Chapter 4. An analogue of W for discrete Markov chains.

4.0 Introduction.

In this chapter, we construct for Markov chains some σ-finite measures which enjoy similar properties as the measure **W** studied in Chapter 1. Very informally, these σ-finite measures are obtained by "conditioning a recurrent Markov process to be transient".

Our construction applies to discrete versions of one- and two-dimensional Brownian motion, i.e. simple random walk on $\mathbb{Z}$ and $\mathbb{Z}^2$, but it can also be applied to a much larger class of Markov chains.

This chapter is divided into three sections; in Section 4.1, we give the construction of the σ-finite measures mentioned above ; in Section 4.2, we study the main properties of these measures, and in Section 4.3, we study some examples in more details.

4.1 Construction of the σ-finite measures $(\mathbb{Q}_x, x \in E)$

4.1.1 Notation and hypothesis.

Let E be a countable set, $(X_n)_{n\geq 0}$ the canonical process on $E^{\mathbb{N}}$, $(\mathcal{F}_n)_{n\geq 0}$ its natural filtration, and $\mathcal{F}_\infty$ the σ-field generated by $(X_n)_{n\geq 0}$.

Let us denote by $(\mathbb{P}_x)_{x\in E}$ the family of probability measures on $(E^{\mathbb{N}}, (\mathcal{F}_n)_{n\geq 0}, \mathcal{F}_\infty)$ associated to a Markov chain ($\mathbb{E}_x$ below denotes the expectation with respect to $\mathbb{P}_x$) ; more precisely, we suppose there exist probability transitions $(p_{y,z})_{y,z\in E}$ such that :

$$\mathbb{P}_x(X_0 = x_0, X_1 = x_1, ..., X_k = x_k) = \mathbf{1}_{x_0=x} p_{x_0,x_1} p_{x_1,x_2} ... p_{x_{k-1},x_k} \tag{4.1.1}$$

for all $k \geq 0$, $x_0, x_1, ..., x_k \in E$.

We assume three more hypotheses :

- For all $x \in E$, the set of $y \in E$ such that $p_{x,y} > 0$ is finite (i.e. the graph associated to the Markov chain is locally finite).
- For all $x, y \in E$, there exists $n \in \mathbb{N}$ such that $\mathbb{P}_x(X_n = y) > 0$ (i.e. the graph of the Markov chain is connected).
- For all $x \in E$, the canonical process is recurrent under the probability $\mathbb{P}_x$.

4.1.2 A family of new measures.

From the family of probabilities $(\mathbb{P}_x)_{x\in E}$, we will construct families of σ-finite measures which should be informally considered to be the law of $(X_n)_{n\geq 0}$ under $\mathbb{P}_x$, after conditionning this process to be transient.

More precisely, let us fix a point $x_0 \in E$ and let us suppose there exists a function $\phi : E \to \mathbb{R}_+$ such that :

- $\phi(x) \geq 0$ for all $x \in E$, and $\phi(x_0) = 0$.
- ϕ is harmonic with respect to $\mathbb{P}$, except at the point x_0, i.e. :
  for all $x \neq x_0$, $\sum_{y\in E} p_{x,y}\phi(y) = \mathbb{E}_x[\phi(X_1)] = \phi(x)$.
- ϕ is unbounded.

As we will see in Section 4.2 (Lemma 4.2.9), if ϕ satisfies the two first conditions, the third one is equivalent to the following (a priori weaker):

- ϕ is not identically zero.

In Section 4.3 (Proposition 4.3.1), we give some sufficient conditions for the existence of ϕ. We also study some examples. Generally, ϕ is not unique, but it will be fixed in this section. For any $r \in]0,1[$, let us define:

$$\psi_r(x) = \frac{r}{1-r}\mathbb{E}_{x_0}[\phi(X_1)] + \phi(x). \tag{4.1.2}$$

From this definition, the following properties hold :

- For all $x \neq x_0$, $\psi_r(x) = \mathbb{E}_x[\psi_r(X_1)]$. (4.1.3)
- $\psi_r(x_0) = r\mathbb{E}_{x_0}[\psi_r(X_1)]$ (4.1.4)

Now, for $y \in E$ and $k \geq -1$, let us denote by L_k^y the local time of X at point y and time k, i.e. :

$$L_k^y = \sum_{m=0}^{k} \mathbf{1}_{X_m = y} \tag{4.1.5}$$

(in particular, $L_{-1}^y = 0$ and $L_0^y = \mathbf{1}_{X_0 = y}$). The properties of ψ_r imply the following result :

Proposition 4.1.1 *For every $x \in E$, $(\psi_r(X_n) r^{L_{n-1}^{x_0}}, n \geq 0)$ is a martingale under $\mathbb{P}_x$.*

Proof of Proposition 4.1.1 For every $n \geq 0$, by Markov property :

$$\begin{aligned}\mathbb{E}_x\left[\psi_r(X_{n+1}) r^{L_n^{x_0}} | \mathcal{F}_n\right] &= r^{L_n^{x_0}} \mathbb{E}_x[\psi_r(X_{n+1})|\mathcal{F}_n] \\ &= r^{L_n^{x_0}} \psi_r(X_n)\left(\mathbf{1}_{X_n \neq x_0} + \frac{1}{r}\mathbf{1}_{X_n = x_0}\right) = r^{L_{n-1}^{x_0}}\psi_r(X_n).\end{aligned} \tag{4.1.6}$$

$\big($from (4.1.3) and (4.1.4)$\big)$.

Corollary 4.1.2

There exists a finite measure $\mu_x^{(r)}$ on $(E^{\mathbb{N}}, \mathcal{F}_\infty)$ such that :

$$\mu_{x\,|\mathcal{F}_n}^{(r)} = \psi_r(X_n) r^{L_{n-1}^{x_0}}.\mathbb{P}_{x\,|\mathcal{F}_n} \tag{4.1.7}$$

At this point, we remark that, for all σ, $0 < \sigma < 1/r$:

- $\psi_r(x) \leq \sup\left(\frac{1-\sigma r}{\sigma(1-r)}, 1\right).\psi_{\sigma r}(x)$ for all $x \in E$.
- Consequently, for $n \geq 1$:

$$\begin{aligned}&\mu_x^{(r)}(\sigma^{L_{n-1}^{x_0}}) = \mathbb{P}_x[\psi_r(X_n)(r\sigma)^{L_{n-1}^{x_0}}] \quad \big(\text{from } (4.1.7)\big) \\ &\leq \sup\left(\frac{1-\sigma r}{\sigma(1-r)}, 1\right)\mathbb{P}_x[\psi_{\sigma r}(X_n)(r\sigma)^{L_{n-1}^{x_0}}] \\ &\leq \sup\left(\frac{1-\sigma r}{\sigma(1-r)}, 1\right)\mu_x^{(\sigma r)}(1) = C\end{aligned} \tag{4.1.8}$$

where $C < \infty$ does not depend on n.

Therefore, $\mu_x^{(r)}(\sigma^{L_\infty^{x_0}}) < \infty$, with

$$L_\infty^{x_0} := \sum_{m=0}^{\infty} \mathbf{1}_{X_m = x_0} = \lim_{k\to\infty} L_k^{x_0}.$$

In particular, $L_\infty^{x_0} < \infty$, $\mu_x^{(r)}$-a.s. It is now possible to define a measure $\mathbb{Q}_x^{(r)}$, by : $\mathbb{Q}_x^{(r)} = \left(\frac{1}{r}\right)^{L_\infty^{x_0}} .\mu_x^{(r)}$; this measure is σ-finite since the sets $\{L_\infty^{x_0} \leq m\}$ increase to $\{L_\infty^{x_0} < \infty\}$; moreover $\{L_\infty^{x_0} = \infty\}$ is $\mathbb{Q}_x^{(r)}$-negligible, and

$$\mathbb{Q}_x^{(r)}(L_\infty^{x_0} \leq m) \leq \left(\frac{1}{r}\right)^m \mu_x^{(r)}(1) < \infty \tag{4.1.9}$$

4.1.3 Definition of the measures $(\mathbb{Q}_x,\ x \in E)$.
Here is a remarkable result, which explains the interest of this construction :
Theorem 4.1.3 *The two following properties hold :*
i) For all $x \in E$, $\mathbb{Q}_x^{(r)}$ does not depend on $r \in]0,1[$.
ii) Let $\mathbb{Q}_x$ denote the measure equal to $\mathbb{Q}_x^{(r)}$ for all $r \in]0,1[$, and $F_n \geq 0$ a $\mathcal{F}_n$-measurable functional. If q is a function from E to $[0,1]$, such that $\{q < 1\}$ is a finite set, then :

$$\mathbb{Q}_x\left[F_n \prod_{k=0}^{\infty} q(X_k)\right] = \mathbb{E}_x\left[F_n \psi_q(X_n) \prod_{k=0}^{n-1} q(X_k)\right] \tag{4.1.10}$$

where for $y \in E$, $\psi_q(y) := \mathbb{Q}_y\left[\prod_{k=0}^{\infty} q(X_k)\right]. \qquad (4.1.11)$

Remark 4.1.4 If we denote by $\mu_x^{(q)}$ the measure defined by :

$$\mu_x^{(q)} = \left(\prod_{k=0}^{\infty} q(X_k)\right) .\mathbb{Q}_x \tag{4.1.12}$$

we obtain :

$$\mu_{x\,|\mathcal{F}_n}^{(q)} = \psi_q(X_n)\left(\prod_{k=0}^{n-1} q(X_k)\right) .\mathbb{P}_{x\,|\mathcal{F}_n}. \tag{4.1.13}$$

These relations are similar to relations between **W** and Feynman-Kac penalisations of Wiener measure W (see Chap. 1, Th. 1.1.2, formulae (1.1.7), (1.1.8), (1.1.16)).
Moreover, ψ_q satisfies the "Sturm-Liouville equation" :

$$\psi_q(x) = q(x)\mathbb{E}_x[\psi_q(X_1)] \tag{4.1.14}$$

The analogy between this situation and the Brownian case described in Chapter 1 can be represented by the following correspondance :

Markov chain	Brownian motion
$\mathbb{P}_{x_0}$	W_0
$\mathbb{P}_x$	W_x
$\mu_x^{(q)}$	$W_{x,\infty}^{(q)}$
$M_n^{(q)} = \psi_q(X_n)\prod_{k=0}^{n-1} q(X_k)$	$M_t^{(q)} = \frac{\varphi_q(X_t)}{\varphi_q(x)} \exp\left(-\frac{1}{2}A_t^{(q)}\right)$
$\psi_q(x) = q(x)\mathbb{E}_x(\psi_q(X_1))$	$\varphi_q''(x) = q(x)\varphi_q(x)$
$\mu_x^{(q)}{}_{\mid\mathcal{F}_n} = M_n^{(q)}.\mathbb{P}_{x\mid\mathcal{F}_n}$	$W_{x,\infty}^{(q)}{}_{\mid\mathcal{F}_t} = M_t^{(q)}.W_{x\mid\mathcal{F}_t}$
$\mathbb{Q}_x$	$\mathbf{W}_x$
$\mu_x^{(q)} = \left(\prod_{k=0}^{\infty} q(X_k)\right).\mathbb{Q}_x$	$W_{x,\infty}^{(q)} = \frac{1}{\varphi_q(x)} \exp\left(-\frac{1}{2}A_\infty^{(q)}\right).\mathbf{W}_x$

Proof of Theorem 4.1.3 To begin with, let us prove the point *ii)* (with $\mathbb{Q}_x^{(r)}$ instead of $\mathbb{Q}_x$) for a function q such that $q(x_0) < 1$. Under the hypotheses of Theorem 4.1.3, for all $n \geq 0$, $F_n \prod_{k=0}^{N-1} q(X_k) \left(\frac{1}{r}\right)^{L_{N-1}^{x_0}}$ tends to $F_n \prod_{k=0}^{\infty} q(X_k) \left(\frac{1}{r}\right)^{L_\infty^{x_0}}$ as $N \to \infty$ and is dominated by $\left(\frac{q(x_0)}{r} \vee 1\right)^{L_\infty^{x_0}}$, which is $\mu_x^{(r)}$-integrable because $\frac{q(x_0)}{r} \vee 1 < \frac{1}{r}$. (from (4.1.8)).
By dominated convergence, if for $y \in E$, $k \geq 0$, we define :

$$\chi_q^{r,k}(y) := \mathbb{E}_y\left[\psi_r(X_k)\prod_{m=0}^{k-1} q(X_m)\right], \tag{4.1.15}$$

for all $x \in E$:

$$\begin{aligned}
&\mathbb{E}_x\left[F_n\,\chi_q^{r,N-n}(X_n)\prod_{k=0}^{n-1} q(X_k)\right] = \mathbb{E}_x\left[F_n\,\psi_r(X_N)\prod_{k=0}^{N-1} q(X_k)\right] \\
&= \mu_x^{(r)}\left[F_n\prod_{k=0}^{N-1} q(X_k)\left(\frac{1}{r}\right)^{L_{N-1}^{x_0}}\right] \\
&\underset{N\to\infty}{\to} \mu_x^{(r)}\left[F_n\prod_{k=0}^{\infty} q(X_k)\left(\frac{1}{r}\right)^{L_\infty^{x_0}}\right] = \mathbb{Q}_x^{(r)}\left[F_n\prod_{k=0}^{\infty} q(X_k)\right].
\end{aligned} \tag{4.1.16}$$

In particular, if we take $n = 0$ and $F_0 = 1$:

$$\chi_q^{r,N}(y) \underset{N\to\infty}{\to} \mathbb{Q}_y^{(r)}\left[\prod_{k=0}^{\infty} q(X_k)\right] \tag{4.1.17}$$

for all $y \in E$.
Moreover :

$$\begin{aligned}
\chi_q^{r,N-n}(y) &\leq \mathbb{E}_y\left[(q(x_0))^{L_{N-n-1}^{x_0}}\psi_r(X_{N-n})\right] \\
&\leq \sup\left(\frac{r}{q(x_0)}\left(\frac{1-q(x_0)}{1-r}\right), 1\right)\mathbb{E}_y\left[(q(x_0))^{L_{N-n-1}^{x_0}}\,\psi_{q(x_0)}(X_{N-n})\right] \\
&= \sup\left(\frac{r}{q(x_0)}\left(\frac{1-q(x_0)}{1-r}\right), 1\right)\psi_{q(x_0)}(y)
\end{aligned} \tag{4.1.18}$$

where

$$\begin{aligned}
\mathbb{E}_x\left[\psi_{q(x_0)}(X_n)\prod_{k=0}^{n-1} q(X_k)\right] &\leq \mathbb{E}_x\left[\psi_{q(x_0)}(X_n)(q(x_0))^{L_{n-1}^{x_0}}\right] \\
&= \psi_{q(x_0)}(x) < \infty.
\end{aligned} \tag{4.1.19}$$

By dominated convergence :

$$\mathbb{E}_x\left[F_n\,\chi_q^{r,N-n}(X_n)\prod_{k=0}^{n-1}q(X_k)\right]\underset{N\to\infty}{\to}\mathbb{E}_x\left[F_n\,\psi_q^{(r)}(X_n)\prod_{k=0}^{n-1}q(X_k)\right], \tag{4.1.20}$$

where $\psi_q^{(r)}(y)=\mathbb{Q}_y^{(r)}\left[\prod_{k=0}^{\infty}q(X_k)\right]$.
The two previous limits are equal; therefore :

$$\mathbb{Q}_x^{(r)}\left[F_n\prod_{k=0}^{\infty}q(X_k)\right]=\mathbb{E}_x\left[F_n\,\psi_q^{(r)}(X_n)\prod_{k=0}^{n-1}q(X_k)\right], \tag{4.1.21}$$

as written in point *ii)* of Theorem 4.1.3 (with $\mathbb{Q}_x^{(r)}$ instead of $\mathbb{Q}_x$).
Now we can prove point *i)*, by taking for any $s\in]0,1[$, $q(x)=\mathbf{1}_{x\neq x_0}+s\mathbf{1}_{x=x_0}$.
Let us first observe that $\dfrac{\psi_r(X_n)}{\psi_s(X_n)}$ is $\mu_y^{(s)}$-a.s. well-defined for all $n\geq 0$; therefore, $\mu_y^{(s)}\left[\dfrac{\psi_r(X_n)}{\psi_s(X_n)}\right]$ is well-defined and :

$$\begin{aligned}\mu_y^{(s)}\left[\frac{\psi_r(X_n)}{\psi_s(X_n)}\right]&=\mathbb{E}_y\left[s^{L_{n-1}^{x_0}}\psi_r(X_n)\right]=\mu_y^{(r)}\left[\left(\frac{s}{r}\right)^{L_{n-1}^{x_0}}\right]\\ &\underset{n\to\infty}{\to}\mu_y^{(r)}\left[\left(\frac{s}{r}\right)^{L_{\infty}^{x_0}}\right]=\mathbb{Q}_y^{(r)}[s^{L_\infty^{x_0}}]=\psi_q^{(r)}(y).\end{aligned} \tag{4.1.22}$$

Moreover, for all $A>0$:

$$\mu_y^{(s)}\left[\frac{\psi_r(X_n)}{\psi_s(X_n)}\right]=\mu_y^{(s)}\left[\frac{\psi_r(X_n)}{\psi_s(X_n)}\mathbf{1}_{\psi_s(X_n)\geq A}\right]+K_A, \tag{4.1.23}$$

where :

$$K_A\leq\sup\left(\frac{\psi_r}{\psi_s}\right).\mu_y^{(s)}[\psi_s(X_n)\leq A]\leq A\,\sup\left(\frac{\psi_r}{\psi_s}\right)\mathbb{E}_y[s^{L_{n-1}^{x_0}}]\underset{n\to\infty}{\to}0, \tag{4.1.24}$$

(from the definition (4.1.7) of $\mu_y^{(s)}$ and the fact that $(X_n)_{n\geq 0}$ is recurrent under $\mathbb{P}_y$). Hence :

$$\begin{aligned}&\liminf_{n\to\infty}\left(\inf_{\psi_s(x)\geq A}\frac{\psi_r(x)}{\psi_s(x)}\right)\mu_y^{(s)}[\psi_s(X_n)\geq A]\\ &\leq\liminf_{n\to\infty}\mu_y^{(s)}\left[\frac{\psi_r(X_n)}{\psi_s(X_n)}\right]\leq\limsup_{n\to\infty}\mu_y^{(s)}\left[\frac{\psi_r(X_n)}{\psi_s(X_n)}\right]\\ &\leq\limsup_{n\to\infty}\left(\sup_{\psi_s(x)\geq A}\frac{\psi_r(x)}{\psi_s(x)}\right)\mu_y^{(s)}[\psi_s(X_n)\geq A].\end{aligned} \tag{4.1.25}$$

Now, since ϕ (and hence, ψ_s) is unbounded, $\inf_{\psi_s(x)\geq A}\dfrac{\psi_r(x)}{\psi_s(x)}$ and $\sup_{\psi_s(x)\geq A}\dfrac{\psi_r(x)}{\psi_s(x)}$ tend to 1 when A goes to infinity and :

$$\mu_y^{(s)}[\psi_s(X_n)\geq A]\to\mu_y^{(s)}(1)=\psi_s(y). \tag{4.1.26}$$

Hence, $\mu_y^{(s)}\left[\dfrac{\psi_r(X_n)}{\psi_s(X_n)}\right]\underset{n\to\infty}{\to}\psi_s(y)$, which implies that $\psi_q^{(r)}(y)=\psi_s(y)$.

By (4.1.21) :

$$\begin{aligned}\mathbb{Q}_x^{(r)}[F_n s^{L_\infty^{x_0}}] &= \mathbb{E}_x\left[F_n s^{L_{n-1}^{x_0}}\psi_q^{(r)}(X_n)\right] = \mathbb{E}_x\left[F_n s^{L_{n-1}^{x_0}}\psi_s(X_n)\right] \\ &= \mu_x^{(s)}(F_n) = \mathbb{Q}_x^{(s)}[F_n s^{L_\infty^{x_0}}].\end{aligned} \tag{4.1.27}$$

By monotone class theorem, if F is $\mathcal{F}_\infty$-measurable and positive :

$$\mathbb{Q}_x^{(r)}(F.s^{L_\infty^{x_0}}) = \mathbb{Q}_x^{(s)}(F.s^{L_\infty^{x_0}}) \tag{4.1.28}$$

for all $r, s \in]0,1[$. Now, for all $r, s, t < 1$:

$$\mathbb{Q}_x^{(r)}(F.t^{L_\infty^{x_0}}) = \mathbb{Q}_x^{(t)}(F.t^{L_\infty^{x_0}}) = \mathbb{Q}_x^{(s)}(F.t^{L_\infty^{x_0}}). \tag{4.1.29}$$

Recall that $L_\infty^{x_0} < \infty$, $\mathbb{Q}_x^{(r)}$ and $\mathbb{Q}_x^{(s)}$-a.s. Therefore, by monotone convergence, $\mathbb{Q}_x^{(r)}(F) = \mathbb{Q}_x^{(s)}(F)$; point *i)* of Theorem 4.1.3 is proven, and $\mathbb{Q}_x$ is well-defined. By (4.1.21), point *ii)* is proven if $q(x_0) < 1$. It is easy to extend this formula to the case $q(x_0) = 1$, again by monotone convergence ; the proof of Theorem 4.1.3 is now complete. ■

Remark 4.1.5 The family $(\mathbb{Q}_x)_{x\in E}$ of σ-finite measures depends on x_0 and ϕ, which were assumed to be fixed in this section. In the sequel of the chapter, these parameters may vary; if some confusion is possible, we will write $(\mathbb{Q}_x^{(\phi,x_0)})_{x\in E}$ instead of $(\mathbb{Q}_x)_{x\in E}$.

4.2 Some more properties of $(\mathbb{Q}_x, x \in E)$.

4.2.1 Martingales associated with $(\mathbb{Q}_x, x \in E)$.

At the beginning of this section, we extend the second point of Theorem 4.1.3 to more general functionals than functionals of the form $F_n \prod_{k=0}^{\infty} q(X_k)$. More precisely, the following result holds :

Theorem 4.2.1

Let F be a positive $\mathcal{F}_\infty$-measurable functional. For $n \geq 0$, $y_0, y_1, ..., y_n \in E$, let us define the quantity :

$$M(F, y_0, y_1, ..., y_n) := \mathbb{Q}_{y_n}\left[F(y_0, y_1, ..., y_n = X_0, X_1, X_2, ...)\right]. \tag{4.2.1}$$

Then, for every $(\mathcal{F}_n)_{n\geq 0}$-stopping time T, one has :

$$\mathbb{Q}_x(F.\mathbf{1}_{T<\infty}) = \mathbb{E}_x\left[M(F, X_0, X_1, ..., X_T)\mathbf{1}_{T<\infty}\right]. \tag{4.2.2}$$

Proof of Theorem 4.2.1: To begin with, let us suppose that $T = n$ for $n \geq 0$, and $F = r^{L_\infty^{x_0}} f_0(X_0)f_1(X_1)...f_N(X_N)$ for $N > n$, $0 \leq f_i \leq 1$, $0 < r < 1$.

One has :

$$\begin{aligned}\mathbb{Q}_x(F) &= \mu_x^{(r)}\left[f_0(X_0)...f_N(X_N)\right] \\ &= \mathbb{E}_x\left[f_0(X_0)...f_N(X_N)r^{L_{N-1}^{x_0}}\psi_r(X_N)\right] \\ &= \mathbb{E}_x\left[f_0(X_0)...f_{n-1}(X_{n-1})r^{L_{n-1}^{x_0}}K(X_n)\right],\end{aligned} \tag{4.2.3}$$

where :

$$\begin{aligned} K(y) &= \mathbb{E}_y \left[f_n(X_0)...f_N(X_{N-n}) r^{L^{x_0}_{N-n-1}} \psi_r(X_{N-n}) \right] \\ &= \mu_y^{(r)} \left[f_n(X_0)...f_N(X_{N-n}) \right] \\ &= \mathbb{Q}_y \left[f_n(X_0)...f_N(X_{N-n}) r^{L^{x_0}_\infty} \right]. \end{aligned} \tag{4.2.4}$$

Hence, for all $y_0, ..., y_n$:

$$\begin{aligned} & f_0(y_0)...f_{n-1}(y_{n-1}) r^{\sum_{k=0}^{n-1} \mathbf{1}_{y_k = x_0}} K(y_n) \\ &= \mathbb{Q}_{y_n} \left[f_0(y_0)...f_{n-1}(y_{n-1}) f_n(X_0)...f_N(X_{N-n}) r^{\sum_{k=0}^{n-1} \mathbf{1}_{y_k = x_0} + L^{x_0}_\infty} \right] \\ &= \mathbb{Q}_{y_n} \left[F(y_0, ..., y_n = X_0, X_1, ...) \right] = M(F, y_0, y_1, ..., y_n). \end{aligned} \tag{4.2.5}$$

Therefore :

$$\mathbb{Q}_x(F) = \mathbb{E}_x \left[M(F, X_0, ..., X_n) \right], \tag{4.2.6}$$

which proves Theorem 4.2.1 for these particular functionals F and for $T = n$.
By monotone class theorem, we can extend (4.2.6) to the functionals $F = r^{L^{x_0}_\infty}.G$, where G is any positive functional, and by monotone convergence (r increasing to 1), Theorem 4.2.1 is proven for all F and $T = n$.
Now, let us suppose that T is a stopping time.
For $n \geq 0$, $M(F\mathbf{1}_{T=n}, X_0, X_1, ..., X_n) = \mathbf{1}_{T=n} M(F, X_0, ..., X_n)$, because $\{T = n\}$ depends only on $X_0, X_1, ..., X_n$; hence,

$$\mathbb{Q}_x(F\mathbf{1}_{T=n}) = \mathbb{E}_x \left[\mathbf{1}_{T=n} M(F, X_0, ..., X_n) \right]. \tag{4.2.7}$$

Summing from $n = 0$ to infinity, we obtain the general case of Theorem 4.2.1. ■

Corollary 4.2.2 *For any functional $F \in L^1(\mathbb{Q}_x)$, $(M(F, X_0, X_1, ..., X_n))_{n \geq 0}$ is a $\mathcal{F}_n$-martingale (with expectation $\mathbb{Q}_x(F)$).*
The correspondance with the Brownian case is the following :

Markov chain	Brownian motion
$F \in L^1_+(\mathbb{Q}_x, \mathcal{F}_\infty)$	$F \in L^1_+(\mathbf{W}_x, \mathcal{F}_\infty)$
$(M(F, X_0, ..., X_n),\ n \geq 0)$ a $(\mathcal{F}_n,\ n \geq 0, \mathbb{P}_x)$ martingale such that $(*)$ $\mathbb{Q}_x[\Gamma_n F] = \mathbb{P}_x[\Gamma_n M(F, X_0, ..., X_n)]$ $(\Gamma_n \in \mathcal{F}_n)$ $\mathbb{Q}_x(F) = \mathbb{P}_x[M(F, X_0, ..., X_n)]$	$(M_t(F),\ t \geq 0)$ a $(\mathcal{F}_t,\ t \geq 0, W_x)$ martingale such that $\mathbf{W}_x[\Gamma_t F] = W_x[\Gamma_t M_t(F)]$ $(\Gamma_t \in \mathcal{F}_t)$ $\mathbf{W}_x(F) = W_x(M_t(F))$

Here, $(*)$ is a consequence of (4.2.2) with $T = n.\mathbf{1}_{\Lambda_n} + (+\infty).\mathbf{1}_{\Lambda_n^c}$.
Now, we are able to describe the properties of the canonical process under $\mathbb{Q}_x$.

4.2.2. Properties of the canonical process under $(\mathbb{Q}_x, x \in E)$.
We have already proven that $L^{x_0}_\infty$ is almost surely finite under $\mathbb{Q}_x$. In fact, the following proposition gives a more general result :

Proposition 4.2.3 *Under $\mathbb{Q}_x$, the canonical process is a.s. transient, i.e $L^{y_0}_\infty < \infty$ for all $y_0 \in E$.*

Proof of Proposition 4.2.3: Let y_0 be in E, and r be in $]0,1[$. If, for $k \geq 1$, $\tau_k^{(y_0)}$ denotes the k-th hitting time of y_0 for the canonical process X, then for all $n \geq 0$:

$$\begin{aligned} \mu_x^{(r)}[L_{n-1}^{y_0} \geq k] &= \mu_x^{(r)}[\tau_k^{(y_0)} < n] = \mathbb{E}_x \left[\mathbf{1}_{\tau_k^{(y_0)} < n} r^{L_{n-1}^{x_0}} \psi_r(X_n) \right] \\ &= \mathbb{E}_x \left[\mathbf{1}_{\tau_k^{(y_0)} < n} r^{L^{x_0}_{\tau_k^{(y_0)}-1}} \psi_r(y_0) \right] \end{aligned} \tag{4.2.8}$$

by strong Markov property (applied at time $\tau_k^{(y_0)} \wedge n$), and by the fact that $\mathbb{E}_{y_0}[r^{L_{m-1}^{x_0}} \psi_r(X_m)] = \psi_r(y_0)$ for all $m \geq 0$ (from Proposition 4.1.1).
Hence :

$$\mu_x^{(r)}[L_{n-1}^{y_0} \geq k] \leq \psi_r(y_0) \mathbb{E}_x \left[r^{L^{x_0}_{\tau_k^{(y_0)}-1}} \right]; \tag{4.2.9}$$

and by monotone convergence :

$$\mu_x^{(r)}[L_{\infty}^{y_0} \geq k] \leq \psi_r(y_0) \mathbb{E}_x \left[r^{L^{x_0}_{\tau_k^{(y_0)}-1}} \right] \underset{k \to \infty}{\rightarrow} 0 \tag{4.2.10}$$

(since $L^{x_0}_{\tau_k^{(y_0)}} \underset{k \to \infty}{\rightarrow} \infty$, $\mathbb{P}_x$-a.s.); this implies Proposition 4.2.3. ∎

Now, we have the following decomposition result which gives a precise description of the canonical process under $\mathbb{Q}_y$ ($y \in E$) :

Proposition 4.2.4 *For all $y, y_0 \in E$, one has :*

$$\mathbb{Q}_y = \mathbb{Q}_y^{[y_0]} + \sum_{k \geq 1} \mathbb{P}_y^{\tau_k^{(y_0)}} \circ \widetilde{\mathbb{Q}}_{y_0}, \tag{4.2.11}$$

where $\mathbb{Q}_y^{[y_0]} = \mathbf{1}_{\forall n \geq 0, X_n \neq y_0} \mathbb{Q}_y$ is the restriction of $\mathbb{Q}_y$ to trajectories which do not hit y_0, $\widetilde{\mathbb{Q}}_{y_0} = \mathbf{1}_{\forall n \geq 1, X_n \neq y_0} \mathbb{Q}_{y_0}$ is the restriction of $\mathbb{Q}_{y_0}$ to trajectories which do not return to y_0, and $\mathbb{P}_y^{\tau_k^{(y_0)}} \circ \widetilde{\mathbb{Q}}_{y_0}$ denotes the concatenation of $\mathbb{P}_y$ stopped at time $\tau_k^{(y_0)}$ and $\widetilde{\mathbb{Q}}_{y_0}$, i.e. the image of $\mathbb{P}_y \otimes \widetilde{\mathbb{Q}}_{y_0}$ by the functional Φ from $E^{\mathbb{N}} \times E^{\mathbb{N}}$ such that :

$$\Phi((z_0, z_1, ..., z_n, ...), (z'_0, z'_1, ..., z'_n, ...)) = (z_0, z_1, ..., z_{\tau_k^{(y_0)}}, z'_1, ..., z'_n). \tag{4.2.12}$$

This formula (4.2.11) can be compared to (3.2.20) or (1.1.40).

Proof of Proposition 4.2.4 : We apply Theorem 4.2.1 to the stopping time $T = \tau_k^{(y_0)}$, and to the functional :

$$F = GH(X_{\tau_k^{(y_0)}}, X_{\tau_k^{(y_0)}+1}, ...) \mathbf{1}_{\forall u \geq 1, X_{\tau_k^{(y_0)}+u} \neq y_0}, \tag{4.2.13}$$

where G, H are positive functionals such that $G \in \mathcal{F}_{\tau_k^{(y_0)}}$.
For $k \geq 1$, we obtain :

$$\begin{aligned} &\mathbb{Q}_y \left[GH(X_{\tau_k^{(y_0)}}, X_{\tau_k^{(y_0)}+1}, ...) \mathbf{1}_{L_{\infty}^{y_0} = k} \right] \\ &= \mathbb{E}_y \left[\mathbf{1}_{\tau_k^{(y_0)} < \infty} G(X_0, ..., X_{\tau_k^{(y_0)}}) \right] \widetilde{\mathbb{Q}}_{y_0}[H], \end{aligned} \tag{4.2.14}$$

which implies :

$$\mathbb{Q}_y\left[GH(X_{\tau_k^{(y_0)}}, X_{\tau_k^{(y_0)}+1}, ...)\mathbf{1}_{L_\infty^{y_0}=k}\right] = \mathbb{E}_y[G]\,\widetilde{\mathbb{Q}}_{y_0}[H], \tag{4.2.15}$$

because $\tau_k^{(y_0)} < \infty$, $\mathbb{P}_y$-a.s. (the canonical process is recurrent under $\mathbb{P}_y$). Moreover :

$$\mathbb{Q}_y[H\mathbf{1}_{L_\infty^{y_0}=0}] = \mathbb{Q}_y^{[y_0]}(H) \tag{4.2.16}$$

by definition. Now, $L_\infty^{y_0} < \infty$, $\mathbb{Q}_y$-a.s. by Proposition 4.2.3, so there exists $k \geq 0$ such that $L_\infty^{y_0} = k$: the equalities (4.2.15) and (4.2.16) imply the Proposition 4.2.4 by monotone class theorem. ■

4.2.3 Dependence of $\mathbb{Q}_x$ on x_0.

The next Theorem shows that in the construction of the family $(\mathbb{Q}_x)_{x\in E}$, the choice of the point x_0 in E is in fact not so important. More precisely, the following result holds :

Theorem 4.2.5. *For all $y_0 \in E$, let us define the function $\phi^{[y_0]}$ by :*

$$\phi^{[y_0]}(y) = \mathbb{Q}_y^{[y_0]}(1) \tag{4.2.17}$$

Then the following holds :
i) $\phi^{[x_0]}$ is equal to ϕ and for all $y_0 \in E$, $\phi^{[y_0]} - \phi$ is a bounded function.
ii) For all $y_0 \in E$:

- *$\phi^{[y_0]}$ is finite and harmonic outside of y_0, i.e. for all $y \neq y_0$:*

$$\mathbb{E}_y[\phi^{[y_0]}(X_1)] = \phi^{[y_0]}(y).$$

- *$\phi^{[y_0]}(y_0) = 0$.*
- *$\widetilde{\mathbb{Q}}_{y_0}(1) = \mathbb{E}_{y_0}[\phi^{[y_0]}(X_1)]$.*

iii) By point ii), y_0 and the function $\phi^{[y_0]}$ can be used to construct a family $(\mathbb{Q}_x^{(\phi^{[y_0]},y_0)})_{x\in E}$ of σ-finite measures by the method given in Section 4.1. Moreover, this family is equal to the family $(\mathbb{Q}_x = \mathbb{Q}_x^{(\phi,x_0)})_{x\in E}$ constructed with ϕ and x_0.
iv) For all $y_0, y \in E$, the image of the measure $\mathbb{Q}_y$ by the total local time at y_0 is given by the following expressions :

- *$\mathbb{Q}_y[L_\infty^{y_0} = 0] = \phi^{[y_0]}(y)$.*
- *For all $k \geq 1$, $\mathbb{Q}_y[L_\infty^{y_0} = k] = \mathbb{E}_{y_0}[\phi^{[y_0]}(X_1)]$.*

Proof of Theorem 4.2.5. Let y_0 and y be in E. For all $r \in]0,1[$, $n \geq 1$:

$$\begin{aligned}\mu_y^{(r)}[L_{n-1}^{y_0} \geq 1] &= \mu_y^{(r)}[\tau_1^{(y_0)} < n] = \mathbb{E}_y\left[r^{L_{n-1}^{x_0}}.\mathbf{1}_{\tau_1^{(y_0)}<n}.\psi_r(X_n)\right] \\ &= \mathbb{E}_y\left[r^{L^{x_0}_{\tau_1^{(y_0)}-1}}.\mathbf{1}_{\tau_1^{(y_0)}<n}\right]\psi_r(y_0)\end{aligned} \tag{4.2.19}$$

from (4.1.7) and the martingale property. Hence :

$$\mu_y^{(r)}[L_\infty^{y_0} \geq 1] = \psi_r(y_0)\,\mathbb{E}_y\left[r^{L^{x_0}_{\tau_1^{(y_0)}-1}}\right]. \tag{4.2.20}$$

If $y_0 = x_0$, this implies :

$$\mu_y^{(r)}[L_\infty^{x_0} \geq 1] = \psi_r(x_0) \tag{4.2.21}$$

Therefore :

$$\begin{aligned} \phi^{[x_0]}(y) &= \mathbb{Q}_y[L_\infty^{x_0} = 0] = \mu_y^{(r)}[L_\infty^{x_0} = 0] \\ &= \mu_y^{(r)}(1) - \psi_r(x_0) = \psi_r(y) - \psi_r(x_0) = \phi(y) \end{aligned} \tag{4.2.22}$$

as written in Theorem 4.2.5. If $y_0 \neq x_0$, let us define the quantities :

$$p_{y,y_0}^{(x_0)} = \mathbb{P}_y[\tau_1^{y_0} < \tau_1^{x_0}], \tag{4.2.23}$$

and

$$q_{y_0}^{(x_0)} = \mathbb{P}_{x_0}[\tau_1^{y_0} > \tau_2^{x_0}]. \tag{4.2.24}$$

We have :

$$\mathbb{P}_y\left[L^{x_0}_{\tau_1^{(y_0)}-1} = 0\right] = p_{y,y_0}^{(x_0)} \tag{4.2.25}$$

and, for $k \geq 1$, by strong Markov property :

$$\mathbb{P}_y\left[L^{x_0}_{\tau_1^{(y_0)}-1} = k\right] = (1 - p_{y,y_0}^{(x_0)})(q_{y_0}^{(x_0)})^{k-1}(1 - q_{y_0}^{(x_0)}) \tag{4.2.26}$$

Summing all these equalities, one obtains :

$$\mathbb{E}_y\left[r^{L^{x_0}_{\tau_1^{(y_0)}-1}}\right] = p_{y,y_0}^{(x_0)} + \frac{r(1 - p_{y,y_0}^{(x_0)})(1 - q_{y_0}^{(x_0)})}{1 - rq_{y_0}^{(x_0)}} \tag{4.2.27}$$

and from (4.2.21) and (4.2.27) :

$$\begin{aligned} \mu_y^{(r)}[L_\infty^{y_0} \geq 1] &= \left[\frac{r}{1-r}\mathbb{E}_{x_0}[\phi(X_1)] + \phi(y_0)\right] \\ &\times \left[p_{y,y_0}^{(x_0)} + \frac{r(1 - p_{y,y_0}^{(x_0)})(1 - q_{y_0}^{(x_0)})}{1 - rq_{y_0}^{(x_0)}}\right]. \end{aligned} \tag{4.2.28}$$

(from (4.2.20) and (4.1.2)). Moreover :

$$\mu_y^{(r)}(1) = \psi_r(y) = \frac{r}{1-r}\mathbb{E}_{x_0}[\phi(X_1)] + \phi(y). \tag{4.2.29}$$

By hypothesis, there exists $n \geq 0$ such that $\mathbb{P}_{x_0}(X_n = y_0) > 0$; it is easy to check that it implies : $q_{y_0}^{(x_0)} < 1$.
Hence, by considering the difference between (4.2.28) and (4.2.29) and taking $r \to 1$, one obtains :

$$\phi^{[y_0]}(y) = \mathbb{E}_{x_0}[\phi(X_1)]\frac{1 - p_{y,y_0}^{(x_0)}}{1 - q_{y_0}^{(x_0)}} + [\phi(y) - \phi(y_0)]. \tag{4.2.30}$$

Therefore :

$$\phi(y) - \phi(y_0) \leq \phi^{[y_0]}(y) \leq \frac{\mathbb{E}_{x_0}[\phi(X_1)]}{1 - q_{y_0}^{(x_0)}} + [\phi(y) - \phi(y_0)] \tag{4.2.31}$$

which implies point *i)* of the Theorem, and in particular the finiteness of $\phi^{[y_0]}$. By applying Theorem 4.2.1 to $T = 1$ and $F = \mathbf{1}_{L_\infty^{y_0}=0}$, one can easily check that $\phi^{[y_0]}$ is harmonic everywhere except at point y_0 (where it is equal to zero).
By taking $T = 1$ and $F = \mathbf{1}_{L_\infty^{y_0}=1}$, one obtains the formula : $\widetilde{\mathbb{Q}}_{y_0}(1) = \mathbb{E}_{y_0}[\phi^{[y_0]}(X_1)]$. Hence, we obtain point *ii)* of the Theorem, and the point *iv)* by formula (4.2.11). Now, by taking the notation : $\mu_y^{(r),y_0} = r^{L_\infty^{y_0}}.\mathbb{Q}_y$, one has (for all positive and $\mathcal{F}_n$-measurable functionals F_n), by applying Theorem 4.2.1 to $T = n$ and $F = F_n\, r^{L_\infty^{y_0}}$:

$$\mu_y^{(r),y_0}(F_n) = \mathbb{Q}_y[F_n\, r^{L_\infty^{y_0}}] = \mathbb{E}_y\left[F_n\, r^{L_{n-1}^{y_0}}\alpha(X_n)\right], \tag{4.2.32}$$

where $\alpha(z) = \mathbb{Q}_z[r^{L_\infty^{y_0}}]$. By point *iv)* of the Theorem (already proven), one has :

$$\begin{aligned}\alpha(z) &= \phi^{[y_0]}(z) + \left(\sum_{k=1}^{\infty} r^k\right)\mathbb{E}_{y_0}[\phi^{[y_0]}(X_1)] \\ &= \frac{r}{1-r}\,\mathbb{E}_{y_0}[\phi^{[y_0]}(X_1)] + \phi^{[y_0]}(z)\end{aligned} \tag{4.2.33}$$

Hence :

$$\mu_y^{(r),y_0}(F_n) = \mathbb{E}_y\left[F_n\, r^{L_{n-1}^{x_0}}\left(\frac{r}{1-r}\,\mathbb{E}_{y_0}[\phi^{[y_0]}(X_1)] + \phi^{[y_0]}(X_n)\right)\right] \tag{4.2.34}$$

This formula implies that $\mu_y^{(r),y_0}$ is the measure defined in the same way as $\mu_y^{(r)}$, but from the point y_0 and the function $\phi^{[y_0]}$, instead of the point x_0 and the function ϕ. By considering the new measure with density $r^{-L_\infty^{y_0}}$ with respect to $\mu_y^{(r),y_0}$, one obtains the equality :

$$\mathbb{Q}_y = \mathbb{Q}_y^{(\phi^{[y_0]},y_0)} \tag{4.2.35}$$

which completes the proof of Theorem 4.2.5. ■

There is also an important formula, which is a direct consequence of (4.2.1), (4.2.5) and Theorem 4.2.5. :

Corollary 4.2.6 *Let F_n be a positive $\mathcal{F}_n$-measurable functional, y, y_0 be in E and g_{y_0} be the last hitting time of y_0 for the canonical process. Then the following formula holds :*

$$\mathbb{Q}_y\left[F_n \mathbf{1}_{g_{y_0}<n}\right] = \mathbb{E}_y[F_n \phi^{[y_0]}(X_n)] \tag{4.2.36}$$

In particular, one has :

$$\mathbb{Q}_y\left[F_n \mathbf{1}_{g_{x_0}<n}\right] = \mathbb{E}_y[F_n \phi(X_n)] \tag{4.2.37}$$

and $\left(\phi^{[y_0]}(X_n),\ n \geq 0\right)$, $\left(\phi(X_n),\ n \geq 0\right)$ are two $\mathbb{P}$ submartingales.
The correspondance with the Brownian case is the following :

Markov chain	Brownian motion
$\mathbb{Q}_y[F_n \mathbf{1}_{g_{x_0}<n}] = \mathbb{E}_y[F_n \phi(X_n)]$	$\mathbf{W}_x(F_t 1_{g<t}) = W_x(F_t \lvert X_t \rvert)$
$\mathbb{Q}_y[F_n \mathbf{1}_{g_{y_0}<n}] = \mathbb{E}_y[F_n \phi^{[y_0]}(X_n)]$	$\mathbf{W}_x(F_t 1_{\sigma_a<t}) = W_x(F_t(\lvert X_t \rvert - a)_+)$
$F_n \in \mathcal{F}_n$	$F_t \in \mathcal{F}_t$

By Theorem 4.2.5, the construction of a given family $(\mathbb{Q}_x)_{x\in E}$ can be obtained by taking any point y_0 instead of x_0, if the corresponding harmonic function $\phi^{[y_0]}$ is well-chosen.

4.2.4 Dependence of $\mathbb{Q}_x$ on ϕ.
In fact, this family of σ-finite measures depends only upon the equivalent class of the function ϕ, for an equivalence relation which will be described below. More precisely, if α and β are two functions from E to $\mathbb{R}_+$, let us write : $\alpha \simeq \beta$, iff α is equivalent to β when $\alpha + \beta$ tends to infinity ; i.e, for all $\epsilon \in]0,1[$, there exists $A > 0$ such that for all $x \in E$, $\alpha(x) + \beta(x) \geq A$ implies $1 - \epsilon < \dfrac{\alpha(x)}{\beta(x)} < 1 + \epsilon$. With this definition, one has the following result :

Propostion 4.2.7 *The relation $\simeq$ is an equivalence relation.*

Proof of Proposition 4.2.7 The reflexivity and the symmetry are obvious, so let us prove the transitivity.
We suppose that there are three functions α, β, γ such that $\alpha \simeq \beta$ and $\beta \simeq \gamma$.
There exists $\epsilon : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$, tending to zero at infinity, such that $\alpha + \beta \geq A$ implies $\left|\frac{\alpha}{\beta} - 1\right| \leq \epsilon(A)$, and $\beta + \gamma \geq A$ implies $\left|\frac{\beta}{\gamma} - 1\right| \leq \epsilon(A)$. For a given $x \in E$, let us suppose that $\alpha(x) + \gamma(x) \geq A$ for $A > 4 \sup\{z, \epsilon(z) \geq 1/2\}$. There are two cases :

- $\alpha(x) \geq A/2$. In this case, $\alpha(x) + \beta(x) \geq A/2$; hence, $\left|\frac{\alpha(x)}{\beta(x)} - 1\right| \leq \epsilon(A/2) \leq 1/2$, which implies : $\beta(x) + \gamma(x) \geq \beta(x) \geq \alpha(x)/2 \geq A/4$.

  Therefore : $\left|\frac{\beta(x)}{\gamma(x)} - 1\right| \leq \epsilon(A/4)$. Consequently, there exist u and v, $|u| \leq \epsilon(A/2) \leq 1/2$, $|v| \leq \epsilon(A/4) \leq 1/2$, such that $\frac{\alpha(x)}{\gamma(x)} = (1+u)(1+v)$, which implies :

$$\begin{aligned} \left|\frac{\alpha(x)}{\gamma(x)} - 1\right| &\leq |u| + |v| + |uv| \leq \epsilon(A/2) + \epsilon(A/4) + \epsilon(A/2)\epsilon(A/4) \\ &\leq \frac{3}{2}\left(\epsilon(A/2) + \epsilon(A/4)\right) \end{aligned} \tag{4.2.38}$$

- $\alpha(x) \leq A/2$. In this case, $\gamma(x) \geq A/2$, hence we are in the same situation as in the first case if we exchange $\alpha(x)$ and $\gamma(x)$

The above inequality implies : $\alpha \simeq \gamma$, since $\epsilon(A/2) + \epsilon(A/4)$ tends to zero when A goes to infinity. Hence, $\simeq$ is an equivalence relation. ■

This equivalence relation satisfies the following property :

Lemma 4.2.8 *Let ϕ_1 and ϕ_2 be two functions from E to $\mathbb{R}_+$ which are equal to zero at a point $y_0 \in E$ and which are harmonic at the other points i.e. for all $y \neq y_0$, $E_y[\phi_i(X_1)] = \phi_i(y)$, $i = 1, 2$. If $\phi_1 \simeq \phi_2$, then $\phi_1 = \phi_2$.*

Proof of Lemma 4.2.8 By the martingale property, for all $x \in E$, $A > 0$:

$$\begin{aligned} \phi_1(x) &= \mathbb{E}_x\left[\phi_1(X_{n \wedge \tau_1^{(y_0)}})\right] \\ &= \mathbb{E}_x\left[\phi_1(X_{n \wedge \tau_1^{(y_0)}}) \mathbf{1}_{\phi_1(X_{n \wedge \tau_1^{(y_0)}}) + \phi_2(X_{n \wedge \tau_1^{(y_0)}}) \geq A}\right] + K, \end{aligned} \tag{4.2.39}$$

where $|K| \leq A\, \mathbb{P}_x(\tau_1^{(y_0)} > n)$. Now, if $\phi_1(y) + \phi_2(y) \geq A$, one has :

$$(1 - \epsilon(A))\phi_1(y) \leq \phi_2(y) \leq (1 + \epsilon(A))\phi_1(y), \tag{4.2.40}$$

where $\epsilon(A)$ tends to zero when A tends to infinity. Therefore :

$$\phi_1(x) = \alpha\,\mathbb{E}_x\left[\phi_2(X_{n\wedge\tau_1^{(y_0)}})\mathbf{1}_{\phi_1(X_{n\wedge\tau_1^{(y_0)}})+\phi_2(X_{n\wedge\tau_1^{(y_0)}})\geq A}\right] + K, \tag{4.2.41}$$

where $1-\epsilon(A) \leq \alpha \leq 1+\epsilon(A)$. Moreover :

$$\phi_2(x) = \mathbb{E}_x\left[\phi_2(X_{n\wedge\tau_1^{(y_0)}})\mathbf{1}_{\phi_1(X_{n\wedge\tau_1^{(y_0)}})+\phi_2(X_{n\wedge\tau_1^{(y_0)}})\geq A}\right] + K', \tag{4.2.42}$$

where $|K'| \leq A\,\mathbb{P}_x(\tau_1^{(y_0)} > n)$. Hence :

$$\phi_1(x) = \alpha\left(\phi_2(x) - K'\right) + K. \tag{4.2.43}$$

Now, if A is fixed, $|K| + |K'|$ tend to zero when n goes to infinity. Therefore :

$$(1-\epsilon(A))\phi_1(x) \leq \phi_2(x) \leq (1+\epsilon(A))\phi_1(x). \tag{4.2.44}$$

This inequality is true for all $A \geq 0$; hence : $\phi_1 = \phi_2$, which proves Lemma 4.2.8. ■

We now give another lemma, which is quite close to Lemma 4.2.8 :

Lemma 4.2.9 *Let ϕ be a function from E to $\mathbb{R}$ which is equal to zero at a point $y_0 \in E$ and harmonic at the other points. If ϕ is bounded, it is identically zero.*

Proof of Lemma 4.2.9 Since ϕ is bounded, there exists $A > 0$ such that $|\phi(x)| < A$. The harmonicity of ϕ implies, for every $n \geq 0$ and $x \neq y_0$:

$$\phi(x) = \mathbb{E}_x[\phi(X_{n\wedge\tau_1^{y_0}})]$$

Consequently, since $\phi(y_0) = 0$, we get :

$$|\phi(x)| \leq A\,\mathbb{P}_x(\tau_1^{y_0} > n) \underset{n\to\infty}{\longrightarrow} 0$$

since $(X_n,\, n \geq 0)$ is recurrent. Hence, ϕ is identically zero.

If ϕ is bounded and positive, then ϕ is equivalent to zero (by definition of $\simeq$). Hence, in this case, Lemma 4.2.9 may be considered as a particular case of Lemma 4.2.8.

Now, let us state the following result, which explains why we have defined the previous equivalence relation :

Proposition 4.2.10 *Let x_0, y_0 be in E, ϕ a positive function which is harmonic except at x_0 and equal to zero at x_0, ψ a positive function which is harmonic except at y_0 and equal to zero at y_0. In these conditions, the family $(\mathbb{Q}_x^{(\phi,x_0)})_{x\in E}$ of σ-finite measures is identical to the family $(\mathbb{Q}_x^{(\psi,y_0)})_{x\in E}$ if and only if $\phi \simeq \psi$. Therefore this family can also be denoted by $(\mathbb{Q}_x^{[\phi]})_{x\in E}$, where $[\phi]$ is the equivalence class of ϕ.*

Proof of Proposition 4.2.10 If the two families of measures are equal, for all $x \in E$, $\mathbb{Q}_x^{(\phi,x_0)} = \mathbb{Q}_x^{(\psi,y_0)}$. Now, it has been proven that $\psi(x) = \mathbb{Q}_x^{(\psi,y_0)}(L_\infty^{y_0} = 0)$. Hence, if $\phi^{[y_0]}(x) = \mathbb{Q}_x^{(\phi,x_0)}(L_\infty^{y_0} = 0)$, one has $\psi = \phi^{[y_0]}$.

Since $\phi - \phi^{[y_0]}$ is bounded (point *i)* of Theorem 4.2.5), $\phi - \psi$ is bounded, which implies that ϕ is equivalent to ψ. On the other hand, if ϕ is equivalent to ψ, and if $\phi^{[y_0]} = \mathbb{Q}_x^{(\phi,x_0)}(L_\infty^{y_0} = 0)$, ψ and $\phi^{[y_0]}$ are two equivalent functions which are harmonic except at point y_0, and equal

to zero at y_0. Hence, by Lemma 4.2.8, $\psi = \phi^{[y_0]}$, and by Theorem 4.2.5, for all $x \in E$, $\mathbb{Q}_x^{(\phi,x_0)} = \mathbb{Q}_x^{(\phi^{[y_0]},y_0)}$.
Therefore, $\mathbb{Q}_x^{(\phi,x_0)} = \mathbb{Q}_x^{(\psi,y_0)}$, which proves Proposition 4.2.10.
In the next Section, we give some examples of the above construction.

4.3 Some examples.

4.3.1 The standard random walk.
In this case, $E = \mathbb{Z}$ and for all $x \in E$, $\mathbb{P}_x$ is the law of the standard random walk. The functions $\phi_+ : x \to x_+$, $\phi_- : x \to x_-$ and their sum $\phi : x \to |x|$ satisfies the harmonicity conditions above at point $x_0 = 0$.
Let $(\mathbb{Q}_x^+)_{x\in\mathbb{Z}}$, $(\mathbb{Q}_x^-)_{x\in\mathbb{Z}}$ and $(\mathbb{Q}_x)_{x\in\mathbb{Z}}$ be the associated σ-finite measures. For all $a \in \mathbb{Z}$, let us take the notations : $\phi_+^{[a]}(x) = \mathbb{Q}_x^+[L_\infty^a = 0]$, $\phi_-^{[a]}(x) = \mathbb{Q}_x^-[L_\infty^a = 0]$ and $\phi^{[a]}(x) = \mathbb{Q}_x[L_\infty^a = 0]$.
The function $\phi_+^{[a]}$ satisfies the harmonicity conditions at point a and is equivalent to ϕ_+. Now, these two properties are also satisfied by the function $x \to (x-a)_+$; hence, by Lemma 4.2.8, $\phi_+^{[a]}(x) = (x-a)_+$. By the same argument, $\phi_-^{[a]}(x) = (x-a)_-$ and $\phi^{[a]}(x) = |x-a|$.
Therefore, we have the equalities for every positive and $\mathcal{F}_n$-measurable functional F_n, and for every $x, a \in \mathbb{Z}$:

$$\mathbb{Q}_x^+[F_n \mathbf{1}_{g_a<n}] = \mathbb{E}_x[F_n(X_n - a)_+], \tag{4.3.1}$$

$$\mathbb{Q}_x^-[F_n \mathbf{1}_{g_a<n}] = \mathbb{E}_x[F_n(X_n - a)_-], \tag{4.3.2}$$

$$\mathbb{Q}_x[F_n \mathbf{1}_{g_a<n}] = \mathbb{E}_x[F_n|X_n - a|]. \tag{4.3.3}$$

These equations and the fact that the canonical process is transient under $\mathbb{Q}_x^+$, $\mathbb{Q}_x^-$, $\mathbb{Q}_x$ characterize these measures. Moreover, by using equations (4.3.1), (4.3.2) and (4.3.3), it is not difficult to prove that for all $x \in \mathbb{Z}$, these measures are the images of $\mathbb{Q}_0^+$, $\mathbb{Q}_0^-$ and $\mathbb{Q}_0$ by the translation by x.
Now, for all $a, x \in \mathbb{Z}$, and for all positive and $\mathcal{F}_n$-measurable functional F_n :

$$\mathbb{Q}_x^{+,[a]}[F_n] := \mathbb{Q}_x^+[F_n \mathbf{1}_{L_\infty^a=0}] = \mathbb{E}_x[F_n(X_{n\wedge\tau_1^{(a)}} - a)_+]. \tag{4.3.4}$$

Hence, if $x \leq a$, $\mathbb{Q}_x^{+,[a]} = 0$, and if $x > a$, $\mathbb{Q}_x^{+,[a]}$ is $(x-a)$ times the law of a Bessel random walk strictly above a, starting at point x (cf [LG] for a definition of the Bessel random walk).
By the same arguments, if $x \geq a$, $\mathbb{Q}_x^{-,[a]} = 0$, and if $x < a$, $\mathbb{Q}_x^{-,[a]}$ is $(a-x)$ times the law of a Bessel random walk strictly below a, starting at point x. Moreover, $\mathbb{Q}_x^{[a]}$ is the $|x-a|$ times the law of a Bessel random walk above or below a, depending on the sign of $x-a$. The same kind of arguments imply that (with obvious notations) :

- $\widetilde{\mathbb{Q}}_a^+$ is $1/2$ times the law of a Bessel random walk strictly above a.
- $\widetilde{\mathbb{Q}}_a^-$ is $1/2$ times the law of a Bessel random walk strictly below a.
- $\widetilde{\mathbb{Q}}_a$ is the law of a symmetric Bessel random walk, strictly above or below a with equal probability.

The equalities given by Proposition 4.2.4 are the following :

$$\mathbb{Q}_x^+ = \mathbb{Q}_x^{+,[a]} + \sum_{k\geq 1} \mathbb{P}_x^{\tau_k^{(a)}} \circ \widetilde{\mathbb{Q}}_a^+, \tag{4.3.5}$$

$$\mathbb{Q}_x^- = \mathbb{Q}_x^{-,[a]} + \sum_{k\geq 1} \mathbb{P}_x^{\tau_k^{(a)}} \circ \widetilde{\mathbb{Q}}_a^-, \tag{4.3.6}$$

$$\mathbb{Q}_x = \mathbb{Q}_x^{[a]} + \sum_{k\geq 1} \mathbb{P}_x^{\tau_k^{(a)}} \circ \widetilde{\mathbb{Q}}_a. \tag{4.3.7}$$

Moreover, one has :

- $\mathbb{Q}_x^+[L_\infty^a = 0] = (x-a)_+$ and $\mathbb{Q}_x^+[L_\infty^a = k] = 1/2$ for all $k \geq 1$.
- $\mathbb{Q}_x^-[L_\infty^a = 0] = (x-a)_-$ and $\mathbb{Q}_x^-[L_\infty^a = k] = 1/2$ for all $k \geq 1$.
- $\mathbb{Q}_x[L_\infty^a = 0] = |x-a|$ and $\mathbb{Q}_x[L_\infty^a = k] = 1$ for all $k \geq 1$.

Hence, by applying Theorem 4.2.1 and Corollary 4.2.2 to the functional $F = h(L_\infty^a)$ for a positive function h such that $\sum_{n\in\mathbb{N}} h(n) < \infty$, and for $a \in \mathbb{Z}$, one obtains that for all $x \in \mathbb{Z}$:

$$M_n^+ = (X_n - a)_+ \, h(L_{n-1}^a) + \frac{1}{2} \sum_{k=L_{n-1}^a+1}^{\infty} h(k), \tag{4.3.8}$$

$$M_n^- = (X_n - a)_- \, h(L_{n-1}^a) + \frac{1}{2} \sum_{k=L_{n-1}^a+1}^{\infty} h(k), \tag{4.3.9}$$

and their sum

$$M_n = |X_n - a| \, h(L_{n-1}^a) + \sum_{k=L_{n-1}^a+1}^{\infty} h(k) \tag{4.3.10}$$

are martingales under the probability $\mathbb{P}_x$. Other martingales can be obtained by taking other functionals F.

4.3.2 The "bang-bang random walk".

In this case, we suppose that $E = \mathbb{N}$ and that $(\mathbb{P}_x)_{x\in\mathbb{N}}$ is the family of measures associated to transition probabilities : $p_{0,1} = 1$, $p_{y,y+1} = 1/3$ and $p_{y,y-1} = 2/3$ for all $y \geq 1$. Informally, under $\mathbb{P}_x$ (for any $x \in \mathbb{N}$), the canonical process is a Markov process which tends to decrease when it is strictly above zero, and which increases if it is equal to zero.

The family of measures $(\mathbb{Q}_x)_{x\in\mathbb{N}}$ can be constructed by taking $x_0 = 0$ and $\phi(x) = 2^x - 1$ for all $x \in \mathbb{N}$.

For $y \in \mathbb{N}$, the function $\phi^{[y]} : x \to \mathbb{Q}_x[L_\infty^y = 0]$ is harmonic except at y where it is equal to zero, and it is equivalent to ϕ.

Since the function : $x \to (2^x - 2^y).\mathbf{1}_{x\geq y}$ satisfies the same properties, by Lemma 4.2.8, we get : $\phi^{[y]}(x) = (2^x - 2^y).\mathbf{1}_{x\geq y}$. For all $x \in \mathbb{N}$, the measure $\mathbb{Q}_x$ is characterized by the transience of the canonical process, and by the formula :

$$\mathbb{Q}_x[F_n \, \mathbf{1}_{g_a<n}] = \mathbb{E}_x[F_n \, (2^{X_n} - 2^a)_+], \tag{4.3.11}$$

which holds for all a, $n \in \mathbb{N}$ and for every positive $\mathcal{F}_n$-measurable functional F_n.

Adopting obvious notations, it is not difficult to prove the formula :

$$\mathbb{Q}_x^{[a]}(F_n) = \mathbb{E}_x[F_n \, (2^{X_{n\wedge\tau_1^{(a)}}} - 2^a)] \, \mathbf{1}_{x\geq a}, \tag{4.3.12}$$

and for $n \geq 1$:

$$\widetilde{\mathbb{Q}}_a(F_n) = \mathbb{E}_a \left[F_n \left(2^{X_{n \wedge \tau_2^{(a)}}} - 2^a\right) \mathbf{1}_{X_1 = a+1} \right]. \tag{4.3.13}$$

Moreover :

- The total mass of $\mathbb{Q}_x^{[a]}$ is zero if $x \leq a$, and $2^x - 2^a$ if $x > a$.
- The total mass of $\widetilde{\mathbb{Q}}_a$ is 1 if $a = 0$, and $2^a/3$ if $x \geq 1$.
- For $x > a$ and under the probability $\bar{\mathbb{P}}_x^{[a]} = \mathbb{Q}_x^{[a]}/(2^x - 2^a)$, the canonical process is a Markov process with probability transitions : $\bar{p}_{x,x+1} = \frac{2.2^{x-a} - 1}{3.2^{x-a} - 3}$ and $\bar{p}_{x,x-1} = \frac{2^{x-a} - 1}{3.2^{x-a} - 3}$. We remark that $\bar{p}_{x,x+1}$ tends to $2/3$ when x goes to infinity, and $\bar{p}_{x,x-1}$ tends to $1/3$ (the opposite case as the initial transition probabilities).
- Under the probability $\frac{\widetilde{\mathbb{Q}}_a}{(2^a/3)\mathbf{1}_{a \geq 1} + \mathbf{1}_{a=0}}$, the canonical process is a Markov process with the same transition probabilities as under $\bar{\mathbb{P}}_x^{[a]}$, with $X_1 = a + 1$ almost surely.

For all $a, x \in \mathbb{N}$, the image of $\mathbb{Q}_x$ by the total local times is given by the equalities :

$$\mathbb{Q}_x[L_\infty^a = 0] = (2^x - 2^a) \mathbf{1}_{x > a}, \tag{4.3.14}$$

and for all $k \geq 1$:

$$\mathbb{Q}_x[L_\infty^a = k] = K(a), \tag{4.3.15}$$

where $K(0) = 1$ and $K(a) = 2^a/3$ for $a \geq 1$.
Moreover, if h is an integrable function from $\mathbb{N}$ to $\mathbb{R}_+$, and if $a, x \in \mathbb{N}$,

$$M_n = h(L_{n-1}^a)\,(2^{X_n} - 2^a)_+ + K(a) \sum_{k = L_{n-1}^a + 1}^{\infty} h(k) \tag{4.3.16}$$

is a martingale under the initial probability $\mathbb{P}_x$.

4.3.3 The random walk on a tree.

We consider a random walk on a binary tree, which can be represented by the set $E = \{\varnothing, (0), (1), (0,0), (0,1), (1,0), (1,1), (0,0,0), ...\}$ of k-uples of elements in $\{0,1\}$ ($k \in \mathbb{N}$).
Obviously, k is the distance to the origin $\varnothing$ of the tree.
The probability transitions of the Markov process associated to the starting family of probabilities $(\mathbb{P}_x)_{x \in E}$ are $p_{\varnothing,(0)} = p_{\varnothing,(1)} = 1/2$, and for $k \geq 1$: $p_{(x_1,x_2,...,x_k),(x_1,x_2,...,x_{k-1})} = 1/2$, $p_{(x_1,...,x_k),(x_1,...,x_k,0)} = p_{(x_1,...,x_k),(x_1,...,x_k,1)} = 1/4$.
In particular, under $\mathbb{P}_x$ (for all $x \in E$), the distance to the origin is a standard reflected random walk.
If the reference point x_0 is $\varnothing$, we can take for ϕ the distance to the origin of the tree. But there are other possible functions ϕ for the same point x_0. For example, if $(a_0, a_1, a_2, ...)$ is an infinite sequence of elements in $\{0,1\}$ it is possible to take for ϕ the function such that for all $(x_0, x_1, ..., x_k) \in E$, one has $\phi(x_0, x_1, ..., x_k) = 2^p - 1$, where p is the smallest element of $\mathbb{N}$ such that $p > k$ or $x_p \neq a_p$. In particular, if $a_p = 0$ for all p, one has $\phi(\varnothing) = 0$, $\phi((0)) = 1$, $\phi((1)) = 0$, $\phi((0,0)) = 3$, $\phi((0,1)) = 1$, $\phi((1,0)) = \phi((1,1)) = 0$, $\phi((0,0,0)) = 7$, etc.

Each choice of the sequence $(a_p)_{p\in\mathbb{N}}$ gives a different function ϕ and hence a different family $(\mathbb{Q}_x^{[\phi]})_{x\in E}$ of σ-finite measures.

4.3.4 Some more general conditions for existence of ϕ.

The following proposition gives some sufficient conditions for the existence of a function ϕ which satisfies the hypothesis of Section 4.1.2 :

Proposition 4.3.1 *Let $(\mathbb{P}_x)_{x\in E}$ be the family of probabilities associated to a discrete time Markov process on a countable set E. We suppose that for all $x \in E$, the set of $y \in E$ such that the transition probability $p_{x,y}$ is strictly positive is finite. Furthermore, let us consider a function ϕ which satisfies one of the two following conditions (for a given point $x_0 \in E$) :*

- *There exists a function f from $\mathbb{N}$ to $\mathbb{R}_+^*$ such that $f(n)/f(n+1)$ tends to 1 when n goes to infinity, and such that for all $x \in E$:*

$$\mathbb{E}_x[\tau_1^{(x_0)} \geq n] \underset{n\to\infty}{\sim} f(n)\,\phi(x). \tag{4.3.17}$$

  where $\tau_1^{(x_0)}$ is the first hitting time of x_0, for the canonical process.

- *For all $x \in E$, $\mathbb{P}_x(X_k = x_0)$ tends to zero when k tends to infinity, and :*

$$\sum_{k=0}^{N} [\mathbb{P}_{x_0}(X_k = x_0) - \mathbb{P}_x(X_k = x_0)] \underset{N\to\infty}{\to} \phi(x). \tag{4.3.18}$$

In these conditions, ϕ is harmonic, except at point x_0 where this function is equal to zero.

Proof of Proposition 4.3.1 Let us suppose that the first condition is satisfied. For all $x \neq x_0$ and for all $y \in E$ such that $p_{x,y} > 0$:

$$\mathbb{E}_y\left[\tau_1^{(x_0)} \geq n\right] \underset{n\to\infty}{\sim} f(n)\,\phi(y). \tag{4.3.19}$$

By adding the equalities obtained for each point y and multiplied by $p_{x,y}$, we obtain :

$$\sum_{y\in E} p_{x,y}\mathbb{E}_y\left[\tau_1^{(x_0)} \geq n\right] \underset{n\to\infty}{\sim} f(n) \sum_{y\in E} p_{x,y}\,\phi(y), \tag{4.3.20}$$

which implies :

$$\mathbb{E}_x\left[\tau_1^{(x_0)} \geq n+1\right] \underset{n\to\infty}{\sim} f(n)\,\mathbb{E}_x[\phi(X_1)]. \tag{4.3.21}$$

Moreover :

$$\mathbb{E}_x\left[\tau_1^{(x_0)} \geq n+1\right] \underset{n\to\infty}{\sim} f(n+1)\,\phi(x). \tag{4.3.22}$$

By comparing these equivalences and by using the fact that $f(n)$ is equivalent to $f(n+1)$ and is strictly positive, one obtains :

$$\phi(x) = \mathbb{E}_x[\phi(X_1)]. \tag{4.3.23}$$

Since $\phi(x_0)$ is obviously equal to zero ($\mathbb{E}_{x_0}\left[\tau_1^{(x_0)} \geq n\right] = 0$), Proposition 4.3.1 is proven if the first condition holds.

Now let us assume the second condition holds.

If $x \neq x_0$, for all y such that $p_{x,y} > 0$:

$$\sum_{k=0}^{N} [\mathbb{P}_{x_0}(X_k = x_0) - \mathbb{P}_y(X_k = x_0)] \underset{N\to\infty}{\to} \phi(y). \tag{4.3.24}$$

Therefore :

$$\sum_{y\in E} p_{x,y} \left[\sum_{k=0}^{N} (\mathbb{P}_{x_0}(X_k = x_0) - \mathbb{P}_y(X_k = x_0))\right] \underset{N\to\infty}{\to} \sum_{y\in E} p_{x,y}\, \phi(y). \tag{4.3.25}$$

This equality implies :

$$\sum_{k=0}^{N} [\mathbb{P}_{x_0}(X_k = x_0)] - \sum_{k=1}^{N+1} [\mathbb{P}_x(X_k = x_0)] \underset{N\to\infty}{\to} \mathbb{E}_x[\phi(X_1)]. \tag{4.3.26}$$

Now, $\mathbb{P}_x(X_0 = x_0) = 0$ (since $x \neq x_0$) and when N goes to infinity, $\mathbb{P}_x(X_{N+1} = x_0)$ tends to zero by hypothesis. Hence :

$$\sum_{k=0}^{N} [\mathbb{P}_{x_0}(X_k = x_0) - \mathbb{P}_x(X_k = x_0)] \underset{N\to\infty}{\to} \mathbb{E}_x[\phi(X_1)], \tag{4.3.27}$$

which implies :

$$\phi(x) = \mathbb{E}_x[\phi(X_1)], \tag{4.3.28}$$

as written in Proposition 4.3.1. ■

Remark 4.3.2 *If the condition :*

$$\sum_{k=0}^{N} [\mathbb{P}_{x_0}(X_k = x_0) - \mathbb{P}_x(X_k = x_0)] \underset{N\to\infty}{\to} \phi(x) \tag{4.3.29}$$

is satisfied for a function ϕ, then ϕ is automatically positive. Indeed :

$$\sum_{k=0}^{N} [\mathbb{P}_{x_0}(X_k = x_0) - \mathbb{P}_x(X_k = x_0)] = \mathbb{E}_{x_0}\left[\sum_{k=0}^{N} \mathbf{1}_{X_k = x_0}\right] - \mathbb{E}_x\left[\sum_{k=0}^{N} \mathbf{1}_{X_k = x_0}\right], \tag{4.3.30}$$

where, by the strong Markov property :

$$\begin{aligned} \mathbb{E}_{x_0}\left[\textstyle\sum_{k=0}^{N} \mathbf{1}_{X_k = x_0}\right] &\geq \mathbb{E}_x\left[\sum_{k=0}^{\tau_1^{(x_0)}+N} \mathbf{1}_{X_k = x_0}\right] \\ &\geq \mathbb{E}_x\left[\sum_{k=0}^{N} \mathbf{1}_{X_k = x_0}\right]. \end{aligned} \tag{4.3.31}$$

4.3.5 <u>The standard random walk on $\mathbb{Z}^2$.</u>

In this case, $E = \mathbb{Z}^2$ and $(\mathbb{P}_x)_{x \in \mathbb{Z}^2}$ is the family of probabilities associated to the standard random walk. If we take $x_0 = (0,0)$, the problem is to find a function ϕ which satisfies the hypothesis of Section 4.1.2 : it can be solved by using Proposition 4.3.1.
More precisely, by doing some classical computations (see for example [Spi]), we can prove that for all $(x,y) \in \mathbb{Z}^2$, and for all $k \in \mathbb{N}$:

$$\mathbb{P}_{(x,y)} \left[X_k = (0,0) \right] = \mathbf{1}_{k \equiv x+y \, (mod.\, 2)} \, \frac{C}{k+1} \, + \, \epsilon_{(x,y)}(k), \tag{4.3.32}$$

where for all (x,y), $k^2 \, \epsilon_{(x,y)}(k)$ is bounded and C is a universal constant.
Therefore, for all N :

$$\begin{aligned} \textstyle\sum_{k=0}^{N} \mathbb{P}_{(x,y)} \left[X_k = (0,0) \right] &= C \sum_{k \leq N, \, k \equiv x+y \, (mod.\, 2)} \frac{1}{k+1} + \sum_{k=0}^{N} \epsilon_{(x,y)}(k) \\ &= \frac{C}{2} \log(N) + \eta_{(x,y)}(N), \end{aligned} \tag{4.3.33}$$

where for all $(x,y) \in \mathbb{Z}^2$, $\eta_{(x,y)}(N)$ converges to a limit $\eta_{(x,y)}(\infty)$ when N goes to infinity. Therefore :

$$\sum_{k=0}^{N} \left[\mathbb{P}_{(0,0)} \left(X_k = (0,0) \right) - \mathbb{P}_{(x,y)} \left(X_k = (0,0) \right) \right] \underset{N \to \infty}{\to} \phi((x,y)) := \eta_{(0,0)}(\infty) - \eta_{(x,y)}(\infty). \tag{4.3.34}$$

By Proposition 4.3.1, the function ϕ is harmonic except at $(0,0)$, and can be used to construct the family of probabilities $(\mathbb{Q}_{(x,y)})_{(x,y) \in \mathbb{Z}^2}$, as in dimension one. Moreover, it is not difficult to check that $\mathbb{Q}_{(x,y)}$ is the image of $\mathbb{Q}_{(0,0)}$ by the translation of (x,y).