Introduction to Part 3A and Part 3B.[19]

Here, we shall give only rough ideas of problems and results. For the precise formulation of our results, see the main text.

[Indication] To approach e.g., Theorem 10 (§45 [3]), which is one of our main results, the readers are requested to read §37 and §38 for the definition of S-operators, and then §41 [1]~[3] and §45 [1] [2] for the definition of ample fields L/k. (S-operators and ample fields are two main concepts introduced in this study, and are basic for our purpose. G_p-fields are examples of ample fields.)

The Problems . Let $\mathfrak{R}$ be a compact Riemann surface. Suppose that there are given $s + t$ $(0 \leq s, t < \infty)$ distinct points $P_1, \ldots, P_t; Q_1, \ldots, Q_s$ on $\mathfrak{R}$, and s positive integers $e_1, \ldots, e_s$ satisfying

$$\text{(i)} \qquad 2g - 2 + t + \sum_{i=1}^{s}\left(1 - \frac{1}{e_i}\right) > 0,$$

where g is the genus of $\mathfrak{R}$. Then, as is well-known, there is a unique simply connected (unbounded) covering $\widetilde{\mathfrak{R}}$ of $\mathfrak{R} \setminus \{P_1, \ldots, P_t\}$, isomorphic to the complex upper half plane $\mathfrak{H} = \{\tau \in \mathbf{C} \,|\, \operatorname{Im}\tau > 0\}$, which is unramified except at Q_i and ramified at Q_i with index e_i $(1 \leq i \leq s)$:

$$\text{(ii)} \qquad \begin{array}{ccl} \widetilde{\mathfrak{R}} & \cong & \mathfrak{H} = \{\tau \in \mathbf{C} \,|\, \operatorname{Im}\tau > 0\} \\ \downarrow & & \\ \mathfrak{R} & - & \{P_1, \cdots, P_t\}. \end{array}$$

Fix an isomorphism $\widetilde{\mathfrak{R}} \simeq \mathfrak{H}$, and consider τ as a multivalued function on $\mathfrak{R} \setminus \{P_1, \ldots, P_t\}$. Let $dx \neq 0$ be any meromorphic differential (1-form) on $\mathfrak{R}$ (which may not be exact), and put $\tau_x = \frac{d\tau}{dx}$, $\tau_{\underbrace{xx\ldots x}_{i+1}} = \frac{d}{dx}\tau_{\underbrace{xx\ldots x}_{i}}$ $(i \geq 1)$, so that $\tau_x, \tau_{xx}, \ldots$ are multivalued meromorphic functions on $\mathfrak{R} \setminus \{P_1, \ldots, P_t\}$. Put

$$\text{(iii)} \qquad A = -\frac{2\tau_x \cdot \tau_{xxx} - 3\tau_{xx}^2}{\tau_x^2}.$$

Then it is well-known (classically) that A *is a univalent meromorphic function on* $\mathfrak{R}$. Moreover, if we consider A as known, and (iii) as a differential equation for τ, then all the solutions of (iii) are $\frac{a\tau+b}{c\tau+d}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are any elements of $GL_2(\mathbf{C})$. Since $\operatorname{Aut}\mathfrak{H} = PSL_2(\mathbf{R})$, this shows that A depends only on dx, and is independent of the isomorphism $\widetilde{\mathfrak{R}} \simeq \mathfrak{H}$. So, the map

$$\text{(iv)} \qquad dx \longmapsto A$$

[19] The main contents of these Parts are published in Additional References [16]

is well-defined by

(v) $$\dot{\mathfrak{R}} = \{\mathfrak{R}\,;\, P_1,\ldots,P_t\,;\, Q_1,\ldots,Q_s\,;\, e_1,\ldots,e_s\}.$$

Note that the determination of the map (iv) is, in a sense, equivalent to the determination of the uniformization (ii) of $\dot{\mathfrak{R}}$.

For example, let $\mathfrak{R}$ be given by the algebraic function field $L = \mathbf{C}(x,y)$ with the equation $y^5 = x(x-1)$ (hence $g = 2$), and let $s = t = 0$. Then we can determine the map (iv) explicitly. For example, if dx is the differential of "this x," then we have

(vi) $$A = -\frac{24}{25}\frac{x^2-x+1}{x^2(x-1)^2}.$$

Now our problem is, in a simplest and vaguest expression, *to "algebraize" the map* (iv). Actually, this includes several mutually related problems. Some of them are:

(A) Suppose that $\dot{\mathfrak{R}}$ and dx are given algebraically (e.g., in terms of algebraic function fields or algebraic curves, and their differentials). Then is it possible to characterize the map (iv) algebraically?

(A′) Suppose that $\dot{\mathfrak{R}}$, dx are explicitly given algebraically (e.g., by hyperelliptic equations with variable parameters). Then can we give an explicit formula for (iv)?

(B) Suppose that $\dot{\mathfrak{R}}$, dx are given algebraically, defined over an arbitrary constant field k (instead of $\mathbf{C}$). Then can we give a good algebraic definition of the map (iv)?

(B′) Consider the Shimura curve corresponding to a fuchsian group given by a quaternion algebra over a totally real algebraic number field F. Let $\mathfrak{R}$ be its model (given by Shimura) defined over a classfield k over F. Let dx be rational over k. Then is A also rational over k?

The Results. Roughly speaking, our results are as follows (cf. the main text for their precise formulations):

The problem (A) Solved when $\dot{\mathfrak{R}}$ is "arithmetic" (or equivalently, "ample"). (Part 3A, §40, §41).

(A′) Solved in very special cases (Part 3A, §40).

(B) Solved in some sense when $\dot{\mathfrak{R}}$ is "arithmetic" (or "ample") and the characteristic of k is zero (Part 3B, §45).

(B′) Solved completely (and affirmatively) (Part 3B, §48).

Here, we shall give a rough idea of what we mean by "arithmetic" (or "ample"). As is well-known, $\dot{\mathfrak{R}}$ correspond to fuchsian groups Δ (i.e., discrete subgroups of $G_{\mathbf{R}} = PSL_2(\mathbf{R})$ with finite-volume quotients) in a one-to-one manner (cf. §40; Δ is the covering group of (ii)). We shall call $\dot{\mathfrak{R}}$ "arithmetic" (or "ample") if the commensurability group of Δ in $G_{\mathbf{R}}$ is dense in $G_{\mathbf{R}}$. This notion can be easily "algebraized," and it is in this sense that (B) is solved in arithmetic cases (of characteristic 0). The reason that we can solve the problems in arithmetic cases is based on a very simple principle, which can be seen in its most primitive form in §40, [5] (Proposition 12).

In Part 3A, §37, we shall define the symbol $\langle\xi,\eta\rangle$, which is an "algebraization" of the Schwarzian derivatives. In §38, we introduce the notion of S-operators, which is fundamental in our study. In §39, we define the canonical S-operator on automorphic function fields, and in §40, §41, the canonical S-operator on algebraic function fields over $\mathbf{C}$ (in the above notations, $dx \to A(dx)^2$ is the canonical S-operator), and its algebraic characterization in arithmetic (ample) cases. In §43, §44, we deal with the connection between linear differential equations of second order on $\mathfrak{R}$. In part 3B, §45, we formulate one of our main theorems (Theorem 10) on unique existence of an invariant S-operator on ample fields L/k (k: any field of characteristic 0), and give the proof in §46, §47. In §48, we shall give some applications of Theorem 10 (e.g., to the canonical S-operator on Shimura curves).

Remark . It is known classically that if Δ is a fuchsian group and if $f(\tau)$ is a meromorphic automorphic form of weight k with respect to Δ, then $\Phi(\tau) = k \cdot f(\tau)f''(\tau) - (k+1)f'(\tau)^2$ is a meromorphic automorphic form of weight $2k+4$ with respect to Δ (here, $'$ denotes the derivative w.r.t. τ). As is seen in §39, our problem of (algebraic) determination of the map (iv) is equivalent to the (algebraic) determination of the map

$$\text{(vii)} \qquad \varphi_k : \quad f(\tau)(d\tau)^{k/2} \mapsto \Phi(\tau)(d\tau)^{k+2}$$

for the case of $k = 2$. We note here that the cases of $k = 0$ (which is trivial since then $\Phi(\tau)(d\tau)^2 = -\{d(f(\tau))\}^2$) and $k = 2$ are the only fundamental cases. In fact, if we know φ_0 and φ_2, then φ_k for all other k are expressed algebraically by φ_0 and φ_2 (decompose $f(\tau)$ into the product of an automorphic function and the $\frac{1}{2}k$-th power of an automorphic form of weight 2).