Chapter 1. On a remarkable σ-finite measure W on path space, which rules penalisations for linear Brownian motion

1.0 Introduction.

1.0.1 $\big(\Omega, (X_t, \mathcal{F}_t),\ t \geq 0, \mathcal{F}_\infty, W_x(x \in \mathbb{R})\big)$ denotes the canonical realisation of 1-dimensional Brownian motion. $\Omega = \mathcal{C}(\mathbb{R}_+ \to \mathbb{R})$, $(X_t, t \geq 0)$ is the coordinate process on this space and $(\mathcal{F}_t,\ t \geq 0)$ denotes its natural filtration ; $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$. For every $x \in \mathbb{R}$, W_x denotes Wiener measure on $(\Omega, \mathcal{F}_\infty)$ such that $W_x(X_0 = x) = 1$. We write W for W_0 and if Z is a r.v. defined on $(\Omega, \mathcal{F}_\infty)$, we write $W_x(Z)$ for the expectation of Z under the probability W_x.

1.0.2 In a series of papers $\big([\text{RVY}, i],\ i = I, II, \cdots, X\big)$ we have studied various penalisations of Wiener measure with certain positive functionals $(F_t,\ t \geq 0)$; that is for each functional $(F_t,\ t \geq 0)$ in a certain class, we have been able to show the existence of a probability W_∞^F on $(\Omega, \mathcal{F}_\infty)$ such that : for every $s \geq 0$ and every $\Gamma_s \in b(\mathcal{F}_s)$, the space of bounded $\mathcal{F}_s$ measurable variables :

$$\lim_{t\to\infty} \frac{W(\Gamma_s F_t)}{W(F_t)} = W_\infty^F(\Gamma_s) \tag{1.0.1}$$

In this paper, we shall construct a positive and σ-finite measure $\mathbf{W}$ on $(\Omega, \mathcal{F}_\infty)$ which, in some sense, "rules all these penalisations jointly".

1.0.3 In Section 1.1 of this chapter, we show the existence of $\mathbf{W}$ and we describe some of its properties.

In Section 1.2, we show how to associate to $\mathbf{W}$ a family of $\big((\mathcal{F}_t,\ t \geq 0), W\big)$ martingales $\big(M_t(F),\ t \geq 0\big)$ $\big(F \in L^1_+(\mathcal{F}_\infty, \mathbf{W})\big)$. We study the properties of these martingales and give many examples.

In Section 1.3, we describe links between $\mathbf{W}$ and a σ-finite measure $\boldsymbol{\Lambda}$ which is defined as the "law" of the total local time of the canonical process under $\mathbf{W}$ in Chapter 3 of [RY, M]. In particular, we construct an invariant measure $\widetilde{\boldsymbol{\Lambda}}$ for the Markov process $\big((X_t, L_t^\bullet),\ t \geq 0\big)$ (and $\widetilde{\boldsymbol{\Lambda}}$ is intimately related to $\boldsymbol{\Lambda}$). Here, $L_t^\bullet$ denotes the local times process $(L_t^x, x \in \mathbb{R}_+)$, so that this Markov process $(X, L^\bullet)$ takes values in $\mathbb{R} \times \mathcal{C}(\mathbb{R} \longrightarrow \mathbb{R}_+)$.

1.0.4 Notations : As certain σ-finite measures play a prominent role in our paper, we write them, as a rule, in bold characters. Thus, no confusion should arise between the σ-finite measure $\mathbf{W}_x$ and the Wiener measure W_x.

1.1 Existence of W and first properties.

Our aim in this section is to define, via Feynman-Kac type penalisations, a positive and σ-finite measure $\mathbf{W}$ on $(\Omega, \mathcal{F}_\infty)$. Moreover, independently from this penalisation procedure, we give several remarkable descriptions of $\mathbf{W}$.

1.1.1 A few more notations.

$\big(\Omega, (X_t, \mathcal{F}_t)_{t\geq 0}, \mathcal{F}_\infty, W_x(x \in \mathbb{R})\big)$ denotes the canonical realisation of 1-dimensional Brownian motion.

We denote by $\mathcal{I}$ the set of positive Radon measures $q(dx)$ on $\mathbb{R}$, such that :

$$0 < \int_0^\infty \big(1 + |x|\big)\ q(dx) < \infty \tag{1.1.1}$$

For every $q \in \mathcal{I}$, $(A_t^{(q)},\ t \geq 0)$ denotes the additive functional defined by :

$$A_t^{(q)} := \int_{\mathbb{R}} L_t^y\ q(dy) \tag{1.1.2}$$

where $(L_t^y,\ t \geq 0, y \in \mathbb{R})$ denotes the jointly continuous family of local times of Brownian motion $(X_t,\ t \geq 0)$. When the Radon measure q admits a density with respect to the Lebesgue measure on $\mathbb{R}$ (and then we denote again this density by q) the density of occupation formula yields :

$$A_t^{(q)} = \int_{\mathbb{R}} L_t^y \, q(dy) = \int_0^t q(X_s)ds \tag{1.1.3}$$

We denote by $b(\mathcal{F}_s)$ $\big($resp. $b_+(\mathcal{F}_s)\big)$ the vector space of bounded and real valued (resp. the cone of bounded and positive) $\mathcal{F}_s$ measurable r.v.'s.
As our means to construct $\mathbf{W}$, we use a penalisation result obtained in [RVY, I] $\big($see also [RY, M]$\big)$. In the next subsection, we recall this result.

1.1.2 A Feynman-Kac penalisation result.

Theorem 1.1.1. *Let $q \in \mathcal{I}$ and :*

$$D_{x,t}^{(q)} := W_x\left(\exp\left(-\tfrac{1}{2}\, A_t^{(q)}\right)\right) \tag{1.1.4}$$

$$W_{x,t}^{(q)} := \frac{\exp\left(-\dfrac{1}{2}\, A_t^{(q)}\right)}{D_{x,t}^{(q)}} \cdot W_x \tag{1.1.5}$$

1) *For every $s \geq 0$ and $\Gamma_s \in b(\mathcal{F}_s)$, $W_{x,t}^{(q)}(\Gamma_s)$ admits a limit as $t \longrightarrow \infty$, denoted by $W_{x,\infty}^{(q)}(\Gamma_s)$, i.e. :*

$$W_{x,t}^{(q)}(\Gamma_s) \underset{t\to\infty}{\longrightarrow} W_{x,\infty}^{(q)}(\Gamma_s) \tag{1.1.6}$$

We express this property by writing that $W_{x,t}^{(q)}$ converges, as $t \longrightarrow \infty$, to $W_{x,\infty}^{(q)}$ along the filtration $(\mathcal{F}_s,\ s \geq 0)$.

2) *$W_{x,\infty}^{(q)}$ induces a probability on $(\Omega, \mathcal{F}_\infty)$ such that :*

$$W_{x,\infty}^{(q)}{}_{|\mathcal{F}_s} = M_{x,s}^{(q)} \cdot W_x{}_{|\mathcal{F}_s} \tag{1.1.7}$$

where $(M_{x,s}^{(q)},\ s \geq 0)$ is the $\big((\mathcal{F}_s,\ s \geq 0),\ W_x\big)$ martingale defined by :

$$M_{x,s}^{(q)} := \frac{\varphi_q(X_s)}{\varphi_q(x)} \ \exp\left(-\frac{1}{2}\, A_s^{(q)}\right) \tag{1.1.8}$$

In particular, $M_{x,0}^{(q)} = 1 \quad W_x$ a.s.
The function $\varphi_q : \mathbb{R} \longrightarrow \mathbb{R}_+$ which is featured in (1.1.8) is strictly positive, continuous, convex and satisfies :

$$\varphi_q(x) \underset{|x|\to\infty}{\sim} |x| \tag{1.1.9}$$

3) *φ_q may be defined via one or the other of the two following properties :*

i) φ_q is the unique solution of the Sturm-Liouville equation :

$$\varphi'' = \varphi \cdot q \quad \textit{(in the sense of distributions)} \tag{1.1.10}$$

which satisfies the boundary conditions :

$$\varphi'(+\infty) = -\varphi'(-\infty) = 1 \tag{1.1.11}$$

ii) $\sqrt{\frac{\pi t}{2}}\, W_x\left(\exp\left(-\frac{1}{2}\, A_t^{(q)}\right)\right) \underset{t\to\infty}{\longrightarrow} \varphi_q(x)$ (1.1.12)

4) *Under the family of probabilities $(W_{x,\infty}^{(q)},\ x\in\mathbb{R})$, the canonical process $(X_t,\ t\geq 0)$ is a transient time homogeneous diffusion. More precisely, there exists a $\big(\Omega,(\mathcal{F}_t,\ t\geq 0), W_\infty^{(q)}\big)$ Brownian motion $(B_t,\ t\geq 0)$ such that :*

$$X_t = x + B_t + \int_0^t \frac{\varphi_q'(X_s)}{\varphi_q(X_s)}\, ds \tag{1.1.13}$$

In particular, this diffusion process $(X_t,\ t\geq 0)$ admits the following function γ_q as its scale function :

$$\gamma_q(x) := \int_0^x \frac{dy}{\varphi_q^2(y)} \tag{1.1.14}$$

$\big($and : $\big|\gamma_q(\pm\infty)\big| < \infty\big)$.

We note that the function φ_q featured in Theorem 1.1 is not exactly the one found in [RY, M]. It differs from it by the factor $\sqrt{\frac{\pi}{2}}$; we have made this slight change in order to simplify some further formulae.

1.1.3 Definition of **W**.

We now use Theorem 1.1.1 to construct the σ-finite measure $\mathbf{W}$. In fact, we define, for every $x\in\mathbb{R}$, a positive and σ-finite $\mathbf{W}_x$ which is deduced from $\mathbf{W}$ via the following simple translation by x :

$$\mathbf{W}_x\big(F(X_s,\ s\geq 0)\big) = \mathbf{W}\big(F(x+X_s,\ s\geq 0)\big) \tag{1.1.15}$$

for every positive functional F. This formula (1.1.15) explains why, most of the time, we may limit ourselves to consider $\mathbf{W}_0$, which we denote simply by $\mathbf{W}$.

Theorem 1.1.2. *(Existence of* **W***)*
There exists, on $(\Omega,\mathcal{F}_\infty)$ a positive and σ-finite measure $\mathbf{W}$, with infinite total mass, such that, for every $q\in\mathcal{I}$:

$$\mathbf{W} = \varphi_q(0)\exp\left(\frac{1}{2}\, A_\infty^{(q)}\right)\cdot W_\infty^{(q)} \tag{1.1.16}$$

or

$$W_\infty^{(q)} = \frac{1}{\varphi_q(0)}\exp\left(-\frac{1}{2}\, A_\infty^{(q)}\right)\cdot \mathbf{W} \tag{1.1.16$'$}$$

In other terms, the RHS of (1.1.16) does not depend on $q\in\mathcal{I}$. In particular :

$$\mathbf{W}\left(\exp\left(-\frac{1}{2}\, A_\infty^{(q)}\right)\right) = \varphi_q(0) \tag{1.1.17}$$

or more generally, from (1.1.15) :

$$\mathbf{W}_x\left(\exp\left(-\frac{1}{2}\, A_\infty^{(q)}\right)\right) = \varphi_q(x). \tag{1.1.17$'$}$$

As we shall soon see, the measure $\mathbf{W}$ is such that, for every $t > 0$ and for every r.v. $\Gamma_t \in b_+(\mathcal{F}_t)$, $\mathbf{W}(\Gamma_t)$ equals 0 or $+\infty$ depending whether $W(\Gamma_t) = 0$ or is strictly positive. Thus, the measure $\mathbf{W}$, although, as we show later, it is σ-finite on $(\Omega, \mathcal{F}_\infty)$, is not σ-finite on either of the measurable spaces $(\Omega, \mathcal{F}_t)$, $t > 0$.

Proof of Theorem 1.1.2.

i) We shall establish that, for every $q \in \mathcal{I}$, the measure on $(\Omega, \mathcal{F}_\infty)$:

$$\varphi_q(0) \exp\left(\frac{1}{2} A_\infty^{(q)}\right) \cdot W_\infty^{(q)}$$

does not depend on q, which allows to define $\mathbf{W}$ from formula (1.1.16). Then, we shall prove that $\mathbf{W}$, thus defined, is $(\Omega, \mathcal{F}_\infty)$ σ-finite.

ii) **Lemma 1.1.3.** *For every $q \in \mathcal{I}$ and every $x \in \mathbb{R}$:*

$$1) \quad \text{if } \lambda < 1 \quad W_{x,\infty}^{(q)}\left(\exp \frac{\lambda}{2} A_\infty^{(q)}\right) < \infty \tag{1.1.18}$$

$$2) \quad \text{if } \lambda \geq 1 \quad W_{x,\infty}^{(q)}\left(\exp \frac{\lambda}{2} A_\infty^{(q)}\right) = +\infty \tag{1.1.19}$$

Proof of Lemma 1.1.3.

From (1.1.7), for every $\lambda \in]0,1[$:

$$\begin{aligned} W_{x,\infty}^{(q)}\left(\exp \frac{\lambda}{2} A_t^{(q)}\right) &= W_x\left(\frac{\varphi_q(X_t)}{\varphi_q(x)} \exp\left(-\left(\frac{1-\lambda}{2}\right) A_t^{(q)}\right)\right) \\ &= \frac{\varphi_{(1-\lambda)q}(x)}{\varphi_q(x)}\, W_x\left(\frac{\varphi_q(X_t)}{\varphi_{(1-\lambda)q}(X_t)}\, \frac{\varphi_{(1-\lambda)q}(X_t)}{\varphi_{(1-\lambda)q}(x)} \exp\left(-\left(\frac{1-\lambda}{2}\right) A_t^{(q)}\right)\right) \end{aligned} \tag{1.1.20}$$

We have been able to write (1.1.20) because the functions φ_q and $\varphi_{(1-\lambda)q}$ are strictly positive. On the other hand, since for every $q \in \mathcal{I}$, $\varphi_q(x) \underset{|x|\to\infty}{\sim} |x|$, there exist two constants :

$$0 < C_1(\lambda, q) \leq C_2(\lambda, q) < \infty$$

such that :

$$C_1(\lambda, q) \leq \inf_{y\in\mathbb{R}} \frac{\varphi_q(y)}{\varphi_{(1-\lambda)q}(y)} \leq \sup_{y\in\mathbb{R}} \frac{\varphi_q(y)}{\varphi_{(1-\lambda)q}(y)} \leq C_2(\lambda, q) \tag{1.1.21}$$

Thus, from (1.1.20) :

$$W_{x,\infty}^{(q)}\left(\exp \frac{\lambda}{2} A_t^{(q)}\right) \leq \frac{\varphi_{(1-\lambda)q}(x)}{\varphi_q(x)} \sup_{y\in\mathbb{R}} \frac{\varphi_q(y)}{\varphi_{(1-\lambda)q}(y)}\, W_{x,\infty}^{((1-\lambda)q)}(1) \leq \frac{C_2(\lambda, q)}{C_1(\lambda, q)} \tag{1.1.22}$$

We now let $t \longrightarrow \infty$ and we use the monotone convergence Theorem to obtain point 1) of Lemma 1.1.3.

We now write relation (1.1.20) with $\lambda = 1$:

$$W_{x,\infty}^{(q)}\left(\exp \frac{1}{2} A_t^{(q)}\right) = W_x\left(\frac{\varphi_q(X_t)}{\varphi_q(x)}\right) \underset{t\to\infty}{\sim} k(x)\sqrt{t} \tag{1.1.23}$$

with $k(x) = \dfrac{1}{\varphi_q(x)} \cdot \sqrt{\dfrac{2}{\pi}} > 0$, since $\varphi_q(x) \underset{|x|\to\infty}{\sim} |x|$. It then remains to let $t \longrightarrow \infty$ in (1.1.23), then to apply once again the monotone convergence Theorem to obtain point 2) of Lemma 1.1.3.

iii) Formula (1.1.16) is then a consequence of :

Lemma 1.1.4. *The measure* $\varphi_q(x)\exp\left(\frac{1}{2}\, A_\infty^{(q)}\right) \cdot W_{x,\infty}^{(q)}$ *does not depend on* $q \in \mathcal{I}$.

We note that the measure $\varphi_q(x)\exp\left(\frac{1}{2}\, A_\infty^{(q)}\right) \cdot W_{x,\infty}^{(q)}$ is well defined since, from point 1) of Lemma 1.1.3, the r.v. $A_\infty^{(q)}$ is $W_{x,\infty}^{(q)}$ a.s. finite. On the other hand, the measure $\varphi_q(x)\exp\left(\frac{1}{2}\, A_\infty^{(q)}\right) \cdot W_{x,\infty}^{(q)}$ has infinite total mass from point 2) of Lemma 1.1.3.

Proof of Lemma 1.1.4.

Let $q_1, q_2 \in \mathcal{I}$. Then, from (1.1.7), we have for every $\Gamma_u \in b_+(\mathcal{F}_u)$, with $u \le t$:

$$\begin{aligned} W_{x,\infty}^{(q_1)}\left(\Gamma_u \varphi_{q_1}(x)\exp\left(\frac{1}{2}\, A_t^{(q_1)}\right)\right) &= W_x\big(\Gamma_u \varphi_{q_1}(X_t)\big) \\ &= W_x\left(\Gamma_u \frac{\varphi_{q_1}(X_t)}{\varphi_{q_2}(X_t)}\, \varphi_{q_2}(X_t)\right) \\ &= W_{x,\infty}^{(q_2)}\left(\Gamma_u \varphi_{q_2}(x)\frac{\varphi_{q_1}(X_t)}{\varphi_{q_2}(X_t)}\exp\left(\frac{1}{2}\, A_t^{(q_2)}\right)\right) \qquad (1.1.24) \end{aligned}$$

Since the relation (1.1.24) takes place for every $\Gamma_u \in b_+(\mathcal{F}_u)$ for any $u \le t$, we may replace Γ_u by $\Gamma_u\exp\big(-\varepsilon A_t^{(q_1+q_2)}\big)$ $(\varepsilon > 0)$. We obtain :

$$\begin{aligned} & W_{x,\infty}^{(q_1)}\left[\Gamma_u \varphi_{q_1}(x)\, \exp\left(\left(\frac{1}{2} - \varepsilon\right) A_t^{(q_1)}\right) \cdot \exp\big(-\varepsilon A_t^{(q_2)}\big)\right] \\ &= W_{x,\infty}^{(q_2)}\left[\Gamma_u \varphi_{q_2}(x)\frac{\varphi_{q_1}(X_t)}{\varphi_{q_2}(X_t)}\, \exp\left(\left(\frac{1}{2} - \varepsilon\right) A_t^{(q_2)}\right) \cdot \exp\big(-\varepsilon A_t^{(q_1)}\big)\right] \qquad (1.1.25) \end{aligned}$$

However — this is point 4) of Theorem 1.1.1 — $|X_t| \underset{t\to\infty}{\longrightarrow} \infty$, $W_{x,\infty}^{(q_2)}$ a.s. and the function $x \longrightarrow \dfrac{\varphi_{q_1}(x)}{\varphi_{q_2}(x)}$ is bounded and tends to 1 when $|x| \longrightarrow \infty$. The dominated convergence Theorem - which we may apply thanks to Lemma 1.1.3 - implies then, by letting $t \longrightarrow \infty$ in (1.1.25) :

$$\begin{aligned} & \varphi_{q_1}(x) W_{x,\infty}^{(q_1)}\left[\Gamma_u \exp\left(\left(\frac{1}{2} - \varepsilon\right) A_\infty^{(q_1)}\right) \exp\big(-\varepsilon A_\infty^{(q_2)}\big)\right] \\ = \;& \varphi_{q_2}(x) W_{x,\infty}^{(q_2)}\left[\Gamma_u \left(\exp\left(\left(\frac{1}{2} - \varepsilon\right) A_\infty^{(q_2)}\right) \cdot \exp\big(-\varepsilon A_\infty^{(q_1)}\big)\right]\right. \qquad (1.1.26) \end{aligned}$$

Since (1.1.26) holds for every $\Gamma_u \in b_+(\mathcal{F}_u)$ the monotone class Theorem implies that (1.1.26) is still true when we replace $\Gamma_u \in b_+(\mathcal{F}_u)$ by $\Gamma \in b_+(\mathcal{F}_\infty)$. It then remains to let $\varepsilon \longrightarrow 0$ and to use the monotone convergence Theorem to obtain : for every $\Gamma \in b_+(\mathcal{F}_\infty)$:

$$\varphi_{q_1}(x) W_{x,\infty}^{(q_1)}\left(\Gamma \exp\left(\frac{1}{2}\, A_\infty^{(q_1)}\right)\right) = \varphi_{q_2}(x) W_{x,\infty}^{(q_2)}\left(\Gamma \exp\left(\frac{1}{2}\, A_\infty^{(q_2)}\right)\right)$$

This is Lemma 1.1.4 and point 1) of Theorem 1.1.2.

iv) We now show that $\mathbf{W}$ has infinite mass, but is σ-finite on $\mathcal{F}_\infty$.

Firstly, it is clear, from point 2) of Lemma 1.1.3, that :

$$\mathbf{W}(1) = \varphi_q(0) W_\infty^{(q)} \left(\exp\left(\frac{1}{2}\, A_\infty^{(q)}\right)\right) = +\infty \tag{1.1.27}$$

On the other hand, from point 1) of Lemma 1.1.3, $A_\infty^{(q)} < \infty \quad W_\infty^{(q)}$ a.s. Hence :

$$1_{A_\infty^{(q)} \le n} \uparrow 1 \qquad W_\infty^{(q)} \quad \text{a.s.}$$

Thus :

$$\mathbf{W}(A_\infty^{(q)} \le n) = \varphi_q(0)\; W_\infty^{(q)} \left(\left(\exp\left(\frac{1}{2}\, A_\infty^{(q)}\right)\right) \cdot 1_{A_\infty^{(q)} \le n}\right) \le \varphi_q(0) e^{\frac{n}{2}} \tag{1.1.28}$$

which proves that $\mathbf{W}$ is $(\Omega, \mathcal{F}_\infty)$ σ-finite.

v) We now show that, for every $\Gamma_t \in b_+(\mathcal{F}_t)$, $\mathbf{W}(\Gamma_t) = 0$ or $+\infty$.

By definition of $\mathbf{W}$, we have :

$$\begin{aligned}
\mathbf{W}(\Gamma_t) &= \varphi_q(0)\; W_\infty^{(q)} \left(\Gamma_t \exp\left(\frac{1}{2}\, A_\infty^{(q)}\right)\right) \\
&= \varphi_q(0)\; W_\infty^{(q)} \left(\Gamma_t \exp\left(\frac{1}{2}\, A_t^{(q)}\right) W_{X_t,\infty}^{(q)} \left(\exp\left(\frac{1}{2}\, A_\infty^{q}\right)\right)\right)
\end{aligned} \tag{1.1.29}$$

from the Markov property. But, from Lemma 1.1.3, $W_{x,\infty}^{(q)} \left(\exp\left(\frac{1}{2}\, A_\infty^{(q)}\right)\right) = +\infty$ for every $x \in \mathbb{R}$. Thus, $\mathbf{W}(\Gamma_t)$ equals 0 or $+\infty$ according to whether $W_\infty^{(q)}(\Gamma_t)$ is 0 or is strictly positive, i.e. according to whether $W(\Gamma_t)$ equals 0 or is strictly positive since, from (1.1.7) and (1.1.8), the probabilities W and $W_\infty^{(q)}$ are equivalent on $\mathcal{F}_t$.

The careful reader may have been surprised about our use in the proof of Lemma 1.1.4 of the r.v. $\exp\big(-\varepsilon\, A_t^{(q_1+q_2)}\big)$. This is purely technical and "counteracts" the fact that $\mathbf{W}$ takes only the values 0 and $+\infty$ on $\mathcal{F}_t$.

We shall now give several other descriptions of the measure $\mathbf{W}$. In order to obtain these descriptions we use a particular case of Theorem 1.1.1, which shall play a key role in our study. This particular case is that of $q = \delta_0$ (or more generally $q = \lambda\delta_0$), the Dirac measure in 0. We begin by recalling a result in this case.

1.1.4 Study of the canonical process under $W_\infty^{(\lambda\delta_0)}$.

Theorem 1.1.5 below has been obtained in [RVY, II], Theorem 8, p. 339, with $h^+(x) = h^-(x) = \exp\left(-\frac{\lambda x}{2}\right)$ $(\lambda, x \ge 0)$.

Theorem 1.1.5. $\big($*A particular case of Theorem 1.1.1, with* $q = \lambda\delta_0$, *hence* $A_t^{(q)} = \lambda L_t,\ t \ge 0$, *where* $(L_t,\ t \ge 0)$ *is the Brownian local time at* 0.$\big)$

1) *The function* $\varphi_{\lambda\delta_0}$ *defined by (1.1.10), (1.1.11) equals :*

$$\varphi_{\lambda\delta_0}(x) = |x| + \frac{2}{\lambda}\,; \quad \textit{hence,} \quad \varphi_{\lambda\delta_0}(0) = \frac{2}{\lambda} \tag{1.1.30}$$

while the martingale $(M_s^{(\lambda\delta_0)},\ s \geq 0)$ $\big($*see (1.1.8)*$\big)$ *equals :*

$$M_s^{(\lambda\delta_0)} = \left(1 + \frac{\lambda}{2}|X_s|\right)\ \exp\left(-\frac{\lambda}{2}\ L_s\right) \tag{1.1.31}$$

2) *Under* $W_\infty^{(\lambda\delta_0)}$ *:*

i) The r.v. $g := \sup\{u \geq 0\ ;\ X_u = 0\}$ *is* $W_\infty^{(\lambda\delta_0)}$ *a.s. finite and* $L_\infty (= L_g)$ *has density :*

$$f_{L_\infty}^{W_\infty^{(\lambda\delta_0)}}(l) = \frac{\lambda}{2}\ e^{-\frac{\lambda}{2}l}\ \ 1_{[0,\infty[}\ (l) \tag{1.1.32}$$

ii) The processes $(X_u,\ u \leq g)$ *and* $(X_{g+u}, u \geq 0)$ *are independent.*

iii) The process $(X_{g+u}, u \geq 0)$ *is distributed with* $P_0^{(3,\text{ sym})}$ *where :*

$$P_0^{(3,\text{ sym})} = \frac{1}{2}\ (P_0^{(3)} + \widetilde{P}_0^{(3)}) \tag{1.1.33}$$

with $P_0^{(3)}$ *(resp.* $\widetilde{P}_0^{(3)}$*) denoting the law of 3-dimensional Bessel process (resp. its opposite) starting from 0.*

iv) Conditionally on $L_\infty (= L_g) = l$*,* $(X_u,\ u \leq g)$ *is a Brownian motion starting from* 0*, considered until its local time at* 0 *reaches level* l*, that is up to the stopping time :*

$$\tau_l := \inf\{t \geq 0\ ;\ L_t > l\} \tag{1.1.34}$$

We write $W_0^{\tau_l}$ *for the law of this process.*

3)

$$W_\infty^{(\lambda\delta_0)} = \frac{\lambda}{2}\int_0^\infty e^{-\frac{\lambda}{2}l}\big(W_0^{\tau_l} \circ P_0^{(3,\text{ sym})}\big)dl \tag{1.1.35}$$

In (1.1.35), we write $W_0^{\tau_l} \circ P_0^{(3,\text{ sym})}$ for the image of the probability $W_0^{\tau_l} \otimes P_0^{(3,\text{ sym})}$ by the concatenation operation $\circ$:

$$\circ : \Omega \times \Omega \longrightarrow \Omega$$

defined by (note that $X_{\tau_l(\omega)} = 0$) :

$$X_t(\omega \circ \widetilde{\omega}) = \begin{cases} X_t(\omega) & \text{if} \quad t \leq \tau_l(\omega) \\ X_{t-\tau_l(\omega)}(\widetilde{\omega}) & \text{if} \quad t \geq \tau_l(\omega) \end{cases} \tag{1.1.36}$$

Such a notation $\circ$ has been used by Biane-Yor [BY] to whom we refer the reader. Let us note that formula (1.1.35) is nothing else but the translation of the results of point 2) of Theorem 1.1.5.

1.1.5 Some remarkable properties of **W**.

We may now describe the measure **W** independently from any penalisation. We introduce :

$$g := \sup\{t\ ;\ X_t = 0\}, \qquad g_a := \sup\{t\ ;\ X_t = a\} \tag{1.1.37}$$

$$\sigma_{a,b} := \sup\big\{t,\ X_t \in [a,b]\big\} \qquad (a < b) \tag{1.1.38}$$

$$\sigma_a := \sup\big\{t,\ X_t \in [-a,a]\big\} \qquad (a \geq 0) \tag{1.1.39}$$

Theorem 1.1.6. *The following identities hold :*

1) $\quad \mathbf{W} = \int_0^\infty dl\,(W_0^{\tau_l} \circ P_0^{(3,\mathrm{sym})})$ (1.1.40)

2) *i) For every $(\mathcal{F}_t,\ t \geq 0)$ stopping time T and for any r.v. Γ_T which is positive and $\mathcal{F}_T$ measurable :*

$$\mathbf{W}(\Gamma_T\, 1_{g<T}\, 1_{T<\infty}) = W\big(\Gamma_T |X_T| 1_{T<\infty}\big) \tag{1.1.41}$$

ii) The law of g under $\mathbf{W}$ is given by :

$$\mathbf{W}(g \in dt) = \frac{dt}{\sqrt{2\pi t}} \qquad (t \geq 0) \tag{1.1.42}$$

iii) Conditionally on $g = t$, the process $(X_u,\ u \leq g)$ under $\mathbf{W}$ is a Brownian bridge with length t. We denote by $\Pi_{0,0}^{(t)}$ the law of this bridge.

iv) $\quad \mathbf{W} = \int_0^\infty \frac{dt}{\sqrt{2\pi t}} \big(\Pi_{0,0}^{(t)} \circ P_0^{(3,\mathrm{sym})}\big)$ (1.1.43)

v) For every previsible and positive process $(\phi_s,\ s \geq 0)$ we have :

$$\mathbf{W}(\phi_g) = W\left(\int_0^\infty \phi_s dL_s\right) \tag{1.1.44}$$

3) *i) For every $(\mathcal{F}_t,\ t \geq 0)$ stopping time T, the law under $\mathbf{W}$ of $L_\infty - L_T$, on $T < \infty$ is given by :*

$$\begin{aligned}\mathbf{W}(L_\infty - L_T \in dl,\ T < \infty) &= W\big(T < \infty\big)\, 1_{[0,\infty]}(l)dl + W\big(|X_T| 1_{T<\infty}\big)\delta_0(dl) \\ &= W\big(T < \infty\big)\, 1_{[0,\infty]}(l)dl + \mathbf{W}(g \leq T < \infty)\delta_0(dl)\end{aligned}$$

In particular, for $T = t$:

$$\mathbf{W}(L_\infty - L_t \in dl) = 1_{[0,\infty[}(l)dl + \sqrt{\frac{2t}{\pi}}\ \delta_0(dl) \tag{1.1.45}$$

ii) For every $l > 0$, conditionally on $L_\infty - L_T = l$, $T < \infty$, $(X_u,\ u \leq T)$ is a Brownian motion indexed by $[0,T]$ (1.1.46)

iii) The density of (g, L_∞) under $\mathbf{W}$ equals :

$$f_{g,L_\infty}^{\mathbf{W}}(u,l) = \frac{l \exp\left(-\dfrac{l^2}{2u}\right)}{\sqrt{2\pi u^3}}\ 1_{[0,\infty[}(u) 1_{[0,\infty[}(l) \tag{1.1.47}$$

Remark 1.1.7.

1) We deduce from formulae (1.1.43) and (1.1.17) that :

$$\begin{aligned}\varphi_q(0) &= \mathbf{W}\left(\exp\left(-\frac{1}{2}\, A_\infty^{(q)}\right)\right) \\ &= \int_0^\infty \frac{dt}{\sqrt{2\pi t}}\, \Pi_{0,0}^{(t)}\left(\exp\left(-\frac{1}{2}\, A_t^{(q)}\right)\right) \cdot P_0^{(3,\mathrm{sym})}\left(\exp\left(-\frac{1}{2}\, A_\infty^{(q)}\right)\right)\end{aligned} \tag{1.1.48}$$

2) It is proven in Biane-Yor $\big($[BY], see also [Bi]$\big)$ that :

$$\int_0^\infty dl\ W_0^{\tau_l} = \int_0^\infty \frac{dt}{\sqrt{2\pi t}}\, \Pi_{0,0}^{(t)}$$

Thus, from this identity, we deduce easily that (1.1.40) implies (1.1.43).

3) Formula (1.1.41) $\big($see also formulae (1.1.52), (1.1.54), (1.1.55), (1.1.56), (1.1.73)$\big)$ yields a "representation" of the Brownian sub-martingale $\big(|X_t|,\ t \geq 0\big)$ in terms of the increasing process $(1_{g\leq t},\ t\geq 0)$. $\big($By a "representation" of a $\big(P,\ (\mathcal{F}_t,\ t\geq 0)\big)$ submartingale $(Z_t,\ t\geq 0)$, we mean a couple $\big(Q, (C_t,\ t\geq 0)\big)$ where Q is a σ-finite measure and $(C_t,\ t\geq 0)$ is a increasing process such that, for every $\Gamma_t \in b(\mathcal{F}_t)\ :\ Q(\Gamma_t \cdot C_t) = E_P[\Gamma_t \cdot Z_t].\big)$ Here, $(\mathbf{W}, 1_{g\leq t}, t\geq 0)$ is a representation of the submartingale $\big(|X_t|,\ t\geq 0\big)$.

Before we prove Theorem 1.1.6, we present a slightly different version of it. We shall not prove this version, whose proof relies on close arguments to those we needed to obtain Theorem 1.1.6.

Theorem 1.1.8. *Let $a \geq 0$; the following formulae hold :*

1) *For every $(\mathcal{F}_t,\ t\geq 0)$ stopping time T and for every r.v. Γ_T positive and $\mathcal{F}_T$ measurable :*

$$\mathbf{W}\big(\Gamma_T\, 1_{(\sigma_a < T < \infty)}\big) = W\big(\Gamma_T\big(|X_T| - a\big)_+ 1_{T<\infty}\big) \tag{1.1.49}$$

2) *i)* $$\mathbf{W}(\sigma_a \in dt) = \frac{e^{-\frac{a^2}{2t}}}{\sqrt{2\pi t}}\, dt \quad (t\geq 0) \tag{1.1.49'}$$

ii) $$\mathbf{W} = \int_0^\infty \frac{dt}{\sqrt{2\pi t}}\, e^{-\frac{a^2}{2t}} \frac{1}{2}\big(\Pi^{(t)}_{0,a} \circ P^{(a,3)} + \Pi^{(t)}_{0,-a} \circ P^{(-a,3)}\big) \tag{1.1.50}$$

where $\Pi^{(t)}_{\alpha,\beta}$ denotes the law of the Brownian bridge of length t starting from α and ending in β and where $P^{(a,3)}$ (resp. $P^{(-a,3)}$) is the law of the process $(a + R_t,\ t\geq 0)$ $\big($resp. $(-a - R_t,\ t\geq 0)\big)$ where $(R_t,\ t\geq 0)$ is a 3-dimensional Bessel process starting from 0. In particular, the law of $(X_u,\ u\leq \sigma_a)$, conditionally on $\sigma_a = t$ is $\frac{1}{2}(\Pi^{(t)}_{0,a} + \Pi^{(t)}_{0,-a})$

iii) For every positive and previsible process $(\phi_u, u\geq 0)$, we have :

$$\mathbf{W}(\phi_{\sigma_a}) = W\left(\int_0^\infty \phi_u\ d_u(\widetilde{L}^a_u)\right) \tag{1.1.51}$$

with $\widetilde{L}^a_u := \dfrac{1}{2}(L^a_u + L^{-a}_u)$.

We note that points 1) and 2) of Theorem 1.1.6 are particular cases of the corresponding ones in Theorem 1.1.8 when $a = 0$. On the other hand, in the same spirit as for (1.1.49) we have, with the same kind of notation :

$$W\big(\Gamma_T (X_T - a)_+ 1_{T<\infty}\big) = \mathbf{W}^+(\Gamma_T 1_{g_a < T < \infty}) \tag{1.1.52$_+$}$$

$$W\big(\Gamma_T (X_T - a)_- 1_{T<\infty}\big) = \mathbf{W}^-(\Gamma_T 1_{g_a < T < \infty}) \tag{1.1.52$_-$}$$

where :

$$\mathbf{W}^+ = \frac{1}{2}\int_0^\infty \frac{dt}{\sqrt{2\pi t}}\ \Pi^{(t)}_{0,0} \circ P^{(3)}_0 \tag{1.1.53$_+$}$$

$$\mathbf{W}^- = \frac{1}{2}\int_0^\infty \frac{dt}{\sqrt{2\pi t}}\ \Pi^{(t)}_{0,0} \circ \widetilde{P}^{(3)}_0 \tag{1.1.53$_-$}$$

Adding $(1.1.52)_+$ and $(1.1.52)_-$ yields :

$$W(\Gamma_T |X_T - a| 1_{T<\infty}) = \mathbf{W}(\Gamma_T 1_{g_a < T < \infty}) \tag{1.1.54}$$

and also, with $a < b$:

$$W\big(\Gamma_T\big((X_T-b)_+ + (a-X_T)_+\big)1_{T<\infty}\big) = \mathbf{W}(\Gamma_T 1_{\sigma_{a,b}<T<\infty}) \tag{1.1.55}$$

and

$$W\big(\Gamma_T\big(|X_T|-a\big)_+ 1_{T<\infty}\big) = \mathbf{W}(\Gamma_T 1_{\sigma_a<T<\infty}) \qquad (a \geq 0) \tag{1.1.56}$$

Proof of Theorem 1.1.6.
Here is the plan of our proof. We shall use formula (1.1.16) with $q = \delta_0$:

$$\mathbf{W} = \varphi_{\delta_0}(0)\, e^{\frac{1}{2}\,L_\infty}\,.\, W_\infty^{(\delta_0)} = 2\, e^{\frac{1}{2}\,L_\infty}\,.\, W_\infty^{(\delta_0)} \tag{1.1.57}$$

(from (1.1.30)), as well as the properties of $W_\infty^{(\delta_0)}$ recalled in Theorem 1.1.5.

i) We prove (1.1.40) .
Let F and G be two positive functionals. We have, from (1.1.57) :

$$\begin{aligned}
&\mathbf{W}\big(F(X_s,\ s\leq g)\cdot G(X_{g+s},\ s\geq 0)\big)\\
&= 2\,W_\infty^{(\delta_0)}\big(e^{\frac{1}{2}L_\infty}F(X_s,\ s\leq g)\,G(X_{g+s},\ s\geq 0)\big)\\
&= 2\,W_\infty^{(\delta_0)}\big(e^{\frac{1}{2}L_g}F(X_s,\ s\leq g)\,G(X_{g+s},\ s\geq 0)\big)\\
&\quad (\text{since } L_\infty = L_g)\\
&= 2\,W_\infty^{(\delta_0)}\big(e^{\frac{1}{2}L_g}F(X_s,\ s\leq g)\big)\cdot P_0^{(3,\text{sym})}\big(G(X_s,\ s\geq 0)\big)\\
&\quad \big(\text{from Point 2)ii) of Theorem 1.1.5 and from (1.1.33)}\big)\\
&= \left(2\int_0^\infty W_\infty^{(\delta_0)}\big(e^{\frac{1}{2}L_g}F(X_s,\ s\leq g)\big|L_g=l\big)\frac{1}{2}\,e^{-\frac{l}{2}}dl\right)\cdot P_0^{(3,\text{sym})}\big(G(X_s,\ s\geq 0)\big)\\
&\quad \big(\text{from (1.1.32)}\big)\\
&= \left(2\int_0^\infty e^{\frac{l}{2}}W\big(F(X_s,\ s\leq \tau_l)\big)\frac{1}{2}\,e^{-\frac{l}{2}}dl\right)\cdot P_0^{(3,\text{sym})}\big(G(X_s,\ s\geq 0)\big)\\
&= \int_0^\infty dl(W_0^{\tau_l}\circ P_0^{(3,\text{sym})})\,\big(F(X_s,\ s\leq g)\cdot G(X_{g+s},\ s\geq 0)\big)
\end{aligned}$$

from point 2, *iv)* of Theorem 1.1.5.
ii) We now prove (1.1.41).
For this purpose, we apply formula with $q = \lambda\delta_0$. Thus :

$$A_t^{(q)} = \lambda L_t \quad \text{and, from (1.1.30),} \quad \varphi_{\lambda\delta_0}(x) = \frac{2}{\lambda} + |x|.$$

Thus, from (1.1.7), (1.1.30), (1.1.31), (1.1.16) and Doob's optional stopping Theorem :

$$\begin{aligned}
W\left(\Gamma_T\left(\frac{2}{\lambda}+|X_T|\right)1_{T<\infty}\right) &= \frac{2}{\lambda}\,W_\infty^{(\lambda\delta_0)}\big(e^{\frac{\lambda}{2}L_T}\Gamma_T\,1_{T<\infty}\big)\\
&= \mathbf{W}(\Gamma_T\,1_{g\leq T<\infty}) + \mathbf{W}\big(\Gamma_T\,1_{g>T}e^{-\frac{\lambda}{2}(L_\infty-L_T)}\big)
\end{aligned} \tag{1.1.58}$$

We then let $\lambda \longrightarrow \infty$ in (1.1.58) and note that $L_\infty - L_T > 0$ on $g > T$. The monotone convergence Theorem implies :

$$W\big(\Gamma_T|X_T|1_{T<\infty}\big) = \mathbf{W}(\Gamma_T\,1_{g\leq T<\infty})$$

This is precisely relation (1.1.41). Relation (1.1.42) is an easy consequence of (1.1.41).

iii) We prove (1.1.45) and (1.1.46).

We note that (1.1.41) and (1.1.58) imply :

$$\begin{aligned} \frac{2}{\lambda} W(\Gamma_T 1_{T<\infty}) &= \mathbf{W}\left(\Gamma_T 1_{g>T} \exp\left(-\frac{\lambda}{2}(L_\infty - L_T)\right)\right) \\ &= W(\Gamma_T 1_{T<\infty}) \left(\int_0^\infty e^{-\frac{\lambda}{2} l} dl\right) \end{aligned} \tag{1.1.59}$$

Thus, by injectivity of the Laplace transform, for every function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ Borel and integrable :

$$W(\Gamma_T 1_{T<\infty}) \left(\int_0^\infty \psi(l) dl\right) = \mathbf{W}\big(\Gamma_T \psi(L_\infty - L_T) 1_{g>T}\big) \tag{1.1.60}$$

and

$$W\big(|X_T| 1_{T<\infty}\big) = \mathbf{W}(g \le T < \infty) = \mathbf{W}(L_\infty - L_T = 0,\ T < \infty) \tag{1.1.61}$$

In other terms, we have :

$$\mathbf{W}(L_\infty - L_T \in dl,\ T < \infty) = W\big(T < \infty\big) 1_{[0,\infty[}(l) dl + W\big(|X_T| 1_{T<\infty}\big)\delta_0(dl)$$

and, under $\mathbf{W}$, conditionally on $L_\infty - L_T = l \ \ (l > 0)$, $\ (X_u,\ u \le T)$ is a Brownian motion indexed by $[0, T]$. This is (1.1.45) and (1.1.46).

iv) We now prove point 2, *iii)* of Theorem 1.1.6.

For this purpose, we write (1.1.41), choosing for Γ_t a r.v. of the form $\Phi_{g^{(t)}}$, where $(\Phi_u,\ u \ge 0)$ is a previsible positive process, and where $g^{(t)} := \sup\{s \le t,\ X_s = 0\}$.

The RHS of (1.1.41) becomes, with $T = t$:

$$\begin{aligned} W\big(|X_t| \Phi_{g^{(t)}}\big) &= W\left(\int_0^t \Phi_s\, dL_s\right) \\ &\quad \Big(\text{from the balayage formula } \big(\text{cf [ReY], Chap. VI, p. 260}\big)\Big) \\ &= \int_0^t W\big(\Phi_s | X_s = 0\big) W(dL_s) \\ &= \int_0^t W\big(\Phi_s | X_s = 0\big) \frac{ds}{\sqrt{2\pi s}} \qquad\qquad (1.1.62) \\ &\quad \left(\text{since } W(L_s) = W\big(|X_s|\big) = \sqrt{\frac{2s}{\pi}}\right) \end{aligned}$$

The LHS of (1.1.41) writes :

$$\begin{aligned} \mathbf{W}(\Phi_{g^{(t)}} 1_{g\le t}) &= \mathbf{W}(\Phi_g\, 1_{g\le t}) \\ &\quad \big(\text{since } g = g^{(t)} \text{ on the set } \{g \le t\}\big) \\ &= \int_0^t \mathbf{W}\big(\Phi_g | g = s\big) \frac{ds}{\sqrt{2\pi s}} \qquad\qquad (1.1.63) \\ &\quad \text{from (1.1.42). Thus :} \end{aligned}$$

$$\int_0^t W\big(\Phi_s | X_s = 0\big) \frac{ds}{\sqrt{2\pi s}} = \int_0^t \mathbf{W}\big(\Phi_g | g = s\big) \frac{ds}{\sqrt{2\pi s}}$$

This relation implies $\mathbf{W}\big(\Phi_s|g=s\big) = W\big(\Phi_s|X_s=0\big)$, i.e. point 2, *iii)* of Theorem 1.1.6.
We also note that we deduce from the equality between (1.1.62) and (1.1.63) :

$$W\left(\int_0^t \Phi_s\, dL_s\right) = \int_0^t \mathbf{W}\big(\Phi_g|g=s\big)\frac{ds}{\sqrt{2\pi s}}$$

that :

$$\begin{aligned} W\left(\int_0^\infty \Phi_s\, dL_s\right) &= \int_0^\infty \mathbf{W}\big(\Phi_g|g=s\big)\mathbf{W}(g\in ds) \\ &= \mathbf{W}(\Phi_g) \end{aligned} \tag{1.1.64}$$

i.e. point 2, *v)* of Theorem 1.1.6.

v) <u>We now prove point 2, *iv)* of Theorem 1.1.6.</u>
We obtain, with the help of (1.1.57), for two positive functionals F and G :

$$\begin{aligned} &\mathbf{W}\big(F(X_s,\ s\le g)\, G(X_{g+s},\ s\ge 0)\big) \\ &= 2\,W_\infty^{(\delta_0)}\big(F(X_s,\ s\le g)\, e^{\frac12 L_g} G(X_{g+s,\, s\ge 0})\big) \\ &= 2\,W_\infty^{(\delta_0)}\big(F(X_s,\ s\le g)\, e^{\frac12 L_g}\big)\ P_0^{(3,\mathrm{sym})}\big(G(X_s,\ s\ge 0)\big) \end{aligned}$$

(from point 2 *ii)* and 2 *iii)* of Theorem 1.1.5)

$$= \mathbf{W}\big(F(X_s,\ s\le g)\big)\cdot P_0^{(3,\mathrm{sym})}\big(G(X_s,\ s\ge 0)\big)$$

(using once again (1.1.57))

$$= \left(\int_0^\infty \mathbf{W}\big(F(X_s,\ s\le g)|g=t\big)\frac{dt}{\sqrt{2\pi t}}\right)\cdot P_0^{(3,\mathrm{sym})}\big(G(X_s,\ s\ge 0)\big)$$

(from (1.1.42))

$$= \int_0^\infty \frac{dt}{\sqrt{2\pi t}}\ \Pi_{0,0}^{(t)}\big(F(X_s,\ s\le t)\big)\cdot P_0^{(3,\mathrm{sym})}\big(G(X_s,\ s\ge 0)\big)$$

(from point 2 *iii)* of Theorem 1.1.6)

$$= \int_0^\infty \frac{dt}{\sqrt{2\pi t}}\big(\Pi_{0,0}^{(t)}\circ P_0^{(3,\mathrm{sym})}\big)\ \big(F(X_s,\ s\le g)\, G(X_{g+s},\ s\ge 0)\big)\cdot$$

vi) Formula (1.1.47) is a consequence of (1.1.42), (1.1.43) and the fact that :
Under $\Pi_{0,0}^{(t)}$, L_t is distributed as $\sqrt{2t\mathfrak{e}}$, where $\mathfrak{e}$ is a standard exponential r.v.

Remark 1.1.9.

1) We have, from (1.1.16) and Theorem 1.1.5 :

$$\frac{\lambda}{2}\, e^{-\frac{\lambda}{2}L_\infty}\cdot \mathbf{W} = W_\infty^{(\lambda\delta_0)} \tag{1.1.65}$$

But, from Theorem 1.1.1, under $W_\infty^{(\lambda\delta_0)}$:

$$X_t = B_t + \int_0^t \frac{\operatorname{sgn} X_s}{\frac{2}{\lambda}+|X_s|}\, ds$$

Hence (see [RY, M], Chap. 4) : $W_\infty^{(\lambda\delta_0)} \underset{\lambda\to\infty}{\longrightarrow} P_0^{(3,\,\mathrm{sym})}$.

Thus $\displaystyle \frac{\lambda}{2}\,(e^{-\frac{\lambda}{2}L_\infty})\,\mathbf{W} \underset{\lambda\to\infty}{\longrightarrow} P_0^{(3,\,\mathrm{sym})}$ (1.1.66)

This convergence holds in the sense of weak convergence with respect to the topology of uniform convergence on compacts in $\mathcal{C}\big([0,\infty[\to\mathbb{R})$.

2) Formula (1.1.41) may be proven in a different manner than by the way we have indicated. Indeed, from (1.1.57) (where, to simplify, we choose $T=t$)

$$\begin{aligned}\mathbf{W}(\Gamma_t\, 1_{g\le t}) &= 2\,W_\infty^{(\delta_0)}(\Gamma_t\, 1_{g\le t}\, e^{\frac12 L_\infty}) \\ &= 2\,W_\infty^{(\delta_0)}(\Gamma_t\, 1_{g\le t}\, e^{\frac12 L_t}) \\ &\quad \big(\text{since } L_\infty = L_t \ \text{ on the set } (g\le t)\big) \\ &= 2\,W_\infty^{(\delta_0)}(\Gamma_t\, e^{\frac12 L_t} W_\infty^{(\delta_0)}\big(1_{g\le t}|\mathcal{F}_t\big)\big) \end{aligned} \tag{1.1.67}$$

But

$$\begin{aligned} W_\infty^{(\delta_0)}\big(1_{g\le t}|\mathcal{F}_t\big) &= W_\infty^{(\delta_0)}\,\big(T_0\circ\theta_t=\infty|\mathcal{F}_t\big) \\ &= W_{X_t,\infty}^{(\delta_0)}\,(T_0=\infty) \end{aligned} \tag{1.1.68}$$

with $T_0=\inf\{t\ge 0\ ;\ X_t=0\}$, by Markov property. But, from (1.1.14), the scale function of the process $(X_t,\ t\ge 0)$ under $(W_{x,\infty}^{(\delta_0)},\ x\in\mathbb{R})$ equals :

$$\gamma_{\delta_0}(x)=\frac{x}{2\big(2+|x|\big)} \tag{1.1.69}$$

We deduce from (1.1.69) :

$$W_{x,\infty}^{(\delta_0)}(T_0=\infty)=\frac{|x|}{2+|x|} \tag{1.1.70}$$

Plugging (1.1.70) and (1.1.68) in (1.1.67), we obtain :

$$\begin{aligned}\mathbf{W}(\Gamma_t\, 1_{g\le t}) &= 2\,W_\infty^{(\delta_0)}\left(\Gamma_t\, e^{\frac12 L_t}\,\frac{|X_t|}{2+|X_t|}\right) \\ &= 2\,W\left(\Gamma_t\ e^{\frac12 L_t}\ \frac{|X_t|}{2+|X_t|}\ \frac{2+|X_t|}{2}\ e^{-\frac12 L_t}\right) \\ &\quad (\text{from (1.1.31), with } \lambda=1, \text{ and (1.1.7)}) \\ &= W\big(\Gamma_t|X_t|\big)\end{aligned}$$

Formulae (1.1.54), (1.1.56), (1.1.57) may be proven following the same arguments.

3) Let $q\in\mathcal{I}$ such that the convex hull of its support equals the interval $[a,b]$ $(a\le b)$. From (1.1.7) and (1.1.6) we have :

$$\begin{aligned} W\big(\varphi_{\lambda q}(X_t)\cdot\Gamma_t\big) &= \varphi_{\lambda q}(0)\,W_\infty^{(\lambda q)}\big(\Gamma_t\, e^{-\frac{\lambda}{2}A_t^{(q)}}\big) \\ &= \mathbf{W}\big(\Gamma_t\, e^{-\frac{\lambda}{2}(A_\infty^{(q)}-A_t^{(q)})}\big) \\ &= \mathbf{W}(\Gamma_t\, 1_{\sigma_{a,b}\le t}) + \mathbf{W}\big(\Gamma_t e^{-\frac{\lambda}{2}(A_\infty^{(q)}-A_t^{(q)})}1_{\sigma_{a,b}>t}\big)\end{aligned} \tag{1.1.71}$$

On the other hand, we have proven in [RY, IX] (see also [RY, M], Chap. 2) that there exists, for every $x\in\mathbb{R}$, a positive and σ-finite measure $\boldsymbol{\nu}_x^{(q)}$ such that :

$$\int_0^\infty e^{-\frac{\lambda y}{2}}\boldsymbol{\nu}_x^{(q)}(dy)=\varphi_{\lambda q}(x) \tag{1.1.72}$$

It remains to let $\lambda \to \infty$ in (1.1.72) to obtain, since $A_\infty^{(q)} - A_t^{(q)} > 0$ on the set $(\sigma_{a\,b} > t)$:

$$W\big(\Gamma_t\, \boldsymbol{\nu}_{X_t}^{(q)}(\{0\}) = \mathbf{W}(\Gamma_t\, 1_{\sigma_{a,b} \le t}) \tag{1.1.73}$$

Hence, $\boldsymbol{\nu}_x^{(q)}(\{0\})$ depends only on supp(q) and $\big(\boldsymbol{\nu}_{X_t}^{(q)}(\{0\}),\ t \ge 0\big)$ is a sub-martingale. Formula (1.1.55) (with $T = t$) is a particular case of (1.1.73), since :

$$\boldsymbol{\nu}_x^{(\delta_a+\delta_b)}(\{0\}) = (x-b)_+ + (a-x)_+ \tag{1.1.74}$$

$\big($see [RY, IX]$\big)$.

1.1.6. Another approach to Theorem 1.1.6.

Let, for $q \in \mathcal{I}$, the probability $W_\infty^{(q)}$ be defined by (1.1.7). Then :

$$W_\infty^{(q)}{}_{|\mathcal{F}_t} = \frac{\varphi_q(X_t)}{\varphi_q(0)}\, e^{-\frac{1}{2} A_t^{(q)}} \cdot W_{|\mathcal{F}_t} \tag{1.1.75}$$

In Theorem 1.1.2, we have defined the measure $\mathbf{W}$ from the formula :

$$\mathbf{W} = \varphi_q(0)\, e^{\frac{1}{2} A_\infty^{(q)}} W_\infty^{(q)} \tag{1.1.76}$$

then, we have shown that :

$$\mathbf{W} = \int_0^\infty \frac{dt}{\sqrt{2\pi t}}\, (\Pi_{0,0}^{(t)} \circ P_0^{(3,\,\mathrm{sym})}) \tag{1.1.77}$$

$\big($cf Theorem 1.1.6, relation (1.1.43)$\big)$. We now "forget" our previous results and proceed in a reverse way. For this purpose, we define, for the time being, the measure :

$$\widetilde{\mathbf{W}} = \int_0^\infty \frac{dt}{\sqrt{2\pi t}} (\Pi_{0,0}^{(t)} \circ P_0^{(3,\,\mathrm{sym})}) \tag{1.1.78}$$

We shall show that, for every $q \in \mathcal{I}$:

$$\frac{1}{\varphi_q(0)}\, e^{-\frac{1}{2} A_\infty^{(q)}} \cdot \widetilde{\mathbf{W}} = W_\infty^{(q)}$$

Theorem 1.1.10.
Let $\widetilde{\mathbf{W}}$ be defined by (1.1.78) and $W_\infty^{(q)}$ be defined by (1.1.75). Then, for every $q \in \mathcal{I}$:

$$\frac{1}{\varphi_q(0)}\, e^{-\frac{1}{2} A_\infty^{(q)}} \cdot \widetilde{\mathbf{W}} = W_\infty^{(q)} \tag{1.1.79}$$

Proof of Theorem 1.1.10.
We compute the value of $W_\infty^{(q)}$ when integrating the following general class of functionals which are $\mathcal{F}_t$-measurable and positive :

$$F(X_u,\ u \le g^{(t)}) \cdot G(X_{g^{(t)}+u},\ u \le t - g^{(t)}) \tag{1.1.80}$$

We have :

$$\begin{aligned}
&W_\infty^{(q)}\big(F(X_u,\ u \le g^{(t)})\ G(X_{g^{(t)}+u}\ ;\ u \le t - g^{(t)})\big)\\
&= \frac{1}{\varphi_q(0)}\, W\left[F(X_u,\ u \le g^{(t)})\ G(X_{g^{(t)}+v}\ ;\ v \le t - g^{(t)}) \exp\left(-\frac{1}{2}\, A_t^{(q)}\right) \varphi_q(X_t)\right]\\
&\big(\text{from}(1.1.75)\big)\\
&= \frac{1}{\varphi_q(0)}\, W\left[F(X_u,\ u \le g^{(t)}) \exp\left(-\frac{1}{2}\, A_{g^{(t)}}^{(q)}\right) \cdot G(X_{g^{(t)}+u},\ u \le t - g^{(t)})\right.\\
&\qquad\left. \cdot\varphi_q(X_t)\, \exp\left(-\frac{1}{2}\big(A_t^{(q)} - A_{g^{(t)}}^{(q)}\big)\right)\right]
\end{aligned} \tag{1.1.81}$$

We now consider the probability W restricted to $\mathcal{F}_t$, denoted as $W^{(t)}$, which we disintegrate with respect to the law of $g^{(t)}$:

$$W^{(t)} = \int_0^t \frac{du}{\pi\sqrt{u(t-u)}}\, \big(\Pi_{0,0}^{(u)} \circ M^{(t-u,\mathrm{sym})}\big) \tag{1.1.82}$$

with :

$$W(g^{(t)} \in du) = \frac{du}{\pi\sqrt{u(t-u)}} \qquad u \le t$$

and where $\Pi_{0,0}^{(u)}$ denotes the law of the Brownian bridge with length u and $M^{(t,\,\mathrm{sym})}$ is the law of a symmetric Brownian meander of length t. Thus, (1.1.81) becomes :

$$\begin{aligned}
&W_\infty^{(q)}\big[F(X_u,\ u \le g^{(t)})\ G(X_{g^{(t)}+v},\ v \le t - g^{(t)})\big]\\
&= \frac{1}{\varphi_q(0)} \int_0^t \frac{du}{\pi\sqrt{u(t-u)}}\, \Pi_{0,0}^{(u)}\big(F(X_v,\ v \le u) e^{-\frac{1}{2}\, A_u^{(q)}}\big)\\
&\qquad \cdot M^{(t-u,\,\mathrm{sym})}\big(\varphi_q(X_{t-u}) e^{-\frac{1}{2}\, A_{t-u}^{(q)}} \cdot G(X_l,\ l \le t-u)\big)
\end{aligned}$$

Using now Imhof's relation $\big($see [RY, M], Chap. 1, Item $G\big)$:

$$M^{(t,\,\mathrm{sym})} = \sqrt{\frac{\pi t}{2}}\, \frac{1}{|X_t|}\, P_0^{(3,\,\mathrm{sym})}|_{\mathcal{F}_t} \tag{1.1.83}$$

we obtain :

$$\begin{aligned}
&W_\infty^{(q)}\big[F(X_u,\ u \le g^{(t)})\, G(X_{g^{(t)}+v},\ v \le t - g^{(t)})\big]\\
&= \frac{1}{\varphi_q(0)} \int_0^t \frac{du}{\pi\sqrt{u(t-u)}}\, \Pi_{0,0}^{(u)}\big(F(X_v\ v \le u)\, e^{-\frac{1}{2}\, A_u^{(q)}}\big)\\
&\qquad \cdot P_0^{(3,\mathrm{sym})}\left(\varphi_q(X_{t-u}) \frac{G(X_l,\ l \le t-u)}{|X_{t-u}|}\, \sqrt{\frac{\pi}{2}(t-u)}\ e^{-\frac{1}{2}\, A_{t-u}^{(q)}}\right)
\end{aligned} \tag{1.1.84}$$

We observe that the factor $\sqrt{t-u}$ simplifies on the RHS of (1.1.84). We then let $t \longrightarrow \infty$ in (1.1.84), by using the fact that $\varphi_q(x) \underset{|x|\to\infty}{\sim} |x|$. We obtain, since $g^{(t)} \underset{t\to\infty}{\longrightarrow} g$ a.s. under $W_\infty^{(q)}$ (cf Theorem 1.1.1) :

$$\begin{aligned} & W_\infty^{(q)}\big(F(X_u,\ u \le g) \cdot G(X_{g+v},\ v \ge 0)\big) \\ = & \ \frac{1}{\varphi_q(0)} \left(\int_0^\infty \frac{du}{\sqrt{2\pi u}}\, \Pi_{0,0}^{(u)}\big(F(X_v,\ v \le u) e^{-\frac{1}{2} A_u^{(q)}}\big)\right) \cdot P_0^{(3,\mathrm{sym})}\big(G(X_l,\ l \ge 0)\, e^{-\frac{1}{2} A_\infty^{(q)}}\big) \\ = & \ \frac{1}{\varphi_q(0)}\, \widetilde{\mathbf{W}}\big(e^{-\frac{1}{2} A_\infty^{(q)}} F(X_u,\ u \le g)\, G(X_{g+l},\ l \ge 0)\big) \end{aligned}$$

This is the statement of Theorem 1.1.10.

1.1.7 Relations between $\mathbf{W}$ and other penalisations (than the Feynman-Kac ones).

We have shown - this is Theorem 1.1.2 - that for every $q \in \mathcal{I}$:

$$\begin{aligned} \mathbf{W} &= \varphi_q(0) \exp\left(\frac{1}{2} A_\infty^{(q)}\right) \cdot W_\infty^{(q)} \\ &= \mathbf{W}\left(\exp\left(-\frac{1}{2} A_\infty^{(q)}\right)\right) \cdot \exp\left(\frac{1}{2} A_\infty^{(q)}\right) \cdot W_\infty^{(q)} \end{aligned} \tag{1.1.85}$$

Of course, this formula is very much linked with the penalisation of the Wiener measure by the multiplicative functional $\left(F_t = \exp\left(-\frac{1}{2} A_t^{(q)}\right),\ t \ge 0\right)$. Here, we shall prove that formulae analogous to (1.1.85) are true for other penalisations than these Feynman-Kac ones.

We now fix some notations :

$$S_t := \sup_{s\le t} X_s, \quad I_t := \inf_{s \le t} X_s \tag{1.1.86}$$

$$\Gamma_+ := \{\omega \in \Omega\ ;\ \lim_{t\to\infty} X_t(\omega) = +\infty\},\ \Gamma_- := \{\omega \in \Omega, \lim_{t\to\infty} X_t(\omega) = -\infty\} \tag{1.1.87}$$

$$\mathbf{W}^+ := 1_{\Gamma_+} \cdot \mathbf{W}, \quad \mathbf{W}^- := 1_{\Gamma_-} \cdot \mathbf{W} \tag{1.1.88}$$

$$\theta_+ := \sup\{t\ ;\ S_t < S_\infty\}, \qquad \theta_- := \sup\{t\ ;\ I_t > I_\infty\} \tag{1.1.89}$$

Let ψ_+ (resp. ψ_-) a Borel and integrable function from $\mathbb{R}_+$ to $\mathbb{R}_+$ (resp. from $\mathbb{R}_-$ to $\mathbb{R}_+$). We denote by $(M_s^{\psi_+(S)},\ s \ge 0)$ $\big($resp. $(M_s^{\psi_-(I)},\ s \ge 0)\big)$ the Azéma-Yor martingale defined by :

$$M_s^{\psi_+(S)} := \frac{1}{\left(\int_0^\infty \psi_+(y)dy\right)} \left(\psi_+(S_s)\,(S_s - X_s) + \int_{S_s}^\infty \psi_+(y)dy\right) \tag{1.1.90}$$

$$M_s^{\psi_-(I)} := \frac{1}{\left(\int_{-\infty}^0 \psi_-(y)dy\right)} \left(\psi_-(I_s)\,(X_s - I_s) + \int_{-\infty}^{I_s} \psi_-(y)dy\right) \tag{1.1.91}$$

Let $W_\infty^{\psi_+(S)}$ (resp $W_\infty^{\psi_-(I)}$) denote the probability on $(\Omega, \mathcal{F}_\infty)$ characterized by :

$$W_\infty^{\psi_+(S)}{}_{|\mathcal{F}_t} = M_t^{\psi_+(S)} \cdot W_{|\mathcal{F}_t}, \qquad W_\infty^{\psi_-(I)} \cdot_{|\mathcal{F}_t} = M_t^{\psi_-(I)} \cdot W_{|\mathcal{F}_t} \tag{1.1.92}$$

$\big($see [RVY, II] for more informations about these probabilities$\big)$.

The analogue of formulae (1.1.85) and (1.1.41) is here :

Theorem 1.1.11. *Let ψ_+, ψ_- as above, with $\psi_+(\infty) = \psi_-(-\infty) = 0$.*

1)
$$W_\infty^{\psi_+(S)} = \frac{1}{\mathbf{W}\big(\psi_+(S_\infty)\big)} \cdot \psi_+(S_\infty) \cdot \mathbf{W}^- \tag{1.1.93}$$
$$W_\infty^{\psi_-(I)} = \frac{1}{\mathbf{W}\big(\psi_-(I_\infty)\big)} \cdot \psi_-(I_\infty) \cdot \mathbf{W}^+ \tag{1.1.94}$$

2) *For every $t \geq 0$ and $\Gamma_t \in b_+(\mathcal{F}_t)$:*
$$W\big(\Gamma_t(S_t - X_t)\big) = \mathbf{W}^-(\Gamma_t\, 1_{\theta_+ \leq t}) \tag{1.1.95}$$
$$W\big(\Gamma_t(X_t - I_t)\big) = \mathbf{W}^+(\Gamma_t\, 1_{\theta_- \leq t}) \tag{1.1.96}$$

Proof of Theorem 1.1.11.

i) We have, from (1.1.85), for $q \in \mathcal{I}$, and $\Gamma_t \in b_+(\mathcal{F}_t)$:

$$\begin{aligned}
&\mathbf{W}(e^{-\frac{1}{2}A_\infty^{(q)}} \cdot \Gamma_t) \\
&= \varphi_q(0)\, W_\infty^{(q)}(\Gamma_t) \\
&= W\big(\Gamma_t \varphi_q(X_t) e^{-\frac{1}{2}A_t^{(q)}}\big) \\
&\quad \big(\text{from (1.1.7) and (1.1.8)}\big) \\
&= \left(\int_0^\infty \psi(y)dy\right) \cdot W_\infty^{\psi(S)} \left(\Gamma_t \frac{\varphi_q(X_t)e^{-\frac{1}{2}A_t^{(q)}}}{\psi(S_t)(S_t - X_t) + \displaystyle\int_{S_t}^\infty \psi(y)dy}\right)
\end{aligned} \tag{1.1.97}$$

from (1.1.92) and (1.1.90), and we have written, to simplify, ψ for ψ_+. Formula (1.1.97) being true for every $\Gamma_t \in b_+(\mathcal{F}_t)$, we may take $\Gamma_t = \Gamma_u\, 1_{\psi(S_t)>a} \cdot 1_{S_t - X_t > b|X_t|} \cdot 1_{\int_{S_t}^\infty \psi(y)dy > c}$ with $0 < b < 1$, $a, c > 0$ for any $\Gamma_u \in \mathcal{F}_u$, $u \leq t$. We obtain thus :

$$\begin{aligned}
&\mathbf{W}\big[\Gamma_u e^{-\frac{1}{2}A_\infty^{(q)}}\, 1_{\psi(S_t)>a} 1_{S_t - X_t > b|X_t|}\, 1_{\int_{S_t}^\infty \psi(y)dy > c}\big] \\
&= \left(\int_0^\infty \psi(y)dy\right) \cdot W_\infty^{\psi(S)} \Bigg[\Gamma_u \frac{\varphi_q(X_t)e^{-\frac{1}{2}A_t^{(q)}}}{\psi(S_t)(S_t - X_t) + \int_{S_t}^\infty \psi(y)dy}\, 1_{\psi(S_t)>a} \\
&\qquad\qquad 1_{S_t - X_t > b|X_t|}\, 1_{\int_{S_t}^\infty \psi(y)dy > c}\Bigg]
\end{aligned} \tag{1.1.98}$$

We shall now let $t \to \infty$ in (1.1.98) with u being fixed. On the LHS, we have :

$$\begin{aligned}
&\mathbf{W}^+ \text{ a.s.} && 1_{\psi(S_t)>a} \underset{t\to\infty}{\longrightarrow} 0 \qquad \big(\text{since } S_t \longrightarrow +\infty \text{ and } \psi(S_t) \underset{t\to\infty}{\longrightarrow} 0, \text{ since } \psi(+\infty) = 0\big) \\
&\mathbf{W}^- \text{ a.s.} && 1_{\psi(S_t)>a} \underset{t\to\infty}{\longrightarrow} 1_{\psi(S_\infty)>a} \\
& && 1_{S_t - X_t > b|X_t]} \underset{t\to\infty}{\longrightarrow} 1 \\
& && 1_{\int_{S_t}^\infty \psi(y)dy > c} \longrightarrow 1_{\int_{S_\infty}^\infty \psi(y)dy > c}
\end{aligned} \tag{1.1.99}$$

Thus, from Lebesgue's dominated convergence Theorem, the LHS of (1.1.98) converges, as $t \to \infty$, towards L, with :

$$L = \mathbf{W}\big(\Gamma_u 1_{\Gamma_-} e^{-\frac{1}{2}A_\infty^q} 1_{\psi(S_\infty)>a}\, 1_{\int_{S_\infty}^\infty \psi(y)dy > c}\big) \tag{1.1.100}$$

We now consider the RHS of (1.1.98). On the set :

$$\big(\psi(S_t) > a\big) \cap \big(S_t - X_t > b|X_t|\big) \cap \left(\int_{S_t}^{\infty} \psi(y)dy > c\right),$$

we have :

$$\frac{\varphi_q(X_t)}{\psi(S_t)(S_t - X_t) + \int_{S_t}^{\infty} \psi(y)dy} \leq \frac{d + |X_t|}{ab|X_t| + c} \leq k$$

since $|\varphi_q(x)| \leq d + |x|$; thus, we may apply the dominated convergence Theorem to obtain, since under $W_\infty^{\psi(S)}$ (see [RVY, II]) : $X_t \underset{t\to\infty}{\longrightarrow} -\infty$ and $S_t \underset{t\to\infty}{\longrightarrow} S_\infty$ a.s., the convergence of the RHS of (1.1.98) to R, with :

$$R = \left(\int_0^\infty \psi(y)dy\right) \cdot W_\infty^{\psi(S)} \left(\frac{\Gamma_u}{\psi(S_\infty)}\, e^{-\frac{1}{2}A_\infty^{(q)}} 1_{\psi(S_\infty)>a}\, 1_{\int_{S_\infty}^{\infty} \psi(y)dy>c}\right) \tag{1.1.101}$$

$\big($since $\varphi_q(x) \underset{|x|\to\infty}{\sim} |x|\big)$. Hence, letting $a, c \to 0$ and applying the monotone class Theorem, the equality between (1.1.100) and (1.1.101) implies, for every $\Gamma \in b_+(\mathcal{F}_\infty)$:

$$\mathbf{W}^-(\Gamma e^{-\frac{1}{2}A_\infty^{(q)}}) = \left(\int_0^\infty \psi(y)dy\right) W_\infty^{\psi(S)} \left(\frac{\Gamma}{\psi(S_\infty)}\, e^{-\frac{1}{2}A_\infty^{(q)}}\right)$$

then, replacing $\Gamma e^{-\frac{1}{2}A_\infty^{(q)}}$ by Γ :

$$\begin{aligned}
\mathbf{W}^-(\Gamma) &= \left(\int_0^\infty \psi(y)dy\right) \cdot W_\infty^{\psi(S)} \left(\frac{\Gamma}{\psi(S_\infty)}\right) \\
&= \mathbf{W}^-\big(\psi(S_\infty)\big)\, W_\infty^{\psi(S)} \left(\frac{\Gamma}{\psi(S_\infty)}\right) \\
&= \mathbf{W}\big(\psi(S_\infty)\big)\, W_\infty^{\psi(S)} \left(\frac{\Gamma}{\psi(S_\infty)}\right)
\end{aligned}$$

since $\psi(\infty) = 0$ and $S_\infty = +\infty \quad \mathbf{W}^+$ a.s.

We note that there is no problem to divide by $\psi(S_\infty)$ since $\psi(S_\infty) > 0 \quad W_\infty^{\psi(S)}$ a.s. $\big($under $W_\infty^{\psi(S)}$, S_∞ admits ψ as density (see [RVY, II])$\big)$.

We have proven (1.1.93), and the proof of (1.1.94) is similar.

ii) We now prove (1.1.95).

For this purpose, we use the penalisation by $(e^{-\frac{\lambda}{2}S_t},\ t \geq 0)$ i.e. (1.1.90) and (1.1.92), with $\psi_+(x) = e^{-\lambda x}$. We obtain :

$$M_t^{\psi_+(S)} = \left(1 + \frac{\lambda}{2}\,(S_t - X_t)\right) e^{-\frac{\lambda}{2}S_t} \tag{1.1.102}$$

Hence, for every $t \geq 0$ and $\Gamma_t \in b_+(\mathcal{F}_t)$:

$$\begin{aligned}
W\left(\Gamma_t\left(\frac{2}{\lambda} + (S_t - X_t)\right)\right) &= \frac{2}{\lambda}\, W_\infty^{\psi_+(S)}\big[e^{\frac{\lambda S_t}{2}}\Gamma_t\big] \\
&= \mathbf{W}^-\big(e^{-\frac{\lambda}{2}(S_\infty - S_t)}\Gamma_t\big) \qquad \big(\text{from (1.1.93)}\big) \\
&= \mathbf{W}^-(\Gamma_t\, 1_{\theta_+\leq t}) + \mathbf{W}^-(\Gamma_t\, e^{-\frac{\lambda}{2}(S_\infty - S_t)} 1_{\theta_+>t})
\end{aligned} \tag{1.1.103}$$

We then let $\lambda \to +\infty$ in (1.1.103) by noting that $S_\infty - S_t > 0$ on $(\theta_+ > t)$. We obtain :

$$W\big(\Gamma_t(S_t - X_t)\big) = \mathbf{W}^-(\Gamma_t\, 1_{\theta_+ \le t})$$

This is (1.1.95). By symmetry, (1.1.96) now follows.

Remark 1.1.12 We deduce from(1.1.103) and (1.1.95) that :

$$W(\Gamma_t)\,\left(\int_0^\infty e^{-\frac{\lambda}{2}y}dy\right) = \mathbf{W}^-\big(\Gamma_t\, e^{-\frac{\lambda}{2}(S_\infty - S_t)} 1_{\theta_+ > t}\big)$$

and operating as in the proof of point 3), *i)* of Theorem 1.1.6, we obtain :

$$\begin{aligned} \mathbf{W}^-(S_\infty - S_t \in dl) &= 1_{[0,\infty[}(l)dl + W(S_t - X_t)\,\delta_0(dl) \\ &= 1_{[0,\infty[}(l)dl + \sqrt{\frac{2t}{\pi}}\,\delta_0(dl) \end{aligned} \tag{1.1.104}$$

and, conditionally on $S_\infty - S_t = l,\ l > 0,\ (X_u,\ u \le t)$ is, under $\mathbf{W}^-$, a Brownian motion indexed by $[0,t]$. Theorem 1.1.11 is the prototype of similar results which we may obtain for other penalisations. Here are, without proof, some examples.

Theorem 1.1.11'.

1) *Let $h^+, h^- : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\int_0^\infty (h^+ + h^-)(y)dy < \infty$. Let $W_\infty^{h^+,h^-}$ denote the probability defined by (see [RVY, II]) :*

$$W_\infty^{h^+,h^-}{}_{|\mathcal{F}_t} = M_t^{h^+,h^-} \cdot W_{|\mathcal{F}_t} \tag{1.1.105}$$

with

$$\begin{aligned} M_t^{h^+,h^-} &= \frac{1}{\frac{1}{2}\int_0^\infty (h^+ + h^-)(y)dy} \Big\{\Big(X_t^+\, h^+(L_t) + X_t^-\, h^-(L_t) \\ &\qquad + \int_{L_t}^\infty \frac{1}{2}(h^+ + h^-)(y)dy\Big)\Big\} \end{aligned} \tag{1.1.106}$$

Then :

$$\mathbf{W} = \Big\{\mathbf{W}^+\big(\big(h^+(L_\infty)\big) + \mathbf{W}^-(h^-(L_\infty)\big)\Big\}\left(1_{\Gamma_+}\frac{1}{h^+(L_\infty)} + 1_{\Gamma_-}\frac{1}{h^-(L_\infty)}\right) W_\infty^{h^+,h^-} \tag{1.1.107}$$

In other words :

$$1_{\Gamma_+}.W_\infty^{h^+,h^-} = \frac{1}{\mathbf{W}^+\big(\big(h^+(L_\infty)\big) + \mathbf{W}^-(h^-(L_\infty)\big)}\, h^+(L_\infty)\,.\mathbf{W}^+ \tag{1.1.108}$$

$$1_{\Gamma_-}.W_\infty^{h^+,h^-} = \frac{1}{\mathbf{W}^+\big(\big(h^+(L_\infty)\big) + \mathbf{W}^-(h^-(L_\infty)\big)}\, h^-(L_\infty)\,.\mathbf{W}^- \tag{1.1.109}$$

2) *Let $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be Borel and integrable, and let us define :*

$$M_t^{\psi(S_g)} := \left(\frac{1}{2}\,\psi(S_{g(t)})|X_t| + \psi(S_t)(S_t - X_t^+) + \int_{S_t}^\infty \psi(y)dy\right)\cdot\frac{1}{\int_0^\infty \psi(y)dy} \tag{1.1.110}$$

with $g^{(t)} := \sup\{s \le t,\ X_s = 0\}$. *If* $W_\infty^{\psi(S_g)}$ *is given by :*

$$W_\infty^{\psi(S_g)}{}_{|\mathcal{F}_t} = M_t^{\psi(S_g)} \cdot W_{|\mathcal{F}_t} \tag{1.1.111}$$

(*see [RY, VIII]*), *then :*

i) $$W_\infty^{\psi(S_g)} = \frac{\psi(S_g)}{\mathbf{W}\big(\psi(S_g)\big)} \cdot \mathbf{W} \tag{1.1.112}$$

ii) If $\rho := \sup\{u \le g,\ S_{g^{(u)}} < S_g\}$, *then, for all* t *and for all* $\Gamma_t \in b_+(\mathcal{F}_t)$ *:*

$$\mathbf{W}(\Gamma_t\, 1_{\rho \le t}) = W\left(\Gamma_t \left(\frac{1}{2}\,|X_t| + (S_t - X_t^+) 1_{S_t = S_g^{(t)}}\right)\right) \tag{1.1.113}$$

We could also present analogous results for penalisations associated to the numbers of down-crossings (see [RVY, II]) or the length of the longest excursion before $g^{(t)}$ (see [RVY, VII]), etc...
We use, in Section 2, Theorem 1.1.11 and 1.1.11' to give explicit examples of martingales $\big(M_t(F),\ t \ge 0\big)$, $F \in L^1_+(\mathbf{W})$. These martingales are defined in Theorem 1.2.1

1.2 W-Brownian martingales associated to $\mathbf{W}$.

The notation in this Section 1.2 is the same as in Section 1.1. Our aim here is to associate to every r.v. in $L^1_+(\Omega, \mathcal{F}_\infty, \mathbf{W})$ a W-martingale and to study a few of its properties. Thus, $\mathbf{W}$ appears as "a machine to construct W-martingales". We shall also prove (see Theorem 1.2.5) a decomposition Theorem which is valid for every positive Brownian supermartingale.

1.2.1 Definition of the martingales $\big(M_t(F),\ t \ge 0\big)$.

Theorem 1.2.1. *Let* $F \in L^1_+(\Omega, \mathcal{F}_\infty, \mathbf{W})$. *There exists a positive (necessarily continuous)* $\big((\mathcal{F}_t,\ t \ge 0), W\big)$ *martingale* $\big(M_t(F),\ t \ge 0\big)$ *such that :*

1) *For every* $t \ge 0$ *and* $\Gamma_t \in b(\mathcal{F}_t)$ *:*

$$\mathbf{W}(F \cdot \Gamma_t) = W\big(M_t(F) \cdot \Gamma_t\big) \tag{1.2.1}$$

In particular, for every $t \ge 0$ *:*

$$\mathbf{W}(F) = W\big(M_t(F)\big) \tag{1.2.2}$$

2) $(M_t(F), t \ge 0)$ *may be computed via the "characteristic formula" :*

$$M_t(F) = \widehat{\mathbf{W}}_{X_t}\big(F(\omega_t, \widehat{\omega}^t)\big) \tag{1.2.3}$$

(cf Point 1 of Remark 1.2.2 for this notation)

3) $$M_t(F) \underset{t\to\infty}{\longrightarrow} 0 \quad W \text{ a.s.} \tag{1.2.4}$$

In particular, the martingale $\big(M_t(F),\ t \ge 0\big)$ *is not uniformly integrable.*

4) *For every* $q \in \mathcal{I}$ *:*

$$M_t(F) = \varphi_q(0)\, M_t^{(q)}\, W_\infty^{(q)}(F\, e^{\frac{1}{2} A_\infty^{(q)}} | \mathcal{F}_t) \tag{1.2.5}$$

where $M_t^{(q)}$, φ_q *and* $W_\infty^{(q)}$ *are defined in Theorem 1.1.1.*

Remark 1.2.2.

1. We now give some explanation about the notation in (1.2.3). If $\omega \in \mathcal{C}(\mathbb{R}_+ \to \mathbb{R})$, then ω_t (resp. ω^t) denotes the part of ω before t (resp. after t) :

$$\omega = (\omega_t, \omega^t)$$

that is, precisely :

$$X_u(\omega) = \begin{cases} X_u(\omega_t) & \text{if} \quad u \le t \\ X_{u-t}(\omega^t) & \text{if} \quad u \ge t \end{cases}$$

and our notation $\widehat{\mathbf{W}}_{X_t}\big(F(\omega_t, \widehat{\omega}^t)\big)$ stands for the expectation of $F(\omega_t, \bullet)$ with respect to $\mathbf{W}_{X_t(\omega)}$.

2. To every r.v. G in $L^1_+(\Omega, \mathcal{F}_\infty, W)$ we may of course associate the positive martingale $\big(\widetilde{M}_t(G) := W(G|\mathcal{F}_t),\ t \ge 0\big)$. But, contrarily to the description for $M_t(F)$ given in Theorem 1.2.1, this is a uniformly integrable martingale.

3. Formula (1.2.5) may seem ambiguous, since the r.v. $W^{(q)}_\infty\big(F\, e^{\frac{1}{2}A^{(q)}_\infty}|\mathcal{F}_t\big)$ is only defined $W^{(q)}_\infty$ a.s. But since from (1.1.7), the probability $W^{(q)}_\infty$ is absolutely continuous on $\mathcal{F}_t$ with respect to W, there is in fact no ambiguity. On the other hand, from (1.1.16) :

$$\mathbf{W}(F) = \varphi_q(0)\, W^{(q)}_\infty \left(F \exp\left(\frac{1}{2}\, A^{(q)}_\infty\right)\right) < \infty \tag{1.2.6}$$

as soon as $F \in L^1(\mathbf{W})$. Thus, the $\big((\mathcal{F}_t,\ t \ge 0), W^{(q)}_\infty\big)$ martingale $\Big(W^{(q)}_\infty\Big(F \exp(\frac{1}{2}\, A^{(q)}_\infty)|\mathcal{F}_t\Big), t \ge 0\Big)$ is $W^{(q)}_\infty$-uniformly integrable.

4. Of course, $\big(M_t(F),\ t \ge 0\big)$ is continuous, as it is a $\big((\mathcal{F}_t,\ t \ge 0), W\big)$ martingale.

5. On the injectivity of $F \longrightarrow \big(M_t(F),\ t \ge 0\big)$: assume that, for F_1 and F_2 belonging to $L^1(\Omega, \mathcal{F}_\infty, \mathbf{W})$ we have : $M_t(F_1) = M_t(F_2)$ a.s., for every $t \ge 0$. Then $F_1 = F_2$ $\mathbf{W}$ a.s. Indeed, from (1.2.1) :

$$W\big(\Gamma_t(M_t(F_1) - M_t(F_2)\big) = 0 = \mathbf{W}\big(\Gamma_t(F_1 - F_2)\big)$$

As this relation is true for every $t \ge 0$ and $\Gamma_t \in b(\mathcal{F}_t)$, the monotone class Theorem implies that, for every $\Gamma \in b(\mathcal{F}_\infty)$:

$$\mathbf{W}\big(\Gamma(F_1 - F_2)\big) = 0, \quad \text{i.e.} \quad F_1 = F_2 \ \ \mathbf{W}\ a.s.$$

Later in this Section (see Lemma 1.2.8), we shall obtain a more direct "construction" of F from $\big(M_t(F),\ t \ge 0\big)$.

Proof of Theorem 1.2.1.

i) We show point 1.

We denote by W^F the finite positive measure on $(\Omega, \mathcal{F}_\infty)$ defined by :

$$W^F(G) := \mathbf{W}(F \cdot G) \tag{1.2.7}$$

Let $\Gamma_t \in b_+(\mathcal{F}_t)$ such that $W(\Gamma_t) = 0$. From (1.1.7), for every $q \in \mathcal{I}$, $W^{(q)}_\infty(\Gamma_t) = 0$ hence, from (1.1.16) :

$$W^F(\Gamma_t) = \mathbf{W}(F \cdot \Gamma_t) = \varphi_q(0)\ W^{(q)}_\infty(F\, e^{\frac{1}{2}A^{(q)}_\infty}\Gamma_t) = 0$$

from (1.2.6). Thus :

$$W^F_{|\mathcal{F}_t} \ll W_{|\mathcal{F}_t}$$

Consequently, from the Radon-Nikodym Theorem, there exists a W integrable, positive r.v. $M_t(F)$, such that

$$W^F_{|\mathcal{F}_t} = M_t(F) \cdot W_{|\mathcal{F}_t} \tag{1.2.8}$$

This is a rewriting of formula (1.2.1). Formula (1.2.2) is obtained from (1.2.1) by taking $\Gamma_t \equiv 1$. The fact that $\big(M_t(F),\ t \geq 0\big)$ is a $\big((\mathcal{F}_t,\ t \geq 0),\ W\big)$ martingale follows from (1.2.8). We also note that, as every Brownian martingale, the process $\big(M_t(F),\ t \geq 0\big)$ admits a continuous version (which we shall always consider).

ii) We show point 4.
From (1.2.1), (1.1.16) and (1.1.7), we have for every $\Gamma_t \in b_+(\mathcal{F}_t)$

$$\begin{aligned}
\mathbf{W}(\Gamma_t F) &= W\big(\Gamma_t M_t(F)\big) \\
&= \varphi_q(0) W^{(q)}_\infty(\Gamma_t F\, e^{\frac{1}{2}A^{(q)}_\infty}) \qquad \big(\text{from (1.1.16)}\big) \\
&= \varphi_q(0) W^{(q)}_\infty\big(\Gamma_t W^{(q)}_\infty(F\, e^{\frac{1}{2}A^{(q)}_\infty} | \mathcal{F}_t)\big) \\
&= \varphi_q(0) W\big(\Gamma_t M^{(q)}_t W^{(q)}_\infty\big(F\, e^{\frac{1}{2}A^{(q)}_\infty} | \mathcal{F}_t\big)\big) \qquad \big(\text{from (1.1.7)}\big)
\end{aligned}$$

(1.2.5) follows.
iii) We show point 3.
• For every $s \geq 0$ and $\Gamma_s \in b(\mathcal{F}_s)$, we have for $s \leq t$ from (1.2.1) :

$$\mathbf{W}(\Gamma_s \cdot F) = W\big(\Gamma_s \cdot M_t(F)\big) \tag{1.2.9}$$

Since the $\big((\mathcal{F}_t,\ t > 0), W\big)$ martingale $\big(M_t(F),\ t \geq 0\big)$ is positive, it converges W a.s. towards $M_\infty(F)$. Letting $t \to \infty$ in (1.2.9) and using Fatou's Lemma, we have :

$$W\big(\Gamma_s M_\infty(F)\big) \leq \mathbf{W}(\Gamma_s \cdot F)$$

Choosing $\Gamma_s = 1_{g^{(s)} \geq a}$, with $g^{(s)} := \sup\{u \leq s, X_u = 0\}$ we obtain :

$$W\big(1_{g^{(s)} \geq a} \cdot M_\infty(F)\big) \leq \mathbf{W}(1_{g^{(s)} \geq a} \cdot F) \tag{1.2.10}$$

Letting $s \to \infty$ in (1.2.10) and noting that :

$$\begin{aligned}
&1_{g^{(s)} \geq a} \longrightarrow 1 \qquad W \ \text{a.s.} \\
&1_{g^{(s)} \geq a} \longrightarrow 1_{g \geq a} \ \ \mathbf{W} \ \text{a.s..}
\end{aligned}$$

we obtain :

$$W\big(M_\infty(F)\big) \leq \mathbf{W}(1_{g \geq a} \cdot F)$$

Now, from Theorem 1.1.6 we know that $g < \infty$ $\mathbf{W}$ a.s., hence we get : $\mathbf{W}(1_{g \geq a} \cdot F) \underset{a \to \infty}{\longrightarrow} 0$. Thus :

$$W\big(M_\infty(F)\big) = 0 \qquad \text{and} \qquad M_\infty(F) = 0 \qquad W \text{ a.s.}$$

• Another way to prove point 3. consists in writing, for $s \le t$:

$$\begin{aligned} W\big(\Gamma_s M_t(F)\big) &= \varphi_q(0)\, W\big(\Gamma_s M_t^{(q)} W_\infty^{(q)}\big(F\, e^{\frac{1}{2}A_\infty^{(q)}}|\mathcal{F}_t\big)\big) \quad \big(\text{from (1.2.5)}\big) \\ &= \varphi_q(0) W_\infty^{(q)}\big(\Gamma_s W_\infty^{(q)}\big(F\, e^{\frac{1}{2}A_\infty^{(q)}}|\mathcal{F}_t\big)\big) \quad \big(\text{from (1.1.7)}\big) \qquad (1.2.11) \end{aligned}$$

But, since the $W_\infty^{(q)}$ martingale $\big(W_\infty^{(q)}\big(F\, e^{\frac{1}{2}A_\infty^{(q)}}|\mathcal{F}_t\big),\ t \ge 0\big)$ is uniformly integrable it converges a.s. and in $L^1(W_\infty^{(q)})$ towards $F\, e^{\frac{1}{2}A_\infty^{(q)}}$ as $t \to \infty$. Thus, letting $t \to \infty$ in (1.2.11) and using again Fatou's Lemma, we obtain :

$$W\big(\Gamma_s M_\infty(F)\big) \le \varphi_q(0)\, W_\infty^{(q)}(\Gamma_s F\, e^{\frac{1}{2}A_\infty^{(q)}}) \qquad (1.2.12)$$

We then choose $\Gamma_s = \mathbf{1}_{\{A_s^{(q)} \ge a\}}$ and obtain

$$W\big(1_{(A_s^{(q)} \ge a)}\, M_\infty(F)\big) \le \varphi_q(0)\, W_\infty^{(q)}\big(1_{A_s^{(q)} \ge a}\, F\, e^{-\frac{1}{2}A_\infty^{(q)}}\big) \qquad (1.2.13)$$

We then let $s \to \infty$ and note that :

$$\begin{aligned} &1_{A_s^{(q)} \ge a} \longrightarrow 1 \qquad W \text{ a.s. (since Brownian motion is recurrent)} \\ &1_{A_s^{(q)} \ge a} \longrightarrow 1_{A_\infty^{(q)} \ge a}\ W_\infty^{(q)} \text{ a.s.} \end{aligned}$$

Hence :

$$W\big(M_\infty(F)\big) \le \varphi_q(0)\, W_\infty^{(q)}\big(1_{A_\infty^{(q)} \ge a}\, F\, e^{-\frac{1}{2}A_\infty^{(q)}}\big) \qquad (1.2.14)$$

It now suffices to let $a \to \infty$, using the fact that $A_\infty^{(q)} < \infty$ $W_\infty^{(q)}$ a.s., and that $F\, e^{-\frac{1}{2}A_\infty^{(q)}} \in L^1(W_\infty^{(q)})$ $\big(\text{from (1.2.6)}\big)$ to obtain :

$$W\big(M_\infty(F)\big) = 0 \quad \text{and hence :} \quad M_\infty(F) = 0 \quad W \text{ a.s.}$$

iv) We prove point 2, i.e. the "characteristic formula" for $M_t(F)$.
We have, from (1.2.5) :

$$\begin{aligned} M_t(F) &= \varphi_q(0)\, M_t^{(q)}\, W_\infty^{(q)}\big(F\, e^{-\frac{1}{2}A_\infty^{(q)}}|\mathcal{F}_t\big) \\ &= \varphi_q(X_t) e^{-\frac{1}{2}A_t^{(q)}} W_\infty^{(q)}\big(F\, e^{-\frac{1}{2}A_\infty^{(q)}}|\mathcal{F}_t\big) \\ &\quad \big(\text{from the definition (1.1.8) of } M_t^{(q)}\big) \\ &= \varphi_q(X_t) \widehat{W}_{X_t,\infty}^{(q)}\big(e^{\frac{1}{2}(A_\infty^q - A_t^q)} F(\omega_t, \widehat{\omega}^t)\big) \\ &\quad \text{(from the Markov property)} \\ &= \widehat{\mathbf{W}}_{X_t}\big(F(\omega_t, \widehat{\omega}^t)\big), \qquad \text{from (1.1.16)} \end{aligned}$$

1.2.2 Examples of martingales $\big(M_t(F),\ t \ge 0\big)$.

Formula (1.2.3) which provides an "explicit" expression for $M_t(F)$ is not always, practically, easy to compute.

1.2.2.1 A first method to obtain examples of $\big(M_t(F),\ t \ge 0\big)$.

To begin with, we present a "computation principle" to obtain $M_t(F)$.

"**Computation principle**"

Let $(N_t,\ t \geq 0)$ denote a $\big((\mathcal{F}_t,\ t \geq 0),\ W\big)$ positive martingale such that $N_0 = 1$. Let W_∞^N be the probability on $(\Omega, \mathcal{F}_\infty)$ which is characterized by :

$$W^N_{\infty\,|\mathcal{F}_t} = N_t \cdot W_{|\mathcal{F}_t} \tag{1.2.15}$$

Let us assume that there exists a r.v. $F \in L^1_+(\Omega, \mathcal{F}_\infty, \mathbf{W})$ such that :

$$F \cdot \mathbf{W} = \mathbf{W}(F) \cdot W_\infty^N \tag{1.2.16}$$

Then

$$M_t(F) = \mathbf{W}(F) \cdot N_t \tag{1.2.17}$$

Proof of the "Computation principle".
We have, for every $t \geq 0$ and $\Gamma_t \in b(\mathcal{F}_t)$, from (1.2.1) :

$$\mathbf{W}(F \cdot \Gamma_t) = W\big(M_t(F) \cdot \Gamma_t\big)$$

On the other hand, from the hypothesis (1.2.16) :

$$\mathbf{W}(F \cdot \Gamma_t) = \mathbf{W}(F)\, W_\infty^N(\Gamma_t)$$

Hence, this quantity also equals :

$$\mathbf{W}(F)\, W(\Gamma_t \cdot N_t) \tag{1.2.18}$$

from (1.2.15). Since Γ_t denotes any $\mathcal{F}_t$ measurable set in (1.2.18), one obtains :

$$M_t(F) = \mathbf{W}(F) \cdot N_t \quad W \ \text{a.s.}$$

Example 1. Let $q \in \mathcal{I}$ and $N_t := \dfrac{\varphi_q(X_t)}{\varphi_q(0)} \exp\!\left(-\dfrac{1}{2}\, A_\infty^{(q)}\right)$.
From (1.1.16) and (1.1.7), the hypotheses of the "Computation principle" are satisfied with $F = \exp\left(-\dfrac{1}{2}\, A_\infty^{(q)}\right)$. Thus :

$$\begin{aligned} M_t(e^{-\frac{1}{2}A_\infty^{(q)}}) &= \mathbf{W}(e^{-\frac{1}{2}A_\infty^{(q)}}) \cdot \frac{\varphi_q(X_t)}{\varphi_q(0)} \exp\left(-\frac{1}{2}\, A_t^{(q)}\right) \\ &= \varphi_q(X_t) \exp\left(-\frac{1}{2}\, A_t^{(q)}\right) \end{aligned} \tag{1.2.19}$$

since, from (1.1.17), $\mathbf{W}\left(\exp\left(-\frac{1}{2}\, A_\infty^{(q)}\right)\right) = \varphi_q(0)$.
Example 2. Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ Borel and integrable and :

$$N_t := \frac{1}{\int_0^\infty h(y)dy} \cdot \left(h(L_t)|X_t| + \int_{L_t}^\infty h(y)dy\right) \tag{1.2.20}$$

From Theorem 1.1.11', the hypotheses of the "Computation principle" are satisfied with $F = h(L_\infty)$ $\big($we note from point 3, *i)* of Theorem 1.1.6 : $\mathbf{W}\big(h(L_\infty)\big) = \displaystyle\int_0^\infty h(l)dl < \infty\big)$.
Thus :

$$M_t\big(h(L_\infty)\big) = h(L_t)|X_t| + \int_{L_t}^\infty h(y)dy \tag{1.2.21}$$

$\big($cf [RVY, II] for the use of this martingale$\big)$.

Example 3. Let $S_t := \sup_{s\le t} X_s$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ Borel and integrable, such that $\psi(+\infty)=0$. Due to Theorem 1.1.11, the "Computation principle" applies with $F=\psi(S_\infty)$ and

$$N_t := \frac{1}{\int_0^\infty \psi(y)dy}\ \left(\psi(S_t)(S_t-X_t)+\int_{S_t}^\infty \psi(y)dy\right)$$

We note that, from (1.1.104) (taken with $t=0$) :

$$\mathbf{W}\big(\psi(S_\infty)\big)=\int_0^\infty \psi(l)dl<\infty. \tag{1.2.22}$$

Thus :

$$M_t\big(\psi(S_\infty)\big)=\psi(S_t)(S_t-X_t)+\int_{S_t}^\infty \psi(y)dy \tag{1.2.23}$$

Another manner to obtain (1.2.23) may be to invoke Lévy's Theorem :

$$\big((S_t,S_t-X_t),\ t\ge 0\big) \overset{\text{(law)}}{=} \big((L_t,|X_t|),\ t\ge 0\big)$$

then to use (1.2.21).

The reader may refer to [RVY, II] for links between the Azéma-Yor martingale $\left(\psi(S_t)(S_t-X_t) + \int_{S_t}^\infty \psi(y)dy,\ t\ge 0\right)$ and the penalisation problem with the process $\big(\psi(S_t),\ t\ge 0\big)$.

Example 4. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ a Borel, integrable function with $\psi(\infty)=0$. The "Computation principle", with the help of Theorem 1.1.11', yields to, with $F=\psi(S_g)$:

$$\begin{aligned} M_t\big(\psi(S_g)\big) &= \frac{1}{2}\ \psi(S_{g^{(t)}})|X_t|+\psi(S_t)(S_t-X_t^+)+\int_{S_t}^\infty \psi(y)dy \\ &= \frac{1}{2}\ \psi(S_{g^{(t)}})\cdot X_t + M_t\big(\psi(S_\infty)\big)\end{aligned} \tag{1.2.24}$$

where $M_t\big(\psi(S_\infty)\big)$ is defined by (1.2.23). We note that, from (1.1.104), since $\psi(+\infty)=0$:

$$\mathbf{W}^-\big(\psi(S_g)\big)=\mathbf{W}\big(\psi(S_\infty)\big)=\int_0^\infty \psi(l)dl<\infty \tag{1.2.25}$$

Example 5. Let $a<b$ and :

$$\begin{array}{llll} T^{(1)} & := \inf\{t\ge 0\ ;\ X_t>b\}, & T^{(2)} & := \inf\{t\ge T^{(1)}\ ;\ X_t<a\} \\ T^{(2n+1)} & := \inf\{t\ge T^{(2n)}\ ;\ X_t>b\}, & T^{(2n+2)} & := \inf\{t\ge T^{(2n+1)}\ ;\ X_t<a\}\end{array}$$

Define :

$$D_t^{[a,b]} := \sum_{n\ge 1} 1_{(T^{(2n)}\le t)}$$

$D_t^{[a,b]}$ is the number of down-crossings on the interval $[a,b]$ before time t. Let $h:\mathbb{N}\to\mathbb{R}_+$ such that h is decreasing, $h(0)=1, h(+\infty)=0$ and denote $\Delta h(n):=h(n)-h(n+1)$. The "Computation principle" and an extension to this situation of Theorem 1.1.11' lead to :

$$M_t\big(\Delta h(D_\infty^{[a,b]})\big)=\sum_{n\geq 0}\Big\{1_{[T^{(2n)},T^{(2n+1)}[}(t)\left[\frac{h(n)}{2}\left(1+\frac{b-X_t}{b-a}\right)+\frac{h(n+1)}{2}\left(\frac{X_t-a}{b-a}\right)\right]$$
$$+1_{[T^{(2n+1)},T^{(2n+2)}[}(t)\left[\frac{h(n+1)}{2}\left(1+\frac{b-X_t}{b-a}\right)+\frac{h(n)}{2}\left(\frac{X_t-a}{b-a}\right)\right]\Big\} \quad (1.2.26)$$

The reader may refer to [RVY, II] for more information relative to this martingale.

Example 6. Let $\Sigma_{g^{(t)}}$ denote the length of the longest excursion of Brownian motion $(X_u,\ u\geq 0)$ before $g^{(t)}:=\sup\{s\leq t\ ;\ X_s=0\}$. Let $h:\mathbb{R}_+\to\mathbb{R}_+$ such that $\int_0^\infty z|h'(z)|dz<\infty$. Then, the "Computation principle" and an extension of Theorem 1.1.11', lead to :

$$M_t\big(\sqrt{h}(\Sigma_g)\big)=\sqrt{h}(\Sigma_{g^{(t)}})\cdot|X_t|+h_1(A_t)\Phi\left(\frac{|X_t|}{\sqrt{(\Sigma_{g^{(t)}}-A_t)_+}}\right)$$
$$+\int_0^{\frac{|X_t|}{\sqrt{(\Sigma_{g^{(t)}}-A_t)_+}}}h_1\left(A_t+\frac{X_t^2}{v^2}\right)\left(\exp\left(-\frac{v^2}{2}\right)\right)dv \quad (1.2.27)$$

with

$$A_t:=t-g^{(t)},\ \Phi(x):=\int_x^\infty\exp\left(-\frac{v^2}{2}\right)dv$$
$$h_1(x):=-\int_{\sqrt{x}}^\infty zh'(z)dz$$

(see [RY, VIII] or [RY, M], Chap. 3).

1.2.2.2 A second manner to compute explicitly martingales of the form $\big(M_t(F),\ t\geq 0\big)$.

This method hinges upon the following Theorem 1.2.3. $(F_u,\ u\geq 0)$ denotes a positive predictable process such that :

$$\mathbf{W}(F_g)<\infty \quad (1.2.28)$$

We note that from Theorem 1.1.6, this condition is equivalent to :

$$W\left(\int_0^\infty F_s\,dL_s\right)<\infty \quad (1.2.29)$$

or equivalently after the change of variable $l=L_s$, to :

$$\int_0^\infty W(F_{\tau_l})dl=W\left(\int_0^\infty F_s\,dL_s\right)<\infty \quad (1.2.30)$$

with :

$$\tau_l:=\inf\{t>0\ ;\ L_t>l\} \quad (1.2.31)$$

Theorem 1.2.3. *Let $(F_u,\ u \geq 0)$ denote a positive predictable process such that :*

$$\mathbf{W}(F_g) = \int_0^\infty W(F_{\tau_l})dl < \infty \tag{1.2.32}$$

Then, the martingale $\big(M_t(F_g),\ t \geq 0\big)$ may be expressed as :

$$\begin{aligned} M_t(F_g) &= F_{g^{(t)}} \cdot |X_t| + \int_t^\infty p_{u-t}(X_t)\, \Pi_{0,0}^{(u)}(F_u|\mathcal{F}_t)\, du && (1.2.33)\\ &= F_{g^{(t)}} \cdot |X_t| + \int_{L_t}^\infty W(F_{\tau_l}|\mathcal{F}_t)\, dl && (1.2.34)\\ &= \int_0^t F_{g^{(s)}}\, \mathrm{sgn}(X_s)\, dX_s + W\left(\int_0^\infty F_{\tau_l}\, dl|\mathcal{F}_t\right) && (1.2.35)\end{aligned}$$

In (1.2.33), $\Pi_{0,0}^{(u)}$ denotes the law of Brownian bridge of length u and :

$$p_s(x) := \frac{1}{\sqrt{2\pi s}}\, e^{-\frac{x^2}{2s}} \tag{1.2.36}$$

Proof of Theorem 1.2.3.
i) We first prove (1.2.33).
For every $t \geq 0$ and $\Gamma_t \in b(\mathcal{F}_t)$ we have by (1.2.1) :

$$\begin{aligned} \mathbf{W}(\Gamma_t F_g) &= W\big(\Gamma_t\, M_t(F_g)\big)\\ &= \mathbf{W}(\Gamma_t\, F_g\, 1_{g\leq t}) + \mathbf{W}(\Gamma_t\, F_g\, 1_{g>t})\\ &= \mathbf{W}(\Gamma_t\, F_{g^{(t)}}\, 1_{g\leq t}) + \mathbf{W}(\Gamma_t\, F_g\, 1_{g>t})\\ &\quad \big(\text{since } g = g^{(t)} \text{ on the set } (g \leq t)\big)\\ &:= (1_t) + (2_t) && (1.2.37)\end{aligned}$$

We study successively (1_t) and (2_t) :

$$\begin{aligned} (1_t) &= \mathbf{W}(\Gamma_t\, F_{g^{(t)}}\, 1_{g\leq t})\\ &= W\big(\Gamma_t\, F_{g^{(t)}}\, |X_t|\big) && (1.2.38)\\ &\quad \big(\text{ from point 2, } i) \text{ of Theorem 1.1.6.}\big)\\ (2_t) &= \mathbf{W}(\Gamma_t\, F_g\, 1_{g>t})\\ &= \int_t^\infty \frac{du}{\sqrt{2\pi u}}\, \mathbf{W}\big(\Gamma_t\, F_g|g=u\big) \qquad (\text{from (1.1.42)})\\ &= \int_t^\infty \frac{du}{\sqrt{2\pi u}}\, \Pi_{0,0}^{(u)}(\Gamma_t\, F_u) \qquad (\text{from point 2)iii) of Theorem 1.1.6})\\ &= \int_t^\infty \frac{du}{\sqrt{2\pi u}}\, \Pi_{0,0}^{(u)}\big(\Gamma_t\, \Pi_{0,0}^{(u)}(F_u|\mathcal{F}_t)\big)\end{aligned}$$

We now use the (partial) absolute continuity formula for the law of the Brownian bridge with respect to that of Brownian motion :

$$\Pi_{0,0\,|\mathcal{F}_t}^{(u)} = \frac{p_{u-t}(X_t)}{p_u(0)} \cdot W_{|\mathcal{F}_t} \qquad (u > t) \tag{1.2.39}$$

to obtain :

$$\begin{aligned}(2_t) &= \int_t^\infty \frac{du}{\sqrt{2\pi u}}\, W\left(\frac{\Gamma_t\, p_{u-t}(X_t)}{p_u(0)}\, \Pi_{0,0}^{(u)}\big(F_u|\mathcal{F}_t\big)\right) \\ &= \int_t^\infty du\, W\big(\Gamma_t\, p_{u-t}(X_t)\, \Pi_{0,0}^{(u)}\big(F_u|\mathcal{F}_t\big)\big) \end{aligned} \tag{1.2.40}$$

since $p_u(0) = \dfrac{1}{\sqrt{2\pi u}}\cdot$ Gathering (1.2.40), (1.2.37) and (1.2.38), we obtain (1.2.33).

ii) We now prove (1.2.34).
Of course, (1.2.34) is equivalent to :

$$\int_t^\infty p_{u-t}(X_t)\, \Pi_{0,0}^{(u)}\big(F_u|\mathcal{F}_t\big) du = \int_{L_t}^\infty W\big(F_{\tau_l}|\mathcal{F}_t\big) dl \tag{1.2.41}$$

or to :

$$W\left(\Gamma_t \int_t^\infty p_{u-t}(X_t)\, \Pi_{0,0}^{(u)}\big(F_u|\mathcal{F}_t\big) du\right) = W\left(\Gamma_t \cdot \int_{L_t}^\infty F_{\tau_l}\, dl\right) \tag{1.2.42}$$

for any $\Gamma_t \in b(\mathcal{F}_t)$. But we have :

$$\begin{aligned} W\left(\Gamma_t \cdot \int_{L_t}^\infty F_{\tau_l} dl\right) &= W\left(\Gamma_t \int_t^\infty F_u\, dL_u\right) \\ &\qquad \text{(after the change of variable } l = L_u\text{)} \\ &= \int_t^\infty \frac{du}{\sqrt{2\pi u}}\, \Pi_{0,0}^{(u)}\,(F_u\, \Gamma_t) \\ &= \int_t^\infty \frac{du}{\sqrt{2\pi u}}\, \Pi_{0,0}^{(u)}\, \Big(\Gamma_t\, \Pi_{0,0}^{(u)}\big(F_u|\mathcal{F}_t\big)\Big) \\ &= \int_t^\infty \frac{du}{\sqrt{2\pi u}}\, W\left(\Gamma_t\, \frac{p_{u-t}(X_t)}{p_u(0)}\, \Pi_{0,0}^{(u)}\big(F_u|\mathcal{F}_t\big)\right) \\ &\qquad \big(\text{by the absolute continuity formula (1.2.39)}\big) \\ &= W\left(\Gamma_t \int_t^\infty p_{u-t}(X_t)\, \Pi_{0,0}^{(u)}\big(F_u|\mathcal{F}_t\big) du\right) \end{aligned}$$

ii') We give now a direct proof - i.e. without using (1.2.33) - of (1.2.34). We have, for every $t \geq 0$ and $\Gamma_t \in b_+(\mathcal{F}_t)$:

$$\begin{aligned} \mathbf{W}(F_g\, \Gamma_t) &= \mathbf{W}(F_g\, \Gamma_t\, 1_{g\leq t}) + \mathbf{W}(F_g\, \Gamma_t\, 1_{g>t}) \\ &= \mathbf{W}(F_{g^{(t)}}\, \Gamma_t\, 1_{g\leq t}) + \mathbf{W}(\widetilde{\Gamma}_g\, F_g) \end{aligned}$$

(since $g = g^{(t)}$ on the set $(g \leq t)$), and we have used the notation :

$$\begin{aligned} &(\widetilde{\Gamma}_u,\ u \geq 0) := \big(\Gamma_t\, 1_{]t,\infty[}(u),\ u \geq 0\big) \\ &\quad = W(\Gamma_t\, F_{g^{(t)}} |X_t| + W\left(\int_0^\infty \widetilde{\Gamma}_{\tau_l}\, F_{\tau_l}\, dl\right) \end{aligned}$$

$\big($from point 2 *i)* of Theorem 1.1.6 and from formula (1.1.44)$\big)$.

Hence :

$$\begin{aligned}\mathbf{W}(F_g\,\Gamma_t) &= W\big(\Gamma_t\,F_{g^{(t)}}|X_t|\big) + W\left(\Gamma_t\int_0^\infty 1_{t<\tau_l}\,F_{\tau_l}\,dl\right)\\ &= W\big(\Gamma_t\,F_{g^{(t)}}|X_t|\big) + W\left(\Gamma_t\int_{L_t}^\infty F_{\tau_l}\,dl\right)\\ &= W\big(\Gamma_t\,F_{g^{(t)}}|X_t|\big) + W\left(\Gamma_t\int_{L_t}^\infty W\big(F_{\tau_l}|\mathcal{F}_t\big)\,dl\right)\end{aligned}$$

which implies (1.2.34).

iii) We now prove (1.2.35).

To go from (1.2.34) to (1.2.35), we use the balayage formulae, which yields :

$$F_{g^{(t)}}\cdot|X_t| = \int_0^t F_{g^{(s)}}\,\mathrm{sgn}(X_s)\,dX_s + \int_0^t F_u\,dL_u$$

and we add this expression to $\displaystyle\int_{L_t}^\infty W\big(F_{\tau_l}|\mathcal{F}_t\big)\,dl = W\left(\int_t^\infty F_u\,dL_u|\mathcal{F}_t\right)$ on the RHS. It is now clear that (1.2.34) implies (1.2.35).

Corollary 1.2.4.

1) *Formula (1.2.34) expresses the martingale* $\big(M_t(F_g),\ t\geq 0\big)$ *as the sum of a submartingale* $(F_{g^{(t)}}\cdot|X_t|,\ t\geq 0)$ *and a supermartingale* $\left(W\left(\int_0^\infty F_{\tau_l}\,1_{\tau_l>t}\,dl|\mathcal{F}_t\right),\ t\geq 0\right)$ *both of which converge to 0 a.s., as* $t\to\infty$.

2) *The variable* $\displaystyle\int_0^\infty F^2_{g^{(u)}}du$ *is finite a.s. but it satisfies :*

$$W\left(\left(\int_0^\infty F^2_{g^{(u)}}\,du\right)^{\frac{1}{2}}\right) = +\infty \qquad (1.2.43)$$

unless $F_g = 0$, $\mathbf{W}$ *a.s.*

Proof of Corollary 1.2.4.

The first statement is obvious since $F_{g^{(t)}}|X_t|$ is the absolute value of the martingale $F_{g^{(t)}}\cdot X_t$. Moreover, $|F_{g^{(t)}}\cdot X_t|\leq M_t(F_g)$, hence since $M_t(F_g)\underset{t\to\infty}{\longrightarrow}0$ a.s. (see Theorem 1.2.1) the same is true for $F_{g^{(t)}}\cdot X_t$. To prove the second item, assume that :

$$W\left(\left(\int_0^\infty F^2_{g^{(u)}}\,du\right)^{\frac{1}{2}}\right) < \infty$$

Then, the martingale $\left(\int_0^t F_{g^{(s)}}\,\mathrm{sgn}(X_s)\,dX_s,\ t\geq 0\right)$ would be in H^1 ; a fortiori it would be uniformly integrable. From (1.2.35), since $W\left(\int_0^\infty F_{\tau_l}\,dl\right)<\infty$, $\big(M_t(F_g),\ t\geq 0\big)$ would also be uniformly integrable ; but this is only possible, since this martingale converges a.s. to 0 (see Theorem 1.2.1) if it is identically equal to 0, that is $F_g = 0$ $\mathbf{W}$ a.s. (see point 5 of Remark 1.2.2).

Of course, if we want to compute $\big(M_t(F),\ t \geq 0\big)$ in a completely explicit manner, we need to compute $\Pi_{0,0}^{(u)}\big(F_u|\mathcal{F}_t\big)$, for $t \leq u$ $\left(\text{or } W\left(\int_0^\infty F_{\tau_l}\, dl|\mathcal{F}_t\right)\right)$. This is what has been done in the Examples 4 and 6 above. Here is an example where this computation is immediate.

Example 7. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ Borel such that :

$$\int_0^\infty \psi(t)\, \frac{dt}{\sqrt{2\pi t}} < \infty \tag{1.2.44}$$

Then :

$$M_t\big(\psi(g)\big) = \psi(g^{(t)})|X_t| + \int_0^\infty \frac{du}{\sqrt{2\pi u}}\, e^{-\frac{X_t^2}{2u}}\psi(t+u) \tag{1.2.45}$$

To obtain (1.2.45), we apply Theorem 1.2.3 with the (deterministic) process $(F_u,\ u \geq 0) := \big(\psi(u),\ u \geq 0\big)$ and we use :

$$\Pi_{0,0}^{(u)}\big(F_u|\mathcal{F}_t\big) = \Pi_{0,0}^{(u)}\big(\psi(u)|\mathcal{F}_t\big) = \psi(u)$$

We then make the change of variable $u - t = v$ in (1.2.33).
More generally (see Theorem 1.1.8), with $g_a := \sup\{t\ ;\ X_t = a\}$, we have :

$$M_t\big[\psi(g_a)\big] = \psi(g_a^{(t)})|X_t - a| + \int_0^\infty \frac{du}{\sqrt{2\pi u}}\, e^{-\frac{(X_t-a)^2}{2u}}\psi(t+u) \tag{1.2.46}$$

with :

$$g_a^{(t)} := \sup\{s \leq t\ ;\ X_s = a\} \tag{1.2.47}$$

Back to Example 2. Formula (1.2.21) is a particular case of (1.2.34). Indeed, if we apply (1.2.34) with $(F_u,\ u \geq 0) := \big(h(L_u),\ u \geq 0\big)$, we obtain :

$$\begin{aligned} M_t\big(h(L_\infty)\big) &= M_t\big(h(L_g)\big) \\ &= h(L_{g^{(t)}})|X_t| + W\left(\int_{L_t}^\infty h(L_{\tau_l})dl|\mathcal{F}_t\right) \\ &= h(L_t)|X_t| + \int_{L_t}^\infty h(l)\, dl \end{aligned}$$

since $L_{g^{(t)}} = L_t$ and $L_{\tau_l} = l$.
In the same spirit, for $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ Borel such that :

$$\int_0^\infty \int_0^\infty h(l,u)\, \frac{l\, e^{-\frac{l^2}{2u}}}{\sqrt{2\pi u^3}}\, dl\, du < \infty \tag{1.2.48}$$

then $\big(\text{see (1.1.47)}\big)$ $\mathbf{W}\big(h(L_\infty, g)\big) < \infty$ and

$$\begin{aligned} M_t\big(h(L_\infty, g)\big) &= h(L_{g^{(t)}}, g^{(t)}) \cdot |X_t| + W\left(\int_{L_t}^\infty h(L_{\tau_l}, \tau_l)\, dl|\mathcal{F}_t\right) \\ &= h(L_t, g^{(t)}) \cdot |X_t| + \widehat{W}_{X_t}\left(\int_0^\infty h(L_t + \widehat{L}_v, t+v)d\widehat{L}_v\right) \end{aligned} \tag{1.2.49}$$

1.2.2.3 A third manner to obtain explicit examples of martingales $\big(M_t(F),\ t \geq 0\big)$.

• We begin with a definition. We shall say that a family of r.v.'s $(F_t,\ t \geq 0)$ converges, as $t \to \infty$, towards F **W** a.s. if for some $G > 0$, $G \in L^1_+(\mathcal{F}_\infty, \mathbf{W})$ $F_t \underset{t\to\infty}{\longrightarrow} F$ W^G a.s. We recall : $W^G(\Gamma) := \mathbf{W}(G\,\Gamma)$, $\Gamma \in b(\mathcal{F}_\infty)$. Clearly, this definition does not depend on the r.v. G chosen in the above class. In particular, it may be convenient to take for G the r.v. $\exp\left(-\frac{1}{2}A^{(q)}_\infty\right)$ for some $q \in \mathcal{I}$; hence, the a.s. **W**-convergence is precisely the $W^{(q)}_\infty$ a.s. convergence.
This definition may seem complicated. However, its aim is to take care of the difficulty arising from the fact that for every $\Gamma_t \in b_+(\mathcal{F}_t)$, $\mathbf{W}(\Gamma_t)$ equals either 0 or $+\infty$ (see point *v)* of the proof of Theorem 1.1.2).
Equivalently, $F_t \underset{t\to\infty}{\longrightarrow} F$ **W** a.s. if and only if $\mathbf{W}(\Delta) = 0$ with $\Delta = \big\{\omega\ ;\ F_t(\omega) \underset{t\to\infty}{\not\longrightarrow} F(\omega)\big\}$

• In Section 1.2.3 below we shall obtain the following result : (it is a Corollary of Theorem 1.2.5, in the same Section 1.2.3)
Corollary 1.2.6. *A positive* $\big((\mathcal{F}_t,\ t \geq 0), W\big)$ *martingale* $M_t,\ t \geq 0$ *is of the form* $\big(M_t(F),\ t \geq 0\big)$ *for some* $F \in L^1_+(\mathcal{F}_\infty, \mathbf{W})$ *if and only if :*

$$\lim_{t\to\infty} \frac{M_t}{1+|X_t|} \text{ exists } \mathbf{W}\text{-a.s.}$$

and

$$M_0 = \mathbf{W}\left(\lim_{t\to\infty} \frac{M_t}{1+|X_t|}\right)$$

and, in this case :

$$F = \lim_{t\to\infty} \frac{M_t}{1+|X_t|} \quad \mathbf{W} \text{ a.s.}$$

• We now illustrate with 3 examples how due to this Corollary, we may compute explicitly $\big(M_t(F),\ t \geq 0\big)$ for some $F \in L^1_+(\mathcal{F}_\infty, \mathbf{W})$.

Back to Example 1. Let $q \in \mathcal{I}$ and $M_t := \varphi_q(X_t)\ \exp\left(-\frac{1}{2}A^{(q)}_t\right)$. Since

$$\varphi_q(x) \underset{|x|\to\infty}{\sim} |x| \text{ and } |X_t| \underset{t\to\infty}{\longrightarrow} \infty \quad \mathbf{W} \text{ a.s.}$$

we have :

$$\frac{M_t}{1+|X_t|} \underset{t\to\infty}{\longrightarrow} \exp\left(-\frac{1}{2}A^{(q)}_\infty\right) := F \quad \mathbf{W} \text{ a.s.}$$

On the other hand,

$$M_0 = \varphi_q(0) = \mathbf{W}\left(\exp\left(-\frac{1}{2}A^{(q)}_\infty\right)\right) \qquad \big(\text{from (1.1.17)}\big)$$

Thus, from Corollary 1.2.6. :

$$M_t\left(\exp\left(-\frac{1}{2}A^{(q)}_\infty\right)\right) = \varphi_q(X_t)\ \exp\left(-\frac{1}{2}A^{(q)}_t\right)$$

Back to Example 2. Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ Borel and integrable and :

$$M_t := h(L_t)|X_t| + \int_{L_t}^{\infty} h(y)dy$$

It is clear that :

$$\frac{M_t}{1+|X_t|} \underset{t\to\infty}{\longrightarrow} h(L_\infty) \quad \mathbf{W} \text{ a.s.}$$

and that from point 3)i) of Theorem 1.1.6. :

$$M_0 = \int_0^{\infty} h(y)dy = \mathbf{W}\big(h(L_\infty)\big)$$

Thus, from Corollary 1.2.6 :

$$M_t\big(h(L_\infty)\big) = h(L_t)|X_t| + \int_{L_t}^{\infty} h(y)dy$$

Back to Example 3. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ Borel and integrable, with $\psi(\infty) = 0$. Let

$$M_t := \psi(S_t)(S_t - X_t) + \int_{S_t}^{\infty} \psi(y)dy$$

Then :

$$\frac{M_t}{1+|X_t|} \underset{t\to\infty}{\longrightarrow} \psi(S_\infty) \qquad \mathbf{W} \text{ a.s. } \big(\text{see (1.1.99)}\big)$$

From (1.2.22) :

$$\mathbf{W}\big(\psi(S_\infty)\big) = \int_0^{\infty} \psi(l)dl = M_0$$

Hence :

$$M_t\big(\psi(S_\infty)\big) = \psi(S_t)(S_t - X_t) + \int_{S_t}^{\infty} \psi(y)dy$$

1.2.3 <u>A decomposition Theorem for positive Brownian supermartingales.</u>
Here is the most inportant result of this Section 1.2.
Theorem 1.2.5. *Let $(Z_t,\ t \geq 0)$ denote a positive $\big((\mathcal{F}_t,\ t\geq 0), W\big)$ supermartingale. We denote $Z_\infty := \lim_{t\to\infty} Z_t$ $(W$ a.s.$)$. Then :*
1)

$$z_\infty := \lim_{t\to\infty} \frac{Z_t}{1+|X_t|} \text{ exists } \mathbf{W} \text{ a.s.} \tag{1.2.50}$$

$$\text{and} \qquad \mathbf{W}(z_\infty) < \infty \tag{1.2.51}$$

2) $(Z_t,\ t \geq 0)$ *decomposes in a unique manner in the form :*

$$Z_t = M_t(z_\infty) + W\big(Z_\infty|\mathcal{F}_t\big) + \xi_t, \qquad t \geq 0 \tag{1.2.52}$$

where $\big(M_t(z_\infty),\ t \geq 0\big)$ *and* $\big(W\big(Z_\infty|\mathcal{F}_t\big),\ t \geq 0\big)$ *denote two* $\big((\mathcal{F}_t,\ t \geq 0), W\big)$ *martingales and :*

$$(\xi_t,\ t \geq 0) \text{ is a } \big((\mathcal{F}_t,\ t \geq 0), W\big) \text{ positive supermartingale}$$

such that :
i) $Z_\infty \in L^1_+(\mathcal{F}_\infty, W)$*, hence* $W\big(Z_\infty|\mathcal{F}_t\big)$ *converges* W *a.s. and in* $L^1(\mathcal{F}_\infty, W)$ *towards* Z_∞*.*

ii) $$\frac{W\big(Z_\infty|\mathcal{F}_t\big)+\xi_t}{1+|X_t|} \underset{t\to\infty}{\longrightarrow} 0 \qquad \mathbf{W} \text{ a.s.} \tag{1.2.53}$$

iii) $$M_t(z_\infty) + \xi_t \underset{t\to\infty}{\longrightarrow} 0 \qquad W \text{ a.s.} \tag{1.2.54}$$

After proving Theorem 1.2.5, we shall give a number of examples of $\big((\mathcal{F}_t,\ t \geq 0), W\big)$ supermartingales for which we can compute explicitly the decomposition (1.2.52).
We refer the reader to subsection 1.2.2.3 for the definition of the a.s. $\mathbf{W}$ convergence.

Corollary 1.2.6. $\big($*Characterisation of martingales of the form* $\big(M_t(F),\ t \geq 0\big)\big)$.
A $\big((\mathcal{F}_t,\ t \geq 0),\ W\big)$ *positive martingale* $(Z_t,\ t \geq 0)$ *is equal to* $\big(M_t(F),\ t \geq 0\big)$ *for an* $F \in L^1_+(\mathcal{F}_\infty, \mathbf{W})$ *if and only if :*

$$Z_0 = \mathbf{W}\left(\lim_{t\to\infty} \frac{Z_t}{1+|X_t|}\right) \tag{1.2.55}$$

Note that $\displaystyle\lim_{t\to\infty} \frac{Z_t}{1+|X_t|}$ *exists* $\mathbf{W}$ *a.s. from (1.2.50).*

Proof of Corollary 1.2.6.
We write, from (1.2.52) :

$$Z_t = M_t(z_\infty) + W(Z_\infty|\mathcal{F}_t) + \xi_t$$

$\big($where, in this situation, $(\xi_t,\ t \geq 0)$ is a positive martingale$\big)$. Hence :

$$Z_0 = W\big(M_0(z_\infty)\big) + W\big(W(Z_\infty|\mathcal{F}_0)\big) + W(\xi_0)$$

i.e., from (1.2.55) and (1.2.2) :

$$Z_0 = \mathbf{W}(z_\infty) = \mathbf{W}(z_\infty) + W(Z_\infty) + W(\xi_0)$$

hence :

$$W(Z_\infty) = W(\xi_0) = 0 \quad \text{and} \quad W(Z_\infty|\mathcal{F}_t) = \xi_t = 0, \quad \text{i.e. } Z_t = M_t(z_\infty)$$

Proof of Theorem 1.2.5.
This proof hinges on the three following Lemmas.

Lemma 1.2.7. *Let* $F, G \in L^1_+(\mathcal{F}_\infty, \mathbf{W})$ *and* $G > 0$ $\mathbf{W}$ *a.s. Then :*

$$\frac{M_t(F)}{M_t(G)} = W^G\left(\frac{F}{G}\Big|\mathcal{F}_t\right) \quad W^G \text{ a.s.} \tag{1.2.56}$$

Consequently :

$$\frac{M_t(F)}{M_t(G)} \underset{t\to\infty}{\longrightarrow} \frac{F}{G} \quad W^G \text{ a.s.} \quad (\text{hence } \mathbf{W} \text{ a.s.}) \tag{1.2.57}$$

Lemma 1.2.8. *Let $F \in L^1_+(\mathcal{F}_\infty, \mathbf{W})$. Then :*

$$\frac{M_t(F)}{1+|X_t|} \underset{t\to\infty}{\longrightarrow} F \qquad \mathbf{W} \text{ a.s.} \tag{1.2.58}$$

Lemma 1.2.9. *Let $(Z_t,\ t \geq 0)$ denote a positive $\big((\mathcal{F}_t,\ t\geq 0), W\big)$ supermartingale. Then :*

1) $\quad z_\infty := \lim_{t\to\infty} \frac{Z_t}{1+|X_t|} \quad$ exists $\mathbf{W}$ a.s. (1.2.59)

Furthermore :

$$\mathbf{W}(z_\infty) < \infty \tag{1.2.60}$$

2) *For every $t \geq 0$:* $\quad M_t(z_\infty) \leq Z_t \qquad W$ *a.s.* (1.2.61)

Proof of Lemma 1.2.7.

We have, for every $t \geq 0$ and $\Gamma_t \in b(\mathcal{F}_t)$:

$$\begin{aligned}
W^G\left(\Gamma_t \frac{M_t(F)}{M_t(G)}\right) &= \mathbf{W}\left(\Gamma_t\, G\, \frac{M_t(F)}{M_t(G)}\right) && \big(\text{by definition of } W^G\big)\\
&= W\left(\Gamma_t\, M_t(G)\, \frac{M_t(F)}{M_t(G)}\right) && \big(\text{by definition of } M_t(G)\big)\\
&= W\big(\Gamma_t\, M_t(F)\big)\\
&= \mathbf{W}(\Gamma_t\, F) && \big(\text{by definition of } M_t(F)\big)\\
&= W^G\left(\Gamma_t\, \frac{F}{G}\right) && (\text{by definition of } W^G)\\
&= W^G\left(\Gamma_t\, W^G\left(\frac{F}{G}\Big|\mathcal{F}_t\right)\right)
\end{aligned}$$

This is (1.2.56). Now, (1.2.57) is an immediate consequence of (1.2.56) since $\dfrac{F}{G} \in L^1(W^G)$. Indeed : $W^G\left(\dfrac{F}{G}\right) = \mathbf{W}\left(G \cdot \dfrac{F}{G}\right) = \mathbf{W}(F) < \infty$.

Proof of Lemma 1.2.8.

i) We first apply Lemma 1.2.7 with $G := \exp\left(-\frac{1}{2}\, A_\infty^{(q)}\right)$, for any $q \in \mathcal{I}$. Then, recall that (Example 1) $M_t(G) = \varphi_q(X_t) \exp\left(-\frac{1}{2}\, A_t^{(q)}\right)$ and, since $\varphi_q(x) \sim |x|$ as $|x| \to \infty$, we get :

$$\frac{M_t(G)}{1+|X_t|} \underset{t\to\infty}{\longrightarrow} \exp\left(-\frac{1}{2}\, A_\infty^{(q)}\right) = G \qquad \mathbf{W} \text{ a.s.}$$

which is the statement of Lemma 1.2.8 with $F = \exp\left(-\frac{1}{2}\, A_\infty^{(q)}\right)$.

ii) For a general $F \in L^1_+(\mathcal{F}_\infty, \mathbf{W})$, we write :

$$\frac{M_t(F)}{1+|X_t|} = \frac{M_t(F)}{M_t(G)} \cdot \frac{M_t(G)}{1+|X_t|} \underset{t\to\infty}{\longrightarrow} \frac{F}{G} \cdot G \qquad \mathbf{W} \text{ a.s.}$$

by applying Lemma 1.2.7, and the result of point *i)* above.

Proof of Lemma 1.2.9.

i) We begin with an argument similar to the one we used to prove Lemma 1.2.8, that is : we write :

$$\frac{Z_t}{1+|X_t|} = \frac{Z_t}{M_t(G)} \, \frac{M_t(G)}{1+|X_t|}$$

We now use the fact that $\left(\frac{Z_t}{M_t(G)},\ t \geq 0\right)$ is a $\big((\mathcal{F}_t,\ t\geq 0),\ W^G\big)$ positive supermartingale; hence it converges W^G a.s. to a r.v. ζ ; consequently :

$$z_\infty := \lim_{t\to\infty} \frac{Z_t}{1+|X_t|} \qquad \text{exists } W^G \text{ a.s.}$$

and we have :

$$z_\infty = \zeta \cdot G$$

ii) Since $\zeta := \lim_{t\to\infty} \frac{Z_t}{M_t(G)}$, W^G a.s., is the limit as $t \to \infty$ of a W^G supermartingale, we have :

$$W^G(\zeta) \leq \frac{Z_0}{M_0(G)} \qquad \text{hence :}$$

$$\mathbf{W}(z_\infty) = W^G(\zeta) \leq \frac{Z_0}{M_0(G)} < \infty$$

iii) For any $t \geq 0$ and $\Gamma_t \in b_+(\mathcal{F}_t)$, we have :

$$\begin{aligned}
\mathbf{W}(\Gamma_t\, z_\infty) &= \mathbf{W}\left(\Gamma_t \lim_{u\to\infty} \frac{Z_u}{1+|X_u|}\right) \\
&= \mathbf{W}\left(\Gamma_t \lim_{u\to\infty} \frac{Z_u}{1+|X_u|} \cdot 1_{g\leq u}\right) \\
&\leq \varliminf_{u\to\infty} \mathbf{W}\left(\Gamma_t \frac{Z_u}{1+|X_u|}\, 1_{g\leq u}\right) \quad \text{(from Fatou's Lemma)} \\
&= \varliminf_{u\to\infty} \quad W\left(\Gamma_t \frac{Z_u}{1+|X_u|}|X_u|\right) \quad \big(\text{from point 2 } i) \text{ of Theorem 1.1.6}\big) \\
&\leq \varliminf_{u\to\infty} \quad W(\Gamma_t\, Z_u) \quad \left(\text{since } \frac{|X_u|}{1+|X_u|} \leq 1\right) \\
&\leq W(\Gamma_t\, Z_t)
\end{aligned}$$

since $(Z_t,\ t\geq 0)$ is a supermartingale. Finally :

$$\mathbf{W}(\Gamma_t\, z_\infty) = W\big(\Gamma_t\, M_t(z_\infty)\big) \leq W(\Gamma_t \cdot Z_t)$$

which is equivalent to point 2 of Lemma 1.2.9.
<u>We may now end the proof of Theorem 1.2.5.</u>

Let $\widetilde{Z}_t := Z_t - M_t(z_\infty) \qquad (t \geq 0)$

Since $\big(M_t(z_\infty),\ t \geq 0\big)$ is a $\big((\mathcal{F}_t,\ t \geq 0),\ W\big)$ martingale, the process $(\widetilde{Z}_t,\ t \geq 0)$ is still a $\big((\mathcal{F}_t,\ t \geq 0),\ W\big)$ positive $\big($from (1.2.61)$\big)$ supermartingale, and since $M_t(z_\infty) \underset{t\to\infty}{\longrightarrow} 0 \quad W$ a.s. from Theorem 1.2.1, we obtain :

$$\widetilde{Z}_t \underset{t\to\infty}{\longrightarrow} Z_\infty \qquad W \text{ a.s.}$$

Since $(\widetilde{Z}_t,\ t \geq 0)$ is a positive supermartingale, we obtain :

$$W(Z_\infty|\mathcal{F}_t) \leq \widetilde{Z}_t$$

We now write :

$$\xi_t := \widetilde{Z}_t - W(Z_\infty|\mathcal{F}_t) \qquad t \geq 0$$

This is a positive supermartingale such that $\lim\limits_{t\to\infty} \xi_t = 0 \quad W$ a.s. On the other hand, $\mathbf{W}$ a.s. :

$$\lim_{t\to\infty} \frac{\xi_t}{1+|X_t|} = \lim_{t\to\infty} \frac{\widetilde{Z}_t}{1+|X_t|} = z_\infty - z_\infty = 0$$

The uniqueness of decomposition (1.2.52) being immediate, Theorem 1.2.5 is proven.

1.2.4 A decomposition result for the martingale $\big(M_t(F),\ t \geq 0\big)$.

A difference with the preceding subsection is that the r.v.'s F which we now consider belong to $L^1(\mathcal{F}_\infty, W)$, but are not necessarily positive.

We shall now prove a decomposition result of the $\big((\mathcal{F}_t,\ t \geq 0),\ W\big)$ martingale $\big(M_t(F),\ t \geq 0\big)$. For this purpose, we shall use the following lemma.

Lemma 1.2.10. *Let $F \in L^1(\mathcal{F}_\infty, \mathbf{W})$*

1) *There exists a predictable process $\big(k_s(F),\ s \geq 0\big)$ which is defined $dL_s(\omega)W(d\omega)$ a.s., and is positive if F is positive, such that :*

$$W\left(\int_0^\infty |k_s(F)|dL_s\right) = \mathbf{W}\big(|k_g(F)|\big) \leq \mathbf{W}\big(|F|\big) < \infty \tag{1.2.62}$$

and such that for every bounded predictable process $(\Phi_s,\ s \geq 0)$

$$\mathbf{W}(\Phi_g F) = W\left(\int_0^\infty \Phi_s\, k_s(F) dL_s\right) \tag{1.2.63}$$

$$= \mathbf{W}\big(\Phi_g\, k_g(F)\big) \tag{1.2.64}$$

Thus :

$$\mathbf{W}(F|\mathcal{F}_g) = k_g(F) \tag{1.2.65}$$

2) *We have $\mathbf{W}\big(|k_g(F)|\big) < \infty$ $\big($from (1.2.62)$\big)$*

$$\mathbf{W}\big(|k_g(F)|\big) \leq \mathbf{W}(|F|) < \infty \tag{1.2.66}$$

and

$$\big(k_s(k_g(F),\ s \geq 0)\big) = (k_s(F),\ s \geq 0) \quad dL_s(\omega)\, W(d\omega) \qquad \text{a.s.} \tag{1.2.67}$$

3) *If* $(h_s,\ s \geq 0)$ *is a predictable process such that* $\mathbf{W}(|h_g|) < \infty$*, then :*

$$\big(k_s(h_g),\ s \geq 0\big) = (h_s,\ s \geq 0) \quad dL_s(\omega)W(d\omega) \qquad \text{a.s.} \tag{1.2.68}$$

Proof of Lemma 1.2.10.
It suffices, by linearity, to prove this Lemma when $F \geq 0$.
i) Formula (1.2.64), written for $F \equiv 1$ and $k_s(F) \equiv 1$:

$$\mathbf{W}(\Phi_g) = W\left(\int_0^\infty \Phi_s\, dL_s\right) \tag{1.2.69}$$

is formula (1.1.44). Let us define the measure μ_F, on the predictable σ-field, and more generally on the set of positive predictable processes by :

$$\mu_F(\Phi) = \mathbf{W}(\Phi_g \cdot F) \tag{1.2.70}$$

Clearly, μ_F is absolutely continuous, on the predictable σ-field, with respect to μ_1, which is the measure μ_F for $F \equiv 1$. Thus, from (1.2.69), μ_F is absolutely continuous on the predictable σ-field with respect to the measure $dL_s(\omega)W(d\omega)$. Thus, there exists, from the Radon-Nikodym Theorem, a process $\big(k_s(F),\ s \geq 0\big)$ which is predictable such that, for every $\Phi \geq 0$ predictable :

$$\mu_F(\Phi) = \mathbf{W}(\Phi_g \cdot F) = W\left(\int_0^\infty \Phi_s\, k_s(F) dL_s\right)$$

This is relation (1.2.64). The further relations (1.2.65) and (1.2.66) follow immediately.
ii) The other points of Lemma 1.2.10 are elementary. We show, for example, (1.2.68). We have, from (1.2.63) and (1.2.69), for every predictable and bounded process Φ :

$$\begin{aligned}\mathbf{W}(\Phi_g\, h_g) &= W\left(\int_0^\infty \Phi_s\, k_s(h_g) dL_s\right)\\ &= W\left(\int_0^\infty \Phi_s\, h_s\, dL_s\right)\end{aligned}$$

Hence, Φ being arbitrary, (1.2.68). Relation (1.2.67) is obtained by application of (1.2.68) with $(h_s,\ s \geq 0) = \big(k_s(F),\ s \geq 0\big)$.
Here is now the announced decomposition Theorem.
Theorem 1.2.11. *Let* $F \in L^1(\mathcal{F}_\infty, \mathbf{W})$*. There exist two continuous positive processes* $\big(\Sigma_t(F),\ t \geq 0\big)$ *and* $\big(\Delta_t(F),\ t \geq 0\big)$ *such that, for every* $t \geq 0$ *:*

$$M_t(F) = \Sigma_t(F) + \Delta_t(F) \qquad (t \geq 0) \tag{1.2.71}$$

with :
1)*i) For every* $t \geq 0$ *and* $\Gamma_t \in b(\mathcal{F}_t)$ *:*

$$\mathbf{W}(\Gamma_t\, 1_{g \leq t}\, F) = W\big(\Gamma_t \Sigma_t(F)\big) \tag{1.2.72}$$

ii) $\big(\Sigma_t(F),\ t \geq 0\big)$ *is a quasimartingale (a positive submartingale if* $F \geq 0$*) which vanishes on the zero set of* $(X_u,\ u \geq 0)$*. Its Doob-Meyer decomposition writes :*

$$\Sigma_t(F) = -M_t^{\Sigma(F)} + \int_0^t k_s(F)\, dL_s \tag{1.2.73}$$

In particular, the bounded variation part of this decomposition is absolutely continuous with respect to dL_s*. In (1.2.73),* $\big(M_t^{\Sigma(F)},\ t \geq 0\big)$ *is a* $\big((\mathcal{F}_t,\ t \geq 0),\quad W\big)$ *martingale satisfying, if* $F \geq 0$ *:*

$$\sup_{s \leq t} M_s^{\Sigma(F)} = \int_0^t k_s(F) dL_s \tag{1.2.74}$$

$$\lim_{t\to\infty} M_t^{\Sigma(F)} := M_\infty^{\Sigma(F)} = \int_0^\infty k_s(F)\, dL_s = \sup_{t\geq 0} M_t^{\Sigma(F)} \tag{1.2.75}$$

In particular, this martingale is not uniformly integrable.

iii) We have the "explicit formula" :

$$\Sigma_t(F) = |X_t| \cdot \widehat{E}_{X_t}^{(3)}\big(F(\omega_t, \widehat{\omega}_t)\big) \tag{1.2.76}$$

(see point 1 of Remark 1.2.2 for such a notation).
In (1.2.76), the expectation is taken with respect to $\widehat{\omega}_t$*, the letter* ω_t*, and* X_t*, being frozen ;* $\widehat{E}_{X_t}^{(3)}$ *denotes the expectation relatively to a 3-dimensional Bessel process starting from* X_t*, if* $X_t > 0$*, and the expectation with respect to the opposite of a 3-dimensional Bessel process, if* $X_t < 0$*.*

iv) The application $F \to \big(\Sigma_t(F),\ t \geq 0\big)$ *is injective since :*

$$\frac{\Sigma_t(F)}{1 + |X_t|} \underset{t\to\infty}{\longrightarrow} F \qquad \mathbf{W} \text{ a.s.} \tag{1.2.77}$$

v) We have, for every $t \geq 0$ *:*

$$W\big\{\Sigma_t(F) - \Sigma_t\big(k_g(F)\big)\big|\mathcal{F}_{g^{(t)}}\big\} = 0 \tag{1.2.78}$$

2)*i) For every* $t \geq 0$ *and* $\Gamma_t \in b(\mathcal{F}_t)$ *:*

$$\mathbf{W}(\Gamma_t\, 1_{g>t}\, F) = W\big(\Gamma_t\, \Delta_t(F)\big) \tag{1.2.79}$$

ii) $\big(\Delta_t(F),\ t \geq 0\big)$ *is a quasimartingale (a positive supermartingale if* $F \geq 0$*). Its Doob-Meyer decomposition writes :*

$$\Delta_t(F) = M_t^{\Delta(F)} - \int_0^t k_s(F)\, dL_s \tag{1.2.80}$$

where $\big(M_t^{\Delta(F)},\ t \geq 0\big)$ *is the* $\big((\mathcal{F}_t,\ t \geq 0),\quad W\big)$ *martingale given by :*

$$M_t^{\Delta(F)} = W\left(\int_0^\infty k_s(F)\, dL_s|\mathcal{F}_t\right) \tag{1.2.81}$$

In particular, since from (1.2.62), $\displaystyle\int_0^\infty k_s(F)\,dL_s \in L^1(\mathcal{F}_\infty, W)$*, this martingale is uniformly integrable.*

iii) The application $F \to \big(\Delta_t(F),\ t \geq 0\big)$ *is not injective since :*

$$\big(\Delta_t(F),\ t \geq 0\big) = \big(\Delta_t(k_g(F)),\ t \geq 0\big) \tag{1.2.82}$$

$\big($*and* $k_g(F) \neq F$ *when* F *is not* $\mathcal{F}_g$ *measurable*$\big)$.

3) *The martingale* $\big(M_t(F),\ t \geq 0\big)$ *satisfies :*

$$\Big(W\big(M_t(F)|\mathcal{F}_{g^{(t)}}\big),\ t \geq 0\Big) = \Big(W\big(M_t(k_g(F))|\mathcal{F}_{g^{(t)}}\big),\ t \geq 0\Big) \tag{1.2.83}$$

The following Theorem is an important consequence of Theorem 1.2.11.

Theorem 1.2.12. *Let $F \in L^1(\mathcal{F}_\infty, \mathbf{W})$.*
Then, the $\big((\mathcal{F}_t,\ t \geq 0), W\big)$ martingale $\big(M_t(F),\ t \geq\ 0\big)$ vanishes on the zeros of $(X_u,\ u \geq 0)$ if and only if $k_g(F) = 0$.

Remark 1.2.13

1) If $F = F_g$, with $(F_u,\ u \geq 0)$ a positive previsible process Theorem 1.2.3 implies, in this particular case :

$$\Sigma_t(F_g) = F_{g^{(t)}} \cdot |X_t|, \quad \Delta_t(F_g) = \int_{L_t}^{\infty} W(F_{\tau_l}|\mathcal{F}_t)\, dl.$$

2) If $F \geq 0$, the supermartingale $\big(\Delta_t(F),\ t \geq 0\big)$ satisfies :

$$\Delta_t(F) \underset{t\to\infty}{\longrightarrow} 0 \qquad W \text{ a.s.,} \quad \text{since } 0 \leq \Delta_t(F) \leq M_t(F)$$

and

$$\frac{\Delta_t(F)}{1+|X_t|} = \frac{M_t(F)}{1+|X_t|} - \frac{\Sigma_t(F)}{1+|X_t|} \underset{t\to\infty}{\longrightarrow} F - F = 0 \qquad \mathbf{W} \text{ a.s.}$$

from Lemma 1.2.8 and (1.2.77). Hence, in the decomposition (1.2.52) of the supermartingale $\Delta_t(F)$, there remains uniquely the term $(\xi_t,\ t \geq 0)$.

3) When $F \geq 0$, gathering the terms (1.2.71), (1.2.73), (1.2.80) and (1.2.81), we have :

$$M_t(F) = -M_t^{\Sigma(F)} + W\left(\int_0^\infty k_s(F) dL_s | \mathcal{F}_t\right)$$

This formula implies $\big($from (1.2.75)$\big)$ that $(M_t^{\Sigma(F)},\ t \geq 0)$ is not uniformly integrable since if it were, then $\big(M_t(F),\ t \geq 0\big)$ would be null.

4) From relation (1.2.83) there exists an application

$$\begin{array}{llll} m: & L^1(\mathcal{F}, \mathbf{W}) & \longrightarrow & \mathcal{M}\big((\mathcal{F}_{g^{(t)}},\ t \geq 0), \quad W\big) \\ & F & \longrightarrow & \big(m_t(F),\ t \geq 0\big) \end{array}$$

where $\mathcal{M}\big((\mathcal{F}_{g^{(t)}},\ t \geq 0),\ W\big)$ denotes the set of $\big((\mathcal{F}_{g^{(t)}},\ t \geq 0),\ W\big)$ martingales ; this application m is such that :

$$m_t(F) = W\big(M_t(k_g(F))|\mathcal{F}_{g^{(t)}}\big) \tag{1.2.84}$$

with

$$m_t(F) := \sigma_t(F) + \delta_t(F)$$

and

$$\begin{aligned} \sigma_t(F) &= \sqrt{\frac{\pi}{2}}\, k_{g^{(t)}}(F) \sqrt{t - g^{(t)}} \\ \delta_t(F) &= W\left(\int_0^\infty k_s(F)\, dL_s | \mathcal{F}_{g^{(t)}}\right) \end{aligned}$$

If $F \geq 0$, $\big(\sigma_t(F),\ t \geq 0\big)$ resp. $\big(\delta_t(F),\ t \geq 0\big)$ is a $\big((\mathcal{F}_{g^{(t)}},\ t \geq 0), \quad W\big)$ submartingale $\big($resp. $\big((\mathcal{F}_{g^{(t)}},\ t \geq 0), \quad W\big)$ supermartingale $\big)$.

5) We recall that by definition, a process $(Z_t,\ t \geq 0)$ is a quasimartingale if, for every $t \geq 0$:

$$\sup\ W\left(\sum_{i=1}^{n-1} \left|W\big(Z_{t_{i+1}} - Z_{t_i}\big)\right| \left|\mathcal{F}_{t_i}\right.\right) < \infty$$

the sup being taken over the set of subdivisions $0 \leq t_1 < \cdots < t_n < t$. In fact, such a process is the difference of two supermartingales $\big($see [R]$\big)$. On the other hand, the Föllmer measure $\big($see [F]$\big)$ μ_Z - with finite mass - of a supermartingale $(Z_t,\ t \geq 0)$ (or of a quasimartingale) is the measure defined on the predictable σ-field and characterised by :

$$\mu_Z(\Gamma_t\, 1_{]t,\infty]}) = W(\Gamma_t \cdot Z_t) \qquad \big(\Gamma_t \in b(\mathcal{F}_t)\big)$$

Hence formulae (1.2.65), (1.2.70) and (1.2.79) imply that the measure μ_F defined by (1.2.70) is the Föllmer measure of the quasimartingale $\big(\Delta_t(F),\ t \geq 0\big)$.

Proof of Theorem 1.2.11.

i) We define $\Sigma_t(F)$ via :

$$\Sigma_t(F) = M_t(F\, 1_{g \leq a})\Big|_{a=t} \tag{1.2.85}$$

Hence, for every $\Gamma_t \in b(\mathcal{F}_t)$:

$$\mathbf{W}(\Gamma_t\, 1_{g \leq t} \cdot F) = W\big(\Gamma_t\, \Sigma_t(F)\big) \tag{1.2.86}$$

It is easy to deduce from (1.2.86) that $\big(\Sigma_t(F) = \Sigma_t(F^+) - \Sigma_t(F^-),\ t \geq 0\big)$ is a semimartingale, as the difference of two submartingales and we shall show below (see point *vi)* of this proof) that it is in fact a quasimartingale which admits a continuous version.

ii) We show (1.2.73).

By linearity, it suffices to prove (1.2.73) for $F \geq 0$. From (1.2.86), we have for $s \leq t$ and $\Gamma_s \in b(\mathcal{F}_s)$:

$$\begin{aligned} \mathbf{W}(\Gamma_s\, 1_{s \leq g \leq t}\, F) &= W\Big(\Gamma_s\big(\Sigma_t(F) - \Sigma_s(F)\big)\Big) \\ &= W\left(\Gamma_s \cdot \int_s^t k_u(F)\, dL_u\right) \end{aligned} \tag{1.2.87}$$

by using Lemma 1.2.10 with $(\Phi_u := \Gamma_s\, 1_{]s,t]}\,(u),\ u \geq 0)$. (1.2.73) follows immediately from (1.2.87).

iii) We show (1.2.74) and (1.2.75).

Since, if $F \geq 0$, then $\Sigma_s(F) \geq 0$, we have :

$$\sup_{s \leq t} M_s^{\Sigma(F)} \leq \int_0^t k_u(F) dL_u \qquad \text{and}$$

$$\sup_{s \leq t} M_s^{\Sigma(F)} \geq \sup_{s \leq g^{(t)}} M_s^{\Sigma(F)} = \int_0^{g^{(t)}} k_u(F) dL_u = \int_0^t k_u(F) dL_u$$

since $\Sigma_{g^{(t)}}(F) = 0$ from (1.2.76) (which is proven below).

On the other hand, since $0 \leq \Sigma_t(F) \leq M_t(F)$ and since $M_t(F) \underset{t\to\infty}{\longrightarrow} 0$ $\quad W$ a.s. from Theorem 1.2.1, we have $\Sigma_t(F) \underset{t\to\infty}{\longrightarrow} 0$ $\quad W$ a.s., and thus, from (1.2.73) :

$$\lim_{t\to\infty} M_t^{\Sigma(F)} := M_\infty^{\Sigma(F)} = \int_0^\infty k_s(F)\, dL_s = \sup_{t \geq 0} M_t^{\Sigma(F)}$$

which, in particular, proves, that $(M_t^{\Sigma(F)},\ t \geq 0)$ is not uniformly integrable.

iv) We show (1.2.76).

For this purpose, we shall use the notation and results of subsection 1.1.4. We have, for every $t \geq 0$ and $\Gamma_t \in b(\mathcal{F}_t)$:

$$\begin{aligned}
&W\big(\Gamma_t\,\Sigma_t(F)\big)\\
&= \mathbf{W}(\Gamma_t\,1_{g\leq t}\,F) \qquad \big(\text{from (1.2.86)}\big)\\
&= 2\,W_\infty^{(\delta_0)}(\Gamma_t\,1_{g\leq t}\,F\,e^{\frac{1}{2}L_\infty})\\
&= 2\,W_\infty^{(\delta_0)}(\Gamma_t\,e^{\frac{1}{2}L_t}\,1_{g\leq t}\,F) \quad \big(\text{since } L_\infty = L_t \text{ on the set } (g \leq t)\big)\\
&= 2\,W_\infty^{(\delta_0)}(\Gamma_t\,e^{\frac{1}{2}L_t}W_\infty^{(\delta_0)}\big(1_{g\leq t}\cdot F|\mathcal{F}_t)\big)\\
&= 2\,W_\infty^{(\delta_0)}\big(\Gamma_t\,e^{\frac{1}{2}L_t}W_\infty^{(\delta_0)}(1_{T_0\circ\theta_t=\infty}\cdot F|\mathcal{F}_t)\big) \quad \big(\text{since } (g\leq t) = (T_0\circ\theta_t=\infty)\big)\\
&= 2\,W_\infty^{(\delta_0)}\big(\Gamma_t\,e^{\frac{1}{2}L_t}\,\widehat{W}_{X_t,\infty}^{(\delta_0)}\big(1_{T_0=\infty}F(\omega_t,\widehat{\omega}^t)\big)\\
&\quad \text{(by the Markov property)}\\
&= 2\,W_\infty^{(\delta_0)}\big(\Gamma_t\,e^{\frac{1}{2}L_t}\,\widehat{W}_{X_t,\infty}^{(\delta_0)}\big(F(\omega_t,\widehat{\omega}^t)|T_0=\infty\big)\cdot W_{X_t,\infty}^{(\delta_0)}(T_0=\infty)\big) \qquad (1.2.88)
\end{aligned}$$

But, from (1.1.70) :

$$W_{X_t,\infty}^{(\delta_0)}(T_0=\infty) = \frac{|X_t|}{2+|X_t|}$$

and, from Theorem 1.1.5, conditionally on $(T_0=\infty)$, $W_{\infty,x}^{(\delta_0)}$ is the law of a Bessel (3) process (resp. of the opposite of a Bessel (3) process) started at x if $x>0$ (resp. if $x<0$). Then :

$$\begin{aligned}
W\big(\Gamma_t\,\Sigma_t(F)\big) &= 2\,W_\infty^{(\delta_0)}\left(\Gamma_t\,e^{\frac{1}{2}L_t}\frac{|X_t|}{2+|X_t|}\,\widehat{E}_{X_t}^{(3)}\big(F(\omega_t,\widehat{\omega}^t)\big)\right)\\
&= W\left(\Gamma_t\,e^{\frac{1}{2}L_t}\,\frac{|X_t|}{2+|X_t|}\,\widehat{E}_{X_t}^{(3)}\big(F(\omega_t,\widehat{\omega}^t)\big)\,e^{-\frac{1}{2}L_t}\big(2+|X_t|\big)\right)
\end{aligned}$$

$\big($from (1.1.31) and (1.1.7)$\big)$.

Finally $W\big(\Gamma_t\,\Sigma_t(F)\big) = W\Big(\Gamma_t|X_t|\,\widehat{E}_{X_t}^{(3)}\big(F(\omega_t,\widehat{\omega}^t)\big)\Big)$

It is relation (1.2.76). Observe that this relation implies $\big(\Sigma_t(F),\ t\geq 0\big)$ vanishes on the zeros of $(X_t,\ t\geq 0)$. On the other hand, (1.2.76) implies (1.2.77), since, under $\mathbf{W}$, $|X_t| \underset{t\to\infty}{\longrightarrow} \infty$ a.s.

v) We show (1.2.83) and (1.2.78).

For every positive, bounded and predictable process $(\Phi_u,\ u\geq 0)$, we have :

$$W\big(\Phi_{g^{(t)}}\,M_t(F)\big) = \mathbf{W}(\Phi_{g^{(t)}}\cdot F) \qquad (1.2.89)$$

by definition of $M_t(F)$. But, the σ-algebra $\mathcal{F}_{g^{(t)}}$ is contained in $\mathcal{F}_g$. Hence the RHS of (1.2.89) equals from (1.2.64) :

$$\mathbf{W}\big(\Phi_{g^{(t)}}\,k_g(F)\big) = W\Big(\Phi_{g^{(t)}}\,M_t\,\big(k_g(F)\big)\Big)$$

Finally :

$$W\big(\Phi_{g^{(t)}}\,k_g(F)\big) = W\Big(\Phi_{g^{(t)}}\,M_t\,\big(k_g(F)\big)\Big)$$

Thus $W\big(M_t(F) - M_t(k_g(F))\big|\mathcal{F}_{g^{(t)}}\big) = 0$ i.e. (1.2.83) is satisfied. (1.2.78) is proven by using the same arguments.

vi) <u>We show (1.2.79).</u>
We define $\Delta_t(F)$ by :

$$\Delta_t(F) := M_t(F\,1_{g>a})|_{a=t}$$

It is clear that :

$$M_t(F) = \Sigma_t(F) + \Delta_t(F)$$

and that, for every $t \geq 0$ and $\Gamma_t \in b(\mathcal{F}_t)$:

$$\mathbf{W}(\Gamma_t\,1_{g>t}\,F) = W\big(\Gamma_t\,\Delta_t(F)\big)$$

Then writing $\Delta_t(F) = \Delta_t(F^+) - \Delta_t(F^-)$ we deduce easily from this formula that $\big(\Delta_t(F^\pm),\ t \geq 0\big)$ are two positive supermartingales and then $\big(\Delta_t(F),\ t \geq 0\big)$ is a quasimartingale. Since $\Sigma_t(F) = M_t(F) - \Delta_t(F) = M_t(F^+) - M_t(F^-) - \Delta_t(F^+) + \Delta_t(F^-)$, it is clear that $\big(\Sigma_t(F),\ t \geq\ 0\big)$ is still a quasimartingale. Formula (1.2.80) then results from (1.2.73) and (1.2.71). Finally, thanks to (1.2.80) and (1.2.73), $\big(\Delta_t(F),\ t \geq 0\big)$ and $\big(\Sigma_t(F),\ t \geq 0\big)$ admit continuous versions.

vii) <u>We show (1.2.81).</u>
We have, from (1.2.79), for every $\Gamma_t \in b(\mathcal{F}_t)$

$$\begin{aligned}
\mathbf{W}(\Gamma_t\,1_{g>t}\,F) &= W\big(\Gamma_t\,\Delta_t(F)\big)\\
&= \mathbf{W}(\widetilde{\Gamma}_g\cdot F)\\
\Big(\text{with}\quad (\widetilde{\Gamma}_u,\ u\geq 0) &:= \big(\Gamma_t\,1_{]t,\infty[}(u),\ u \geq 0\big)\Big)\\
&= W\left(\int_0^\infty \widetilde{\Gamma}_u\,k_u(F)dL_u\right)\quad \text{(from Lemma 1.2.10)}\\
&= W\left(\Gamma_t\cdot\int_t^\infty k_u(F)dL_u\right)\\
&= W\left(\Gamma_t\,W\left(\int_t^\infty k_u(F)dL_u|\mathcal{F}_t\right)\right)\\
\text{Hence : }\ \Delta_t(F) &= W\left(\int_t^\infty k_u(F)dL_u|\mathcal{F}_t\right)\\
&= W\left(\int_0^\infty k_u(F)dL_u\Big|\mathcal{F}_t\right) - \int_0^t k_u(F)dL_u
\end{aligned}$$

This equality implies (1.2.80) and (1.2.81).

viii) <u>We show (1.2.82).</u>
It suffices, to prove (1.2.82), to show that for every $t \geq 0$ and $\Gamma_t \in b(\mathcal{F}_t)$, we have :

$$W\big(\Gamma_t\,\Delta_t(F - k_g(F))\big) = 0$$

But :

$$\begin{aligned}
&W\big(\Gamma_t\,\Delta_t(F - k_g(F))\big) = \mathbf{W}\big(\Gamma_t\,1_{g>t}(F - k_g(F)\big)\\
&\qquad = \mathbf{W}\big(\widetilde{\Gamma}_g(F - k_g(F)\big)
\end{aligned}$$

$\Big($with $\quad (\widetilde{\Gamma}_u := \Gamma_t\, 1_{]t,\infty[}\,(u),\ u \geq 0)\Big)$

$$= W\left(\Gamma_t \int_t^\infty \big(k_u(F) - k_u(k_g(F))\big)\, dL_u\right) \quad \big(\text{from } (1.2.63)\big)$$
$$= 0$$

since $\quad k_u(F) = k_u(k_g(F))$, from (1.2.67).

ix) Observe that, by using (1.2.82), (1.2.83) is a consequence of (1.2.78). Indeed :

$$\begin{aligned} M_t\big(F - k_g(F)\big) &= \Sigma_t\big(F - k_g(F)\big) + \Delta_t\big(F - k_g(F)\big) \\ &= \Sigma_t\big(F - k_g(F)\big) \qquad \text{from } (1.2.82) \end{aligned}$$

Thus :

$$W\big(M_t\big(F - k_g(F)\big)\big|\mathcal{F}_{g^{(t)}}\big) = W\big(\Sigma_t\big(F - k_g(F)\big)\big|\mathcal{F}_{g^{(t)}}\big) = 0 \quad \text{from } (1.2.78)$$

This ends the proof of Theorem 1.2.11.

Proof of Theorem 1.2.12.
For this purpose, we need the following result, due to Azéma and Yor $\big($see [AY2]$\big)$: a $\big((\mathcal{F}_t,\ t \geq\ 0),\ W\big)$ martingale $(M_t,\ t \geq 0)$ vanishes on the zeros of $(X_u,\ u \geq 0)$ if and only if for every $t \geq 0$:

$$W(M_t|\mathcal{F}_{g^{(t)}}) = 0. \tag{1.2.90}$$

Suppose $k_g(F) = 0$
From (1.2.83), we have : $\quad W(M_t(F)|\mathcal{F}_{g^{(t)}}) = W\big(M_t(k_g(F))|\mathcal{F}_{g^{(t)}}\big) = 0$. Thus, from (1.2.90), $\big(M_t(F),\ t \geq 0\big)$ vanishes on the zeros of $(X_u,\ u \geq 0)$.
Conversely, suppose that $\big(M_t(F),\ t \geq\ 0\big)$ vanishes on the zeros of $(X_u \geq 0)$. Then we have from (1.2.90) and (1.2.83), for every s and t, $s \leq t$ and $\Gamma_s \in b(\mathcal{F}_s)$, since $\Gamma_s\, 1_{s \leq g^{(t)}}$ is a $\mathcal{F}_{g^{(t)}}$ measurable r.v. :

$$\begin{aligned} 0 &= W\big(\Gamma_s\, 1_{s<g^{(t)}}\, M_t(k_g(F))\big) \\ &= \mathbf{W}\big(\Gamma_s\, 1_{s<g^{(t)}}\, k_g(F)\big) \underset{t\to\infty}{\longrightarrow} \mathbf{W}\big(\Gamma_s\, 1_{s\leq g}\, k_g(F)\big) \end{aligned}$$

since $g^{(t)} \underset{t\to\infty}{\longrightarrow} g \quad \mathbf{W}$ a.s.
Thus :

$$\mathbf{W}\big(\Gamma_s\, 1_{s\leq g}\, k_g(F)\big) = 0$$

We deduce from the monotone class Theorem that, for every bounded $\mathcal{F}_g$ measurable r.v. Φ :

$$\mathbf{W}\big(\Phi\, k_g(F)\big) = 0. \tag{1.2.91}$$

i.e. $\ k_g(F) = 0 \quad$ since $\quad k_g(F)$ is $\mathcal{F}_g$-measurable. ■

We end this subsection with some examples of decomposition (1.2.71).

Example 8. Let $F := \exp\left(-\dfrac{\lambda}{2}\, L_\infty\right)$. We have shown (Example 2) that :

$$M_t(F) = \left(\frac{2}{\lambda} + |X_t|\right)\, e^{-\frac{\lambda}{2}\, L_t} \tag{1.2.92}$$

We then have :

$$\begin{aligned}
&M_t(F) = \Sigma_t(F) + \Delta_t(F) \qquad \text{with} \\
&\Sigma_t(F) = |X_t|\, e^{-\frac{\lambda}{2}\,L_t},\ \Delta_t(F) = \frac{2}{\lambda}\, e^{-\frac{\lambda}{2}\,L_t}
\end{aligned} \tag{1.2.93}$$

Indeed, from (1.2.72) :

$$\begin{aligned}
\mathbf{W}(\Gamma_t\, 1_{g\le t}\, e^{-\frac{\lambda}{2}\,L_t}) &= W\big(\Gamma_t\, \Sigma_t(e^{-\frac{\lambda}{2}\,L_\infty})\big) \\
&= \mathbf{W}(\Gamma_t\, 1_{g\le t}\, e^{-\frac{\lambda}{2}\,L_t}) \ \big(\text{since } L_\infty = L_t \text{ on the set } (g\le t)\big) \\
&= W(\Gamma_t |X_t| e^{-\frac{\lambda}{2}\,L_t})
\end{aligned}$$

from point 2 *i)* of Theorem 1.1.6.
Thus :

$$\Sigma_t(e^{-\frac{\lambda}{2}\,L_\infty}) = |X_t|\, e^{-\frac{\lambda}{2}\,L_t}$$

Example 9. This example generalises Example 8. Let $q \in \mathcal{I}$ and $F := \exp\left(-\frac{1}{2}\,A_\infty^{(q)}\right)$. We know (see Example 1) that :

$$M_t(e^{-\frac{1}{2}\,A_\infty^{(q)}}) = \varphi_q(X_t)\, \exp\left(-\frac{1}{2}\,A_t^{(q)}\right) \tag{1.2.94}$$

Then :

$$\Sigma_t(e^{-\frac{1}{2}\,A_\infty^{(q)}}) = \psi_q(X_t)\, e^{-\frac{1}{2}\,A_t^{(q)}},\ \Delta_t(e^{-\frac{1}{2}\,A_\infty^{(q)}}) = (\varphi_q - \psi_q)(X_t) e^{-\frac{1}{2}\,A_t^{(q)}} \tag{1.2.95}$$

with ψ_q solution of :

$$\begin{aligned}
&\psi'' = q\,\psi \quad \text{on } \mathbb{R}\backslash\{0\} \\
&\psi(x) \underset{|x|\to\infty}{\sim} |x|, \quad \psi(0) = 0
\end{aligned} \tag{1.2.96}$$

Proof of (1.2.95). We have :

$$\begin{aligned}
\mathbf{W}(\Gamma_t\, 1_{g\le t}\, e^{-\frac{1}{2}\,A_\infty^{(q)}}) &= \varphi_q(0)\ W_\infty^{(q)}(\Gamma_t\, 1_{g\le t}) \qquad \big(\text{from } (1.1.16)\big), \\
&\quad \text{(with the notation of Theorems 1.1.1 and 1.1.2)} \\
&= \varphi_q(0)\, W_\infty^{(q)}\big(\Gamma_t\, W_\infty^{(q)}(1_{g\le t}|\mathcal{F}_t)\big) \\
&= \varphi_q(0)\, W_\infty^{(q)}\big(\Gamma_t\, W_{\infty,X_t}^{(q)}(T_0 = \infty)\big)
\end{aligned} \tag{1.2.97}$$

But, by using the scale function γ_q of the Markov process $(X_t,\ t\ge 0)$ under $W_\infty^{(q)}$, we have, with γ_q given by (1.1.14) :

$$\begin{aligned}
W_{\infty,x}^{(q)}(T_0 = \infty) &= \frac{\gamma_q(x) - \gamma_q(0)}{\gamma_q(\infty) - \gamma_q(0)} \qquad \text{if } x > 0 \\
&= \frac{\gamma_q(0) - \gamma_q(x)}{\gamma_q(0) - \gamma_q(-\infty)} \quad \text{if } x < 0 \\
&:= \lambda_q(x)
\end{aligned} \tag{1.2.98}$$

Hence, by definition of $\Sigma_t(e^{-\frac{1}{2}A_\infty^{(q)}})$:

$$\begin{aligned} W\big(\Gamma_t\,\Sigma_t\,(e^{-\frac{1}{2}A_\infty^{(q)}})\big) &= \varphi_q(0)\,W_\infty^{(q)}\big(\Gamma_t\,\lambda_q(X_t)\big) \\ &= W\big(\Gamma_t\,\varphi_q(X_t)\lambda_q(X_t)e^{-\frac{1}{2}A_t^{(q)}})\big) \end{aligned} \tag{1.2.99}$$

Thus :

$$\Sigma_t(e^{-\frac{1}{2}A_\infty^{(q)}}) = \psi_q(X_t)\,e^{\frac{1}{2}A_t^{(q)}}$$

with

$$\psi_q(x) := \lambda_q(x)\,\varphi_q(x) \tag{1.2.100}$$

It is clear, from (1.2.100), (1.2.98) and since $\varphi_q(x) \underset{|x|\to\infty}{\sim} |x|$ that :

$$\psi_q(x) \underset{|x|\to\infty}{\sim} |x| \quad \text{and} \quad \psi_q(0) = 0.$$

On the other hand, the relation $\psi_q'' = q\,\psi_q$ on $\mathbb{R}$ is the consequence of direct calculation using the explicit form of γ_q given by (1.1.14) (see Lemma 1.3.3 below for such a computation). We deduce from (1.2.95) and from Itô-Tanaka :

$$\begin{aligned} \Sigma_t(e^{-\frac{1}{2}A_\infty^{(q)}}) &= \int_0^t \psi_q'(X_s)\,e^{-\frac{1}{2}A_s^{(q)}}dX_s + \frac{1}{2}\int_0^t \big(\psi_q'(0_+) - \psi_q'(0_-)\big)\,e^{-\frac{1}{2}A_s^{(q)}}dL_s \\ \text{i.e.} \qquad & M_t^{\Sigma(F)} = -\int_0^t \psi_q'(X_s)e^{-\frac{1}{2}A_s^{(q)}}\,dX_s \\ & k_s(e^{-\frac{1}{2}A_\infty^{(q)}}) = \frac{1}{2}\big(\psi_q'(0_+) - \psi_q'(0_-)\big)\,e^{-\frac{1}{2}A_s^{(q)}}. \end{aligned}$$

Example 10. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ Borel and integrable with $\psi(\infty) = 0$ and $F := \psi(S_\infty)$. We know (see Example 3) that :

$$M_t\big(\psi(S_\infty)\big) = \psi(S_t)(S_t - X_t) + \int_{S_t}^\infty \psi(y)dy \tag{1.2.101}$$

We have :

$$\Sigma_t\big(\psi(S_\infty)\big) = \psi(S_t)X_t^-, \quad \Delta_t\big(\psi(S_\infty)\big) = \psi(S_t)(S_t - X_t^+) + \int_{S_t}^\infty \psi(y)dy \tag{1.2.102}$$

Indeed :

$$\mathbf{W}\big(\Gamma_t\,1_{g\le t}\,\psi(S_\infty)\big) = \mathbf{W}^-\big(\Gamma_t\,1_{g\le t}\,\psi(S_g)\big)$$

$\big($since $\psi(\infty) = 0, S_\infty = \infty$ on Γ_+, $S_\infty = S_g$ on $\Gamma_-\big)$.

$$\begin{aligned} &= \mathbf{W}^-\big(\Gamma_t\,1_{g\le t}\,\psi(S_{g^{(t)}})\big) && \big(\text{since } g^{(t)} = g \text{ on } (g\le t)\big) \\ &= W\big(\Gamma_t\,\psi(S_{g^{(t)}})X_t^-\big) && \text{(from (1.1.52))} \\ &= W\big(\Gamma_t\,\psi(S_t)\,X_t^-\big) && (\text{since } S_{g^{(t)}} = S_t \quad \text{if} \quad X_t < 0) \end{aligned}$$

Thus $\qquad \Sigma_t\big(\psi(S_\infty)\big) = \psi(S_t)\,X_t^-$

Example 11. In some sense, the present example stands midway between Examples 9 and 10. Let $q : \mathbb{R} \to \mathbb{R}_+$ such that $q(x) = 0$ if $x < 0$, $q(x) > 0$ if $x > 0$, $\underline{\lim}_{x\to\infty} q(x) > 0$. We have shown, in [RY, IX] (see also [RY, M]) the existence for every $x \in \mathbb{R}$ of a σ-finite measure $\boldsymbol{\nu}_x^{(q)}$, on $\mathbb{R}_+$ such that :

$$M_t\big(h(A_\infty^{(q)})\big) = \int_{\mathbb{R}_+} h(A_t^{(q)} + y)\, \boldsymbol{\nu}_{X_t}^{(q)}(dy) \tag{1.2.103}$$

for $h : \mathbb{R}_+ \to \mathbb{R}_+$ sub-exponential at infinity.
We then have :

$$\begin{aligned} \Sigma_t\big(h(A_\infty^{(q)})\big) &= h(A_t^{(q)}) \cdot X_t^- && (1.2.104) \\ \Delta_t\big(h(A_\infty^{(q)})\big) &= \int_{\mathbb{R}_+} h(A_t^{(q)} + y)\big(\boldsymbol{\nu}_{X_t}^{(q)}(dy) - X_t^- \delta_0(dy)\big) \\ &= \int_{\mathbb{R}_+} h(A_t^{(q)} + y)\, \boldsymbol{\nu}_{X_t}^{(q),a}(dy) && (1.2.105) \end{aligned}$$

where $\boldsymbol{\nu}_{X_t}^{(q),a}$ denotes the absolute continuous part of $\boldsymbol{\nu}_{X_t}^{(q)}$. Relation (1.2.104) is obtained from the same arguments as those used for relation (1.2.102) by noting that $1_{X_t \le 0}\ dA_t^{(q)} = 0$ and (1.2.105) results from :

$$\begin{aligned} \text{if } x < 0,\ \boldsymbol{\nu}_x^{(q)}(dy) &= \boldsymbol{\nu}_x^{(q),a}(dy) + x^- \delta_0(dy) \\ \text{if } x > 0,\ \boldsymbol{\nu}_x^{(q)}(dy) &= \boldsymbol{\nu}_x^{(q),a}(dy) \qquad \text{(see [RY, IX])} \end{aligned}$$

Example 12. Let $q : \mathbb{R} \to \mathbb{R}_+$ such that :

$$\int_{-\infty}^0 \big(1 + |x|\big) q(x) dx < \infty\ ;\ \underline{\lim}_{x\to\infty} x^{2\alpha} q(x) > 0 \quad \text{for some } \alpha < 1$$

and $A_t^{(q)} := \int_0^t q(X_s) ds$. Let φ_q the solution of $\varphi'' = q\,\varphi$, $\varphi'(-\infty) = -1$, $\varphi(+\infty) = 0$. Then, we have :

$$M_t\left(\exp - \frac{1}{2} A_\infty^{(q)}\right) = \varphi_q(X_t) \exp\left(-\frac{1}{2} A_t^{(q)}\right) \tag{1.2.106}$$

$$\text{and} \qquad e^{-\frac{1}{2} A_\infty^{(q)}} \cdot \mathbf{W}^- = \mathbf{W}(e^{-\frac{1}{2} A_\infty^{(q)}}) \cdot W_\infty^{(q)} \tag{1.2.107}$$

where the probability $W_\infty^{(q)}$ is characterised by

$$W_\infty^{(q)}|_{\mathcal{F}_t} = \frac{\varphi_q(X_t)}{\varphi_q(0)} \exp\left(-\frac{1}{2} A_t^{(q)}\right) \cdot W|_{\mathcal{F}_t} \tag{1.2.108}$$

(see [RVY, I], the one-sided case, p. 209). We then have :

$$\Sigma_t(e^{-\frac{1}{2} A_\infty^{(q)}}) = \psi_q(X_t) e^{-\frac{1}{2} A_t^{(q)}} \tag{1.2.109}$$

with

$$\begin{aligned} &\psi_q(x) = 0 \quad \text{if } x \ge 0 \\ &\psi_q(x) \underset{x\to-\infty}{\sim} |x| \quad \text{and} \quad \psi_q'' = q\,\psi_q \ \text{ on } \mathbb{R}_- \end{aligned}$$

Hence

$$\Delta_t(e^{-\frac{1}{2}A_\infty^{(q)}}) = \begin{cases} M_t(e^{-\frac{1}{2}A_\infty^{(q)}}) & \text{if } X_t \geq 0 \\ (\varphi_q - \psi_q)(X_t)\, e^{-\frac{1}{2}A_t^{(q)}} & \text{if } X_t \leq 0 \end{cases}$$

(1.2.109) is obtained by following the same arguments as those in Example 9. What changes is that, under the probability $W_{x,\infty}^{(q)}$, we have $X_t \to -\infty$ a.s., for every x (see Theorem 5.1 in [RVY, I]).

Example 13. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ Borel and integrable, such that $\displaystyle\int_0^\infty \psi(y)dy = 1$. Then we have, from (1.2.24)

$$M_t\big(\psi(S_g)\big) = \frac{1}{2}\,\psi(S_{g^{(t)}})|X_t| + \psi(S_t)(S_t - X_t^+) + \int_{S_t}^\infty \psi(y)dy \tag{1.2.110}$$

(see Example 4, (1.2.24) and (1.2.25)) ;

$$\mathbf{W}^-\big(\psi(S_g)\big) = \mathbf{W}\big(\psi(S_\infty)\big) = \int_0^\infty \psi(l)dl$$

On the other hand, we have :

$$\begin{aligned} \mathbf{W}\big(\Gamma_t\, 1_{g\leq t}\, \psi(S_g)\big) &= W\big(\Gamma_t\, \Sigma_t\big(\psi(S_g)\big)\big) \\ &= \mathbf{W}\big(\Gamma_t\, 1_{g\leq t}\, \psi(S_{g^{(t)}})\big) \quad \big(\text{since } g = g^{(t)} \text{ on } (g \leq t)\big) \\ &= W\big(\Gamma_t\, \psi(S_{g^{(t)}})|X_t|\big) \quad \text{(from point 2 } i) \text{ of Theorem 1.1.6.)} \end{aligned}$$

Hence :

$$\Sigma_t\big(\psi(S_g)\big) = \psi(S_{g^{(t)}})|X_t| \tag{1.2.111}$$

and, from (1.2.110) :

$$\Delta_t\big(\psi(S_g)\big) = \psi(S_t)(S_t - X_t^+) + \int_{S_t}^\infty \psi(y)dy \tag{1.2.112}$$

1.2.5 A penalisation Theorem, for functionals in class $\mathcal{C}$

In Section 1 of this Chapter, we constructed the measure **W** from the penalisation results, and more particularly from Feynman-Kac type penalisations. We shall now operate in a reverse order : starting from the existence and the properties of the measure **W** which we just established, we shall obtain penalisation results.

Here is the class of functionals $(F_t,\ t \geq 0)$ for which we shall obtain such a penalisation result.

Definition 1.2.13. Let $(F_t,\ t \geq 0)$ denote an adapted, positive process. We shall say that this process belongs to the class $\mathcal{C}$ if

i) $(F_t,\ t \geq 0)$ is a decreasing process, i.e. if $s \leq t$:

$$0 \leq F_t \leq F_s \quad W \text{ a.s.} \tag{1.2.113}$$

In particular, since $0 \leq F_t \leq F_0$ and since F_0 is a.s. constant, this process is bounded by a constant $C = F_0$.

ii) There exists $a \geq 0$ such that for every $t \geq \sigma_a$, with :

$$\sigma_a := \sup\{t \geq 0 \; ; \; X_t \in [-a, a]\}$$

we have :

$$F_t = F_{\sigma_a} = F_\infty \tag{1.2.114}$$

iii)

$$\mathbf{W}(F_\infty) = \mathbf{W}(F_{\sigma_a}) < \infty \tag{1.2.115}$$

One of the advantages of this class $\mathcal{C}$ is that it contains a large number of processes $(F_t,\ t \geq 0)$ for which we have already obtained a penalisation result. More precisely, let $\varphi : \mathbb{R}^n \to \mathbb{R}_+$ Borel. Then :

$$F_t := \varphi(L_t^{a_1}, \cdots L_t^{a_r}, A_t^{(q_1)}, \cdots A_t^{(q_s)}, D_t^{[\alpha_1,\beta_1]}, \cdots D_t^{[\alpha_u,\beta_u]}, S_{g^{(t)}}, -I_{g^{(t)}})$$

(see Examples 1 to 9 for these notations) belongs to the class $\mathcal{C}$ $\big($if (1.2.115) is satisfied$\big)$ as soon as $q_1, \cdots, q_s$ are elements of $\mathcal{I}$ with compact support (if we choose a large enough) and φ is a function which is decreasing with respect to each of its arguments. We may add S_t and $(-I_t)$ to the list of the arguments of φ, if φ has compact support in these arguments.
One can give some examples of functionals $(F_t,\ t \geq 0)$ which are not in the class $\mathcal{C}$ and for which the statement of Theorem 1.2.14 below does not apply. One of these examples is the functional :

$$\left(F_t = \exp\left(-\int_{-\infty}^{\infty} (L_t^y)^2\, dy\right),\ t \geq 0\right)$$

(see [N3] for a study of this functional).
Here is the first step towards a penalisation result.
Theorem 1.2.14. *Let $(F_t,\ t \geq 0)$ be a process which belongs to $\mathcal{C}$. Then :*
1)

$$\sqrt{\frac{\pi t}{2}}\, W(F_t) \underset{t\to\infty}{\longrightarrow} \mathbf{W}(F_\infty) \tag{1.2.116}$$

2)

$$W(F_t.|X_t|) \underset{t\to\infty}{\longrightarrow} \mathbf{W}(F_\infty) \tag{1.2.117}$$

Proof of Theorem 1.2.14.
1) We start with the proof of point 1)
We write F_t in the form :

$$F_t = F_t\, \frac{(|X_t| - a)_+}{1 + |X_t|} + F_t\, \frac{1 + |X_t| - (|X_t| - a)_+}{(1 + |X_t|)^2}\, (|X_t| - a)_+$$
$$+ F_t\, \frac{\big[(1 + |X_t|) - (|X_t| - a)_+\big]^2}{(1 + |X_t|)^2} := F_t^{(1)} + F_t^{(2)} + F_t^{(3)} \tag{1.2.118}$$

and we study each term of this decomposition of F_t.
i) Study of $W(F_t^{(1)})$.

For $\lambda > 0$, we have :

$$\int_0^\infty e^{-\lambda t} W(F_t^{(1)}) dt = \int_0^\infty e^{-\lambda t} W\left(F_t \, \frac{(|X_t| - a)_+)}{1 + |X_t|}\right) dt$$

$$= \int_0^\infty e^{-\lambda t} \mathbf{W}\left(\frac{F_t}{1+|X_t|} \, 1_{\sigma_a \le t}\right) dt$$

(by Theorem 1.1.8, relation (1.1.49))

$$= \int_0^\infty e^{-\lambda t} \mathbf{W}\left(F_{\sigma_a} \, \frac{1_{\sigma_a \le t}}{1+|X_t|}\right) dt \qquad \text{(from (1.2.114))}$$

$$= \mathbf{W}\left(F_{\sigma_a} \, e^{-\lambda \sigma_a} \int_0^\infty e^{-\lambda u} \frac{du}{1 + |X_{\sigma_a + u}|}\right)$$

(after the change of variable $t = \sigma_a + u$)

$$= \mathbf{W}(F_\infty e^{-\lambda \sigma_a}) \, E_0^{(3)} \left(\int_0^\infty e^{-\lambda u} \frac{du}{1 + a + R_u}\right) \tag{1.2.119}$$

from point 2 of Theorem 1.1.8, where in (1.2.119) $(R_u, \; u \ge 0)$ denotes a Bessel process of dimension 3 started at 0. But

$$E_0^{(3)} \left[\frac{1}{1 + a + R_u}\right] \underset{u \to \infty}{\sim} \sqrt{\frac{2}{\pi u}}$$

and is a decreasing function of u. By the (easy part of the) Tauberian Theorem (see [Fe]) :

$$\int_0^\infty e^{-\lambda t} \, W(F_t^{(1)}) dt \underset{\lambda \to 0}{\sim} \mathbf{W}(F_\infty) \sqrt{\frac{2}{\lambda}} \tag{1.2.120}$$

ii) Study of $W(F_t^{(2)})$.
For $\lambda > 0$, we have :

$$\int_0^\infty e^{-\lambda t} \, W(F_t^{(2)}) dt \le (1 + a) \int_0^\infty e^{-\lambda t} W\left(F_t \, \frac{(|X_t| - a)_+}{(1 + |X_t|)^2}\right) dt$$

(from (1.2.118) and since : $0 \le 1 + |X_t| - (|X_t| - a)_+ \le 1 + a$)

$$= (1 + a) \mathbf{W}(F_\infty \, e^{-\lambda \sigma_a}) \, E_0^{(3)} \left(\int_0^\infty e^{-\lambda u} \frac{du}{(1 + a + R_u)^2}\right)$$

(by using the same argument as in point *i)*)

$$\le (1 + a) \, \mathbf{W}(F_\infty) \int_0^\infty e^{-\lambda u} E_0^{(3)} \left(\frac{1}{(1 + a + R_u)^{\frac{3}{2}}}\right) du$$

$$\le (1 + a) \, \mathbf{W}(F_\infty) O\left(\frac{1}{\lambda^{\frac{1}{4}}}\right) = o\left(\frac{1}{\sqrt{\lambda}}\right) \qquad (\lambda \to 0) \tag{1.2.121}$$

iii) Study of $W(F_t^{(3)})$.

$$W(F_t^{(3)}) \le (1 + a)^2 C \, W\left(\frac{1}{1 + |X_t|^2}\right)$$

from (1.2.118). Hypothesis *i)* : $0 \le F_t \le C$ imply :

$$\sqrt{\frac{\pi t}{2}}\, W(F_t^{(3)}) \underset{t\to\infty}{\longrightarrow} 0 \tag{1.2.122}$$

Thus :

$$\int_0^\infty e^{-\lambda t} W(F_t^{(3)})dt = o\left(\frac{1}{\sqrt{\lambda}}\right) \qquad (\lambda \to 0) \tag{1.2.123}$$

Gathering (1.2.120), (1.2.121) and (1.2.123) we obtain :

$$\int_0^\infty e^{-\lambda t} W(F_t)dt \underset{\lambda\to 0}{\sim} \sqrt{\frac{2}{\lambda}}\, \mathbf{W}(F_\infty) \tag{1.2.124}$$

$W(F_t)$ being by hypothesis a decreasing function in t, the Tauberian Theorem implies :

$$\sqrt{\frac{\pi t}{2}}\, W(F_t) \underset{t\to\infty}{\longrightarrow} \mathbf{W}(F_\infty)$$

This is precisely the statement of point 1) of Theorem 1.2.14.

2) We now prove point 2 of Theorem 1.2.14

We write

$$\begin{aligned} W\big(F_t \cdot |X_t|\big) &= W\big(F_t\big(|X_t|-a\big)_+\big) + W\big(F_t\big(|X_t| - \big(|X_t|-a\big)_+\big) \\ &:= (1_t) + (2_t) \end{aligned}$$

and we study successively (1_t) and (2_t).

$$\begin{aligned} \cdot(1_t) &= W\big(F_t(|X_t|-a)_+\big) = \mathbf{W}(F_t\, 1_{\sigma_a \le t}) \qquad \text{(from Theorem 1.1.8)} \\ &= \mathbf{W}(F_\infty\, 1_{\sigma_a\le t}) \qquad \text{(from (1.2.114))} \\ &\underset{t\to\infty}{\longrightarrow} \mathbf{W}(F_\infty) \qquad \text{(by the monotone convergence Theorem)} \\ &\qquad \big(\text{since } F_\infty \in L^1_+(\mathcal{F}_\infty, \mathbf{W})\big) \\ \cdot(2_t) &= W\big(F_t\big(|X_t| - \big(|X_t|-a\big)_+\big) \le a\, W(F_t) \end{aligned}$$

We now write :

$$W(F_t) = W\left(F_t\, \frac{1+|X_t| - \big(|X_t|-a\big)_+}{1+|X_t|}\right) + W\left(F_t\, \frac{\big(|X_t|-a\big)_+}{1+|X_t|}\right) \tag{1.2.125}$$

$$\begin{aligned} &= (3_t) + (4_t) \qquad \text{and we have} \\ (3_t) &= W\left(F_t\, \frac{1+|X_t| - \big(|X_t|-a\big)_+}{1+|X_t|}\right) \le (1+a) W\left(\frac{F_t}{1+|X_t|}\right) \\ &\le (1+a)\, C\, W\left(\frac{1}{1+|X_t|}\right) \underset{t\to\infty}{\longrightarrow} 0 \\ \text{since} \quad & (F_t,\ t\ge 0) \text{ is bounded} \\ (4_t) &= W\left(F_t\, \frac{\big(|X_t|-a\big)_+}{1+|X_t|}\right) = \mathbf{W}\left(\frac{F_t}{1+|X_t|}\, 1_{\sigma_a\le t}\right) \qquad \text{(from Th. 1.1.8.)} \\ &= \mathbf{W}\left(\frac{F_\infty}{1+|X_t|}\, 1_{\sigma_a \le t}\right) \qquad \text{(from (1.2.114))} \\ &\underset{t\to\infty}{\longrightarrow} 0 \qquad \text{since } |X_t| \underset{t\to\infty}{\longrightarrow} +\infty \quad \mathbf{W} \text{ a.s. and we apply the dominated convergence Theorem.} \end{aligned}$$

This ends the proof of Theorem 1.2.14. We are now able to state the announced penalisation Theorem.

Theorem 1.2.15. *(General penalisation Theorem)*

Let $(F_t,\ t \geq 0)$ be a process belonging to $\mathcal{C}$. Then, for every $s \geq 0$ and $\Gamma_s \in b(\mathcal{F}_s)$:

1) *The limit, as $t \to \infty$, $\dfrac{W(\Gamma_s F_t)}{W(F_t)}$ exists* (1.2.126)

2) *This limit equals :*

$$\lim_{t\to\infty} \frac{W(\Gamma_s F_t)}{W(F_t)} = \frac{W\big(\Gamma_s M_s(F_\infty)\big)}{\mathbf{W}(F_\infty)} := W_\infty^F(\Gamma_s) \tag{1.2.127}$$

The probability W_∞^F, which is characterised by (1.2.127) satisfies :

$$W_\infty^F = \frac{F_\infty}{\mathbf{W}(F_\infty)} \cdot \mathbf{W} \tag{1.2.128}$$

By comparing (1.2.128) with (1.1.16'), (1.1.93), (1.1.94), (1.1.108), (1.1.109) and (1.1.112), one can see that Theorem 1.2.15 is a general Theorem which implies many results given in Section 1.1 of this monograph, for example Theorems 1.1.1, 1.1.2, 1.1.11 and 1.1.11'.

Proof of Theorem 1.2.15.

i) We shall use the following notations : let $\omega_s \in \mathcal{C}\big([0,s] \to \mathbb{R}\big)$ and $(F_t^{(\omega_s)},\ t \geq 0)$ the functional defined by :

$$F_t^{(\omega_s)}(X_u,\ u \geq 0) := F_{t+s}\big(\omega_s \circ \big(\omega_s(s) + X_u,\ u \geq 0\big)\big) \tag{1.2.129}$$

With this notation, we have the following Lemma.

Lemma 1.2.16. *If $(F_t,\ t \geq 0) \in \mathcal{C}$, then, for W-almost every $\omega_s \in \mathcal{C}\big([0,s] \to \mathbb{R}\big)$ $(F_t^{(\omega_s)},\ t \geq 0) \in \mathcal{C}$.*

Proof of Lemma 1.2.16.

i) It is clear that $(F_t^{(\omega_s)},\ t \geq 0)$ is a monotone function of t and that, from (1.2.129) and (1.2.114) we have, for $t \geq \sigma_{|\omega_s(s)|+a}$:

$$F_t^{(\omega_s)}(X_u,\ u \geq 0) = F^{(\omega_s)}_{\sigma_{|\omega_s(s)|+a}}(X_u,\ u \geq 0) = F_\infty^{(\omega_s)}(X_u,\ u \geq 0)$$

ii) We need to prove that $\mathbf{W}(F_\infty^{(\omega_s)}) < \infty$. We note that :

$$\begin{aligned} \mathbf{W}(F_\infty^{(\omega_s)}) &= \mathbf{W}\big(F_\infty\big(\omega_s \circ \big(\omega_s(s) + X_u,\ u \geq 0\big)\big)\big) \\ &= M_s(F_\infty)(\omega_s) \qquad \big(\text{from}(1.2.3)\big) \end{aligned}$$

Hence :

$$W\big(\mathbf{W}(F_\infty^{(\omega_s)})\big) = W\big(M_s(F_\infty)\big) = \mathbf{W}(F_\infty) < \infty \qquad \big(\text{from } (1.2.2)\big)$$

In particular :

$$\mathbf{W}(F_\infty^{(\omega_s)}) < \infty \qquad W \ \text{ a.s.}$$

This is Lemma 1.2.16.

ii) We may now end the proof of Theorem 1.2.15. We have, for $t \geq s$:

$$\begin{aligned} \frac{W\big(F_t|\mathcal{F}_s\big)}{W(F_t)} &= \frac{\widehat{W}(F_{t-s}^{(\omega_s)})}{W(F_t)} \quad \text{(from the Markov property)} \\ &= \frac{\sqrt{\frac{\pi t}{2}}\,\widehat{W}(F_{t-s}^{(\omega_s)})}{\sqrt{\frac{\pi t}{2}}\,W(F_t)} \underset{t\to\infty}{\longrightarrow} \frac{\mathbf{W}(F_\infty^{(\omega_s)})}{\mathbf{W}(F_\infty)} \qquad \text{a.s.} \end{aligned} \tag{1.2.130}$$

$\big($from Theorem 1.2.14 applied to $(F_t,\ t \geq 0)$ and to $(F_t^{(\omega_s)},\ t \geq 0)$ due to Lemma 1.2.16.$\big)$

$$= \frac{M_s(F_\infty)}{\mathbf{W}(F_\infty)}$$

(from point 2 of Theorem 1.2.1.)

To show Theorem 1.2.15, it now suffices to see that the convergence is (1.2.130) also holds in $L^1(\mathcal{F}_\infty, W)$. However, from Scheffé's Lemma $\big($see [M], T. 21$\big)$ this is implied by the equality : $W\left(\frac{M_s(F_\infty)}{\mathbf{W}(F_\infty)}\right) = 1$ for every $s \geq 0$, which follows immediately from Theorem 1.2.1 (equality (1.2.2)).

Remark 1.2.17.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ Borel such that : $\displaystyle\int_0^\infty \varphi(x)(1+x^2)dx < \infty$ and let :

$$\begin{aligned} F_t^{(1)} &:= \varphi(S_t)\ 1_{(X_t>0)} \qquad (t \geq 0) \\ F_t^{(2)} &:= \varphi(S_{d_t})1_{(X_t>0)} \qquad (t \geq 0) \end{aligned}$$

It is shown in [RY, VIII] that :

i) $$E\big(F_t^{(1)}\big) \underset{t\to\infty}{\sim} \frac{3}{2}\sqrt{\frac{2}{\pi t^3}}\int_0^\infty \varphi(x)x^2dx \tag{1.2.131}$$

$$E\big(F_t^{(2)}\big) \underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi t^3}}\int_0^\infty \varphi(x)x^2dx \tag{1.2.132}$$

ii) for every $s \geq 0$ and $\Gamma_s \in b(\mathcal{F}_s)$

$$\frac{E[\Gamma_s\,F_t^{(i)}]}{E(F_t^{(i)})} \underset{t\to\infty}{\longrightarrow} E(\Gamma_s\,M_s^\psi) \qquad (i=1,2) \tag{1.2.133}$$

where the martingale $\big(M_s^\psi,\ s \geq 0\big)$ is defined by :

$$M_s^\psi = \psi(S_s)(S_s - X_s) + \int_{S_s}^\infty \psi(y)dy$$

$$\text{and} \qquad \psi(x) := \varphi(x)x^2 + 2\int_x^\infty \varphi(y)y\,dy \qquad (x \geq 0)$$

We now inspect Theorem 1.2.15 in the light of this result. If we assume that $\lim\limits_{y\to+\infty} \varphi(y) = 0$, we obtain :

$$\lim_{t\to\infty} F_t^{(i)} = 0 \qquad \mathbf{W} \qquad \text{a.s.}$$

and, from (1.2.131) and (1.2.132).

$$\lim_{t\to\infty} \sqrt{t}\,E[F_t^{(i)}] = 0 \quad (i=1,2)$$

Thus, we are working here in a degenerate case of Theorem 1.2.15 and of Theorem 1.2.14, i.e. : in a case where $F_\infty \equiv 0$. However, from (1.2.133), this situation is not so "degenerate", since it allows to obtain a non-trivial penalisation.

1.2.6 Some other results about the martingales $(M_t(F),\ t \geq 0)$.

Let us first state the following definition :

Definition 1.2.18. Let $(F_t,\ t \geq 0)$ denote an adapted, positive process. We shall say that this process belongs to the class $\widetilde{\mathcal{C}}$ if :

i) $(F_t,\ t \geq 0)$ is a decreasing process, i.e., if $s \leq t$:

$$0 \leq F_t \leq F_s \quad W \text{ a.s.} \tag{1.2.134}$$

In particular, since $0 \leq F_t \leq F_0$ and since F_0 is a.s. constant, this process is bounded by a constant $C = F_0$.

ii) There exists $a \geq 0$ such that, for every $t \geq \sigma_a$, with

$$\sigma_a := \sup\{t \geq 0;\ X_t \in [-a,a]\}$$

$$F_t = F_{\sigma_a} = F_\infty \tag{1.2.135}$$

and there exists $k > 0$ such that

$$\sup_{x\in[-a,a]} \mathbf{W}_x(F_\infty) \leq k \tag{1.2.136}$$

iii) For every random time $T < \infty$ a.s. and every $u \geq 0$:

$$F_{T+u}(\omega) \leq F_u(\theta_T\omega) \tag{1.2.136$'$}$$

Of course, there is the inclusion $\widetilde{\mathcal{C}} \subset \mathcal{C}$. As the class $\mathcal{C}$, the class $\widetilde{\mathcal{C}}$ contains many interesting functionals $(F_t,\ t \geq 0)$. The following result holds :

Theorem 1.2.19. *Let $(F_t,\ t \geq 0)$ be a process in the class $\widetilde{\mathcal{C}}$ and $F_\infty := \lim\limits_{t\to\infty} F_t$. Then, there exists a bounded process $(Y_t,\ t \geq 0)$:*

$$0 \leq |Y_t| \leq c \tag{1.2.137}$$

such that :

$$M_t(F_\infty) = F_t|X_t| + Y_t \quad W \ a.s. \tag{1.2.138}$$

Examples

1) Let $(F_t := h(L_t),\ t \geq 0)$ with $h : \mathbb{R}_+ \to \mathbb{R}_+$ Borel, such that

$$\int_0^\infty h(y)dy = 1.$$

Then, (see (1.2.21)) :

$$M_t(h(L_\infty)) = h(L_t)|X_t| + \int_{L_t}^\infty h(y)dy$$

i.e. this is (1.2.138) with

$$Y_t = \int_{L_t}^\infty h(y)dy.$$

2) Let $\left(F_t := \exp\left(-\frac{1}{2}A_t^{(q)}\right),\ t \geq 0\right)$ with $q \in \mathcal{I}$ and q with compact support. Then (see (1.2.19)) :

$$\begin{aligned} M_t\left(e^{-\frac{1}{2}A_\infty^{(q)}}\right) &= \varphi_q(X_t)\, e^{-\frac{1}{2}A_t^{(q)}} \\ &= e^{-\frac{1}{2}A_t^{(q)}}\,|X_t| + e^{-\frac{1}{2}A_t^{(q)}}\,(\varphi_q(X_t) - |X_t|) \end{aligned}$$

i.e. (1.2.138) with

$$Y_t = e^{-\frac{1}{2}A_t^{(q)}}\,(\varphi_q(X_t) - |X_t|)$$

and we note that

$$0 \leq |Y_t| \leq |\varphi_q(X_t) - |X_t|| \leq k$$

since φ_q is convex and $\varphi_q(x)$ is equivalent to $|x|$ as $|x|$ goes to infinity.

Proof of Theorem 1.2.19.

i) It is sufficient to prove

$$M_t(F_\infty) = F_t.(|X_t| - a)_+ + \widetilde{Y}_t \tag{1.2.139}$$

with $|\widetilde{Y}_t| \leq c'$. Indeed, if (1.2.139) is satisfied, then :

$$\begin{aligned} |M_t(F_\infty) - F_t.|X_t|| &= |F_t(|X_t| - a)_+ + \widetilde{Y}_t - F_t.|X_t|| \\ &= |\widetilde{Y}_t + F_t((|X_t| - a)_+ - |X_t|)| \\ &\leq |\widetilde{Y}_t| + ak \leq c' + ak = c''. \end{aligned}$$

ii) We now prove (1.2.139)

From point 2) of Theorem 1.2.1, we know that :

$$\begin{aligned} M_t(F_\infty) &= \widehat{\mathbf{W}}_{X_t}\big(F_\infty(\omega_t, \hat{\omega}^t)\big) && (1.2.140) \\ &= \widehat{\mathbf{W}}_{X_t}\big(F_\infty(\omega_t, \hat{\omega}^t)\, 1_{\sigma_a(\omega_t, \hat{\omega}^t) < t}\big) \\ &+ \widehat{\mathbf{W}}_{X_t}\big(F_\infty(\omega_t, \hat{\omega}^t)\, 1_{\sigma_a(\omega_t, \hat{\omega}^t) > t}\big) && (1.2.141) \\ &:= (1)_t + (2)_t \end{aligned}$$

Study of $(1)_t$

$$\begin{aligned} (1)_t &= \widehat{\mathbf{W}}_{X_t}\big(F_\infty(\omega_t, \hat{\omega}^t)\, 1_{\sigma_a(\omega_t, \hat{\omega}^t) < t}\big) \\ &= \widehat{\mathbf{W}}_{X_t}\big(F_\infty(\omega_t)\, 1_{\sigma_a(\omega_t, \hat{\omega}^t) < t}\big) \end{aligned}$$

since, on $\sigma_a < t$, $F_\infty = F_t$ (from (1.2.135)). Hence :

$$(1)_t = F_t(\omega_t)\widehat{\mathbf{W}}_{X_t}\big(1_{\sigma_a(\omega_t, \hat{\omega}^t) < t}\big).$$

But one can easily check that :

$$\widehat{\mathbf{W}}_{X_t}\big(1_{\sigma_a(\omega_t, \hat{\omega}^t) < t}\big) = (|X_t| - a)_+. \tag{1.2.142}$$

Indeed, we have :

$$\mathbf{W}_x(\sigma_a = 0) = (|x| - a)_+. \tag{1.2.142'}$$

Since, from (1.1.17') and relation (1.1.30) of Theorem 1.1.5, we have :

$$\mathbf{W}_x \left(\exp \left(-\frac{1}{2} \lambda L_\infty \right) \right) = \frac{2}{\lambda} + |x|.$$

Letting $\lambda \to \infty$, we have

$$\mathbf{W}_x(T_0 = \infty) = |x|.$$

Hence, we obtain (1.2.142') by translation. By (1.2.142), we deduce :

$$(1)_t = \widehat{\mathbf{W}}_{X_t}\big(F_\infty(\omega_t, \hat{\omega}^t)\, 1_{\sigma_a(\omega_t,\hat{\omega}^t)<t}\big) = F_t.(|X_t| - a)_+). \tag{1.2.143}$$

Study of $(2)_t$

$$(2)_t = \widehat{\mathbf{W}}_{X_t}\big(F_\infty(\omega_t, \hat{\omega}^t)\, 1_{\sigma_a(\omega_t,\hat{\omega}^t)>t}\big)$$

- If $|X_t| < a$, then $\sigma_a(\omega_t, \hat{\omega}^t) > t$ and

$$\begin{aligned}\widehat{\mathbf{W}}_{X_t}\left(F_\infty(\omega_t, \hat{\omega}^t) 1_{\sigma_a(\omega_t,\hat{\omega}^t)>t}\right) &= \widehat{\mathbf{W}}_{X_t}\left(F_\infty(\omega_t, \hat{\omega}^t)\right) \\ &\le \sup_{x\in[-a,a]} \mathbf{W}_x(F_\infty) \le k \end{aligned} \tag{1.2.144}$$

(from (1.2.136)).

- If $X_t\,(= x) \notin [-a, a]$:

$$\widehat{\mathbf{W}}_x\left(F_\infty(\omega_t, \hat{\omega}^t) 1_{\sigma_a(\omega_t,\hat{\omega}^t)>t}\right) = \widehat{\mathbf{W}}_x\left(F_\infty(\omega_t, \hat{\omega}^t) 1_{\widehat{T}_a<\infty}\right)$$

where $\widehat{T}_a$ is the hitting time for $\hat{\omega}^t$ of a or $-a$ (it does not depend on ω_t).

Hence

$$\widehat{\mathbf{W}}_x\left(F_\infty(\omega_t, \hat{\omega}^t) 1_{\sigma_a(\omega_t,\hat{\omega}^t)>t}\right) \le \widehat{\mathbf{W}}_x\left(F_\infty\left(\theta_{\widehat{T}_a}(\hat{\omega}^t)\right) 1_{\widehat{T}_a<\infty}\right)$$

since, from (1.2.136') :

$$F_\infty(\omega_t, \hat{\omega}^t) \le F_\infty\big(\theta_{\widehat{T}_a}(\hat{\omega}^t)\big)$$

on the event $\{(\sigma_a(\omega_t, \hat{\omega}^t) > t) \cap \widehat{T}_a(\hat{\omega}^t) < \infty\}$. Hence,

$$\begin{aligned}
&\widehat{\mathbf{W}}_x\left(F_\infty(\omega_t, \hat{\omega}^t) 1_{\sigma_a(\omega_t,\hat{\omega}^t)>t}\right) \\
&\le \varphi_{\delta_0}(x)\, W_{x,\infty}^{(\delta_0)}\left(e^{\frac{1}{2}L_\infty} F(\theta_{T_a}\omega)\, 1_{T_a<\infty}\right) \\
&\quad \big(\text{from } (1.1.57)\big) \\
&= \varphi_{\delta_0}(x)\, W_{x,\infty}^{(\delta_0)}\left(e^{\frac{1}{2}L_{T_a}} 1_{T_a<\infty} W_{X_{T_a},\infty}^{(\delta_0)}(e^{\frac{1}{2}L_\infty} F_\infty)\right) \\
&\quad (\text{from the Markov property}) \\
&= \frac{\varphi_{\delta_0}(x)}{\varphi_{\delta_0}(a)}\, W_{x,\infty}^{(\delta_0)}\ (1_{T_a<\infty}\, \mathbf{W}_a(F_\infty)) \\
&\quad \big(\text{from } (1.1.57) \text{ and since } L_{T_a} = 0\ W_{x,\infty}^{(\delta_0)} \text{ a.s. for } |x| > a\big) \\
&= \mathbf{W}_a(F_\infty)\, \frac{\varphi_{\delta_0}(x)}{\varphi_{\delta_0}(a)}\, W_{x,\infty}^{(\delta_0)}(T_a < \infty).
\end{aligned}$$

But, $\varphi_{\delta_0}(x) = 2 + |x|$ and

$$W_{x,\infty}^{(\delta_0)}(T_a < \infty) \underset{|x|\to\infty}{\sim} \frac{2}{2+|x|}$$

(see (1.1.70)).
Hence :

$$\sup_{x\in[-a,a]} \varphi_{\delta_0}(x)\, W_{x,\infty}^{(\delta_0)}\left(F(\omega_t, \hat{\omega}^t) 1_{\sigma_a(\omega_t,\hat{\omega}^t)>t}\right) \leq c'' \tag{1.2.145}$$

Gathering (1.2.145), (1.2.144) and (1.2.143), we obtain Theorem 1.2.19.

Corollary 1.2.20.
Let $(F_t,\ t \geq 0)$ and $(G_t,\ t \geq 0)$ be two processes in $\widetilde{\mathcal{C}}$. Then :
1)

$$W\left(\frac{M_t(F_\infty)M_t(G_\infty)}{1+|X_t|}\right) \underset{t\to\infty}{\longrightarrow} \mathbf{W}(F_\infty.G_\infty) \tag{1.2.146}$$

2)

$$\frac{1}{2}\sqrt{\frac{\pi}{2t}}\, W\big(M_t(F_\infty)M_t(G_\infty)\big) \underset{t\to\infty}{\longrightarrow} \mathbf{W}(F_\infty.G_\infty) \tag{1.2.147}$$

Note that, since $(F_t,\ t \geq 0)$ and $(G_t,\ t \geq 0)$ are in $\widetilde{\mathcal{C}}$, one has :

$$\mathbf{W}(F_\infty.G_\infty) \leq k\, \mathbf{W}(G_\infty) < \infty.$$

Proof of Corollary 1.2.20.
1) We start with point 1)
We have :

$$\begin{aligned}
&W\left(\frac{M_t(F_\infty)M_t(G_\infty)}{1+|X_t|}\right)\\
=&\mathbf{W}\left(F_\infty \frac{M_t(G_\infty)}{1+|X_t|}\right)\\
&\text{(from Theorem 1.2.1)}\\
=&\mathbf{W}\left(F_\infty \frac{G_t|X_t| + Y_t^G}{1+|X_t|}\right)\\
&\text{(from Theorem 1.2.19)}\\
&\underset{t\to\infty}{\longrightarrow} \mathbf{W}(F_\infty G_\infty)
\end{aligned}$$

since $\frac{G_t|X_t|}{1+|X_t|} \leq G_t \leq k$, $F_\infty \in L^1(\mathbf{W})$, G_t decreases to G_∞ when $t \to \infty$, and $|Y_t^G| < c$.
2) We now prove point 2) (briefly)
By polarization, it is sufficient to prove (1.2.147) when $F_\infty = G_\infty$. In this case, $t \to W(M_t^2(F_\infty))$ is an increasing function of t and one can apply the Tauberian Theorem. Let us compute :

$$\int_0^\infty e^{-\lambda t} W(M_t^2(F_\infty))dt = \int_0^\infty e^{-\lambda t} W\left[(F_t.|X_t| + Y_t)^2\right] dt.$$

It is not difficult to see that in this expression, the terms Y_t^2 and $F_t|X_t|.Y_t$ are negligible, so we only need to deal with the term $F_t^2|X_t|^2$. By doing as in the proof of the point 1) of Theorem 1.2.14, one has :

$$\begin{aligned}
F_t^2|X_t|^2 &= F_t^2\,\frac{|X_t|^2(|X_t|-a)_+}{1+|X_t|} \\
&+ F_t^2|X_t|^2\,\frac{1+|X_t|-(|X_t|-a)_+}{(1+|X_t|)^2}\,(|X_t|-a)_+ \\
&+ F_t^2\,\frac{|X_t|^2(1+|X_t|-(|X_t|-a)_+)}{(1+|X_t|)^2} \\
&= (\tilde{1})_t + (\tilde{2})_t + (\tilde{3})_t \qquad (1.2.148)
\end{aligned}$$

Now :

$$\begin{aligned}
(1)_t &:= \int_0^\infty e^{-\lambda t} W((\tilde{1})_t)dt \\
&= \int_0^\infty e^{-\lambda t} W\left(F_t^2\,\frac{|X_t|^2(|X_t|-a)_+}{1+|X_t|}\right)dt \\
&= \int_0^\infty e^{-\lambda t}\mathbf{W}\left(\frac{F_t^2|X_t|^2}{1+|X_t|}\,1_{\sigma_a<t}\right)dt \\
&\text{(from Theorem 1.1.16)} \\
&= \int_0^\infty e^{-\lambda t}\mathbf{W}\left(F_{\sigma_a}^2\,\frac{|X_t|^2}{1+|X_t|}\,1_{\sigma_a<t}\right)dt \\
&\big(\text{from (1.2.135)}\big) \\
&= \mathbf{W}\left(F_{\sigma_a}^2 e^{-\lambda\sigma_a}\int_0^\infty e^{-\lambda u}du\,\frac{|X_{\sigma_a+u}|^2}{1+|X_{\sigma_a+u}|}\right) \\
&\text{(after the change of variables } t=\sigma_a+u) \\
&= \mathbf{W}\left(F_{\sigma_a}^2 e^{-\lambda\sigma_a}\right)E_0^{(3)}\left(\int_0^\infty e^{-\lambda u}\,\frac{(a+R_u)^2}{1+a+R_u}\,du\right) \\
&\text{(from Theorem 1.2.1)}
\end{aligned}$$

Since, by scaling :

$$E_0^{(3)}\left(\int_0^\infty e^{-\lambda u}\,\frac{(a+R_u)^2}{1+a+R_u}\,du\right)\underset{\lambda\to0}{\sim}\frac{\sqrt{2}}{\lambda^{3/2}},$$

one has :

$$(1)_t\underset{\lambda\to0}{\sim}\mathbf{W}(F_\infty)\,\frac{\sqrt{2}}{\lambda^{3/2}}.$$

It is now easy, by using the same arguments as in the proof of point 1) of Theorem 1.2.14, to see that $(2)_t$ and $(3)_t$ are, when λ tends to zero, negligible with respect to $(1)_t$. Finally, from Tauberian Theorem :

$$\frac{1}{2}\sqrt{\frac{\pi}{2t}}\,W\left(M_t(F_\infty)M_t(G_\infty)\right)\underset{t\to\infty}{\longrightarrow}\mathbf{W}(F_\infty.G_\infty).$$

Remark 1.2.21.

1) By using the same arguments as in Corollary 1.2.20, one can see that if $F^{(1)}, ..., F^{(k)}$ are k processes in the class $\widetilde{\mathcal{C}}$, then :

$$W\left(\frac{\prod_{i=1}^{k} F_\infty^{(i)}}{(1+|X_t|)^{k-1}}\right) \underset{t\to\infty}{\longrightarrow} \mathbf{W}\left(\prod_{i=1}^{k} F_\infty^{(i)}\right) \tag{1.2.149}$$

and

$$t^{(k-1)/2}\, W\left(\prod_{i=1}^{k} F_\infty^{(i)}\right) \underset{t\to\infty}{\longrightarrow} c_k\, \mathbf{W}\left(\prod_{i=1}^{k} F_\infty^{(i)}\right) \tag{1.2.150}$$

where c_k is a universal constant.
Note that, at first sight, (1.2.149) and (1.2.150) seem quite strange since one knows (from Theorem 1.2.1) that $M_t(F_\infty^{(i)}) \underset{t\to\infty}{\longrightarrow} 0$, W a.s. for all $i = 1, ..., k$.
2) Let $(F_t,\ t \geq 0)$ and $(G_t,\ t \geq 0)$ be two processes in $\widetilde{\mathcal{C}}$. We penalise Wiener measure by the process $(F_t,\ t \geq 0)$ (see Theorem 1.2.15) and we denote by W_∞^F the probability obtained with this penalisation. Now, let us penalise the probability W_∞^F by $(G_t,\ t \geq 0)$: we obtain the probability $W_\infty^{F,G}$. On the other hand, if we penalise Wiener measure by the functional $(F_t.G_t,\ t \geq 0)$, we obtain the probability $W_\infty^{F.G}$. It is not difficult to see, by using Theorem 1.2.19, that $W_\infty^{F,G} = W_\infty^{F.G}$.
3) Let $(F_t,\ t \geq 0)$ be an adapted, positive and increasing process, such that, for some $\lambda_0 > 0$, $(e^{-\lambda_0 F_t},\ t \geq 0)$ is in $\mathcal{C}$, and such that for all x, $\mathbf{W}_x(e^{-\lambda_0 F_\infty}) < \infty$. Then, for all $x \in \mathbb{R}$, there exists a positive and σ-finite measure $\boldsymbol{\nu}_x^{(F_\infty)}$, carried on $\mathbb{R}_+$, and such that for all continuous functions h with compact support :

$$\sqrt{t}\, W_x[h(F_t)] \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}_+} h(y) \boldsymbol{\nu}_x^{(F_\infty)}(dy) \tag{1.2.151}$$

This Theorem is a generalization of a result in [RY, IX]. In [RY, IX], it is obtained when $(F_t,\ t \geq 0)$ is an additive functional. In fact, the measure $\boldsymbol{\nu}_x^{(F_\infty)}$ is the image of $\mathbf{W}_x$ by $F_\infty : \Omega \to \mathbb{R}_+$. The proof of (1.2.151) is essentially a consequence of Theorem 1.2.14.

1.3 Invariant measures related to $\mathbf{W}_x$ and $\boldsymbol{\Lambda}_x$.

We shall now show that the measure $\mathbf{W}$, and the measure $\boldsymbol{\Lambda}$ which we shall define very soon, are closely related to invariant measures of some Markov process taking values in certain functional spaces.

1.3.1 The process $(\mathcal{X}_t,\ t \geq 0)$.

As before, $\big(\Omega, (X_t, \mathcal{F}_t)_{t\geq 0}, \mathcal{F}_\infty, W_x(x \in \mathbb{R})\big)$ denotes the canonical realisation of Brownian motion, starting at zero. Let $\mathcal{X}_0 \in \Omega = \mathcal{C}(\mathbb{R}_+ \to \mathbb{R})$. We define the process $(\mathcal{X}_t,\ t \geq 0)$ taking values on $\mathcal{C}(\mathbb{R}_+ \to \mathbb{R})$, and issued from $\mathcal{X}_0$, by :

$$\mathcal{X}_t(u) := \begin{cases} \mathcal{X}_0(u-t) & \text{if} \quad u \geq t \\ \mathcal{X}_0(0) + X_{t-u} & \text{if} \quad u \leq t \end{cases} \tag{1.3.1}$$

It is easy enough to see that this process is Markov (we denote by $(P_t,\ t \geq 0)$ the semigroup associated with this Markov Process $(\mathcal{X}_t,\ t \geq 0)$) and that the measure :

$$\widetilde{\mathbf{W}} := \int_{\mathbb{R}} dx\, W_x \tag{1.3.2}$$

is an invariant measure for this process. However, this process admits other invariant measures. More precisely :

Theorem 1.3.1. *Let $a, b \geq 0$, with $a+b>0$, and :*

$$\mathbf{W}_x^{a,b} := a\,\mathbf{W}_x^{+} + b\,\mathbf{W}_x^{-} \tag{1.3.3}$$

Then :

$$\widetilde{\mathbf{W}}^{a,b} := \int_{\mathbb{R}} dx\; \mathbf{W}_x^{a,b} \tag{1.3.4}$$

is an invariant measure for the process $(\mathcal{X}_t,\ t \geq 0)$. Recall that $\mathbf{W}_x^{+}$ and $\mathbf{W}_x^{-}$ are defined in (1.1.88) by :

$$\mathbf{W}_x^{+} = 1_{\Gamma_+} \cdot \mathbf{W}_x,\ \mathbf{W}_x^{-} = 1_{\Gamma_-} \cdot \mathbf{W}_x$$

Proof of Theorem 1.3.1.

By symmetry, it suffices to prove that the measure $\widetilde{\mathbf{W}}^{+}$ defined by $\widetilde{\mathbf{W}}^{+} := \displaystyle\int_{\mathbb{R}} dx\,\mathbf{W}_x^{+}$ is invariant. For every measurable and positive functional $F : \Omega \to \mathbb{R}_+$, we have :

$$\begin{aligned}
&\int_{\mathbb{R}} dx \int_{\Omega} \mathbf{W}_x^{+}(d\mathcal{X}) P_t F(\mathcal{X}) \\
&= \int_{\mathbb{R}} dx \int_{\Omega} \mathbf{W}_x^{+}(d\mathcal{X}) W\big(F(x+X_{t-u},\ u \leq t\ ;\ \mathcal{X}(v-t),\ v \geq t)\big) \qquad \big(\text{from } (1.3.1)\big) \\
&= \int_{\mathbb{R}} dx \int_{\Omega} \mathbf{W}_x^{+}(d\mathcal{X}) W\big(F(x+X_t - X_u,\ u \leq t\ ;\ \mathcal{X}(v-t),\ v \geq t)\big)
\end{aligned}$$

$\big($since $(X_{t-u},\ u \leq t)$ has the same law under W as $(X_t - X_u,\ u \leq t)\big)$

$$= \int_{\mathbb{R}} dy\; W\left(\int_{\Omega} \mathbf{W}_{y-X_t}^{+}(d\mathcal{X}) F(y - X_u,\ u \leq t\ ;\ \mathcal{X}(v-t), v \geq t)\right)$$

$\big($from Fubini and after making the change of variable $x + X_t = y\big)$

$$= \int_{\mathbb{R}} dy\; W\left(\int_{\Omega} \mathbf{W}_{y-X_t}(d\mathcal{X}) F(y - X_u,\ u \leq t\ ;\ \mathcal{X}(v-t),\ v \geq t)\, 1_{\Gamma_+}(\mathcal{X})\right)$$

(from the definition of $\mathbf{W}^{+}$ and since $\mathcal{X} \in \Gamma_+$ if and only if : $\lim\limits_{v \to \infty} \mathcal{X}(v-t) = +\infty$)

$$= \int_{\mathbb{R}} dy\; W_y\left(\int_{\Omega} \mathbf{W}_{X_t}(d\mathcal{X}) F(X_u,\ u \leq t; \mathcal{X}(v-t),\ v \geq t)\, 1_{\Gamma_+}(\mathcal{X})\right)$$

since $(X_u,\ u \geq 0)$ and $(-X_u,\ u \geq 0)$ have the same law under W_0. We now write :

$$\begin{aligned}
&\int_{\Omega} \mathbf{W}_{X_t}(d\mathcal{X}) F(X_u,\ u \leq t; \mathcal{X}(v-t),\ v \geq t)\, 1_{\Gamma_+}(\mathcal{X}) \\
&= \widehat{\mathbf{W}}_{X_t}\left(F 1_{\Gamma_+}(\omega_t, \hat{\omega}^t)\right)
\end{aligned}$$

where $\omega_t \in \mathcal{C}([0,t] \to \mathbb{R})$, $\hat{\omega}^t \in \mathcal{C}(\mathbb{R}_+ \to \mathbb{R})$, and :

$$\omega_t(u) = X_u \quad \text{for } u \leq t,$$

$$\hat{\omega}^t(v) = X_t + \mathcal{X}(v) \quad \text{for } v \geq 0$$

(see point 1 of Remark 1.2.2 for such a notation). In the preceding relation, ω_t is frozen and expectation is taken with respect to $\hat{\omega}^t$. Hence, from the "characteristic formula" (1.2.3) for the martingale $(M_t(F1_{\Gamma_+}),\, t \geq 0)$, we have:

$$\int_\Omega \mathbf{W}_{X_t}(d\mathcal{X}) F(X_u,\, u \leq t;\, \mathcal{X}(v-t),\, v \geq t) 1_{\Gamma_+}(\mathcal{X})$$
$$= M_t(F1_{\Gamma_+})(\omega_t).$$

Hence:

$$\int_{\mathbb{R}} dx \int_\Omega \mathbf{W}_x^+(d\mathcal{X}) P_t F(\mathcal{X})$$
$$= \int_{\mathbb{R}} dy\, W_y\big(M_t(F1_{\Gamma_+})\big)$$
$$= \int_{\mathbb{R}} dy\, W_y\big(M_0(F1_{\Gamma_+})\big)$$
$$= \int_{\mathbb{R}} dy\, \mathbf{W}_y(F1_{\Gamma_+})$$

(from (1.2.2) where we replace $\mathbf{W}$ $(= \mathbf{W}_0)$ by $\mathbf{W}_y$).

$$= \int_{\mathbb{R}} dy\, \mathbf{W}_y^+(F)$$

(from the definition of $\mathbf{W}_y^+$).

$$= \widetilde{\mathbf{W}}^+(F).$$

This is Theorem 1.3.1.

1.3.2 The measure $\mathbf{\Lambda}_x$.

Let $\widetilde{\Omega} = \mathcal{C}(\mathbb{R} \to \mathbb{R}_+)$ and $\mathcal{L} : \Omega \to \widetilde{\Omega}$, the application "total local time" defined by :

$$\mathcal{L}(X_t,\, t \geq 0) = (L_\infty^y,\, y \in \mathbb{R}). \tag{1.3.5}$$

We denote by $\mathbf{\Lambda}_x$ the image of $\mathbf{W}_x$ by $\mathcal{L}$. It is possible to give a very simple description of $\mathbf{\Lambda}_x$ (see [RY, M]). Here is this description :

· Let $u, \alpha, \beta \in \mathbb{R}_+$ and $x \in \mathbb{R}$. We denote by $Q_{x,u}^{\alpha,\beta}$ the law of the process $(Y_v,\, v \in \mathbb{R})$ defined as follows :

$Y_x = u$

$(Y_{x+t},\, t \geq 0)$ is the square of an α-dimensional Bessel process

$(Y_{x-t},\, t \geq 0)$ is the square of a β-dimensional Bessel process, independent from $(Y_{x+t},\, t \geq 0)$.

Then :

$$\mathbf{\Lambda}_x = \frac{1}{2} \int_0^\infty du\, (Q_{x,u}^{0,2} + Q_{x,u}^{2,0}) \tag{1.3.6}$$

Sketch of the proof of (1.3.6).
By translation, it suffices to prove (1.3.6) for $x = 0$. Then, we use (1.1.40) :

$$\mathbf{W} = \int_0^\infty dv\,(W_0^{\tau_v} \circ P_0^{(3,\mathrm{sym})})$$

and the following facts :
• From the second Ray-Knight Theorem $\big($see [ReY], Chap. IX$\big)$ for Brownian motion, the process $(L^y_{\tau_l},\ y \geq 0)$ is a 0-dimensional squared Bessel process, starting from l.
• For a 3-dimensional Bessel process, starting from 0, $(L^y_\infty,\ y \geq 0)$ is a 2-dimensional squared Bessel process, starting from 0. This constitutes the "third" Ray-Knight Theorem.
• If $(Z^i_t,\ t \geq 0),\ i = 1, 2$, are two squared Bessel processes with respective dimensions d_1 and d_2, starting respectively from u_1 and u_2, then $(Z^{(1)}_t + Z^{(2)}_t,\ t \geq 0)$ is a squared Bessel process with dimension $d_1 + d_2$ starting from $u_1 + u_2$.
Other properties about the measure $\mathbf{\Lambda}_x$ may be found in $\big($[RY, M], Chap. 2$\big)$. It is easily deduced from (1.3.6) that the r.v. L^y_∞, under $\mathbf{W}_x$, admits the "law" :

$$\mathbf{W}_x(L^y_\infty \in du) = |y - x|\,\delta_0(du) + du \qquad (u \geq 0) \tag{1.3.7}$$

$\big($see also (1.1.45)$\big)$.

1.3.3 Invariant measures for the process $\big((X_t, L^\bullet_t),\ t \geq 0\big)$.
The process $\big((X_t, L^\bullet_t),\ t \geq 0\big)$, where $L^\bullet_t = (L^y_t,\ y \in \mathbb{R})$ denotes the local times process (in the space variable) at time t, for Brownian motion $(X_t,\ t \geq 0)$ is a Markov process taking values in $\mathbb{R} \times \widetilde{\Omega} = \mathbb{R} \times \mathcal{C}(\mathbb{R} \to \mathbb{R}_+)$. In fact, if $\mathcal{X}_0$ is a function which has a finite total local time at each level, $\big((X_t, L^\bullet_t + L^\bullet_\infty(\mathcal{X}_0),\ t \geq 0\big)$ is the image of the process $(\mathcal{X}_t,\ t \geq 0)$ (see (1.3.1)) by the application :

$$H : \Omega \to \mathbb{R} \times \widetilde{\Omega}$$

defined by :

$$H(Y_t,\ t \geq 0) = (Y_0, L^\bullet_\infty) \tag{1.3.8}$$

Of course, H is only defined a.s. (with respect to the law of the process $(X_t,\ t \geq 0)$), i.e. it is only defined for the trajectories $\omega \in \Omega$ for which local time exists. As a Corollary of Theorem 1.3.1, the image of $\widetilde{\mathbf{W}}^{a,b}$ by H is an invariant measure for the process $\big((X_t,\ L^\bullet_t),\ t \geq 0\big)$. This image, which we denote by $\widetilde{\mathbf{\Lambda}}^{a,b}$ is equal, from (1.3.6), to :

$$\widetilde{\mathbf{\Lambda}}^{a,b} = \frac{1}{2}\int_{\mathbb{R}} dx \int_0^\infty du\ (a\,Q^{2,0}_{x,u} + b\,Q^{0,2}_{x,u}) \tag{1.3.9}$$

Thus, we have obtained :
Theorem 1.3.2. *The measure $\widetilde{\mathbf{\Lambda}}^{a,b}$ is an invariant measure for the process $\big((X_t,\ L^\bullet_t),\ t \geq 0\big)$.*
We shall now give a different proof of Theorem 1.3.2 than the one we have just indicated. This proof has the further advantage that it hinges on arguments which shall be useful in the sequel. We begin with the :
Lemma 1.3.3 *Let $q \in \mathcal{I}$*

1) *Define* $\varphi_q^+(x) := \mathbf{W}_x^+\big(e^{-\frac{1}{2}A_\infty^{(q)}}\big) = \mathbf{W}_x\big(e^{-\frac{1}{2}A_\infty^{(q)}} 1_{\Gamma_+}\big)$*. Then,* φ_q^+ *is the unique solution of Sturm-Liouville equation :*

$$\varphi'' = q\,\varphi \qquad \textit{with boundary conditions :}$$
$$\varphi(x) \underset{x\to+\infty}{\sim} x \qquad \varphi(x) \underset{x\to-\infty}{\longrightarrow} C := \frac{1}{\displaystyle\int_{-\infty}^{\infty} \frac{dy}{\varphi_q^2(y)}} \tag{1.3.10}$$

2) *Define* $\varphi_q^-(x) := \mathbf{W}_x^-\big(e^{-\frac{1}{2}A_\infty^{(q)}}\big) = \mathbf{W}_x\big(e^{-\frac{1}{2}A_\infty^{(q)}} 1_{\Gamma_-}\big)$*. Then,* φ_q^- *is the unique solution of the Sturm-Liouville equation :*

$$\varphi'' = q\,\varphi \qquad \textit{with boundary conditions :}$$
$$\varphi(x) \underset{x\to-\infty}{\sim} |x| \qquad \varphi(x) \underset{x\to+\infty}{\longrightarrow} C := \frac{1}{\displaystyle\int_{-\infty}^{\infty} \frac{dy}{\varphi_q^2(y)}} \tag{1.3.11}$$

Proof of Lemma 1.3.3.
It suffices, by symmetry, to prove point 1. We have

$$\begin{aligned} \mathbf{W}_x\big(e^{-\frac{1}{2}A_\infty^{(q)}} 1_{\Gamma_+}\big) &= \varphi_q(x)\, W_{x,\infty}^{(q)}(\Gamma_+) \qquad \big(\text{from (1.1.16)}\big) \\ &= \lim_{\substack{a\to-\infty \\ b\to+\infty}} \varphi_q(x)\, W_{x,\infty}^{(q)}(T_b < T_a) \end{aligned}$$

But, from (1.1.14), this limit equals :

$$\varphi_q(x)\, \frac{\gamma_q(x) - \gamma_q(-\infty)}{\varphi_q(\infty) - \gamma_q(-\infty)} := \varphi_q(x)\, \frac{\gamma_q(x) - \alpha}{\beta - \alpha} \tag{1.3.12}$$

where γ_q is given by (1.1.14). Hence :

$$\varphi_q^+(x) = \varphi_q(x)\, \frac{\gamma_q(x) - \alpha}{\beta - \alpha}. \tag{1.3.13}$$

It remains to prove that φ_q^+ satisfies the announced conditions. But (with γ for γ_q) :

$$\begin{aligned} (\varphi_q^+)''(x) &= \varphi_q''(x)\left(\frac{\gamma(x)-\alpha}{\beta-\alpha}\right) + 2\varphi_q'(x)\frac{\gamma'(x)}{\beta-\alpha} + \varphi_q(x)\frac{\gamma''(x)}{\beta-\alpha} \\ &= \varphi_q''(x)\left(\frac{\gamma(x)-\alpha}{\beta-\alpha}\right) + \frac{2\varphi_q'(x)}{\beta-\alpha}\,\frac{1}{\varphi_q^2(x)} + \frac{\varphi_q(x)}{\beta-\alpha}\left(-2\frac{\varphi_q'(x)}{\varphi_q^3(x)}\right) \qquad \big(\text{from (1.1.14)}\big) \\ &= \varphi_q''(x)\left(\frac{\gamma(x)-\alpha}{\beta-\alpha}\right) = q(x)\varphi_q(x)\frac{\gamma(x)-\alpha}{\beta-\alpha} = q(x)\varphi_q^+(x) \end{aligned} \tag{1.3.14}$$

On the other hand :

$$\varphi_q^+(x) = \varphi_q(x)\frac{\gamma(x)-\gamma(-\infty)}{\gamma(\infty)-\gamma(-\infty)} \underset{x\to\infty}{\sim} \varphi_q(x) \underset{x\to\infty}{\sim} x \tag{1.3.15}$$

$$\varphi_q^+(x) = \varphi_q(x)\frac{\displaystyle\int_{-\infty}^{x}\frac{dy}{\varphi_q^2(y)}}{\displaystyle\int_{-\infty}^{\infty}\frac{dy}{\varphi_q^2(y)}} \underset{x\to-\infty}{\sim} C\,\frac{\varphi_q(x)}{|x|} \underset{x\to-\infty}{\longrightarrow} C = \frac{1}{\displaystyle\int_{-\infty}^{\infty}\frac{dy}{\varphi_q^2(y)}}$$

(since $\varphi_q(y)$ is equivalent to $|y|$ when y goes to $-\infty$).
This proves Lemma 1.3.3. ∎

We now prove Theorem 1.3.2.
Of course, by symmetry, it suffices to show that the measure : $\widetilde{\mathbf{\Lambda}}^+ := \int_{\mathbb{R}} dx\ \mathbf{\Lambda}_x^+$, where $\mathbf{\Lambda}_x^+$ is the image of $\mathbf{W}_x^+$ by $\mathcal{L}$, is invariant for the process $\big((X_t, L_t^\bullet),\ t \geq 0\big)$. We note that from (1.3.6), we have :

$$\widetilde{\mathbf{\Lambda}}_x^+ = \frac{1}{2}\int_0^\infty du\, Q_{x,u}^{2,0} \tag{1.3.16}$$

We denote by $(Q_t,\ t \geq 0)$ the semi-group which is associated to the Markov process $\big((X_t, L_t^\bullet),\ t \geq 0\big)$, and we consider $F : \mathbb{R} \times \widetilde{\Omega} \to \mathbb{R}_+$ of the form :

$$\begin{aligned} F(x,l) &= f(x)\ \exp\left(-\frac{1}{2} < q, l >\right) \\ &= f(x)\ \exp\left(-\frac{1}{2}\int_{\mathbb{R}} l(y)q(y)dy\right) \end{aligned} \tag{1.3.17}$$

for $q \in \mathcal{I}$ and f Borel, bounded. Then, for such an F, we obtain, by definition of the process $\big((X_t, L_t^\bullet),\ t \geq 0\big)$:

$$Q_tF(x,l) = W\left(f(x+X_t)\ \exp\left\{-\frac{1}{2} < q, l > -\frac{1}{2}\int_0^t q(x+X_s)ds\right\}\right) \tag{1.3.18}$$

Now, from the monotone class theorem, Theorem 1.3.2 shall be obtained once we show that :

$$\int_{\mathbb{R}} dx \int_{\widetilde{\Omega}} \mathbf{\Lambda}_x^+(dl)\, Q_tF(x,l) = \int_{\mathbb{R}} dx \int_{\widetilde{\Omega}} \mathbf{\Lambda}_x^+(dl) F(x,l) \tag{1.3.19}$$

for every $t \geq 0$. But, from Lemma 1.3.3, we have :

$$\begin{aligned} \mathbf{W}_x^+\left(\exp -\frac{1}{2}\, A_\infty^{(q)}\right) &= \mathbf{W}_x\left(\exp\left(-\frac{1}{2}\, A_\infty^{(q)}\right)\cdot 1_{\Gamma_+}\right) \\ &= \int_{\widetilde{\Omega}} \mathbf{\Lambda}_x^+(dl)\exp\left(-\frac{1}{2} < q, l >\right) = \varphi_q^+(x) \end{aligned} \tag{1.3.20}$$

since $\mathbf{\Lambda}_x^+$ is the image of $\mathbf{W}_x^+$ by $\mathcal{L}$.
Thus, the left-hand side of (1.3.19) writes :

$$\begin{aligned} LHS &= < Q_tF, 1 >_{\widetilde{\mathbf{\Lambda}}^+} \\ &= \int_{\mathbb{R}} dx \int_{\widetilde{\Omega}} \mathbf{\Lambda}_x^+(dl) W\big(f(x+X_t)\, e^{-\frac{1}{2}<q,l>-\frac{1}{2}\int_0^t q(x+X_s)ds}\big) \\ &\qquad \big(\text{from (1.3.18)}\big) \\ &= W\left(\int_{\mathbb{R}} dx \varphi_q^+(x) f(x+X_t)\ \exp\left(-\frac{1}{2}\int_0^t q(x+X_s)ds\right)\right) \end{aligned}$$

(from Fubini and (1.3.20))

$$= \int_{\mathbb{R}} f(y)dy\, W\left(\varphi_q^+(y-X_t)\exp\left(-\frac{1}{2}\int_0^t q(y-X_t+X_s)ds\right)\right) \tag{1.3.21}$$

after making the change of variables $x + X_t = y$. On the other hand, the right-hand side of (1.3.19) equals :

$$\begin{aligned} RHS &= \int_{\mathbb{R}} dy \int_{\widetilde{\Omega}} \mathbf{\Lambda}_y^+(dl) f(y) \exp\left(-\frac{1}{2} < l, q >\right) \\ &= \int_{\mathbb{R}} f(y) \ \varphi_q^+(y) dy \end{aligned} \tag{1.3.22}$$

from (1.3.20). Thus, Theorem 1.3.2 is an immediate consequence of the following :

Lemma 1.3.4. *For every $q \in \mathcal{I}$, x real and $t \geq 0$:*

$$W\big(\varphi_q^+(y - X_t) \exp\left(-\frac{1}{2}\int_0^t q(y - X_t + X_s) ds\right)\big) = \varphi_q^+(y) \tag{1.3.23}$$

Furthermore, (1.3.23) is also true when φ_q^+ is replaced by φ_q^- or φ_q.

Proof of Lemma 1.3.4.

$$\begin{aligned} &W\left(\varphi_q^+(y - X_t) \exp\left(-\frac{1}{2}\int_0^t q(y - X_t + X_s) ds\right)\right) \\ &\quad = W\left(\varphi_q^+(y - X_t) \exp\left(-\frac{1}{2}\int_0^t q(y - X_t + X_{t-r}) dr\right)\right) \end{aligned}$$

(after making the change of variables $s = t - r$).

$$= W\left(\varphi_q^+(y - X_t) \exp\left(-\frac{1}{2}\int_0^t q(y - X_r) dr\right)\right)$$

$\big($since the process $(X_t - X_{t-r},\ 0 \leq r \leq t)$ has the same law as $(X_r,\ 0 \leq r \leq t)\big)$

$$= W_y\left(\varphi_q^+(X_t) \exp\left(-\frac{1}{2}\int_0^t q(X_r) dr\right)\right)$$

$\big($since $(-X_r,\ r \geq 0)$ has the same law as $(X_r,\ r \geq 0)\big)$

$$= \varphi_q^+(y)$$

because, from (1.3.10) and Itô's formula, $\left(\varphi_q^+(X_t) \exp\left(-\frac{1}{2}\int_0^t q(X_r) dr\right),\ t \geq 0\right)$ is a $\big((\mathcal{F}_t,\ t \geq 0),\ W_y\big)$ martingale.

Remark 1.3.5.

1) We denote by $\mathcal{G}$ the infinitesimal generator of the process $\big((X_t, , L_t^{\bullet}),\ t \geq 0\big)$. For a function

F of the form given by (1.3.17), we obtain :

$$\begin{aligned}
&\mathcal{G}F(x,l)\\
&= \frac{d}{ds}\Big|_{s=0} Q_sF(x,l)\\
&= \frac{d}{ds}\Big|_{s=0} W\left(f(x+X_s)\exp\left(-\frac{1}{2} < q,l > -\frac{1}{2}\int_0^s q(x+X_r)dr\right)\right)\\
&\big(\text{from } (1.3.18)\big)\\
&= \exp\left(-\frac{1}{2} < q,l >\right)\cdot\left[\frac{1}{2}f''(x) - \frac{1}{2}q(x)f(x)\right] \qquad (1.3.24)\\
&= \frac{1}{2}\,\frac{\partial^2 F}{\partial x^2}(x,l) - \frac{1}{2}q(x)F(x,l) \qquad (1.3.25)
\end{aligned}$$

Another way to prove Theorem 1.3.2 consists in showing that, for every F of the form (1.3.17), we have :

$$< \mathcal{G}F,\, 1 >_{\widetilde{\boldsymbol{\Lambda}}^{a,b}} = 0 \qquad (1.3.26)$$

Let us prove (1.3.26).
By symmetry, it suffices to prove (1.3.26) by replacing $\widetilde{\boldsymbol{\Lambda}}^{a,b}$ by $\widetilde{\boldsymbol{\Lambda}}^{+}$. Now, we obtain, for F of the form (1.3.17) with f of class C^2, with compact support :

$$\begin{aligned}
< \mathcal{G}F,\, 1 >_{\widetilde{\boldsymbol{\Lambda}}^{+}} &= \int_{\mathbb{R}} dx \int_{\widetilde{\Omega}} \widetilde{\boldsymbol{\Lambda}}_x^{+}(dl)\, e^{-\frac{1}{2}<q,l>}\left(\frac{1}{2}f''(x) - \frac{1}{2}q(x)f(x)\right)\\
&\big(\text{from } (1.3.24)\big)\\
&= \int_{\mathbb{R}} \varphi_q^{+}(x)dx\left(\frac{1}{2}f''(x) - \frac{1}{2}q(x)f(x)\right)\\
&(\text{from Lemma 1.3.3})\\
&= \int_{\mathbb{R}} \frac{1}{2}f(x)\left[(\varphi_q^{+})''(x) - q(x)\varphi_q^{+}(x)\right] dx\\
&(\text{after integrating by parts})\\
&= 0 \qquad (\text{from Lemma 1.3.3.})
\end{aligned}$$

2) Theorem 1.3.2 invites to ask the following question : is the process $\big((X_t,\, L_t^{\bullet}),\ t \geq 0\big)$ reversible with respect to the measure $\widetilde{\boldsymbol{\Lambda}}^{a,b}$, i.e. : does the following hold :

$$< Q_sF,\, G >_{\widetilde{\boldsymbol{\Lambda}}^{a,b}} = < F,\, Q_sG >_{\widetilde{\boldsymbol{\Lambda}}^{a,b}} \qquad (1.3.27)$$

for every $F, G : \mathbb{R}\times\Omega \to \mathbb{R}_+$ measurable and positive ? The answer to this question is negative. In particular, the operator $\mathcal{G}$ is not symmetric, i.e., in general :

$$< \mathcal{G}F,\, G >_{\widetilde{\boldsymbol{\Lambda}}^{a,b}} \neq < F,\, \mathcal{G}G >_{\widetilde{\boldsymbol{\Lambda}}^{a,b}} \qquad (1.3.28)$$

We now show (1.3.28), with $F(x,l) = f(x)\ \exp\left(-\frac{1}{2} < q,l >\right)$, $G(x,l) = g(x)$, $\widetilde{\boldsymbol{\Lambda}}^{a,b} = \widetilde{\boldsymbol{\Lambda}}$ $:= \widetilde{\boldsymbol{\Lambda}}^{1,1}$. Assuming that the equality would hold in (1.3.28), we would obtain, after an elementary computation :

$$\begin{aligned}
< \mathcal{G}F,\, G >_{\widetilde{\Lambda}} &= \int_{\mathbb{R}} \varphi_q(x)g(x)\left(\frac{1}{2}f''(x) - \frac{1}{2}q(x)f(x)\right)dx\\
&= \int_{\mathbb{R}} \varphi_q(x)f(x)\frac{1}{2}g''(x)dx = < F,\, \mathcal{G}G >_{\widetilde{\boldsymbol{\Lambda}}}
\end{aligned}$$

Thus, the preceding equality would imply, after integrating by parts and use of the relation $\varphi_q'' = q\,\varphi_q$:

$$-2q(x)\varphi_q(x)f(x) = 2\varphi_q'(x)f'(x)$$

for every f in class C^2, with compact support, which is absurd.

3) Of course, the preceding point implies that the measure $\widetilde{\mathbf{W}}^{a,b}$ which is invariant for the process $(\mathcal{X}_t,\ t \geq 0)$ is not reversible.

4) The following relation, which has been obtained from Lemma 1.3.3 and the definition of $\mathbf{\Lambda}_x^{\pm}$:

$$W_x\left[\varphi_q^{\pm}(X_t)\,\exp\left(-\frac{1}{2}\,A_t^{(q)}\right)\right] = \int_{\widetilde{\Omega}} \widetilde{\mathbf{\Lambda}}_x^{\pm}(dl)\,\exp\left(-\frac{1}{2} < q,l >\right) \tag{1.3.29}$$

is a particular case of the following result, which is found in $\big($[RY, M], Chap. 2$\big)$:

Let $F : \widetilde{\Omega} \to \mathbb{R}_+$ measurable, and "sub-exponential at infinity", $\Big($i.e. : there exists $q \in \mathcal{I}$ and $C > 0$ such that, for every $l \in \widetilde{\Omega}$, $F(l) \leq C\,\exp\big(- < q,l >\big)\Big)$, then :

$$\left(\int_{\widetilde{\Omega}} \widetilde{\mathbf{\Lambda}}_{X_t}^{\pm}(dl)\,F(l + L_t^{\bullet}),\ t \geq 0\right) \tag{1.3.30}$$

is a $\big((\mathcal{F}_t,\ t \geq 0),\ W\big)$ martingale ; hence :

$$\begin{aligned} W_x\left(\int_{\widetilde{\Omega}} \widetilde{\mathbf{\Lambda}}_{X_t}^{\pm}(dl)F(l + L_t^{\bullet})\right) &= W_x\left(\int_{\widetilde{\Omega}} \widetilde{\mathbf{\Lambda}}_{X_0}^{\pm}(dl)F(l)\right) \\ &= \int_{\widetilde{\Omega}} \widetilde{\mathbf{\Lambda}}_x^{\pm}(dl)F(l) \end{aligned} \tag{1.3.31}$$

If $F(l) = \exp\left(-\frac{1}{2} < q,l >\right)$, we have :

$$\begin{aligned} \int_{\widetilde{\Omega}} \widetilde{\mathbf{\Lambda}}_{X_t}^{\pm}(dl)F(l + L_t^{\bullet}) &= \int_{\widetilde{\Omega}} \widetilde{\mathbf{\Lambda}}_{X_t}^{\pm}(dl)\exp\left(-\frac{1}{2} < q,l > -\frac{1}{2}\int_{\mathbb{R}} q(x)L_t^x dx\right) \\ &= \varphi_q^{\pm}(X_t)\exp\left(-\frac{1}{2}\,A_t^{(q)}\right) \end{aligned}$$

Thus, when : $F(l) = \exp\left(-\frac{1}{2} < q,l >\right)$, (1.3.31) is nothing else but (1.3.29) since :

$$W_x\left(\varphi_q^{\pm}(X_t)\exp\left(-\frac{1}{2}\,A_t^{(q)}\right)\right) = \varphi_q^{\pm}(x).$$

5) Theorem 1.3.2 also invites to ask the question : are the measures $(\widetilde{\mathbf{\Lambda}}^{a,b},\ a,b \geq 0)$ the only invariant measures of the process $\big((X_t,\ L_t^{\bullet}),\ t \geq 0\big)$. Here is a partial answer to this question. Let $\widehat{\mathbf{\Lambda}}$ be an invariant measure for this process.

i) Since the first component of $\big((X_t,\ L_t^{\bullet}),\ t \geq 0\big)$ is a Brownian motion, and that process admits as its only invariant measure (up to a multiplicative factor) the Lebesgue measure on $\mathbb{R}$, the measure $\widehat{\mathbf{\Lambda}}$ admits a disintegration of the form :

$$\widehat{\mathbf{\Lambda}}(dx,dl) = dx\,\widehat{\mathbf{\Lambda}}_x(dl) \tag{1.3.32}$$

and, denoting by $\widehat{\varphi}_q$ the function defined by :

$$\widehat{\varphi}_q(x) = \widehat{\mathbf{\Lambda}}_x \left(\exp -\frac{1}{2} < q, l > \right) \tag{1.3.33}$$

the computations which lead to (1.3.21) and to (1.3.22) imply, if $\widehat{\mathbf{\Lambda}}$ is invariant :

$$\widehat{\varphi}_q(x) = W_x \left(\widehat{\varphi}_q(X_t) \exp \left(-\frac{1}{2} \int_0^t q(X_s) ds \right) \right)$$

It follows from this formula, using Itô's Lemma, that :

$$\widehat{\varphi}_q'' = q\, \widehat{\varphi}_q \tag{1.3.34}$$

The vector space of the solutions of the Sturm-Liouville equation has dimension 2 ; hence, there exist two constants $C_\pm(q)$ such that :

$$\widehat{\varphi}_q(x) = C_+(q) \varphi_q^+(x) + C_-(q) \varphi_q^-(x) \tag{1.3.35}$$

ii) The invariant measure $\widetilde{\mathbf{\Lambda}}^{a,b}$ which we described in Theorem 1.3.2, and which writes :

$$\begin{aligned} \widetilde{\mathbf{\Lambda}}^{a,b} &= \frac{1}{2} \int_{\mathbb{R}} dx (a \mathbf{\Lambda}_x^+ + b \mathbf{\Lambda}_x^-) = \int_{\mathbb{R}} dx \mathbf{\Lambda}_x^{a,b} && (1.3.36) \\ \text{with } \mathbf{\Lambda}_x^{a,b} &:= \frac{1}{2} (a \mathbf{\Lambda}_x^+ + b \mathbf{\Lambda}_x^-) && (1.3.37) \end{aligned}$$

enjoys the following property : both limits

$$\lim_{x \to +\infty} \frac{1}{x} \mathbf{\Lambda}_x^{a,b} \left(\exp -\frac{1}{2} < q, l > \right) \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{|x|} \mathbf{\Lambda}_x^{a,b} \left(\exp -\frac{1}{2} < q, l > \right) \tag{1.3.38}$$

do not depend on $q \in \mathcal{I}$. Indeed,

$$\frac{1}{x} \mathbf{\Lambda}_x^{a,b} \left(\exp -\frac{1}{2} < q, l > \right) = \frac{1}{2x} \left(a \varphi_q^+(x) + b \varphi_q^-(x) \right) \underset{x \to \infty}{\longrightarrow} \frac{a}{2}$$

from Lemma 1.3.3 and $\frac{1}{|x|} \mathbf{\Lambda}_x^{a,b} \left(\exp -\frac{1}{2} < q, l > \right) \underset{x \to -\infty}{\longrightarrow} \frac{b}{2}$.

iii) We now assume that the invariant measure $\widehat{\mathbf{\Lambda}}$, which equals : $\widehat{\mathbf{\Lambda}}(dx, dl) = dx \widehat{\mathbf{\Lambda}}_x(dl)$ also satisfies that both limits :

$$\lim_{x \to \infty} \frac{1}{x} \widehat{\mathbf{\Lambda}}_x \left(\exp -\frac{1}{2} < q, l > \right) \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{|x|} \widehat{\mathbf{\Lambda}}_x \left(\exp -\frac{1}{2} < q, l > \right)$$

exist and do not depend on $q \in \mathcal{I}$. Then, there exist a and b positive, such that : $\widehat{\mathbf{\Lambda}} = \widehat{\mathbf{\Lambda}}^{a,b}$. Indeed, from (1.3.35), together with Lemma 1.3.3 and (1.3.33), we have :

$$\begin{aligned} \lim_{x \to \infty} \frac{1}{x} \widehat{\mathbf{\Lambda}}_x \left(\exp \left(-\frac{1}{2} < q, l > \right) \right) &= \lim_{x \to \infty} \frac{\widehat{\varphi}_q(x)}{x} \\ &= \lim_{x \to \infty} \frac{C_+(q) \varphi_q^+(x) + C_-(q) \varphi_q^-(x)}{x} = C_+(q) && (1.3.39) \end{aligned}$$

Thus, $C_+(q)$ $\big($and $C_-(q)$, by symmetry$\big)$ are constants, which we shall denote respectively as $\frac{a}{2}$ and $\frac{b}{2}$. Thus, we have :

$$\begin{aligned}\widehat{\mathbf{\Lambda}}_x\left(\exp -\frac{1}{2} < q, l >\right) &= \frac{a}{2}\,\varphi_q^+(x) + \frac{b}{2}\,\varphi_q^-(x) \\ &= \mathbf{\Lambda}_x^{a,b}(e^{-\frac{1}{2}<q,l>})\end{aligned}$$

Hence : $\widehat{\mathbf{\Lambda}}_x = \mathbf{\Lambda}_x^{a,b}$ and $\widehat{\mathbf{\Lambda}} = \widetilde{\mathbf{\Lambda}}^{a,b}$.

1.3.4 Invariant measures of the process $(L_t^{X_t-\bullet},\ t \geq 0)$.

1.3.4.1 For $t \geq 0$, we define the random measure μ_t via :

$$\mu_t(f) = \int_0^t f(X_t - X_s)ds \tag{1.3.40}$$

with f positive, continuous and bounded. It is proven in [DMY] that $(\mu_t,\ t \geq 0)$ is a Markov process taking values in the space of positive measures on $\big(\mathbb{R},\, \mathcal{B}(\mathbb{R})\big)$. Due to the density of occupation formula, we may write (1.3.40) in the form :

$$\begin{aligned}\mu_t(f) &= \int_{\mathbb{R}} f(X_t - y)L_t^y dy \\ &= \int_{\mathbb{R}} f(z)L_t^{X_t - z}\, dz\end{aligned} \tag{1.3.41}$$

We deduce that :

$$\mu_t(dz) = L_t^{X_t - z}\, dz \tag{1.3.42}$$

Hence, rather than working in the space of measures on $\mathbb{R}$, we shall consider the Markov process $(L_t^{X_t-\bullet},\ t \geq 0)$ which takes values in $\widetilde{\Omega} = \mathcal{C}(\mathbb{R} \to \mathbb{R}_+)$.

1.3.4.2 Of course, this Markov process is the image of the process $\big((X_t, L_t^\bullet),\ t \geq 0\big)$ by the application :

$$\theta : \mathbb{R} \times \widetilde{\Omega} \to \widetilde{\Omega}$$

defined by :

$$\theta(x,l)(y) = l(x-y) \qquad x, y \in \mathbb{R},\ l \in \widetilde{\Omega} \tag{1.3.43}$$

This application θ is not bijective since :

$$\theta(x,l) = \theta(x', l')$$

as soon as :

$$l(x - x' + z) = l'(z) \tag{1.3.44}$$

for every $z \in \mathbb{R}$ i.e. : as soon as l' is an adequate translate of l.

i) We begin by verifying directly, i.e. : without using the result of Donati-Martin-Yor recalled above - that the process $(L_t^{X_t-\bullet},\ t \geq 0)$, which takes values in $\mathcal{C}(\mathbb{R} \to \mathbb{R}_+)$ is Markovian, in the natural filtration of the process $\big((X_t, L_t^\bullet),\ t \geq 0\big)$. For this purpose, using Dynkin's criterion (see [D]), and denoting by $(Q_t,\ t \geq 0)$ the semi-group associated to the process $\big((X_t, L_t^\bullet),\ t \geq 0\big)$, one needs to verify that :

$$Q_t(F \circ \theta)(x,l) = Q_t(F \circ \theta)(x', l') \tag{1.3.45}$$

for every $t \geq 0$ and $F : \widetilde{\Omega} \to \mathbb{R}_+$ measurable, as soon as :

$$\theta(x,l) = \theta(x',l')$$

Of course, from the monotone class theorem, it suffices to prove (1.3.45) for F of the form $F_q,\ q \in \mathcal{I}$, with :

$$F_q(l) := \exp\left(-\frac{1}{2} < q, l >\right) \qquad (l \in \widetilde{\Omega}) \tag{1.3.46}$$

We have, from (1.3.43) :

$$\begin{aligned} F_q \circ \theta(x,l) &= \exp\left(-\frac{1}{2} < q, l(x-\cdot) >\right) \\ &= \exp\left(-\frac{1}{2}\int_{\mathbb{R}} q(y) l(x-y)dy\right) = \exp\left(-\frac{1}{2} < \overset{\vee}{q}_x, l >\right) \end{aligned} \tag{1.3.47}$$

$$\text{with} \qquad \overset{\vee}{q}_x(y) = q(x-y) \tag{1.3.48}$$

Thus, from (1.3.18) :

$$Q_t(F_q \circ \theta)(x,l) = W\left(\exp\left(-\frac{1}{2} < \overset{\vee}{q}_{x+X_t}, l > -\frac{1}{2}\int_0^t q(x+X_t-(x+X_r))dr\right)\right) \tag{1.3.49}$$

However :

$$\begin{aligned} < \overset{\vee}{q}_{x+X_t}, l > &= \int_{\mathbb{R}} q(x+X_t-y) l(y)dy \\ &= \int_{\mathbb{R}} q(X_t+z) l(x-z)dz \end{aligned} \tag{1.3.50}$$

Thus, from (1.3.44), if $\theta(x,l) = \theta(x',l')$, we have :

$$l(x-z) = l'(x'-z) \quad \text{hence} \quad < \overset{\vee}{q}_{x+X_t}, l > = < \overset{\vee}{q}_{x'+X_t}, l' >$$

It now follows from (1.3.49) that :

$$Q_t(F_q \circ \theta)(x,l) = Q_t(F_q \circ \theta)(x',l')$$

1.3.4.3 Invariant measures for the process $(L_t^{X_t-\bullet},\ t \geq 0)$.

Of course, from Theorem 1.3.2, the image of $\widetilde{\mathbf{\Lambda}}^{a,b}$ by θ (defined by (1.3.43)) is an invariant measure for the process $(L_t^{X_t-\bullet},\ t \geq 0)$. Unfortunately, an elementary computation shows that this measure is identically infinite. Thus, we need to find directly - without refering to $\widetilde{\mathbf{\Lambda}}^{a,b}$ - invariant measures for $(L_t^{X_t-\bullet},\ t \geq 0)$.

Theorem 1.3.6. *Let $a, b \geq 0$, and :*

$$\mathbf{\Lambda}^{a,b} := a\mathbf{\Lambda}_0^+ + b\mathbf{\Lambda}_0^- \tag{1.3.51}$$

Then, $\mathbf{\Lambda}^{a,b}$ is an invariant measure for $(L_t^{X_t-\bullet},\ t \geq 0)$.

We recall that $\mathbf{\Lambda}_0^\pm$ is the image of $\mathbf{W}_0^\pm = \mathbf{W}^\pm$ by the application $\mathcal{L}$. In particular :

$$\mathbf{\Lambda}_0^\pm\big(\exp -\frac{1}{2} < q, l >\big) = \varphi_q^\pm(0) \qquad (q \in \mathcal{I}) \tag{1.3.52}$$

We now show Theorem 1.3.6.
We denote by $(\overline{Q}_s,\ s \geq 0)$ the semi-group associated to the Markov process $(L_t^{X_t-\bullet},\ t \geq 0)$. Thus, we have, for (1.3.49) :

$$\overline{Q}_s(F_q)(l) = W\left(\exp\left(-\frac{1}{2} < q(X_s + \cdot), l > -\frac{1}{2}\int_0^s q(X_s - X_r)dr\right)\right) \tag{1.3.53}$$

with : $F_q(l) = \exp\left(-\frac{1}{2} < q, l >\right)$.
On the other hand, by symmetry, it suffices to show that the measure $\mathbf{\Lambda}^+ := \mathbf{\Lambda}_0^+$ is invariant for $(L_t^{X_t-\bullet},\ t \geq 0)$. We compute :

$$\begin{aligned}
&\int_{\widetilde{\Omega}} \mathbf{\Lambda}^+(dl)\big(\overline{Q}_s(F_q)\big)(l) \\
&= \int_{\widetilde{\Omega}} \mathbf{\Lambda}^+(dl) W\left(\exp\left(-\frac{1}{2} < q(X_s + \cdot), l > -\frac{1}{2}\int_0^s q(X_s - X_r)dr\right)\right) \\
&= W\left\{\left(\exp -\frac{1}{2}\int_0^s q(X_s - X_r)dr\right) \cdot \int_{\widetilde{\Omega}} \mathbf{\Lambda}^+(dl)\exp\left(-\frac{1}{2} < q(X_s + \cdot), l >\right)\right\}
\end{aligned}$$

(from Fubini)

$$= W\left\{\exp\left(-\frac{1}{2}\int_0^s q(X_s - X_r)dr\right) \cdot \varphi^+_{q(X_s+\cdot)}(0)\right\}$$

$\big($from (1.3.52)$\big)$. Now, it is easy to check that :

$$\varphi^+_{q(X_s+\cdot)}(0) = \varphi_q^+(X_s) \tag{1.3.54}$$

Thus :

$$\begin{aligned}
\int_{\widetilde{\Omega}} \mathbf{\Lambda}^+(dl)\big(\overline{Q}_s(F_q)(l)\big) &= W\left(\varphi_q^+(X_s)\exp\left(-\frac{1}{2}\int_0^s q(X_s - X_r)dr\right)\right) \\
&= \varphi_q^+(0)
\end{aligned}$$

from Lemma 1.3.4 (replacing $(X_t,\ t \geq 0)$ by $(-X_t,\ t \geq 0)\big)$

$$= \int_{\widetilde{\Omega}} \mathbf{\Lambda}^+(dl) F_q(l) \qquad \big(\text{from (1.3.52)}\big)$$

This is Theorem 1.3.6. ∎

Remark 1.3.7.
1) Arguing as in point 2 of Remark 1.3.5, it is easily shown that none of the measures $\mathbf{\Lambda}^{a,b}$ is reversible for the process $(L_t^{X_t-\bullet},\ t \geq 0)$.
2) Here is another way to prove that $\mathbf{\Lambda}^{a,b}$ is invariant. (We give the details for $\mathbf{\Lambda}^+$). We have, with $F_q(l) = \exp -\frac{1}{2} < q, l >$, from (1.3.53) :

$$\overline{Q}_s(F_q)(l) = W\left(\exp\left(-\frac{1}{2} < q(X_s + \cdot), l > -\frac{1}{2}\int_0^s q(X_r)dr\right)\right) \tag{1.3.55}$$

We proceeded from (1.3.53) to (1.3.55) by making the change of variable $r = s - u$ and using the fact that, under W, $(X_s - X_{s-r},\ r \leq s) \overset{\text{(law)}}{=} (X_r,\ r \leq s)$. Thus, denoting by $\overline{\mathcal{G}}$ the infinitesimal generator of the semi-group $(\overline{Q}_s,\ s \geq 0)$, we obtain :

$$\begin{aligned}
\overline{\mathcal{G}}\, F_q(l) &= \frac{d}{ds}\Big|_{s=0} \overline{Q}_s(F_q)(l) \\
&= \frac{d}{ds}\Big|_{s=0} W\left[g(X_s)\exp\left(-\frac{1}{2}\int_0^s q(X_r)dr\right)\right] \\
\Bigg(\text{with } g(x) &:= \exp\left(-\frac{1}{2} < q(x+\cdot), l >\right)\Bigg) \\
&= \frac{1}{2}\, g''(0) - \frac{1}{2}\, q(0)\, g(0) \\
&= \frac{1}{2}\left[\frac{\partial^2}{\partial x^2}\Big|_{x=0}\left(\exp\left(-\frac{1}{2} < q(x+\cdot), l >\right) - q(0)\exp\left(-\frac{1}{2} < q, l >\right)\right)\right]
\end{aligned}$$

Thus :

$$\begin{aligned}
< \overline{\mathcal{G}}\, F_q, 1 >_{\Lambda^+} &= \int_{\widetilde{\Omega}} \overline{\mathcal{G}}\, F_q(l)\Lambda^+(dl) \\
&= \frac{1}{2}\int_{\widetilde{\Omega}} \mathbf{\Lambda}^+(dl)\left(\frac{\partial^2}{\partial x^2}\Big|_{x=0} \exp\left(-\frac{1}{2} < q(x+\cdot), l >\right) - q(0)\exp\left(-\frac{1}{2} < q, l >\right)\right) \\
&= \frac{1}{2}\big(\varphi_q^{+\prime\prime}(0) - q(0)\varphi_q^+(0)\big) = 0 \qquad (1.3.56)
\end{aligned}$$

after interverting the second derivative and integration with respect to $\mathbf{\Lambda}^+(dl)$, using Lemma 1.3.3 and the fact that $\varphi^+_{q(x+\cdot)}(0) = \varphi_q^+(x)$. From relation (1.3.56), we deduce of course that : $< \overline{Q}_s\, F_q, 1 >_{\mathbf{\Lambda}^+} = < F_q, 1 >_{\mathbf{\Lambda}^+}$, i.e. that $\mathbf{\Lambda}^+$ is invariant.