CHAPTER 3

Part 1. Some properties of Γ.

Throughout Part 1 of this chapter, we assume that the quotient G/Γ is compact. Our main purpose is to prove the following results (i)~(iv) (particularly (iv)).

(i) The commutator subgroup $[\Gamma, \Gamma]$ of Γ is of finite index in Γ. Moreover, if Γ is torsion-free, then the index $(\Gamma : [\Gamma, \Gamma])$ is a divisor of $P(1)^2$, where

$$P(u) = \prod_{i=1}^{g}(1 - \pi_i u)(1 - \pi_i' u)$$

is the numerator of the main factor of $\zeta_\Gamma(u)$ (Theorem 2, §6).

(ii) $\Gamma_{\mathbf{R}}$ has no non-trivial deformation in $G_{\mathbf{C}} = PL_2(\mathbf{C})$ (Theorem 3, §7).

(iii) Γ is residually finite. Moreover, Γ contains a torsion-free subgroup of finite index (Theorem 4, §9).

(iv) The field $F = \mathbf{Q}((\mathrm{tr}\,\gamma_{\mathbf{R}})^2 | \gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}})$ is an algebraic number field. Moreover, there is a quaternion algebra A over F, which is uniquely determined by Γ, such that for any field $K \subset \mathbf{C}$ the following two statements (a), (b) are equivalent:

   (a) There is an element $t \in G_{\mathbf{C}} = PL_2(\mathbf{C})$ such that $t^{-1}\Gamma_{\mathbf{R}} t \subset PL_2(K)$.

   (b) K contains F and $A \otimes_F K \cong M_2(K)$.

   Furthermore, $\Gamma_{\mathbf{R}}$ can be considered as a subgroup of $A^\times / F^\times$ (Theorem 5, Proposition 6;§12, §13).

We begin with some preliminaries; then we shall prove Theorem 1 (§5) which asserts $H^1(\Gamma_{\mathbf{R}}, \rho_n) = \{0\}$ $(n \geq 0)$, where ρ_n is the symmetric tensor representation of $G_{\mathbf{R}}$ of degree $2n$ (see §3). This is a consequence of Eichler-Shimura's isomorphism (see §4), Kuga's lemma (Lemma 10 of Chapter 1), and our remarks on cohomology groups (§1 §2). Now, Theorem 1 is basic for all our results (i)-(iv). In fact, (i) and (ii) are almost direct consequences of $H^1(\Gamma_{\mathbf{R}}, \rho_n) = \{0\}$ for $n = 0$ and $n = 1$ respectively; and (iii), (iv) are results of our study of "deformation variety" of $\Gamma_{\mathbf{R}}$ in $G_{\mathbf{C}}$, of which (ii) is the starting point.

Finally, we remark that some of our results are valid also for more general dense subgroups $\Gamma_{\mathbf{R}}$ of $G_{\mathbf{R}}$ satisfying some conditions (see Remark 1 in §7).

The vanishing of $H^1(\Gamma_{\mathbf{R}}, \rho_n)$ and its consequences.

§1. In general, let A be an additive abelian group, and let ρ be a homomorphism of an arbitrary abstract group Γ into the group of all automorphisms of A;

$$\rho : \Gamma \to \operatorname{Aut} A; \quad \rho(\gamma\gamma')a = \rho(\gamma)(\rho(\gamma')a). \tag{1}$$

As usual, 1-cocycles are A-valued functions $a(\gamma)$ on Γ satisfying

$$a(\gamma\gamma') = a(\gamma) + \rho(\gamma)a(\gamma') \quad (\gamma, \gamma' \in \Gamma),$$

and 1-coboundaries are such 1-cocycles $a(\gamma)$ as can be written as $a(\gamma) = (1 - \rho(\gamma))a$ with some fixed $a \in A$. We denote by $H^1(\Gamma, \rho)$ the 1-cohomology group, i.e., the quotient of the group of all 1-cocycles by that of all 1-coboundaries. If Γ^0 is a subgroup of Γ and if ρ_0 is the restriction of ρ to Γ^0, then we get a restriction homomorphism

$$\varphi : H^1(\Gamma, \rho) \to H^1(\Gamma^0, \rho_0). \tag{2}$$

Lemma 1. *If for every $\gamma \in \Gamma$, $\rho(\Gamma^0 \cap \gamma^{-1}\Gamma^0\gamma)$ has no common fixed element $\neq 0$ of A, then φ is injective.*

Proof. Let $a(\gamma)$ be a cocycle representing a class contained in the kernel of φ. Then we get $a(\gamma_0) = (1 - \rho(\gamma_0))a$ ($a \in A$) for all $\gamma_0 \in \Gamma^0$. Put $a'(\gamma) = a(\gamma) - (1 - \rho(\gamma))a$. Then $a'(\gamma)$ is a 1-cocycle with respect to Γ and ρ, and we have $a'(\gamma_0) = 0$ for all $\gamma_0 \in \Gamma^0$. Let δ be any element of Γ. Then we have

$$\begin{aligned} a'(\delta^{-1}\gamma\delta) &= a'(\delta^{-1}) + \rho(\delta^{-1})a'(\gamma) + \rho(\delta^{-1}\gamma)a'(\delta) \\ &= -\rho(\delta^{-1})a'(\delta) + \rho(\delta^{-1})a'(\gamma) + \rho(\delta^{-1}\gamma)a'(\delta). \end{aligned}$$

Hence if $\gamma \in \Gamma^0 \cap \delta\Gamma^0\delta^{-1}$ so that γ and $\delta^{-1}\gamma\delta$ are both contained in Γ^0, then we get $(\rho(\gamma) - 1)a'(\delta) = 0$. Since this holds for all $\gamma \in \Gamma^0 \cap \delta\Gamma^0\delta^{-1}$, we get $a'(\delta) = 0$ by our assumption. Therefore, $a'(\delta) = 0$ for all $\delta \in \Gamma$; hence we get $a(\gamma) = (1 - \rho(\gamma))a$ for all $\gamma \in \Gamma$. Therefore, $a(\gamma)$ is a coboundary. □

Corollary. *Let Γ^0 be a discrete subgroup of $G_{\mathbf{R}} = PSL_2(\mathbf{R})$ such that the quotient $G_{\mathbf{R}}/\Gamma^0$ is of finite invariant volume, and let Γ be a group with $\Gamma^0 \subset \Gamma \subset G_{\mathbf{R}}$, such that $\gamma^{-1}\Gamma^0\gamma$ and Γ^0 are commensurable with each other for every $\gamma \in \Gamma$. Let $\tilde{\rho}$ be a finite dimensional non-trivial irreducible representation of $G_{\mathbf{R}}$, and let ρ be its restriction to Γ. Then the restriction homomorphism φ given by* (2) *is injective.*

Proof. By Borel's density theorem (see [1] and Supplement §1), $\rho|_{\Gamma^0\cap\gamma^{-1}\Gamma^0\gamma}$ is irreducible for all $\gamma \in \Gamma$. □

§2. Returning to the general situation, let Γ^0 be a subgroup of Γ, and assume now that $\gamma^{-1}\Gamma^0\gamma$ is commensurable with Γ^0 for every $\gamma \in \Gamma$. Let $\mathcal{H}(\Gamma, \Gamma^0)$ be the Hecke ring defined with respect to the left Γ^0-coset decomposition of Γ. For each $\Gamma^0\gamma\Gamma^0 \in \mathcal{H}(\Gamma, \Gamma^0)$, let

$d(\Gamma^0\gamma\Gamma^0)$ be the number of left Γ^0-cosets contained in $\Gamma^0\gamma\Gamma^0$, and for each $X \in \mathcal{H}(\Gamma,\Gamma^0)$, we define $d(X)$ by linearity. Then

$$\mathcal{H}(\Gamma,\Gamma^0) \ni X \mapsto d(X) \in \mathbf{Z}$$

is a homomorphism.

Now we shall define an action ρ^* of $\mathcal{H}(\Gamma,\Gamma^0)$ on $H^1(\Gamma^0,\rho_0)$. Take any double coset $\Gamma^0\gamma\Gamma^0 = \sum_{i=1}^d \Gamma^0\gamma_i$ (disjoint), and for each $\sigma \in \Gamma^0$ and i $(1 \le i \le d)$, put $\gamma_i\sigma = x_{ij}\gamma_j$ with $x_{ij} \in \Gamma^0$. For any 1-cocycle $a(\sigma)$, put

$$\rho^*(\Gamma^0\gamma\Gamma^0)a(\sigma) = \sum_{i=1}^{d} \rho(\gamma_i^{-1})a(x_{ij}). \tag{3}$$

Then, as can be checked directly, this is also a 1-cocycle; and moreover, if $a(\sigma)$ is a coboundary, it is also a coboundary. In fact, if $a(\sigma) = (1-\rho(\sigma))a$, then (3) will be equal to $(1-\rho(\sigma))\sum_{i=1}^d \rho(\gamma_i^{-1})a$. Moreover, if we take another left Γ^0-coset decomposition $\Gamma^0\gamma\Gamma^0 = \sum_{i=1}^d \Gamma^0\gamma_i'$ with $\gamma_i' = \sigma_i\gamma_i$ $(\sigma_i \in \Gamma^0)$, and define $\rho^{*\prime}(\Gamma^0\gamma\Gamma^0)$ with respect to this decomposition, a simple straightforward computation (note that $a(\gamma^{-1}) = -\rho(\gamma^{-1})a(\gamma)$ $(\gamma \in \Gamma)$) shows that

$$\rho^{*\prime}(\Gamma^0\gamma\Gamma^0)a(\sigma) = \rho^*(\Gamma^0\gamma\Gamma^0)a(\sigma) - (1-\rho(\sigma))\sum_{i=1}^{d} \rho(\gamma_i^{-1})a(\sigma_i^{-1}). \tag{4}$$

Therefore, $\rho^*(\Gamma^0\gamma\Gamma^0)$ defines an endomorphism of $H^1(\Gamma^0,\rho^0)$, which is well-defined by $\Gamma^0\gamma\Gamma^0$ and does not depend on the choice of $\gamma_1,\cdots,\gamma_d$. Define $\rho^*(X)$ for any $X \in \mathcal{H}(\Gamma,\Gamma^0)$ by linearity. Thus, for each $X \in \mathcal{H}(\Gamma,\Gamma^0)$, $\rho^*(X)$ is an element of $\operatorname{End} H^1(\Gamma^0,\rho_0)$, the endomorphism ring of $H^1(\Gamma^0,\rho_0)$.

Proposition 1. *The notations being as above,*

(i) ρ^* *is an anti-homomorphism of* $\mathcal{H}(\Gamma,\Gamma^0)$ *into* $\operatorname{End} H^1(\Gamma^0,\rho_0)$*;*

$$\rho^*(X\cdot Y) = \rho^*(Y)\circ\rho^*(X) \quad \forall X,Y \in \mathcal{H}(\Gamma,\Gamma^0). \tag{5}$$

(ii) *If* $a(\sigma)$ *is contained in* $\varphi(H^1(\Gamma,\rho))$*, then*

$$\rho^*(X)a(\sigma) = d(X)a(\sigma) \quad \forall X \in \mathcal{H}(\Gamma,\Gamma^0), \tag{6}$$

holds.

Proof. (i). This can be checked in a straightforward manner and hence is omitted.

(ii) Let $a(\sigma)$ be a cocycle *on* Γ. Then,

$$\begin{aligned}\rho^*(\Gamma^0\gamma\Gamma^0)a(\sigma) &= \sum_{i=1}^{d}\rho(\gamma_i^{-1})a(x_{ij}) = \sum_{i=1}^{d}\rho(\gamma_i^{-1})a(\gamma_i\sigma\gamma_j^{-1}) \\ &= \sum_{i=1}^{d}\rho(\gamma_i^{-1})a(\gamma_i) + \sum_{i=1}^{d}\rho(\gamma_i^{-1})\rho(\gamma_i)a(\sigma) + \sum_{i=1}^{d}\rho(\gamma_i^{-1})\rho(\gamma_i\sigma)a(\gamma_j^{-1}) \\ &= \sum_{i=1}^{d}\rho(\gamma_i^{-1})a(\gamma_i) + d\cdot a(\sigma) - \sum_{j=1}^{d}\rho(\sigma)\rho(\gamma_j^{-1})a(\gamma_j) \\ &= d\cdot a(\sigma) + (1-\rho(\sigma))\sum_{i=1}^{d}\rho(\gamma_i^{-1})a(\gamma_i) \\ &\sim d\cdot a(\sigma) = d(\Gamma^0\gamma\Gamma^0)a(\sigma),\end{aligned}$$

which proves (ii). □

§3. Now let ρ_n $(n = 0, 1, 2, \cdots)$ be the real symmetric tensor representation of $G_{\mathbf{R}} = PSL_2(\mathbf{R})$ of degree $2n$. Namely, put

$$\pm\begin{pmatrix}u_1\\v_1\end{pmatrix} = \pm\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}; \quad \pm\begin{pmatrix}a & b\\c & d\end{pmatrix}\in G_{\mathbf{R}},\ \begin{pmatrix}u\\v\end{pmatrix}\in\mathbf{R}^2, \tag{7}$$

and put

$$\begin{pmatrix}u_1^{2n}\\u_1^{2n-1}v_1\\\vdots\\v_1^{2n}\end{pmatrix} = \rho_n\left(\pm\begin{pmatrix}a & b\\c & d\end{pmatrix}\right)\begin{pmatrix}u^{2n}\\u^{2n-1}v\\\vdots\\v^{2n}\end{pmatrix}. \tag{8}$$

Then ρ_n is an absolutely irreducible representation of $G_{\mathbf{R}}$ in $V_n = \mathbf{R}^{2n+1}$. In particular, ρ_0 is the trivial representation, and as can be easily checked, ρ_1 is equivalent to the adjoint representation Ad of $G_{\mathbf{R}}$ in the Lie algebra $\mathfrak{g}_{\mathbf{R}} = \{X\in M_2(\mathbf{R})|\operatorname{tr}X = 0\}$ of $G_{\mathbf{R}}$;

$$G_{\mathbf{R}}\ni g_{\mathbf{R}} : X\mapsto(\operatorname{Ad}g_{\mathbf{R}})X = g_{\mathbf{R}}^{-1}Xg_{\mathbf{R}}. \tag{9}$$

§4. Let $\Gamma_{\mathbf{R}}^0$ be a discrete subgroup of $G_{\mathbf{R}}$ with compact quotient, let $\rho_{n,0}$ $(n = 0, 1, 2, \cdots)$ be the restriction of ρ_n to $\Gamma_{\mathbf{R}}^0$, and let $\mathfrak{M}_{2n+2}$ be the vector space over $\mathbf{R}$ of all holomorphic automorphic forms of weight $2n+2$ with respect to $\Gamma_{\mathbf{R}}^0$. Then, by Eichler-Shimura [12] [31], the following map ι gives an $\mathbf{R}$-linear isomorphism of $\mathfrak{M}_{2n+2}$ onto $H^1(\Gamma_{\mathbf{R}}^0, \rho_{n,0})$:

$$\iota : \mathfrak{M}_{2n+2}\ni f(z)\mapsto a(\sigma) = \operatorname{Re}\begin{pmatrix}\int_z^{\sigma z}f(\tau)\tau^{2n}d\tau\\\int_z^{\sigma z}f(\tau)\tau^{2n-1}d\tau\\\vdots\\\int_z^{\sigma z}f(\tau)d\tau\end{pmatrix}, \tag{10}$$

where z is an arbitrarily fixed point of $\mathfrak{H}$. Let $\Gamma_{\mathbf{R}} \supset \Gamma^0_{\mathbf{R}}$ be a group contained in the commensurability subgroup of $\Gamma^0_{\mathbf{R}}$ in $G_{\mathbf{R}}$, and let $\widetilde{\rho}_n$ be the anti-representation of $\mathcal{H}(\Gamma_{\mathbf{R}}, \Gamma^0_{\mathbf{R}})$ in $\mathfrak{M}_{2n+2}$ defined by linearity and by

$$\mathfrak{M}_{2n+2} \ni f(z) \mapsto \widetilde{\rho}_n(\Gamma^0_{\mathbf{R}}\gamma\Gamma^0_{\mathbf{R}})f(z) = \sum_{i=1}^{d} f(\gamma_i z)(c_i z + d_i)^{-2n-2} \in \mathfrak{M}_{2n+2}, \tag{11}$$

where $\Gamma^0_{\mathbf{R}}\gamma\Gamma^0_{\mathbf{R}} = \sum_{i=1}^{d} \Gamma^0_{\mathbf{R}}\gamma_i$ (disjoint), and $\gamma_i = \pm\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ $(1 \leq i \leq d)$. Then, as can be checked directly, the anti-representations ρ^*_n of $\mathcal{H}(\Gamma_{\mathbf{R}}, \Gamma^0_{\mathbf{R}})$ in $H^1(\Gamma^0_{\mathbf{R}}, \rho_{n,0})$, and $\widetilde{\rho}_n$ of $\mathcal{H}(\Gamma_{\mathbf{R}}, \Gamma^0_{\mathbf{R}})$ in $\mathfrak{M}_{2n+2}$ are equivalent by the isomorphism ι; i.e., we have

$$\rho^*_n(X) \circ \iota = \iota \circ \widetilde{\rho}_n(X) \quad \forall X \in \mathcal{H}(\Gamma_{\mathbf{R}}, \Gamma^0_{\mathbf{R}}), \tag{12}$$

cf. [31] §8.

§5. Now we are in the situation to prove the following Theorem.

THEOREM 1. *Let $\Gamma^0_{\mathbf{R}}$ be a discrete subgroup of $G_{\mathbf{R}} = PSL_2(\mathbf{R})$ with compact quotient, and let $\Gamma_{\mathbf{R}}$ be a dense subgroup of $G_{\mathbf{R}}$ containing $\Gamma^0_{\mathbf{R}}$ such that $\gamma_{\mathbf{R}}^{-1}\Gamma^0_{\mathbf{R}}\gamma_{\mathbf{R}}$ and $\Gamma^0_{\mathbf{R}}$ are commensurable with each other for all $\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}$. Let ρ_n $(n = 0, 1, 2, \cdots)$ be as in §3, and identify $\rho_n|_{\Gamma_{\mathbf{R}}}$ with ρ_n. Then we have*

$$H^1(\Gamma_{\mathbf{R}}, \rho_n) = \{0\} \quad (n = 1, 2, 3, \cdots). \tag{13}$$

Moreover, if $\Gamma_{\mathbf{R}}$ does not contain a normal subgroup of infinite index containing $\Gamma^0_{\mathbf{R}}$, then we also have

$$H^1(\Gamma_{\mathbf{R}}, \rho_0) = \{0\}. \tag{14}$$

PROOF. *The case $n > 0$.* Let $\rho_{n,0}$ be the restriction of ρ_n to $\Gamma^0_{\mathbf{R}}$. Then the restriction homomorphism φ of $H^1(\Gamma_{\mathbf{R}}, \rho_n)$ into $H^1(\Gamma^0_{\mathbf{R}}, \rho_{n,0})$ is injective. In fact, since ρ_n is irreducible and $\neq 1$, we can apply the Corollary of Lemma 1. So, we shall consider $H^1(\Gamma_{\mathbf{R}}, \rho_n)$ as a subspace of $H^1(\Gamma^0_{\mathbf{R}}, \rho_{n,0})$. Now we have an anti-representation ρ^*_n of $\mathcal{H}(\Gamma_{\mathbf{R}}, \Gamma^0_{\mathbf{R}})$ in $H^1(\Gamma^0_{\mathbf{R}}, \rho_{n,0})$, and by Proposition 1, $H^1(\Gamma_{\mathbf{R}}, \rho_n)$ is contained in the kernel of $\rho^*_n(X) - d(X) \cdot I$ for any $X \in \mathcal{H}(\Gamma_{\mathbf{R}}, \Gamma^0_{\mathbf{R}})$. Let $H_1, \cdots, H_N$ be the set of all subgroups of $G_{\mathbf{R}}$ containing $\Gamma^0_{\mathbf{R}}$ as a subgroup of finite index. By a well-known theorem on fuchsian groups, such subgroups are finite in number. Since $(\Gamma_{\mathbf{R}} : \Gamma^0_{\mathbf{R}}) = \infty$, we can take $\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}$ that is not contained in any H_i $(1 \leq i \leq N)$. Then $\Gamma^0_{\mathbf{R}}$ and $\gamma_{\mathbf{R}}$ generate a subgroup of $\Gamma_{\mathbf{R}}$ which contains $\Gamma^0_{\mathbf{R}}$ as a subgroup of *infinite* index; hence $\Gamma^0_{\mathbf{R}}$ and $\gamma_{\mathbf{R}}$ generate a dense subgroup[1] of $G_{\mathbf{R}}$. Therefore, by Lemma 10 of Chapter 1, if λ is an eigenvalue of $\widetilde{\rho}_n(\Gamma^0_{\mathbf{R}}\gamma_{\mathbf{R}}\Gamma^0_{\mathbf{R}})$ in $\mathfrak{M}_{2n+2}$, then $\lambda \neq d(\Gamma^0_{\mathbf{R}}\gamma_{\mathbf{R}}\Gamma^0_{\mathbf{R}})$. Therefore by (12), we see immediately that if λ is an eigenvalue of $\rho^*_n(\Gamma^0_{\mathbf{R}}\gamma_{\mathbf{R}}\Gamma^0_{\mathbf{R}})$ in $H^1(\Gamma^0_{\mathbf{R}}, \rho_{n,0})$, then $\lambda \neq d(\Gamma^0_{\mathbf{R}}\gamma_{\mathbf{R}}\Gamma^0_{\mathbf{R}})$. This shows that the kernel of $\rho^*_n(\Gamma^0_{\mathbf{R}}\gamma_{\mathbf{R}}\Gamma^0_{\mathbf{R}}) - d(\Gamma^0_{\mathbf{R}}\gamma_{\mathbf{R}}\Gamma^0_{\mathbf{R}})$ is trivial, and hence $H^1(\Gamma_{\mathbf{R}}, \rho_n) = \{0\}$.

[1] See Supplement §1.

The case $n = 0$. The above proof of the injectivity of the restriction map φ does not apply to this case, but the rest of the proof applies to this case also. Therefore, it is enough to prove the injectivity of φ.

Since $\rho_0 = I$, $H^1(\Gamma_{\mathbf{R}}, \rho_0)$ resp. $H^1(\Gamma^0_{\mathbf{R}}, \rho_{0,0})$ can be identified with $\mathrm{Hom}(\Gamma_{\mathbf{R}}, \mathbf{R})$ resp. $\mathrm{Hom}(\Gamma^0_{\mathbf{R}}, \mathbf{R})$. Let $\alpha \in \mathrm{Hom}(\Gamma_{\mathbf{R}}, \mathbf{R})$ with $\alpha|_{\Gamma^0_{\mathbf{R}}} = 0$. Let H be the kernel of α. Then H is a normal subgroup of $\Gamma_{\mathbf{R}}$ containing $\Gamma^0_{\mathbf{R}}$, and if $\alpha \neq 0$, $\Gamma_{\mathbf{R}}/H$ must be infinite. Therefore by our assumption, we get $\alpha = 0$. Hence φ is injective. □

Corollary. *Let Γ be a discrete subgroup of $G = G_{\mathbf{R}} \times G_{\mathfrak{p}}$ with compact quotient and with dense images of projections $\Gamma_{\mathbf{R}}, \Gamma_{\mathfrak{p}}$ in $G_{\mathbf{R}}, G_{\mathfrak{p}}$ respectively. Then we have*

$$H^1(\Gamma_{\mathbf{R}}, \rho_n) = \{0\} \quad (n = 0, 1, 2, \cdots). \tag{15}$$

Proof. Put $\Gamma^0 = \Gamma \cap (G_{\mathbf{R}} \times PSL_2(O_{\mathfrak{p}}))$. Then $\Gamma^0_{\mathbf{R}}$ is maximal in $\Gamma_{\mathbf{R}}$ (Corollary of Lemma 11 in Chapter 1), and obviously is not normal in $\Gamma_{\mathbf{R}}$. Therefore, all conditions in Theorem 1 are satisfied. □

§6. Consequence of $H^1(\Gamma_{\mathbf{R}}, \rho_0) = \{0\}$. Let Γ be as in the above Corollary, and let $[\Gamma, \Gamma]$ be the commutator subgroup. Then, since Γ is finitely generated (see §25, Chapter 2), the quotient $\Gamma/[\Gamma, \Gamma]$ is a finitely generated abelian group, and hence is isomorphic to a direct product of a finite group and a free abelian group of finite rank. But since the above corollary for $n = 0$ asserts that

$$\mathrm{Hom}(\Gamma, \mathbf{R}) = \mathrm{Hom}(\Gamma_{\mathbf{R}}, \mathbf{R}) = H^1(\Gamma_{\mathbf{R}}, \rho_0) = \{0\},$$

we see immediately that $\Gamma/[\Gamma, \Gamma]$ must be finite.

As an exercise, let us give some estimation of the group index $(\Gamma : [\Gamma, \Gamma])$ in the case where Γ is torsion-free. For this purpose, it is more convenient to consider the homology group than the cohomology group. Thus let

$$\Gamma^0 = \Gamma \cap (G_{\mathbf{R}} \times PSL_2(O_{\mathfrak{p}})),$$

and let g (≥ 2) be the genus of $\mathfrak{H}/\Gamma^0_{\mathbf{R}}$. Put $A = \Gamma^0/[\Gamma^0, \Gamma^0]$ and consider it as an additive group. Then $A \cong \mathbf{Z}^{2g}$, and we have an anti-representation ρ_0^* of $\mathcal{H}(\Gamma, \Gamma^0)$ on

$$H^1(\Gamma^0, \mathbf{Z}) = \mathrm{Hom}(\Gamma^0, \mathbf{Z}) = \mathrm{Hom}(A, \mathbf{Z})$$

(see §2).[2] Its dual $\mathfrak{R}$ is a representation of $\mathcal{H}(\Gamma, \Gamma^0)$ on A defined by

$$\mathfrak{R}(\Gamma^0 \gamma \Gamma^0)\bar{\sigma} = \prod_{i=1}^{d} x_{ij}, \tag{16}$$

where $\sigma \in \Gamma^0$,

(i) $\bar{\sigma}$ is the $[\Gamma^0, \Gamma^0]$-coset containing σ,
(ii) $\Gamma^0 \gamma \Gamma^0 = \sum_{i=1}^{d} \Gamma^0 \gamma_i$ (disjoint), and $\gamma_i \sigma = x_{ij} \gamma_j$ with $x_{ij} \in \Gamma^0$ $(1 \leq i \leq d)$.

[2] Since $\mathcal{H}(\Gamma, \Gamma^0)$ is commutative (see Chapter 1, §10), all anti-representations of $\mathcal{H}(\Gamma, \Gamma^0)$ are representations.

Moreover, it can be immediately checked that

(17) $$\{\Re(\Gamma^0\gamma\Gamma^0) - d(\Gamma^0\gamma\Gamma^0)\}A \subset [\Gamma,\Gamma] \cap \Gamma^0/[\Gamma^0,\Gamma^0]$$

holds for any $\gamma \in \Gamma$. Therefore, the group index

$$(\Gamma : [\Gamma,\Gamma]) = (\Gamma^0 : [\Gamma,\Gamma] \cap \Gamma^0)$$

divides $\det\{\Re(\Gamma^0\gamma\Gamma^0) - d(\Gamma^0\gamma\Gamma^0)\}$ for any $\gamma \in \Gamma$. Now let $\widetilde{\rho}_0$ be the anti-representation of $\mathcal{H}(\Gamma,\Gamma^0) \cong \mathcal{H}(\Gamma_{\mathbf{R}},\Gamma^0_{\mathbf{R}})$ in $\mathfrak{M}_2$ defined by (11). Then by the identification $\mathfrak{M}_2 \cong_{\mathbf{R}} \mathrm{Hom}(A \otimes_{\mathbf{Z}} \mathbf{R}, \mathbf{R})$, $\widetilde{\rho}_0$ is equivalent to $\rho_0^* \otimes_{\mathbf{Z}} \mathbf{R}$. Hence the above determinant is also equal to $\det\{\widetilde{\rho}_0(\Gamma^0\gamma\Gamma^0) - d(\Gamma^0\gamma\Gamma^0)\}$. Now consider $\mathfrak{M}_2$ as a vector space over $\mathbf{C}$. Then $\widetilde{\rho}_0$ may also be regarded as a g-dimensional complex anti-representation of $\mathcal{H}(\Gamma,\Gamma^0)$ in $\mathfrak{M}_2$. Call it $\widetilde{\rho}_0^{\mathbf{C}}$. Then by Petersson, $\widetilde{\rho}_0^{\mathbf{C}}$ is a direct sum of one-dimensional representations $\chi_1, \cdots, \chi_g$; and $\chi_i(X)$ $(1 \leq i \leq g)$ are real numbers for all $X \in \mathcal{H}(\Gamma,\Gamma^0)$ (see Chapter 1, §9). Therefore,

(18) $$\begin{aligned}\det\{\widetilde{\rho}_0(\Gamma^0\gamma\Gamma^0) - d(\Gamma^0\gamma\Gamma^0)\} &= |\det\{\widetilde{\rho}_0^{\mathbf{C}}(\Gamma^0\gamma\Gamma^0) - d(\Gamma^0\gamma\Gamma^0)\}|^2 \\ &= \det\{\widetilde{\rho}_0^{\mathbf{C}}(\Gamma^0\gamma\Gamma^0) - d(\Gamma^0\gamma\Gamma^0)\}^2.\end{aligned}$$

Put $\Gamma^0\gamma\Gamma^0 = \Gamma^1$ (see Chapter 1, §10). Then (18) will be equal to

(19) $$\det\{\widetilde{\rho}_0^{\mathbf{C}}(\Gamma^1) - q^2 - q\}^2.$$

But $\widetilde{\rho}_0^{\mathbf{C}}$ is nothing but $\rho = \rho_2$ of Chapter 1. Therefore by (54) of Chapter 1, we finally get

(20) $$\det\{\Re(\Gamma^1) - d(\Gamma^1)\} = P(1)^2,$$

where $P(u) = \prod_{i=1}^{g}(1-\pi_i u)(1-\pi_i' u)$ is the main numerator of $\zeta_\Gamma(u)$. So, we have proved the following.

THEOREM 2. *Let Γ be a discrete subgroup of $G = G_{\mathbf{R}} \times G_{\mathfrak{p}}$ with compact quotient and with dense images of projections $\Gamma_{\mathbf{R}}, \Gamma_{\mathfrak{p}}$ in $G_{\mathbf{R}}, G_{\mathfrak{p}}$ respectively. Then the commutator quotient group $\Gamma/[\Gamma,\Gamma]$ is finite, and if Γ is moreover torsion-free, its group order is a divisor of $P(1)^2$, where*

$$P(u) = \prod_{i=1}^{g}(1-\pi_i u)(1-\pi_i' u)$$

is the main numerator of $\zeta_\Gamma(u)$ (see Chapter 1, §8 (20)).[3]

§7. Consequence of $H^1(\Gamma_{\mathbf{R}},\rho_1) = \{0\}$. Let Γ be as in Theorem 2 (but not assumed to be torsion-free). Then, since ρ_1 is equivalent to the adjoint representation Ad of $G_{\mathbf{R}}$ in $\mathfrak{g}_{\mathbf{R}}$ (see §3 (9)), the corollary of Theorem 1 for $n = 1$ shows that $H^1(\Gamma_{\mathbf{R}}, \mathrm{Ad}) = \{0\}$. Put $G_{\mathbf{C}} = PL_2(\mathbf{C}) \cong PSL_2(\mathbf{C})$, let $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g}_{\mathbf{R}} \otimes \mathbf{C}$ be its Lie algebra, and let $\mathrm{Ad}_{\mathbf{C}}$ be the adjoint representation of $G_{\mathbf{C}}$ in $\mathfrak{g}_{\mathbf{C}}$. Denote its restriction to $\Gamma_{\mathbf{R}}$ also by $\mathrm{Ad}_{\mathbf{C}}$. Then it follows immediately from the equality $H^1(\Gamma_{\mathbf{R}}, \mathrm{Ad}) = \{0\}$ that

(21) $$H^1(\Gamma_{\mathbf{R}}, \mathrm{Ad}_{\mathbf{C}}) = \{0\}.$$

Now, by A. Weil (A. Weil [37]), we have the following:

[3] By Chapter 1, Theorem 2, we have $\pi_i, \pi_i' \neq 1$.

Lemma 2 (A. Weil). *Let X be a real Lie group, let Δ be a finitely generated subgroup of X, and let* Ad *be the adjoint representation of X (or its restriction to Δ). Then, if $H^1(\Delta, \mathrm{Ad}) = \{0\}$, Δ has no non-trivial deformation in X.*

By applying this to $X = G_{\mathbf{C}}$ and $\Delta = \Gamma_{\mathbf{R}}$, we get the following theorem by (21):

Theorem 3. *Let Γ be as in Theorem 2 (but not necessarily torsion-free). Then $\Gamma_{\mathbf{R}}$ has no non-trivial deformation in $G_{\mathbf{C}} = PSL_2(\mathbf{C})$.*

Remark 1. Since we used only Theorem 1 (for $n = 1$) and Lemma 2 to get Theorem 3, it is clear that the triviality of deformation of $\Gamma_{\mathbf{R}}$ in $G_{\mathbf{C}}$ is valid for any subgroup $\Gamma_{\mathbf{R}}$ of $G_{\mathbf{R}}$ satisfying the following three conditions.

(i) $\Gamma_{\mathbf{R}}$ contains a discrete subgroup $\Gamma^0_{\mathbf{R}}$ of $G_{\mathbf{R}}$ with compact quotient $G_{\mathbf{R}}/\Gamma^0_{\mathbf{R}}$, and $\Gamma^0_{\mathbf{R}}$, $\gamma_{\mathbf{R}}^{-1}\Gamma^0_{\mathbf{R}}\gamma_{\mathbf{R}}$ are commensurable with each other for every $\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}$.
(ii)[4] $\Gamma_{\mathbf{R}}$ is dense in $G_{\mathbf{R}}$.
(iii) $\Gamma_{\mathbf{R}}$ is finitely generated.

Remark 2. Theorem 3 is slightly stronger than the Corollary of Lemma 8 in Chapter 2, which asserts the triviality of deformation of $\Gamma_{\mathbf{R}}$ *in $G_{\mathbf{R}}$ only.* While the proof of Lemma 8 and its corollary (Chapter 2) is quite elementary with the elliptic elements of $\Gamma_{\mathbf{R}}$ playing the main role, the proof of Theorem 3 is slightly more sophisticated, based on the inequality (89) (Kuga) of Chapter 1 for automorphic forms of weight 4, Borel's density theorem for fuchsian groups, Eichler-Shimura's isomorphism (10), and Weil's Lemma 2. While Lemma 8 was necessary and sufficient for our purpose in Chapter 2, what we now need is our Theorem 3, the triviality of deformation of $\Gamma_{\mathbf{R}}$ *in* $G_{\mathbf{C}}$.

Applications of Theorem 3 ; the deformation variety.

§8. As before and throughout the following, let Γ be a discrete subgroup of $G = G_{\mathbf{R}} \times G_{\mathfrak{p}}$ with compact quotient and with dense images of projections $\Gamma_{\mathbf{R}}, \Gamma_{\mathfrak{p}}$ in $G_{\mathbf{R}}, G_{\mathfrak{p}}$ respectively. Let $\gamma_1, \cdots, \gamma_n$ be a set of generators of Γ, and let $R_\lambda(\gamma_1, \cdots, \gamma_n) = I$ $(\lambda \in \Lambda)$ be a system of fundamental relations between $\gamma_1, \cdots, \gamma_n$. Let $G_{\mathbf{C}} = PL_2(\mathbf{C})$ be identified with a Zariski open subspace

$$\{(x^{11} : x^{12} : x^{21} : x^{22}) | x^{11}x^{22} - x^{12}x^{21} \neq 0\}$$

of the projective space $\mathbf{P}^3$. Put $G^n_{\mathbf{C}} = G_{\mathbf{C}} \times \cdots \times G_{\mathbf{C}}$ (n-copies), and let $V = V_\Gamma$ be an algebraic subset of $G^n_{\mathbf{C}}$ formed of all points $(x_1, \cdots, x_n) \in G^n_{\mathbf{C}}$ satisfying $R_\lambda(x_1, \cdots, x_n) = 1$ for all $\lambda \in \Lambda$. Then it is clear that for any homomorphism (as abstract groups) φ of Γ into $G_{\mathbf{C}}$, $(\varphi(\gamma_1), \cdots, \varphi(\gamma_n))$ lies on V; and conversely, if $(x_1, \cdots, x_n)$ is on V, then by putting $\varphi(\gamma_1) = x_1, \cdots, \varphi(\gamma_n) = x_n$, we get a homomorphism φ of Γ into $G_{\mathbf{C}}$. In this manner, points on V are in one-to-one correspondence with homomorphisms (as abstract groups)

[4] This is equivalent to $(\Gamma_{\mathbf{R}} : \Gamma^0_{\mathbf{R}}) = \infty$ (see Supplement §1).

of Γ into $G_{\mathbf{C}}$. Therefore, we shall identify them and *call V the deformation variety of* Γ *in* $G_{\mathbf{C}}$.

For any element $x \in G_{\mathbf{C}}$, we put $x = ((x^{ij}))$ $(1 \leq i, j \leq 2)$ with projective coordinates x^{ij}. Then for each $\lambda \in \Lambda$, $((R_\lambda(x_1, \cdots, x_n)^{ij})) \in G_{\mathbf{C}} \subset \mathbf{P}^3$ is well-defined, and $R_\lambda(x_1, \cdots, x_n)^{ij}$ are (multi-homogeneous) polynomials of x_k^{ij} with rational integral coefficients. Therefore, V is a bunch of algebraic varieties in $G_{\mathbf{C}}^n$ and it is *normally algebraic over* $\mathbf{Q}$. Let $\varphi_{\mathbf{R}}$ be the projection [5] $\Gamma \to \Gamma_{\mathbf{R}} \subset G_{\mathbf{R}} \subset G_{\mathbf{C}}$, and let V_0 be an irreducible component of V containing $\varphi_{\mathbf{R}}$. Then, since V is normally algebraic over $\mathbf{Q}$, V_0 is defined over $\bar{\mathbf{Q}}$, i.e., the algebraic closure of $\mathbf{Q}$. On the other hand, $G_{\mathbf{C}}$ acts on V as

$$G_{\mathbf{C}} \ni t : V \ni \varphi \mapsto \mathrm{Int}(t) \circ \varphi \in V, \tag{22}$$

where φ is a homomorphism of Γ into $G_{\mathbf{C}}$ considered as a point on V, and Int(t) denotes the inner automorphism $x \mapsto t^{-1}xt$ of $G_{\mathbf{C}}$. Since $\varphi_{\mathbf{R}}(\Gamma) = \Gamma_{\mathbf{R}}$ is dense in $G_{\mathbf{R}}$, its centralizer in $G_{\mathbf{C}}$ is $\{1\}$; hence the stabilizer of $\varphi_{\mathbf{R}} \in V_0$ in $G_{\mathbf{C}}$ is trivial. Now, by Theorem 3, there exists a neighborhood U of $\varphi_{\mathbf{R}}$ in $G_{\mathbf{C}}^n$ such that $U \cap V$ is contained in the $G_{\mathbf{C}}$-orbit of $\varphi_{\mathbf{R}}$. Therefore, V_0' denoting the $G_{\mathbf{C}}$-orbit of $\varphi_{\mathbf{R}}$, V_0' is obviously irreducible, $\dim V_0' = 3$, and $U \cap V = U \cap V_0'$. Therefore V_0 is the unique irreducible component of V containing $\varphi_{\mathbf{R}}$, $\dim V_0 = 3$, and V_0' is a Zariski dense algebraic subset of V_0. Before going into a detailed study of V_0, we shall give some simple application of this to the structure of Γ.

§9. Subgroups of Γ with finite indices. In general, an abstract group Δ is called *residually finite* if the intersection of all subgroups of Δ with finite indices is $\{1\}$, or equivalently, if the intersection of all normal subgroups of Δ with finite indices is $\{1\}$.

THEOREM 4. *Let* Γ *be a discrete subgroup of* $G = G_{\mathbf{R}} \times G_{\mathfrak{p}}$ *with compact quotient and with dense images of projections in* $G_{\mathbf{R}}$ *and* $G_{\mathfrak{p}}$. *Then*

(i) Γ *is residually finite.*
(ii) Γ *contains a subgroup with finite index which has no elements* $\neq 1$ *of finite order.*

PROOF. Since V_0' is Zariski dense in V_0 and since V_0 is defined over $\bar{\mathbf{Q}}$, there are infinitely many $\bar{\mathbf{Q}}$-rational points on V_0'. Let $(a_1, \cdots, a_n)$ be such a point, and let K be an algebraic number field such that all a_i $(1 \leq i \leq n)$ are K-rational. Let φ be the homomorphism of Γ into $G_{\mathbf{C}}$ defined by $\varphi(\gamma_i) = a_i$ $(1 \leq i \leq n)$. Then, since $(a_1, \cdots, a_n)$ lies on V_0', φ is of the form $\varphi = \mathrm{Int}(t) \circ \varphi_{\mathbf{R}}$ with some $t \in G_{\mathbf{C}}$. In particular, φ is injective. Therefore, Γ is isomorphic to a subgroup $\varphi(\Gamma)$ of $PL_2(K)$. Put $a_k = ((a_k^{ij}))$ with $a_k^{ij} \in K$ $(\forall i, j, k)$, and let $\mathfrak{l}$ be a prime ideal of K such that all a_k^{ij} and all $(a_k^{11}a_k^{22} - a_k^{12}a_k^{21})^{-1}$ are $\mathfrak{l}$-integral. Now denote by $O_{\mathfrak{l}}$ the $\mathfrak{l}$-adic completion of the ring of integers of K. Then, since $a_1, \cdots, a_n$ generate $\varphi(\Gamma)$, this shows that $\varphi(\Gamma)$ can be considered as a subgroup of $PL_2(O_{\mathfrak{l}})$; therefore,

$$\Gamma \cong \text{ a subgroup of } PL_2(O_{\mathfrak{l}}). \tag{23}$$

Now the residual finiteness of Γ follows immediately from that of $PL_2(O_{\mathfrak{l}})$ (take congruence subgroups!). This settles (i). Finally, it is well-known (and easy to prove) that $O_{\mathfrak{l}}$

[5] $\varphi_{\mathbf{R}}$ is injective (see Chapter 1, §2, Proposition 1).

being the ring of I-adic integers of any I-adic number field, there exists some n such that the congruence subgroup

$$\{x \in SL_2(O_I) | x \equiv \pm 1 \pmod{I^n}\} / \pm 1$$

is torsion-free. This settles (ii). □

REMARK . In the proof of Theorem 4, we only used the fact that small deformation of $\Gamma_{\mathbf{R}}$ in $G_{\mathbf{C}}$ is *injective*. This is, of course, a consequence of Theorem 3, but it is much weaker than Theorem 3 and can be proved much more easily.

Study of V_0; the field $F = \mathbf{Q}((\operatorname{tr}\gamma_{\mathbf{R}})^2 | \gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}})$.

§10.

PROPOSITION 2. *The notations being as in §8, we have $V_0 = V_0'$.*

PROOF. We have shown in §8 that V_0' is Zariski dense in V_0. Therefore, it is enough to prove that V_0' is closed in $G_{\mathbf{C}}^n$. For each $t \in G_{\mathbf{C}}$, put

$$x_t = (t^{-1}\gamma_{1,\mathbf{R}}t, \cdots, t^{-1}\gamma_{n,\mathbf{R}}t) \in G_{\mathbf{C}}^n. \tag{24}$$

Then V_0' is the set of all x_t with $t \in G_{\mathbf{C}}$, and the map $t \mapsto x_t$ is one-to-one (see §8). Now let $\mathbf{C}[\Gamma_{\mathbf{R}}]$ be the subalgebra of $M_2(\mathbf{C})$ generated by $\gamma_{1,\mathbf{R}}, \cdots, \gamma_{n,\mathbf{R}}$ over $\mathbf{C}$. Then $\mathbf{C}[\Gamma_{\mathbf{R}}] \supset \pm\Gamma_{\mathbf{R}}$; hence $\mathbf{C}[\Gamma_{\mathbf{R}}] \supset SL_2(\mathbf{R})$; hence we get $\mathbf{C}[\Gamma_{\mathbf{R}}] = M_2(\mathbf{C})$. In particular, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are contained in $\mathbf{C}[\Gamma_{\mathbf{R}}]$. Put $t = ((t_{ij})) \in PSL_2(\mathbf{C}) \cong G_{\mathbf{C}}$, and suppose that $t^{-1}\gamma_{1,\mathbf{R}}t, \cdots, t^{-1}\gamma_{n,\mathbf{R}}t$ are contained in a given compact subset of $G_{\mathbf{C}}$. Then

$$t^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t = \begin{pmatrix} * & t_{22}^2 \\ -t_{21}^2 & * \end{pmatrix} \text{ and } t^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} t = \begin{pmatrix} * & -t_{12}^2 \\ t_{11}^2 & * \end{pmatrix}$$

must also be contained in some compact subset of $M_2(\mathbf{C})$; hence all t_{ij} $(1 \le i, j \le 2)$ must be contained in some compact subset of $\mathbf{C}$. Therefore, the intersection of V_0' with any given compact subset of $G_{\mathbf{C}}^n$ is contained in the image (by $t \mapsto x_t$) of some compact subset of $G_{\mathbf{C}}$. But this implies that V_0' is closed in $G_{\mathbf{C}}^n$, since the map $t \mapsto x_t$ is continuous and $G_{\mathbf{C}}^n$ is locally compact. □

COROLLARY . *V_0 is the connected component of V containing $\varphi_{\mathbf{R}}$.*

PROOF. Since V_0 is irreducible, it is connected. Therefore, it is enough to show that if V_1 is any irreducible component of V with $V_0 \cap V_1 \neq \phi$, then $V_1 = V_0$. Let V_1 be such an irreducible component, and let $\varphi \in V_0 \cap V_1$. Then since $V_0 = V_0'$, there is an element $t \in G_{\mathbf{C}}$, such that $\varphi_{\mathbf{R}} = \operatorname{Int}(t) \circ \varphi$. But then, $\varphi_{\mathbf{R}}$ is contained in $\operatorname{Int}(t) \circ V_1$, which is also an irreducible component of V. But we know that V_0 is the unique irreducible component of V containing $\varphi_{\mathbf{R}}$. Therefore, $\operatorname{Int}(t) \circ V_1 = V_0$; hence $V_1 = \operatorname{Int}(t^{-1})V_0 = V_0$. □

§11. A field of definition of V_0. Let $\Gamma_{\mathbf{R}}^{(e)}$ be the set of all elements $\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}$ which are elliptic (i.e., $|\operatorname{tr}\gamma_{\mathbf{R}}| < 2$) and of infinite order. Put

$$(25) \qquad F = \mathbf{Q}((\operatorname{tr}\gamma_{\mathbf{R}})^2 | \gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}) \supset F_0 = \mathbf{Q}((\operatorname{tr}\gamma_{\mathbf{R}})^2 | \gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}^{(e)}).$$

PROPOSITION 3. *Let k ($\subset \mathbf{C}$) be a field of definition of V_0. Then $F \subset k$.*

PROOF. Let σ be any automorphism of $\mathbf{C}$ which is trivial on k. Then $V_0^\sigma = V_0$. Therefore, the homomorphism $\Gamma \ni \gamma \mapsto \gamma_{\mathbf{R}}^\sigma \in G_{\mathbf{C}}$ is conjugate (in $G_{\mathbf{C}}$) to $\varphi_{\mathbf{R}}$. Hence there exists $t \in G_{\mathbf{C}}$ such that $\gamma_{\mathbf{R}}^\sigma = t^{-1}\gamma_{\mathbf{R}}t$ for all $\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}$. By taking traces of both sides, which are well-defined up to the signs, we get $\pm\operatorname{tr}(\gamma_{\mathbf{R}}^\sigma) = \pm\operatorname{tr}\gamma_{\mathbf{R}}$; hence $\pm\operatorname{tr}(\gamma_{\mathbf{R}})^\sigma = \pm\operatorname{tr}\gamma_{\mathbf{R}}$ for all $\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}$. Therefore, $(\operatorname{tr}\gamma_{\mathbf{R}})^2$ is σ-invariant for any $\sigma \in \operatorname{Aut}_k(\mathbf{C})$. Therefore, $(\operatorname{tr}\gamma_{\mathbf{R}})^2 \in k$ for any $\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}$, which implies $F \subset k$. □

COROLLARY. *The fields F, F_0 are algebraic number fields.*

PROOF. Since V_0 is defined over $\bar{\mathbf{Q}}$ (see §8), we can take k to be an algebraic number field. □

PROPOSITION 4. *V_0 is defined over F_0.*

PROOF. To begin with, we shall prove that $\Gamma_{\mathbf{R}}^{(e)}$ generates $\Gamma_{\mathbf{R}}$. By a remark given in Chapter 1 (§3), the set

$$S = \{\operatorname{tr}\gamma_{\mathbf{R}} \mid \gamma_{\mathbf{R}} : \text{ of finite order}\}$$

is finite. Therefore, if we put

$$X = \{x \in G_{\mathbf{R}} \mid |\operatorname{tr} x| < 2, \operatorname{tr} x \notin S\},$$

then X is an open subset of $G_{\mathbf{R}}$ satisfying $X^{-1} = X$ and $\Gamma_{\mathbf{R}} \cap X = \Gamma_{\mathbf{R}}^{(e)}$. Moreover, since $G_{\mathbf{R}}$ is connected, X generates $G_{\mathbf{R}}$ (as abstract group). Now let $\gamma_{\mathbf{R}}$ be an arbitrary element of $\Gamma_{\mathbf{R}}$, and put $\gamma_{\mathbf{R}} = x_1 \cdots x_n$ with $x_i \in X$ $(1 \le i \le n)$. For each i $(1 \le i \le n-1)$, let $\gamma_{\mathbf{R}}^{(i)} \in \Gamma_{\mathbf{R}}$ be sufficiently near x_i. Then $\gamma_{\mathbf{R}}^{(i)} \in \Gamma_{\mathbf{R}} \cap X = \Gamma_{\mathbf{R}}^{(e)}$ for $1 \le i \le n-1$, and moreover $(\gamma_{\mathbf{R}}^{(1)} \cdots \gamma_R^{(n-1)})^{-1}\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}$ is sufficiently near x_n; hence it is contained in $\Gamma_{\mathbf{R}} \cap X = \Gamma_{\mathbf{R}}^{(e)}$. Therefore, we have $\gamma_{\mathbf{R}} = \gamma_{\mathbf{R}}^{(1)} \cdots \gamma_{\mathbf{R}}^{(n)}$ with $\gamma_{\mathbf{R}}^{(i)} \in \Gamma_{\mathbf{R}}^{(e)}$ for all i $(1 \le i \le n)$. Hence $\Gamma_{\mathbf{R}}^{(e)}$ generates $\Gamma_{\mathbf{R}}$.

Now, let σ be an automorphism of $\mathbf{C}$ which is trivial on F_0. Since V_0 can be considered as the set of all homomorphisms φ_t of $\Gamma_{\mathbf{R}}$ into $G_{\mathbf{C}}$ given by $\varphi_t(\gamma_{\mathbf{R}}) = t^{-1}\gamma_{\mathbf{R}}t$ (with $t \in G_{\mathbf{C}}$), it is clear that V_0^σ can be considered as the set of all homomorphisms φ_t^σ of $\Gamma_{\mathbf{R}}$ into $G_{\mathbf{C}}$ given by $\varphi_t^\sigma(\gamma_{\mathbf{R}}) = \varphi_t(\gamma_{\mathbf{R}})^\sigma = (t^\sigma)^{-1}\gamma_{\mathbf{R}}^\sigma t^\sigma$.

Let φ and φ' be, at the moment, arbitrary elements of V_0 and V_0^σ respectively, and identify $G_{\mathbf{C}} = PL_2(\mathbf{C})$ with $PSL_2(\mathbf{C})$. Then $\operatorname{tr}\varphi(\gamma_{\mathbf{R}}), \operatorname{tr}\varphi'(\gamma_{\mathbf{R}}) \in \mathbf{R}$ are well-defined up to the signs, and since σ is trivial on F_0, we have

$$|\operatorname{tr}(\gamma_{\mathbf{R}}^\sigma)| = |(\operatorname{tr}\gamma_{\mathbf{R}})^\sigma| = |\operatorname{tr}\gamma_{\mathbf{R}}| \quad \text{for any } \gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}^{(e)}.$$

Therefore, *we have $|\operatorname{tr}\varphi'(\gamma_{\mathbf{R}})| = |\operatorname{tr}\varphi(\gamma_{\mathbf{R}})|$ for any $\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}^{(e)}$.* Now, fix any $\delta_{\mathbf{R}} \in \Gamma_{\mathbf{R}}^{(e)}$, and let $\pm(\varepsilon, \varepsilon^{-1})$ be the eigenvalues of $\delta_{\mathbf{R}}$. Then, since $\delta_{\mathbf{R}}^\sigma$ and $\delta_{\mathbf{R}}$ have the same traces,

$\pm(\varepsilon, \varepsilon^{-1})$ are also the eigenvalues of $\delta_{\mathbf{R}}^{\sigma}$. Therefore, there exist $t, t' \in G_{\mathbf{C}}$ such that

$$t^{-1}\delta_{\mathbf{R}}t = t'^{-1}\delta_{\mathbf{R}}^{\sigma}t' = \pm\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}.$$

Therefore, by putting $\varphi = \varphi_t \in V_0$ and $\varphi' = \varphi^{\sigma}_{t'\sigma^{-1}} \in V_0^{\sigma}$, we get $\varphi(\delta_{\mathbf{R}}) = \varphi'(\delta_{\mathbf{R}}) = \pm\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}$. Now, let $\gamma_{\mathbf{R}}$ be any element of $\Gamma_{\mathbf{R}}^{(e)}$, and put

$$\varphi(\gamma_{\mathbf{R}}) = \pm\begin{pmatrix}x & y\\ z & w\end{pmatrix}, \varphi'(\gamma_{\mathbf{R}}) = \pm\begin{pmatrix}x' & y'\\ z' & w'\end{pmatrix}$$

with $x + w = x' + w'$. Take an integer $n \neq 0$ such that ε^n is sufficiently near 1. Then $\delta_{\mathbf{R}}^n$ is sufficiently near 1; hence $\gamma_{\mathbf{R}} \cdot \delta_{\mathbf{R}}^n$ is also contained in $\Gamma_{\mathbf{R}}^{(e)}$ (since $\Gamma_{\mathbf{R}}^{(e)} = \Gamma_{\mathbf{R}} \cap X$ and X is open). Therefore, $|\operatorname{tr}\varphi(\gamma_{\mathbf{R}}\delta_{\mathbf{R}}^n)| = |\operatorname{tr}\varphi'(\gamma_{\mathbf{R}}\delta_{\mathbf{R}}^n)|$; hence $x\varepsilon^n + w\varepsilon^{-n} = \pm(x'\varepsilon^n + w'\varepsilon^{-n})$. If $x + w = x' + w' \neq 0$ and if ε^n is still nearer 1, then $x\varepsilon^n + w\varepsilon^{-n}, x'\varepsilon^n + w'\varepsilon^{-n}$ are sufficiently near $x + w = x' + w' \neq 0$; therefore, we have $x\varepsilon^n + w\varepsilon^{-n} = x'\varepsilon^n + w'\varepsilon^{-n}$. If on the other hand, $x + w = x' + w' = 0$, then we can replace $\begin{pmatrix}x & y\\ z & w\end{pmatrix}$ by $-\begin{pmatrix}x & y\\ z & w\end{pmatrix}$ if necessary and assume that $x\varepsilon^n + w\varepsilon^{-n} = x'\varepsilon^n + w'\varepsilon^{-n}$. Now, by the two equations $x + w = x' + w'$ and $x\varepsilon^n + w\varepsilon^{-n} = x'\varepsilon^n + w'\varepsilon^{-n}$, we get $x = x'$ and $w = w'$. *Therefore, if $\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}^{(e)}$, we can put*

$$\varphi(\gamma_{\mathbf{R}}) = \pm\begin{pmatrix}x & y\\ z & w\end{pmatrix}, \varphi'(\gamma_{\mathbf{R}}) = \pm\begin{pmatrix}x & y'\\ z' & w\end{pmatrix}.$$

Now fix another element $\delta'_{\mathbf{R}} \in \Gamma_{\mathbf{R}}^{(e)}$ such that

$$\varphi(\delta'_{\mathbf{R}}) = \pm\begin{pmatrix}a & b\\ c & d\end{pmatrix}, \text{ and } \varphi'(\delta'_{\mathbf{R}}) = \pm\begin{pmatrix}a & b'\\ c' & d\end{pmatrix} \textit{with } ad \neq 1.$$

It is clear that such $\delta'_{\mathbf{R}}$ exists, since $\Gamma_{\mathbf{R}}^{(e)} = \Gamma_{\mathbf{R}} \cap X$ and X is open in $G_{\mathbf{R}}$. Since their determinants are 1, we have $bc, b'c' \neq 0$. Hence we have $\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \rho^{-1}\begin{pmatrix}a & b'\\ c' & d\end{pmatrix}\rho$, with $\rho = \begin{pmatrix}b' & 0\\ 0 & b\end{pmatrix}$; hence if we put $\varphi'' = \operatorname{Int}(\rho) \circ \varphi' \in V_0^{\sigma}$, we get

$$(26)\quad \begin{aligned}\varphi(\delta_{\mathbf{R}}) &= \varphi''(\delta_{\mathbf{R}}) = \pm\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix} \text{ and}\\ \varphi(\delta'_{\mathbf{R}}) &= \varphi''(\delta'_{\mathbf{R}}) = \pm\begin{pmatrix}a & b\\ c & d\end{pmatrix}, \ ad \neq 1.\end{aligned}$$

(Note that ρ commutes with $\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}$.) Now we shall prove that $\varphi = \varphi''$. First, let $\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}^{(e)}$, and put

$$\varphi(\gamma_{\mathbf{R}}) = \pm\begin{pmatrix}x & y\\ z & w\end{pmatrix}, \ \varphi''(\gamma_{\mathbf{R}}) = \pm\begin{pmatrix}x & y'\\ z' & w\end{pmatrix}.$$

For each integer $n \neq 0$, put $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$, and take n such that this matrix is sufficiently near 1 (recall that $\delta'_{\mathbf{R}}$ is in $\Gamma^{(e)}_{\mathbf{R}}$ and hence this is possible). Then $\delta'^{\,n}_{\mathbf{R}} \cdot \gamma_{\mathbf{R}}$ is contained in $\Gamma^{(e)}_{\mathbf{R}}$. Therefore, if the signs of matrices are suitably chosen, the two matrices $\varphi(\delta'^{\,n}_{\mathbf{R}}\gamma_{\mathbf{R}})$ and $\varphi'(\delta'^{\,n}_{\mathbf{R}}\gamma_{\mathbf{R}})$ must have the common diagonal components. Thus, by applying the similar arguments as before on the signs of matrices (and by changing the sign of $\begin{pmatrix} x & y' \\ z' & w \end{pmatrix}$ if $x = w = 0$ and if necessary), we get $a_n x + b_n z = a_n x + b_n z'$ and $c_n y + d_n w = c_n y' + d_n w$; hence

$$b_n(z - z') = c_n(y - y') = 0. \tag{27}$$

But we have $b_n c_n \neq 0$. In fact, since the centralizer of $\delta'_{\mathbf{R}}$ in $G_{\mathbf{R}}$ is isomorphic to $\mathbf{R}/\mathbf{Z}$, it is topologically generated by any one power $\delta'^{\,n}_{\mathbf{R}}$ ($n \neq 0$) of $\delta'_{\mathbf{R}}$. Moreover, φ is induced by an inner automorphism of $G_{\mathbf{C}}$ and hence is continuous. Therefore, if $n \neq 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be approximated by the powers of $\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. Therefore, $b_n = 0$ ($n \neq 0$) implies $b = 0$, which is a contradiction to $ad \neq 1$. Therefore, we get $b_n \neq 0$, and in the same manner we get $c_n \neq 0$. Therefore by (27), we get $z = z'$ and $y = y'$. Therefore, $\varphi(\gamma_{\mathbf{R}}) = \varphi''(\gamma_{\mathbf{R}})$ holds for all $\gamma_{\mathbf{R}} \in \Gamma^{(e)}_{\mathbf{R}}$. But since $\Gamma^{(e)}_{\mathbf{R}}$ generates $\Gamma_{\mathbf{R}}$, $\varphi(\gamma_{\mathbf{R}}) = \varphi''(\gamma_{\mathbf{R}})$ holds for all $\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}$; hence we get $\varphi = \varphi''$. Since $\varphi \in V_0$ and $\varphi'' \in V_0^{\sigma}$, this implies $V_0 \cap V_0^{\sigma} \neq \phi$. But V_0, V_0^{σ} are $G_{\mathbf{C}}$-orbits of any one element of each. Therefore $V_0^{\sigma} = V_0$. Therefore, $V_0^{\sigma} = V_0$ holds for all $\sigma \in \mathrm{Aut}(\mathbf{C})$ which are trivial on F_0. Hence V_0 is defined over F_0. □

Corollary. *We have $F = F_0$; and it is the smallest field of definition of V_0.*

Proof. Since V_0 is defined over F_0 (Proposition 4), Proposition 3 shows $F_0 \supset F$. But $F_0 \subset F$. Therefore, $F_0 = F$. By Proposition 3, if k is a field of definition of V_0, then $k \supset F = F_0$. Therefore, F_0 is the smallest field of definition of V_0. □

§12. V_0 as a principal homogeneous space. Let k be a field of definition of V_0. Then, since $G_{\mathbf{C}} = PL_2(\mathbf{C})$ acts on V_0 in a simply transitive manner and since its action is defined over k, we can regard V_0 as a principal homogeneous space of PL_2 defined over k. Let A_k be the quaternion algebra over k which corresponds to this principal homogeneous space.[6] Then, for any field $K \supset k$, V_0 has a K-rational point if and only if $A_k \otimes_k K \cong M_2(K)$. In particular, let $k = F$ ($= F_0$), and put $A = A_F$. Then A is a quaternion algebra over F, and $A_k = A \otimes_F k$ holds for any field of definition k for V_0 (i.e., for any $k \supset F$). We shall call this *A the quaternion algebra attached to* Γ. Note that if K is a subfield of $\mathbf{C}$ such that V_0 has a K-rational point, then K contains F. In fact, that implies $t^{-1}\Gamma_{\mathbf{R}}t \subset PL_2(K)$ for some $t \in G_{\mathbf{C}}$. Therefore, if $\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}$, we can put $t^{-1}\gamma_{\mathbf{R}}t = \rho \cdot ((a_{ij}))$ with $\rho \in \mathbf{C}^{\times}$, $\forall a_{ij} \in K$.

[6] Cf. e.g. **[34]** for the one-to-one correspondence; principal homogeneous space of PL_n over $k \Leftrightarrow$ central simple algebra of degree n^2 over k.

Taking (trace)^2/determinant of both sides, we get

$$(\mathrm{tr}\,\gamma_{\mathbf{R}})^2 = (\mathrm{tr}((a_{ij})))^2/\det((a_{ij})) \in K.$$

Therefore, $(\mathrm{tr}\,\gamma_{\mathbf{R}})^2 \in K$ holds for all $\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}$; hence $F \subset K$. Therefore, for any subfield K of $\mathbf{C}$, V_0 has a K-rational point if and only if $K \supset F$ and $A \otimes_F K \cong M_2(K)$. By summarizing our results in §8, §10 ~ §12, we get the following Theorem.

THEOREM 5. *Let Γ be a discrete subgroup of $G = G_{\mathbf{R}} \times G_{\mathfrak{p}}$ with compact quotient and with dense projection images $\Gamma_{\mathbf{R}}, \Gamma_{\mathfrak{p}}$ in $G_{\mathbf{R}}, G_{\mathfrak{p}}$ respectively. Let V be the deformation variety of Γ in $G_{\mathbf{C}} = PL_2(\mathbf{C})$ (see §8), and let V_0 be an irreducible component of V, containing the projection map $\varphi_{\mathbf{R}} : \Gamma \to \Gamma_{\mathbf{R}}$. Then V_0 is unique and coincides with the $G_{\mathbf{C}}$-orbit of $\varphi_{\mathbf{R}}$. Moreover, if we put $F = \mathbf{Q}((\mathrm{tr}\,\gamma_{\mathbf{R}})^2 | \gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}})$, then F is an algebraic number field, and it is the smallest field of definition of V_0. Finally, let A be the quaternion algebra over F attached to Γ (see §12). Then for any subfield K of $\mathbf{C}$, V_0 has K-rational point (i.e., $\Gamma_{\mathbf{R}}$ can be realized in $PL_2(K)$) if and only if $K \supset F$ and $A \otimes_F K \cong M_2(K)$.*

Examples will be given in Chapter 4, Part 1.

§13. More about F and A. Throughout the following, we shall denote by Γ^* the intersection of all normal subgroups Γ' of Γ whose quotients Γ/Γ' are finite and of type $(2, 2, \cdots, 2)$. Then Γ^* contains the commutator subgroup $[\Gamma, \Gamma]$ of Γ, and by Theorem 2, $[\Gamma, \Gamma]$ is of finite index in Γ. Therefore, Γ^* is also of finite index in Γ.

PROPOSITION 5. *Let Γ be as in Theorem 5, and let Γ' be a subgroup of Γ of finite index. Put $F = \mathbf{Q}((\mathrm{tr}\,\gamma_{\mathbf{R}})^2 | \gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}})$, $F' = \mathbf{Q}((\mathrm{tr}\,\gamma'_{\mathbf{R}})^2 | \gamma'_{\mathbf{R}} \in \Gamma'_{\mathbf{R}})$, and let A, A' be the quaternion algebras attached to Γ, Γ' respectively. Then $F = F'$, and A is isomorphic to A' over F. Moreover, Γ^* being as above, we have $F = \mathbf{Q}(\mathrm{tr}\,\gamma'_{\mathbf{R}} \mid \gamma'_{\mathbf{R}} \in \Gamma'_{\mathbf{R}})$ for all subgroups Γ' of Γ^* of finite indices.*

PROOF. It is clear that $F' \subset F$. Let V_0 (resp. V'_0) be the connected component of the deformation variety of Γ (resp. Γ') in $G_{\mathbf{C}}$ containing the projection map $\varphi_{\mathbf{R}} : \Gamma \to \Gamma_{\mathbf{R}}$ (resp. $\varphi'_{\mathbf{R}} : \Gamma' \to \Gamma'_{\mathbf{R}}$). We shall show that if K is a subfield of $\mathbf{C}$, then V_0 has a K-rational point if and only if V'_0 has a K-rational point. The "only if" part is trivial. To show the "if" part, suppose that V'_0 has a K-rational point. Then there exists $t \in G_{\mathbf{C}}$ such that $t^{-1}\Gamma'_{\mathbf{R}}t \subset PL_2(K)$. But since Γ' is finitely generated, the intersection $t^{-1}\Gamma'_{\mathbf{R}}t \cap PSL_2(K)$ is of finite index in $t^{-1}\Gamma'_{\mathbf{R}}t$; hence there is a *normal* subgroup $\hat{\Gamma}_{\mathbf{R}}$ of $\Gamma_{\mathbf{R}}$ of finite index such that $t^{-1}\hat{\Gamma}_{\mathbf{R}}t \subset PSL_2(K)$. Put $\Delta_{\mathbf{R}} = t^{-1}\Gamma_{\mathbf{R}}t$, $\hat{\Delta}_{\mathbf{R}} = t^{-1}\hat{\Gamma}_{\mathbf{R}}t$. Since $\widehat{\Gamma}_{\mathbf{R}}$ is dense in $G_{\mathbf{R}}$, $\widehat{\Gamma}_{\mathbf{R}}$ spans $M_2(\mathbf{C})$ over $\mathbf{C}$; hence so does $\widehat{\Delta}_{\mathbf{R}}$. But $\widehat{\Delta}_{\mathbf{R}} \subset PSL_2(K)$. Therefore $\widehat{\Delta}_{\mathbf{R}}$ spans $M_2(K)$ over K. Now let $\delta_{\mathbf{R}} \in \Delta_{\mathbf{R}}$. Then $\delta_{\mathbf{R}}^{-1}\widehat{\Delta}_{\mathbf{R}}\delta_{\mathbf{R}} = \widehat{\Delta}_{\mathbf{R}}$; hence $\delta_{\mathbf{R}}^{-1}M_2(K)\delta_{\mathbf{R}} = M_2(K)$. Therefore, $\delta_{\mathbf{R}} \in PL_2(K)$; hence $\Delta_{\mathbf{R}} \subset PL_2(K)$; hence V_0 has a K-rational point. Therefore, V_0 has a K-rational point if and only if V'_0 has a K-rational point; hence

(28) $$K \supset F,\ A \otimes_F K \cong M_2(K) \Leftrightarrow K \supset F',\ A' \otimes_{F'} K \cong M_2(K).$$

But in general, if B is a quaternion algebra over an algebraic number field k, then there are infinitely many quadratic extensions l of k which split B; i.e., $B \otimes_k l = M_2(l)$. Moreover, if B' is another quaternion algebra over k which is not isomorphic to B over k, then there exists l which splits *one of B or B' but not the other* (however, there may not exist l which splits the given B and which does not split B'). Now, these show that $F = F'$ and that $A \cong A'$ over F. In fact, by our first remark, the intersection of all K containing F (resp. F') and splitting A (resp. A') is F (resp.F'). Therefore, (28) implies $F = F'$. Also, by our second remark, we see immediately that (28) implies $A \cong A'$ over F.

Finally, let K_1, K_2 be two distinct quadratic extensions of F which split A, so that V_0 has K_i-rational points ($i = 1, 2$). Take $t_1, t_2 \in G_{\mathbf{C}}$ such that $t_i^{-1}\Gamma_{\mathbf{R}}t_i \subset PL_2(K_i)$ ($i = 1, 2$). Then, since $\Gamma_{\mathbf{R}} \cap t_i PSL_2(K_i)t_i^{-1}$ are normal subgroups of $\Gamma_{\mathbf{R}}$ whose quotients are finite and of $(2, 2, \cdots, 2)$ type, they contain $\Gamma_{\mathbf{R}}^*$. Hence $t_i^{-1}\Gamma_{\mathbf{R}}^* t_i \subset PSL_2(K_i)$ ($i = 1, 2$). Therefore, if $\gamma_{\mathbf{R}}^* \in \Gamma_{\mathbf{R}}^*$, then we have $\operatorname{tr}\gamma_{\mathbf{R}}^* \in K_1 \cap K_2 = F$. Therefore, if $\Gamma_{\mathbf{R}}'$ is a subgroup of $\Gamma_{\mathbf{R}}^*$ of finite index, then on one hand, we have $\mathbf{Q}(\operatorname{tr}\gamma_{\mathbf{R}}' | \gamma_{\mathbf{R}}' \in \Gamma_{\mathbf{R}}') \subset F$, and on the other hand (by what we have shown already), $\mathbf{Q}((\operatorname{tr}\gamma_{\mathbf{R}}')^2 | \gamma_{\mathbf{R}}' \in \Gamma_{\mathbf{R}}') = F$. Therefore, we get $\mathbf{Q}(\operatorname{tr}\gamma_{\mathbf{R}}' | \gamma_{\mathbf{R}}' \in \Gamma_{\mathbf{R}}') = F$ for all such Γ'. □

REMARK 1. The field $\mathbf{Q}(\operatorname{tr}\gamma_{\mathbf{R}} | \gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}})$ is a finite $(2, \cdots, 2)$ type extension of F, and in general, it does not coincide with F.

PROPOSITION 6. *Let Γ be as in Theorem 5, and let Γ^* be the subgroup of Γ defined at the beginning of this section. Let $A^* = \mathbf{Q}[\Gamma_{\mathbf{R}}^*]$ be the subalgebra of $M_2(\mathbf{R})$ generated over* $\mathbf{Q}$ *by $\Gamma_{\mathbf{R}}^*$. Then its center consists of all scalar matrices $a \cdot I$ with $a \in F = \mathbf{Q}((\operatorname{tr}\gamma_{\mathbf{R}})^2 | \gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}})$, and A^* is isomorphic over F to the quaternion algebra A attached to Γ. Moreover, if $(A^*)^\times/F^\times$ is considered as a subgroup of $PL_2(\mathbf{R})$, then $\Gamma_{\mathbf{R}}$ is contained in $(A^*)^\times/F^\times$.*

PROOF. Remark that, by Proposition 5, we have $F = \mathbf{Q}(\operatorname{tr}\gamma_{\mathbf{R}}^* | \gamma_{\mathbf{R}}^* \in \Gamma_{\mathbf{R}}^*)$. Let F^* be the center of A^*. Then, since

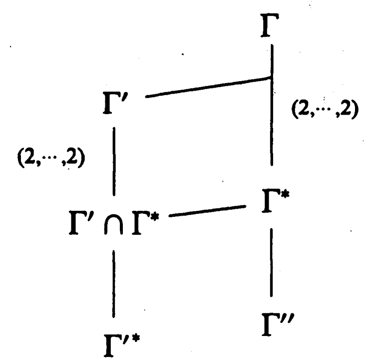
$$A^* \ni \gamma_{\mathbf{R}}^* + \gamma_{\mathbf{R}}^{*\,-1} = (\operatorname{tr}\gamma_{\mathbf{R}}^*) \cdot I$$

for all $\gamma_{\mathbf{R}}^* \in \Gamma_{\mathbf{R}}^*$, F^* contains all scalar matrices $a \cdot I$ with $a \in F$. On the other hand, since $\Gamma_{\mathbf{R}}^*$ is dense in $G_{\mathbf{R}}$, elements of F^* must be scalar matrices. So, let $a^* \cdot I \in F^*$. Then it is a linear combination over $\mathbf{Q}$ of elements of $\Gamma_{\mathbf{R}}^*$. Therefore, its trace $2a^*$ is contained in F; hence $a^* \in F$. Therefore $F^* = \{a \cdot I | a \in F\}$. Now let $\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}$. Then $\gamma_{\mathbf{R}}^2 \in \Gamma_{\mathbf{R}}^*$; hence

$$(\operatorname{tr}\gamma_{\mathbf{R}})\gamma_{\mathbf{R}} = \gamma_{\mathbf{R}}^2 + 1 \in \mathbf{Q}[\Gamma_{\mathbf{R}}^*] = A^*.$$

Therefore if $\operatorname{tr}\gamma_{\mathbf{R}} \neq 0$, then $\gamma_{\mathbf{R}}$ is contained in the subgroup $(A^*)^\times/F^\times$ *of* $PL_2(\mathbf{R})$ (this does not mean that $\gamma_{\mathbf{R}}$ is contained in $(A^*)^\times/F^\times \subset GL_2(\mathbf{R})/F^\times$). But since $\Gamma_{\mathbf{R}}$ is dense in $G_{\mathbf{R}}$ and the set $\{g_{\mathbf{R}} \in G_{\mathbf{R}} | \operatorname{tr} g_{\mathbf{R}} \neq 0\}$ is open in $G_{\mathbf{R}}$, it is clear that $\Gamma_{\mathbf{R}}$ is generated by elements with non-vanishing traces. Therefore we get $\Gamma_{\mathbf{R}} \subset (A^*)^\times/F^\times \subset PL_2(\mathbf{R})$.

Finally, we shall show that A^* is isomorphic to A over F. For this purpose, let K be any field with $K \supset F$ and $A \otimes_F K \cong M_2(K)$. Then there exists $t \in G_{\mathbf{C}}$ such that $t^{-1}\Gamma_{\mathbf{R}}t \subset PL_2(K)$. Since $\Gamma_{\mathbf{R}} \cap tPSL_2(K)t^{-1}$ is a normal subgroup of $\Gamma_{\mathbf{R}}$ with finite $(2, \cdots, 2)$ type quotient, it contains $\Gamma_{\mathbf{R}}^*$; hence $t^{-1}\Gamma_{\mathbf{R}}^* t \subset PSL_2(K)$. Therefore, $x \mapsto t^{-1}xt$ gives an isomorphism over F of A^* into $M_2(K)$. Now $\Gamma_{\mathbf{R}}^*$ contains four elements that are linearly independent over $\mathbf{R}$, and since $\Gamma_{\mathbf{R}}^* \subset PSL_2(\mathbf{R})$, they are also linearly independent over $\mathbf{C}$. Therefore, $t^{-1}\Gamma_{\mathbf{R}}^* t$ contains four elements which are linearly independent over K. Therefore, $t^{-1}A^*t \otimes_F K = M_2(K)$; hence $A^* \otimes_F K \cong M_2(K)$ over F. In particular, A^* is a quaternion algebra over F. Conversely, if K is a field with $K \supset F$ and $A^* \otimes_F K \cong M_2(K)$, then there is an isomorphism φ of A^* into $M_2(K)$ over F; and since φ is trivial on the center F, it is induced by some inner automorphism $\mathrm{Int}(t)$ of $G_{\mathbf{C}}$; hence $t^{-1}A^*t \subset M_2(K)$. Now since $\Gamma_{\mathbf{R}} \subset (A^*)^\times/F^\times \subset PL_2(\mathbf{R})$, we get $t^{-1}\Gamma_{\mathbf{R}}t \subset PL_2(K)$; hence K splits A. Therefore, A^* is isomorphic to A over F. □



REMARK 2. In general, $\mathbf{Q}[\Gamma_{\mathbf{R}}]$ will not give A. It gives $A \otimes_F \mathbf{Q}(\mathrm{tr}\,\gamma_{\mathbf{R}} | \gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}})$.

COROLLARY. *The notations being as in Proposition 6, let Γ' be a subgroup of Γ of finite index. Then* $\mathbf{Q}[\Gamma'^*_{\mathbf{R}}] = \mathbf{Q}[\Gamma^*_{\mathbf{R}}]$. *Moreover,* $\mathbf{Q}[\Gamma^*_{\mathbf{R}}] = \mathbf{Q}[\Gamma''_{\mathbf{R}}]$ *holds for all subgroups Γ'' of Γ^* of finite indices.*

PROOF. Since $\Gamma'/\Gamma' \cap \Gamma^*$ is of type $(2, \cdots, 2)$, Γ'^* is contained in $\Gamma' \cap \Gamma^*$; hence in Γ^*. Therefore, $\mathbf{Q}[\Gamma'^*_{\mathbf{R}}] \subset \mathbf{Q}[\Gamma^*_{\mathbf{R}}]$. But their centers are $\mathbf{Q}((\mathrm{tr}\,\gamma'_{\mathbf{R}})^2 | \gamma'_{\mathbf{R}} \in \Gamma'_{\mathbf{R}})$ and $\mathbf{Q}((\mathrm{tr}\,\gamma_{\mathbf{R}})^2 | \gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}})$ respectively, and they are equal by Proposition 5. Hence if we denote the common center by F, we have

$$F \subset \mathbf{Q}[\Gamma'^*_{\mathbf{R}}] \subset \mathbf{Q}[\Gamma^*_{\mathbf{R}}] \text{ and } [\mathbf{Q}[\Gamma^*_{\mathbf{R}}] : F] = [\mathbf{Q}[\Gamma'^*_{\mathbf{R}}] : F] = 4.$$

Therefore, $\mathbf{Q}[\Gamma^*_{\mathbf{R}}] = \mathbf{Q}[\Gamma'^*_{\mathbf{R}}]$.

Now, we have $\mathbf{Q}[\Gamma''^*_{\mathbf{R}}] \subset \mathbf{Q}[\Gamma''_{\mathbf{R}}] \subset \mathbf{Q}[\Gamma^*_{\mathbf{R}}]$, and $\mathbf{Q}[\Gamma''^*_{\mathbf{R}}] = \mathbf{Q}[\Gamma^*_{\mathbf{R}}]$; hence $\mathbf{Q}[\Gamma''_{\mathbf{R}}] = \mathbf{Q}[\Gamma^*_{\mathbf{R}}]$. □

§14. A remark on F. The following simple remark is needed in Chapter 2, §36. Let F, F_0 be as in §11. We have shown that $F = F_0$ and that it is an algebraic number field. Here, we note that $F = F_0$ holds without the compactness assumption for the quotient G/Γ. (We do not even need the finiteness of volume of G/Γ.) In fact, in our proof of Proposition 4, we have proved that *if $\sigma \in \mathrm{Aut}_{F_0}\mathbf{C}$, then the homomorphism $\Gamma_{\mathbf{R}} \ni \gamma_{\mathbf{R}} \mapsto \gamma^\sigma_{\mathbf{R}} \in G_{\mathbf{C}}$ is induced by an inner automorphism of $G_{\mathbf{C}}$;* and the only properties of $\Gamma_{\mathbf{R}}$ we used in the proof[7] of this assertion are

(i) $\Gamma_{\mathbf{R}}$ is dense in $G_{\mathbf{R}}$, and

(ii) the set $S = \{\mathrm{tr}\,\gamma_{\mathbf{R}} | \gamma_{\mathbf{R}}$ is of finite order$\}$ is finite.

Since on one hand, these properties are satisfied by the projection $\Gamma_{\mathbf{R}}$ (to $G_{\mathbf{R}}$) of any discrete subgroup Γ of $G = G_{\mathbf{R}} \times G_{\mathfrak{p}}$ having a dense image of projection in each component of G (see Chapter 1, §3 for the property (ii)), and on the other hand, the above italicized assertion implies $F = F_0$ at once, it follows that:

If Γ is a discrete subgroup of $G = G_{\mathbf{R}} \times G_{\mathfrak{p}}$ having a dense image of projection in each component of G, then $F = F_0$ holds for such a Γ.

However, we do not know at present whether $F = F_0$ is an algebraic number field in such a general case.

[7] We made use of the language of deformation varieties, but as can be immediately seen, it has nothing to do with the proof.