that the definition (14.6) does not depend on the choice of the frames X_1, X_2, satisfying the normalization condition (14.5), and their metaplectic lifts $\tilde{X}_1$, $\tilde{X}_2$, satisfying the metaplectic normalization condition (14.9) below (this fact is proved in [71], Sec.5.1; cf. also [29], Ch.V,Sec.5).

We define the *BKS-pairing* between different Fock spaces of half-forms $F_{1/2}(M, J_1)$ and $F_{1/2}(M, J_2)$ by the formula

$$(\sigma_1, \sigma_2)_{12} := \int_M < \sigma_1, \sigma_2 > \ . \tag{14.7}$$

Suppose now that our almost complex structures J_1 and J_2 are integrable. Then locally, in a neighborhood U of a point $x \in M$, we can write down an explicit formula for the integrand in the above formula. For that we fix a J_1-frame X_1 and a J_2-frame X_2 in U, satisfying the normalization condition (14.5), and their metaplectic lifts

$\tilde{X}_1$, $\tilde{X}_2$, satisfying the metaplectic normalization condition (14.9), and choose local trivializing holomorphic sections λ_0 of L, ν_1 of $K_{J_1}^{-1/2}$ and ν_2 of $K_{J_2}^{-1/2}$, subject to the conditions

$$< \lambda_0, \lambda_0 >\equiv 1 \ , \quad \tilde{\nu}_1(\tilde{X}_1) \equiv 1 \ , \quad \tilde{\nu}_2(\tilde{X}_2) \equiv 1$$

in U. Then holomorphic sections σ_1 of $L \otimes K_{J_1}^{-1/2}$ and σ_2 of $L \otimes K_{J_2}^{-1/2}$ over U will be written as

$$\sigma_1 = f_1 \cdot \lambda_0 \otimes \nu_1 \ , \quad \sigma_2 = f_2 \cdot \lambda_0 \otimes \nu_2 \ ,$$

where f_1 is a J_1-holomorphic function on U, and f_2 is a J_2-holomorphic function on U. Since J_1 and J_2 are transversal, we can find local J_1-holomorphic coordinates $(z_1, \dots, z_n)$ and J_2-holomorphic coordinates $(w_1, \dots, w_n)$ in U, such that $(\partial/\partial z_1, \dots, \partial/\partial z_n; \partial/\partial \bar{w}_1, \dots, \partial/\partial \bar{w}_n)$ form a local basis of $T^{\mathbb{C}}M$ over M. Then

$$< \sigma_1, \sigma_2 >= \left(\frac{i}{2}\right)^n f_1(z)\overline{f_2(w)} d^n z \wedge d^n \bar{w} \ .$$

Remark 25 ([71]). To describe the metaplectic analogue of (14.5), suppose that our frames X_1 and X_2 are written in terms of a single symplectic frame $(\xi, \eta) := (\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_n)$, as in Subsec. 14.3.1:

$$X_1 = \xi U_1 + \eta V_1 \ , \quad X_2 = \xi U_2 + \eta V_2 \ .$$

Then Eq. (14.5) can be written in the form

$$V_2^* U_1 - U_2^* V_1 = -iI \ .$$

In terms of the matrices

$$W_j = (U_j + iV_j)(U_j - iV_j)^{-1} \ , \quad C_j := U_j - iV_j \ , \quad j = 1, 2 \ ,$$

this condition means that

$$I - W_2^* W_1 = 2(C_2^*)^{-1} C_1^{-1} \ . \tag{14.8}$$

Note that $Z := W_2^* W_1$ belongs to the matrix disc

$$\tilde{D} := \{Z \in \mathrm{L}(n, \mathbb{C}) : I - Z^* Z \text{ is positive definite}\} \ .$$

Consider the map $\tilde{D} \to \mathrm{GL}(n, \mathbb{C})$, given by $Z \mapsto I - Z$. Since $\tilde{D}$ is contractible (moreover, convex), this map can be uniquely extended to a map $Z \mapsto \widetilde{I - Z}$, sending $\tilde{D}$ to $\mathrm{ML}(n, \mathbb{C})$ and taking the value $\tilde{I}$ at $Z = 0$ (where we denote by $\tilde{I}$ the unit element in $\mathrm{ML}(n, \mathbb{C})$). Suppose that the metalinear lifts $\tilde{X}_1, \tilde{X}_2$ of our frames X_1, X_2 are described, according to (14.4), by pairs $(W_1, \tilde{C}_1)$, $(W_2, \tilde{C}_2)$, where $W_j \in D$, $\tilde{C}_j \in \mathrm{ML}(n, \mathbb{C})$ for $j = 1, 2$. Then the metalinear analogue of (14.5) has the form

$$\widetilde{I - W_2^* W_1} = 2(\tilde{C}_2^*)^{-1} \tilde{C}_1^{-1} \ . \tag{14.9}$$

14.5 Blattner–Kostant–Sternberg (BKS) quantization

14.5.1 Lifting the φ_f^t-action

Let (M, ω, J) be a Kähler manifold, and $f \in C^\infty(M, \mathbb{R})$ is an observable on M, for which the Hamiltonian vector field X_f is complete, i.e. the 1-parameter flow $\varphi_f^t = \exp(2\pi i t X_f)$, generated by X_f, is defined for all $t \in \mathbb{R}$. Hence, $\{\varphi_f^t\}$ is a 1-parameter group of symplectomorphisms of M. The flow φ_f^t generates a natural flow on the space of ω-compatible complex structures on M, given by

$$J \longmapsto J_f^t := \varphi_{f,\star}^t \circ J \circ \varphi_{f,\star}^{-t} ,$$

and a natural flow, denoted by the same letter φ_f^t, on the bundle $\mathrm{Fr}_K \to M$ of all J-frames on M.

By the covering homotopy property, this flow can be lifted to a 1-parameter flow $\tilde{\varphi}_f^t$ on the bundle $\widetilde{\mathrm{Fr}_K} \to M$ of all metalinear J-frames on M, yielding a 1-parameter flow of bundle isomorphisms

$$\tilde{\varphi}_f^t : \widetilde{\mathrm{Fr}_J} \longrightarrow \widetilde{\mathrm{Fr}_{J_f^t}} .$$

We are going to define an extension of the φ_f^t-flow to the Fock spaces of half-forms, denoted by

$$\mathcal{H}_t \equiv \mathcal{H}_f^t := F_{1/2}(M, J_f^t) .$$

φ_f^t-action on $K_J^{-1/2}$. Define first a φ_f^t-action on the bundle $K^{-1/2}$ over the space of ω-compatible complex structures on M. Let ν be a section of $K_J^{-1/2}$, identified with the function $\tilde{\nu}$ on the bundle $\widetilde{\mathrm{Fr}_J}$. Denote by $\varphi_f^t \nu$ a section of $K_t^{-1/2} \equiv K_{J_f^t}^{-1/2}$, identified with the function $\widetilde{\varphi_f^t \nu}$, defined by

$$\widetilde{\varphi_f^t \nu}(\tilde{X}) = \tilde{\nu}(\tilde{\varphi}_f^{-t} \tilde{X})$$

for any metalinear frame $\tilde{X} \in \widetilde{\mathrm{Fr}_t} \equiv \widetilde{\mathrm{Fr}_{J_f^t}}$.

φ_f^t-action on sections of L. By Rem. 22, the φ_f^t-flow on M can be lifted to a φ_f^t-flow on sections of L. More precisely, the generator of the φ_f^t-action on L

$$\mathcal{P}_f(\lambda) := i \frac{d}{dt} \left(\varphi_f^t \lambda\right)\Big|_{t=0}$$

is equal to

$$\mathcal{P}_f(\lambda) = r_{\mathrm{KS}}(f)(\lambda) = f\lambda - i\nabla_{X_f}\lambda .$$

φ_f^t-action on the Fock space of half-forms. Recall (cf. Subsec. 14.2.2) that the Fock space of half-forms $\mathcal{H}$ is defined as

$$\mathcal{H} = F_{1/2}(M, J) .$$

Suppose that an element σ of $\mathcal{H}$ is written in the form

$$\sigma = \lambda \otimes \nu ,$$

where $\lambda \in \mathcal{O}(M, L)$, $\nu \in \mathcal{O}(M, K_J^{-1/2})$. Then by definition

$$\varphi_f^t \sigma := \varphi_f^t \lambda \otimes \varphi_f^t \nu \ .$$

By linearity and continuity this definition extends to arbitrary sections in $\mathcal{H}$, so we obtain a Hilbert space isomorphism

$$\varphi_f^t : \mathcal{H} \longrightarrow \mathcal{H}_t$$

with the inverse map, given by φ_f^{-t}. It may be shown (cf. [71]) that $\varphi_f^t : \mathcal{H} \to \mathcal{H}_t$ is unitary.

14.5.2 Quantization of quantizable observables

Let f be an observable on M with a complete Hamiltonian vector field X_f. Suppose first that f is *quantizable*, i.e. the associated flow $\varphi_{f,\star}^t$ preserves the complex structure J, i.e. φ_f^t is a J-holomorphic map. Otherwise speaking, f is quantizable iff $[X_f, T_J^{0,1}M] \subset T_J^{0,1}M$. The quantizable observables form a subalgebra of the Lie algebra $\mathcal{A}$ of all observables. If f is quantizable, then the φ_f^t-flow preserves $\mathcal{H}$, i.e. we have a 1-parameter group of unitary operators $\varphi_f^t : \mathcal{H} \to \mathcal{H}$, and we can define the *quantized observable* $\mathcal{Q}_f$ by

$$\mathcal{Q}_f(\sigma) := i \frac{d}{dt} \left(\varphi_f^t \sigma \right) \big|_{t=0} \tag{14.10}$$

for any $\sigma \in \mathcal{H}$.

We can describe the operator $\mathcal{Q}_f$ in a more explicit way as follows. Define a *partial Lie derivative* L_ξ of half-forms along vector fields ξ on M, preserving J, i.e. $[\xi, T_J^{0,1}M] \subset T_J^{0,1}M$. Namely, for any half-form ν, identified with the function $\tilde{\nu}$ on the bundle $\widetilde{\mathrm{Fr}}_J$, we identify $L_\xi \nu$ with the function $\widetilde{L_\xi \nu}$, given by the formula

$$\widetilde{L_\xi \nu}(\tilde{X})|_x \equiv (L_\xi \tilde{\nu})(\tilde{X})|_x = \frac{d}{dt} \,|_{t=0} \left(\tilde{\nu}(\tilde{\varphi}_f^t \tilde{X})|_{\varphi_f^t x} \right)$$

for any metalinear J-frame $\tilde{X}$. In other words, the L_ξ-derivative of the function $\tilde{\nu}$, evaluated on a metalinear J-frame $\tilde{X}$ at a point x, is equal to the $\frac{d}{dt}$-derivative at $t = 0$ of the function $\tilde{\nu}$, evaluated on the metalinear J-frame $\tilde{\varphi}_f^t \tilde{X}$ at the point $\varphi_f^t x$.

The derivative L_ξ has the properties of the Lie derivative, but it can be taken only along the vector fields ξ, preserving J. The operator $\mathcal{Q}_f$ can be written in terms of partial Lie derivative as

$$\mathcal{Q}_f(\lambda \otimes \nu) = (-i \nabla_{X_f} \lambda + f\lambda) \otimes \nu - i\lambda \otimes L_{X_f} \nu \ .$$

Locally, we can compute the second term on the right as follows. Denote by $X = (X^1, \dots, X^n)$ a local J-frame on an open set U, consisting of Hamiltonian $(0,1)$-vector fields X^j. Then

$$[X_f, X^j](x) = \sum_{k=1}^{n} a_k^j(x) X^k$$

for some smooth matrix function $A := (a^j_k)$ on U. Denote by $\tilde{X}$ a metalinear lift of X over U and choose a local section $\tilde{\nu}_0$ of $K_J^{-1/2}$, so that $\tilde{\nu}_0(\tilde{X}) \equiv 1$. Any $\sigma \in F_{1/2}(U, J)$ can be written in the form

$$\sigma = \lambda \otimes \tilde{\nu}_0$$

for some holomorphic section $\lambda \in \mathcal{O}(U, L)$. Then (cf. [71], Sec.6.2)

$$L_{X_f}\tilde{\nu}_0 = -\frac{1}{2}\mathrm{tr}\, A \cdot \tilde{\nu}_0 \ ,$$

so that

$$\mathcal{Q}_f(\lambda \otimes \tilde{\nu}_0) = \left(-i\nabla_{X_f}\lambda + f\lambda - i\frac{1}{2}\mathrm{tr}\, A \cdot \lambda\right) \otimes \tilde{\nu}_0 \ .$$

It can be shown (cf. [71, 73]) that the map $f \mapsto \mathcal{Q}_f$ is a Lie-algebra representation

$$\{\text{Lie algebra of quantizable observables}\} \xrightarrow{\mathcal{Q}} \mathrm{End}^*\mathcal{H}$$

in the Fock space of half-forms $\mathcal{H} = F_{1/2}(M, J)$.

14.5.3 Quantization of general observables

Assume that for an observable f the integrals, defining the BKS-pairing $\mathcal{H}\times\mathcal{H}_t \to \mathbb{C}$, are finite, so we have a unitary operator

$$U_t : \mathcal{H}_t \longrightarrow \mathcal{H} \ .$$

In its terms the BKS-pairing, defined by formula (14.7), may be written as

$$(\sigma, \sigma_t)_{0t} = (\sigma, U_t\sigma_t)$$

for $\sigma \in \mathcal{H} \equiv \mathcal{H}_0$, $\sigma_t \in \mathcal{H}_t$.

Consider a unitary operator

$$\Phi^t_f := U_t \circ \varphi^t_f : \mathcal{H} \longrightarrow \mathcal{H}$$

and define a self-adjoint *quantized observable* $\mathcal{Q}_f$ by

$$\mathcal{Q}_f := i\frac{d}{dt}\,\Phi^t_f\Big|_{t=0} : \mathcal{H} \longrightarrow \mathcal{H} \ .$$

Then the map $f \mapsto \mathcal{Q}_f$ defines an irreducible Lie-algebra representation

$$\mathcal{Q} : \mathcal{A} \longrightarrow \mathrm{End}^*\mathcal{H}$$

of the algebra of observables $\mathcal{A}$ in the Fock space of half-forms $\mathcal{H}$ (under the assumption that the BKS-pairing is finite for all observables $f \in \mathcal{A}$).

Bibliographic comments

The BKS-quantization is presented in several books on geometric quantization. We follow mainly the Snyatycki book [71], dealing with different kinds of polarizations. We also recommend the Guillemin–Sternberg book [29], devoted mostly to real polarizations, and Tuynman lecture notes [73]. Our goal here was to present the BKS-quantization scheme without going too much into details (which may be found in [71, 29, 73]).